See reference and note at the end.

Chapter 18

Partitioning and Iterating When Teaching and Learning Fraction Addition on Number Lines

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The Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) presented a perspective on teaching fractions that will be new for many teachers. One important difference is that the CCSSM presented a definition for fractions different from the part-whole meaning many teachers currently use with students (Norton et al., 2014). A second important difference is increased emphasis on number line representations of fractions and decreased emphasis on circle representations that are frequently found in instructional materials. These shifts will require teachers to rethink how they teach fractions and fraction arithmetic to their students.

We use the Candy Bar problem to illustrate the shift in meaning for fractions put forward by the CCSSM: Jamal and three of his friends are sharing one candy bar. Everyone gets the same amount. How much of the candy bar do Jamal's friends eat all together? Using the part-whole meaning for fractions, teachers might explain to their students that Jamal's friends eat 3 out of the 4 pieces, or $^3/_4$ of the candy bar. This meaning for fractions relies on counting the number of pieces in all and the number of pieces Jamal's friends eat. Thus, both counts are of the same thing: pieces of the candy bar.

In contrast, the CCSSM definition for fractions comes in two parts (NGA Center & CCSSO, 2010, p. 24). The first part says that just one share is $^1/_4$ of the candy bar, because all four equal-sized pieces create the whole candy bar. The second part says that 3 parts, each of which is $^1/_4$ of the candy bar, is $^3/_4$ of the candy bar. The critical difference is that the CCSSM definition first defines a unit fraction (a fraction whose numerator is 1) and then emphasizes combining copies of that unit fraction. As a consequence, the CCSSM definition makes explicit a distinction between the size of the parts, $^1/_4$ of the candy bar in this example, and the number of parts, 3. This distinction is not made as explicitly with the part-whole meaning discussed above. Attention to the size of parts is critical when comparing fractions and when performing arithmetic with fractions. A further advantage of the CCSSM definition is that it can support reasoning about improper fractions more readily than the part-whole definition: Interpreting $^5/_4$ of a candy bar as 5 out of 4 pieces does not make sense, but thinking of 5 pieces that are all the same size, $^1/_4$ of a candy bar, does, especially when fractions are represented as lengths. We explain this in the following paragraphs.

Representing fractions as lengths on number lines has not been common instructional practice in the United States but is attractive for many reasons. One reason is that using lengths that start at zero emphasizes the fact that fractions like $^1/_4$ are single numbers, not

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pairs of whole numbers like 3 and 4. A second reason is that number lines provide a flexible medium with which to represent individual fractions and to develop meanings and numerical methods for all four arithmetic operations. A third reason is that a solid understanding of number lines that includes fractions provides an important foundation for subsequent topics, such as Cartesian graphs.

Although there are reasons for using lengths on number lines in fractions instruction, many U.S. teachers know that their students have trouble making sense of number lines, and some research also documents students' difficulties with fractions on number lines. For example, Tunç-Pekkan (2015) found that a sample of 656 U.S. fourth- and fifth-grade students were much more proficient using circles and rectangles to represent fractions than using number lines: 80 percent could partition circles and rectangles to show the fractions $^3/_4$ and $^5/_6$, respectively, but only 35 percent could partition a unit interval on the number line to locate $^2/_3$. These findings may reflect U.S. students' limited opportunities to work with lengths on number lines rather than persistent difficulties they might experience after instruction focused on number lines.

Evidence that students can learn to use number lines effectively can be found in international comparisons of student achievement (e.g., Gonzales et al., 2008). In these comparisons, students from Asian countries outperform U.S. students. Furthermore, students from several Asian countries experience systematic development of length-based representations of numbers, first with whole numbers and then with fractions, beginning in early elementary grades and continuing into upper elementary grades (e.g., Tokyo Shoseki, 2006). Furthermore, recent research has demonstrated that U.S. students can make significant gains in their understandings of integers and fractions as lengths or distances on number lines (e.g., Saxe, Diakow, & Gearhart, 2013) when offered instruction specifically designed to support such understandings.

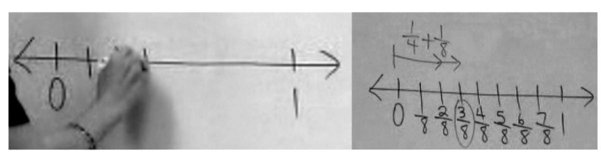
If the CCSSM emphasizes treating fractions as lengths on number lines, and students in some countries that incorporate systematic development of length-based representations attain high levels of achievement relative to students in the United States, then it is natural to ask, How can we help students represent fractions on number lines in ways consistent with the CCSSM? In this chapter, we present some key insights from a study conducted by Izsák, Tillema, and Tunç-Pekkan (2008) for answering this question. The insights highlight the importance of how teachers discuss partitioning and iterating lengths. Partitioning involves subdividing a length into equal-sized sublengths. Researchers have long considered partitioning to be a key understanding in the domain of fractions, but one that is not straightforward for students (e.g., Kieren, 1980; Pothier & Sawada, 1983). Iterating involves concatenating copies of fixed length to create longer and longer lengths. As an example, one could start with one length of $^1/_4$, join a second length of $^1/_4$ to create a length of $^2/_4$, a third to create a length of $^3/_4$, a fourth to create a length of $^4/_4$, a fifth to relate a length of $^5/_4$, and so on. Thus, partitioning and iterating lengths can be used to make sense of both proper and improper fractions. In our *JRME* article (Izsák, Tillema, & Tunç-Pekkan, 2008), we examined ways that iterating and partitioning were important to developing students' understanding of fractions as lengths on a number line in one classroom. In the remainder of this chapter we summarize the study and derive three recommendations for teachers.

Ms. Reese Teaches Fraction Addition

In our *JRME* article, we examined how one experienced sixth-grade teacher, Ms. Reese, and her students (all names of teachers and students used here are pseudonyms) interpreted lessons about fractions and fraction addition in which they participated together. Although the study predated the release of the CCSSM by several years, Ms. Reese's instruction used number lines and emphasized partitioning lengths and iterating unit fractions. The lessons were based on a draft revision of the *Bits and Pieces II* unit from the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2003). A key result from the study was that *how* Ms. Reese generated partitioned number lines in her demonstrated solutions to fraction addition problems had significant consequences for how her students made sense of the lessons.

The following solution to $^{1}/_{4} + {}^{1}/_{8} = {}^{3}/_{8}$ illustrates a consistent sequence of three steps that Ms. Reese and her students discussed for solving fraction addition problems. These three steps summarize Ms. Reese's practice for teaching fraction addition, which in many ways was consistent with the CCSSM. The first step was to determine how many whole numbers to represent. Ms. Reese and her students had worked on estimating sums of fractions in earlier lessons. In the present example, both addends were less than $^{1}/_{2}$, so their sum would be less than 1. Thus, for the current example, the interval from 0 and 1 was needed. In another case, students might estimate that a sum should be between 2 and 3. In such cases, intervals from 0 to larger whole numbers were needed. The locations of whole numbers were benchmarks that guided the subsequent location of fractions on the number line.

The second step was to partition unit intervals (intervals of length one) created in the first step. This required thought in the example of $^{1}/_{4} + {}^{1}/_{8}$, because fourths and eighths are different-sized pieces. Ms. Reese traced the interval from 0 to 1 with her finger and asked her students how to "divide up this amount." Her gestures were consistent with focusing on fractions as lengths. One student suggested a half; another suggested eighths. Ms. Reese took the second suggestion, saying that she needed "eight pieces that look about the same." She made seven tick marks from left to right (fig. 18.1a), labeling them $^{1}/_{8}$, $^{2}/_{8}$, . . . , $^{7}/_{8}$ (fig. 18.1b).



(*a*) (*b*)

Fig. 18.1. Steps 2 and 3 in Ms. Reese's demonstrated method for adding fractions on number lines: (*a*) partition unit intervals; (*b*) draw arrows for each addend and circle the answer

The third step was to draw arrows for each addend and circle the sum (fig. 18.1b). Ms. Reese told the class that she could not see $^{1}/_{4}$ on the number line, but a student pointed out that $^{2}/_{8}$ was the same. Ms. Reese agreed and drew one arrow for each addend as she spoke: "OK. So start at zero and go over to two eighths and then go over one eighth more." Notice that this explanation is based on iterating unit fractions and thus is consistent with representing fractions as lengths on number lines and with the CCSSM definition for fractions.

Across examples, Ms. Reese consistently partitioned unit intervals by adding tick marks from left to right. She was skilled at estimating spacing so that the final tick mark corresponded to the 1. In cases where her final subinterval was slightly off, Ms. Reese moved the location of the 1 to create equal-length subintervals. To illustrate, during her solution to $^{2}/_{5} + {}^{1}/_{10} = {}^{5}/_{10}$, Ms. Reese pointed out that she was moving the 1 and told students:

Ms. Reese: I started putting pieces in here just eyeballing it and trying to space them out, and I didn't have enough spaces, so I moved the one over and made another mark because I need 10 spaces. . . . This is a space [*pointing with thumb and index finger*] and you need 10 of those to be called 10th-size pieces.

Although Ms. Reese consistently counted spaces, focusing on lengths, we learned that adjusting the location of the 1 had unintentional consequences for some of her students. Thus, deciding whether or not to move the 1 is an important piece of expertise when using lengths on number lines to represent fractions for students.

How Students Interpreted Ms. Reese's Instruction

As part of the study, the first author conducted sequences of interviews with pairs of students from Ms. Reese's class. Ms. Reese helped identify students with a range of success but who were not exceptional. During the interviews, the first author asked the students to work tasks similar to those used in the *Bits and Pieces II* lessons and to interpret short excerpts of Ms. Reese's video recorded lessons.

From the interviews we learned that students had different interpretations of moving the 1 when "eyeballing" was a little off. Students who had a strong understanding of a fixed whole unit interpreted the lesson in ways similar to Ms. Reese's intentions, as just a convenient adjustment so the tick marks did not have to be erased and redrawn. Students who did not have a strong understanding of a fixed whole unit, however, interpreted the lessons in ways that Ms. Reese did not intend.

Sonya was one student who interpreted the lessons in ways Ms. Reese did not intend. Ms. Reese identified Sonya and her interview partner, Jenny, as students who were often confused. During their third interview, Sonya and Jenny represented $^{2}/_{3} + ^{3}/_{4}$ using a number line. Sonya's approach paralleled the three-step method that Ms. Reese had demonstrated several times by this point. First, Sonya drew a number line and put "0" on the left hand end, "1" in the middle, and "2" on the right hand end. Second she re-expressed $^{2}/_{3} + ^{3}/_{4}$ first as $^{4}/_{6} + ^{6}/_{8}$ and then as $^{8}/_{12} + ^{12}/_{16}$. Thus, she doubled numerators and denominators to generate correct equivalent fractions but did not make progress toward common denominators. When asked to use the denominator of 12, the students generated $^{8}/_{12}$ and $^{9}/_{12}$ quickly. Sonya put 12 tick marks from left to right between 0 and 1, and Jenny did the same between 1 and 2. Thus, the students made the common error of placing one too many tick marks in each unit interval, indicating they were counting the tick marks rather than attending to the length. Sonya proceeded to label the tick marks, resulting in a number line that showed $^{12}/_{12}$ and 1 in two separate locations. (Sonya's original 1 is the longer tick mark labeled $^{13}/_{12}$ in figure 18.2a.) Third, the students drew arrows to represent the sum, although did so incorrectly.

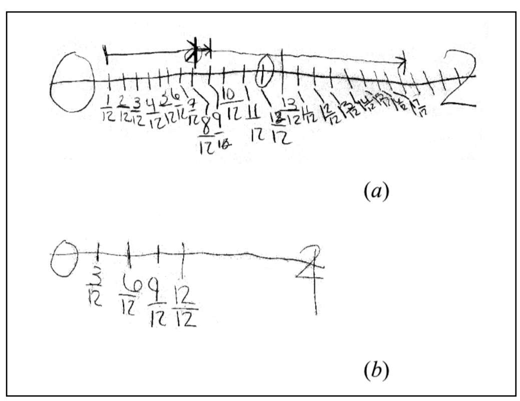


Fig. 18.2. (*a*) Sonya's number line when benchmarks are estimated locations; (*b*) Sonya's number line when benchmarks are exact locations

When the interviewer asked about the locations of 1 and $^{12}/_{12}$, Sonya explained that $^{12}/_{12}$ and 1 were the same because they were equal. The interviewer asked, "Does it make a difference if you have two different places where you've marked the same thing?" Sonya replied, "No. Because both of them equals 1" and explained, "When you draw your number line you have to put 12 of them in each space because that is what the denominator is."

The interviewer then showed lesson video in which Ms. Reese demonstrated $1/4 + 1/8$ on the number line (see fig. 18.1). Sonya noticed that Ms. Reese collocated 1 and $8/8$. (During the lesson excerpt Ms. Reese asked where 8 eighths was on the number line, and several students commented, "On the 1.") When the interviewer returned the students' attention to their work and asked again about $12/12$ and 1 being in two different places, Sonya explained:

Sonya: You are trying, on the 0, 1, and 2 [*pointed to "0," "1," and "2"*], you are trying to put where 8 over 12 and 9 over 12, *you are trying to guess* where, *like a estimate*, you trying to put where it goes here, and then when I drew these [*pointed to 12ths tick marks*], when she drew these, you are like telling, *you know where 12 over 12 is at*. [*emphasis added*]

Sonya's explanation was strikingly similar to those cases where Ms. Reese adjusted the location of the 1, but instead of erasing the original 1, the students now had two 1's. Although at earlier moments during her interviews Sonya placed equivalent fractions at the same location and connected fractions of the form *n*/*n* with the whole unit, we saw no signs that separate locations for the 1 and $12/12$ created a contradiction for her here.

From the interviews, we learned not only that Ms. Reese's adjustment of the 1 unintentionally undermined Sonya's success, but also that Sonya could be considerably more successful with a seemingly small adjustment in instruction. When the interviewer asked the students to draw a new number line thinking of the original location of the 1 as exact, not an estimate, Sonya looked much more proficient. She drew a new number line with "0," "1," and "2" labels, added a "$12/12$" label under the "1," added a tick mark for $6/12$, and explained that "half of 12 is 6" (fig. 18.2b). She then located $3/12$ by partitioning the interval between 0 and $6/12$ in half and similarly located $9/12$ by partitioning the interval between $6/12$ and $12/12$ in half. Sonya continued to fill in additional tick marks, explaining that she could add two tick marks within each fourth to locate "7, and then 8, and then 9, and then 10, 11." Here her work indicated that she was focused on partitioning the fixed length from 0 to 1 into equal pieces. In addition, she accomplished her partition in stages: She first partitioned the length from 0 to 1 in half to locate $6/12$, then partitioned each half in half to locate $3/12$ and $9/12$, and then further partitioned the resulting fourths.

Discussion

A central question that Sonya's difficulties raise is, What can teachers do to strengthen students' understanding and use of number line representations for fractions and fraction addition? We present three recommendations. The first two recommendations are tied directly to the case of Ms. Reese, Sonya, and Jenny and may be more familiar to teachers than the last. The third recommendation reflects broader considerations when using number lines to support instruction in fractions.

The first recommendation is not to change the size of the whole (i.e., the 1) after it has been established on the number line. Like Sonya, many students have an emerging understanding that the whole should not be changed when solving fraction problems. Small actions that seem inconsequential to teachers can throw off students unintentionally. Students who had a strong understanding that the whole had to remain fixed in fraction addition problems did not misinterpret Ms. Reese when she adjusted the location of the 1. Sonya, however, needed more support than other students to maintain a fixed whole when working with a number line. In particular, she benefited from explicit instruction that the location of 1 should not be changed during the solution of a problem.

A second recommendation is to have explicit discussions with students about partitioning intervals into equal-length subintervals because how to partition lengths on number lines is not self-evident to students, especially when they focus on counting tick marks instead of spaces. The case of Ms. Reese and Sonya demonstrates that how teachers produce number line representations can have significant consequences for students' attention to partitioning a fixed interval. Although Ms. Reese focused on equal-sized sublengths, Sonya interpreted left to right markings as a count of the number of tick marks (e.g., $^{12}/_{12}$ meant create 12 tick marks from left to right, not 12 equal-sized pieces). One instructional practice that can help students establish a fixed interval partitioned into sublengths is to have them partition a fixed whole on the number line and adjust the size of the pieces they create within that fixed length, rather than adjusting the whole itself. This activity focuses students' attention on the fact that the size of the whole should not change, and that the goal of the activity is to partition and make equal-sized pieces.

A third recommendation that builds on the first two, and that may be less familiar to teachers, is to provide students with experiences of partitioning a partition. By partitioning a partition, we mean first partitioning a fixed interval into a certain number of equal-sized pieces, and then partitioning each of those equal-sized pieces into still smaller equal-sized pieces. This recommendation supports the first recommendation discussed above, especially when partitioning an interval into a larger number of subintervals. Understandably, Ms. Reese had trouble "eyeballing" tenth-sized pieces when partitioning from left to right. An easier way to partition would have been to partition a fixed whole into halves first and then to partition each of those halves into five equal-sized pieces. Notice that such an approach might well have made sense to a struggling student like Sonya: She demonstrated some capacity to partition in stages when she first partitioned a whole into two equal-sized pieces, labeling her tick mark $^{6}/_{12}$, then partitioned each half into two more equal-sized pieces, labeling her new tick marks $^{3}/_{12}$ and $^{9}/_{12}$, and finally partitioning the resulting fourths (see figure 18.2b). Such activity can focus students' attention on equal-sized sublengths and provides opportunities to think about multiplication factors. To illustrate, Sonya's method creates opportunities to talk about two groups, each with six equal-sized pieces that create the whole unit ($2 \times 6 = 12$), and four groups, each with three equal-sized pieces that create the whole unit ($4 \times 3 = 12$). Connections between multiplication and partitioning can support understandings of equivalent factions, as discussed next.

By partitioning a partition students can use number lines to establish equivalent fractions (see Steffe, 2003; Steffe & Olive, 2010). To illustrate, Sonya created $^{6}/_{12}$ by partitioning the unit interval into two equal parts, and a discussion about equivalent fractions could follow—$^{1}/_{2}$ and $^{6}/_{12}$ name the same length, $^{1}/_{4}$ and $^{3}/_{12}$ name the same length, and $^{3}/_{4}$ and $^{9}/_{12}$ name the same length. In particular, students could begin by partitioning the whole into two halves and then continue by subdividing each of the halves in half again, creating four equal-sized pieces that make up the whole, or $^{1}/_{4}$. Using the fact that $4 \times 3 = 12$, students could partition each of the fourths into three equal-sized pieces. Then three of the twelfths are the same length as one of the fourths. This kind of activity and discussion can help students (a) understand how whole-number multiplication is involved in creating equivalent fractions, (b) use fixed lengths as the basis for understanding numerical notations for equivalent fractions, and (c) connect symbolic and length-based representations of number. Such understandings can form the foundation for studying subsequent topics, like multiplication with fractions, which is different than multiplication with whole numbers because it is based on equal-sized pieces that are smaller than the original whole.

The study reported in this chapter, and our recommendations that follow from it, illustrate subtle nuances that are important for successful instruction with visual representations. That is, how teachers' create and use such representations in classrooms can have significant consequences for the way that students interpret and understand instruction. These consequences can either help or hinder students' future understanding. While Ms. Reese was certainly aware of some of the issues we raised in this paper (e.g., she focused her instruction on creating equal subintervals, not on counting tick marks), she may not have been aware how entrenched these interpretations can be for some students (e.g., counting tick marks). Therefore, it is essential for teachers to engage in instructional practices that provide opportunities for such students to continue working on these issues as they develop a more solid understanding of fractions and fraction addition. Careful attention to such issues, we think, is an important part of implementing the CCSSM successfully.

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Izsák worked on the Children's Math World Project doing research on multidigit multiplication. This research was done after he left the project, but is included here with his permission because it is consistent with the Children's Math Worlds Project approach to fractions (see Fuson and Kalchman, 2002) and helpfully identifies conceptual problems students may have.