Using Math Drawings and Math Talk for Understanding Algorithms

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For more details about the CCSS-M and visual supports, please see the series of visual with audio Teaching Progressions I have made for various math domains. These can be found at karenfusonmath.com

Learning Path Teaching-Learning: Differentiating within Whole-Class Instruction by Using the Math Talk Community

Bridging for teachers and students
by coherent learning supports

Learning
Path

Phase 3: Compact methods for **fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 2: Research-based mathematically-desirable and accessible methods in the middle for **understanding and growing fluency**

Math Sense-Making
Math Structure

Math Drawings
Math Explaining

Phase 1: Students' methods elicited for **understanding** but move rapidly to Phase 2

Common Core Mathematical Practices
Used in a Math Talk Community

Math Sense-Making: Make sense and use appropriate precision 1 Make sense of problems and persevere in solving them. 6 Attend to precision.	**Math Drawings: Model and use tools** 4 Model with mathematics. 5 Use appropriate tools strategically.
Math Structure: See structure and generalize 7 Look for and make use of structure. 8 Look for and express regularity in repeated reasoning.	**Math Explaining: Reason, explain, and question** 2 Reason abstractly and quantitatively. 3 Construct viable arguments and critique the reasoning of others.

Figure 2

The Math Practices in action

A teacher asks every day:

Did I do math sense-making about math structure

using math drawings to support math explaining?

Can I do some part of this better tomorrow?

New Groups Below or Above?

$$\begin{array}{r} 58 \\ +\,36 \\ \hline {}^{1}\ \ \\ 4 \end{array}$$

$$\begin{array}{r} {}^{1}\ \ \\ 58 \\ +\,36 \\ \hline 4 \end{array}$$

The meaningful development of standard algorithms in the CCSS-M

The CCSS-M conceptual approach to computation is deeply mathematical and enables students to make sense of and use the base ten system and properties of operations powerfully. The CCSS-M focus on understanding and explaining such calculations, with the support of visual models, enables students to see mathematical structure as accessible, important, interesting, and useful.

It is crucial to use the Standards of Mathematical Practice throughout the development of computational methods.

Use general methods from the beginning

The critical area for the initial grade in which a type of multidigit computation is introduced specifies that:

Students develop, discuss, and use efficient, accurate, and generalizable methods to [+ - x ÷].

General methods that will generalize to and become standard algorithms can and should be developed, discussed, and explained initially using a visual model.

What is a standard algorithm in the CCSS-M?

The NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

a. decompose numbers into base-ten units and then carry out single-digit computations with those units using the place values to direct the place value of the resulting number; and

b. use the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

To implement a standard algorithm one uses a systematic *written method* for recording the steps of the algorithm.

There are variations in these written methods within a country, across countries, and at different times.

Criteria for emphasized written methods that should be introduced in the classroom

Variations that support and use place value correctly

Variations that make single-digit computations easier, given the centrality of single-digit computations in algorithms

Variations in which all of one kind of step is done first and then the other kind of step is done rather than alternating, because variations in which the kinds of steps alternate can introduce errors and be more difficult.

Variations that keep the initial multidigit numbers unchanged because they are conceptually clearer

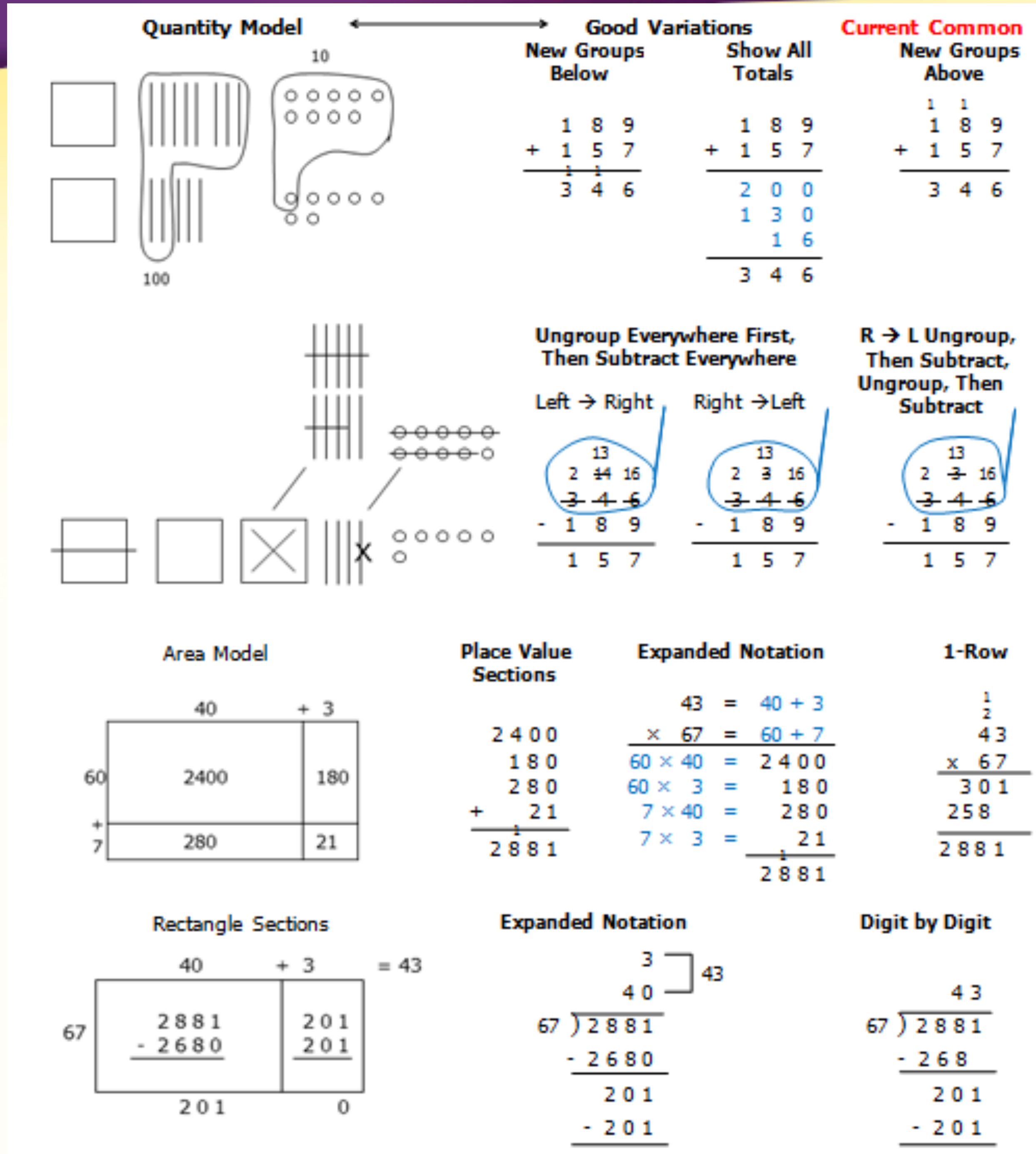
Variations that can be done left to right are helpful to many students because many students prefer to calculate from left to right.

The learning path using helping step variations

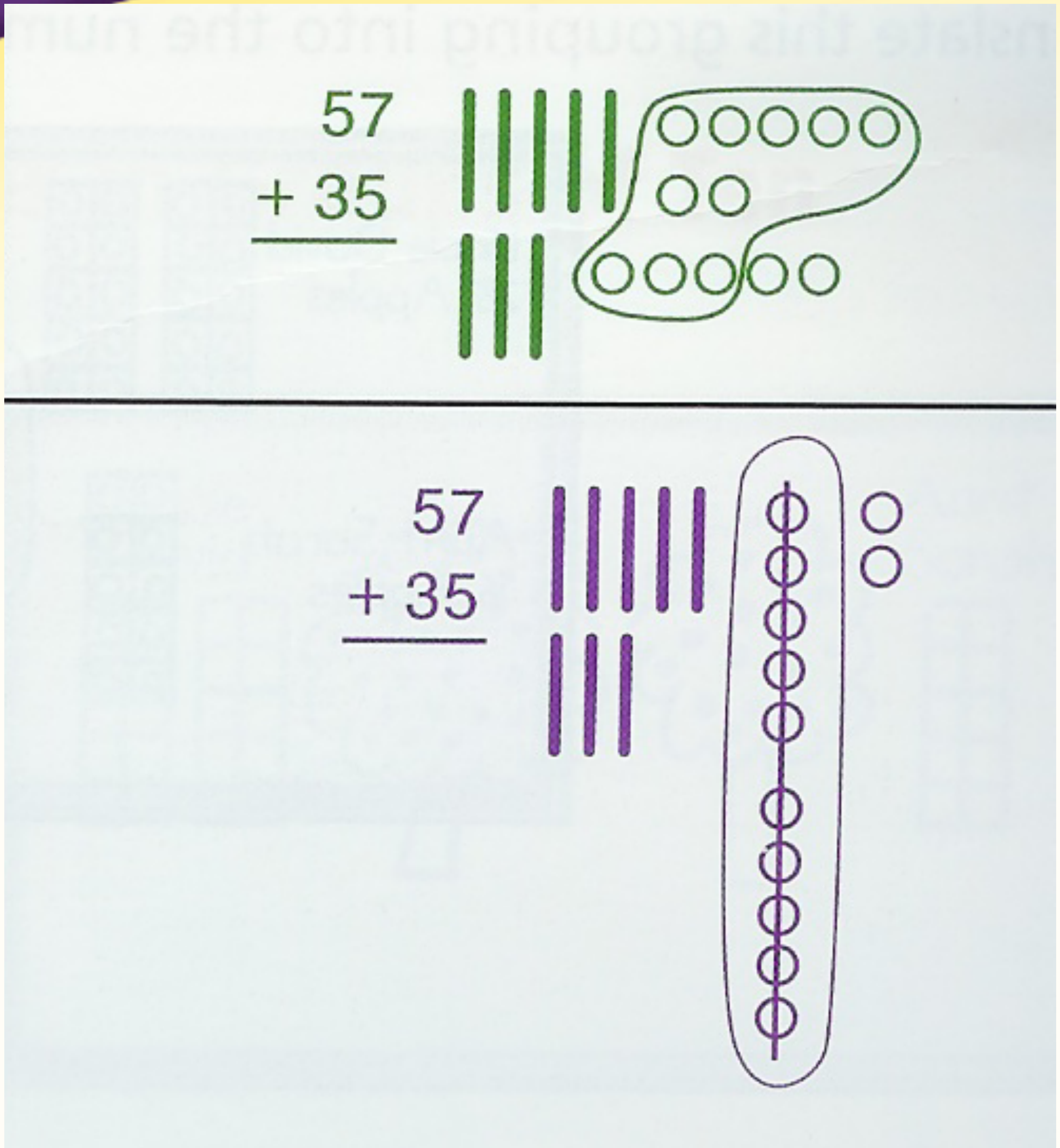
Some variations of a written method include steps or math drawings that help students make sense of and keep track of the underlying reasoning and are an easier place to start. These variations are important initially for understanding.

Over time, these longer written methods can be abbreviated into shorter written methods that are variations of writing the standard algorithm for an operation.

Drawings and Written Variations of Standard Algorithms



G1 Show All Totals



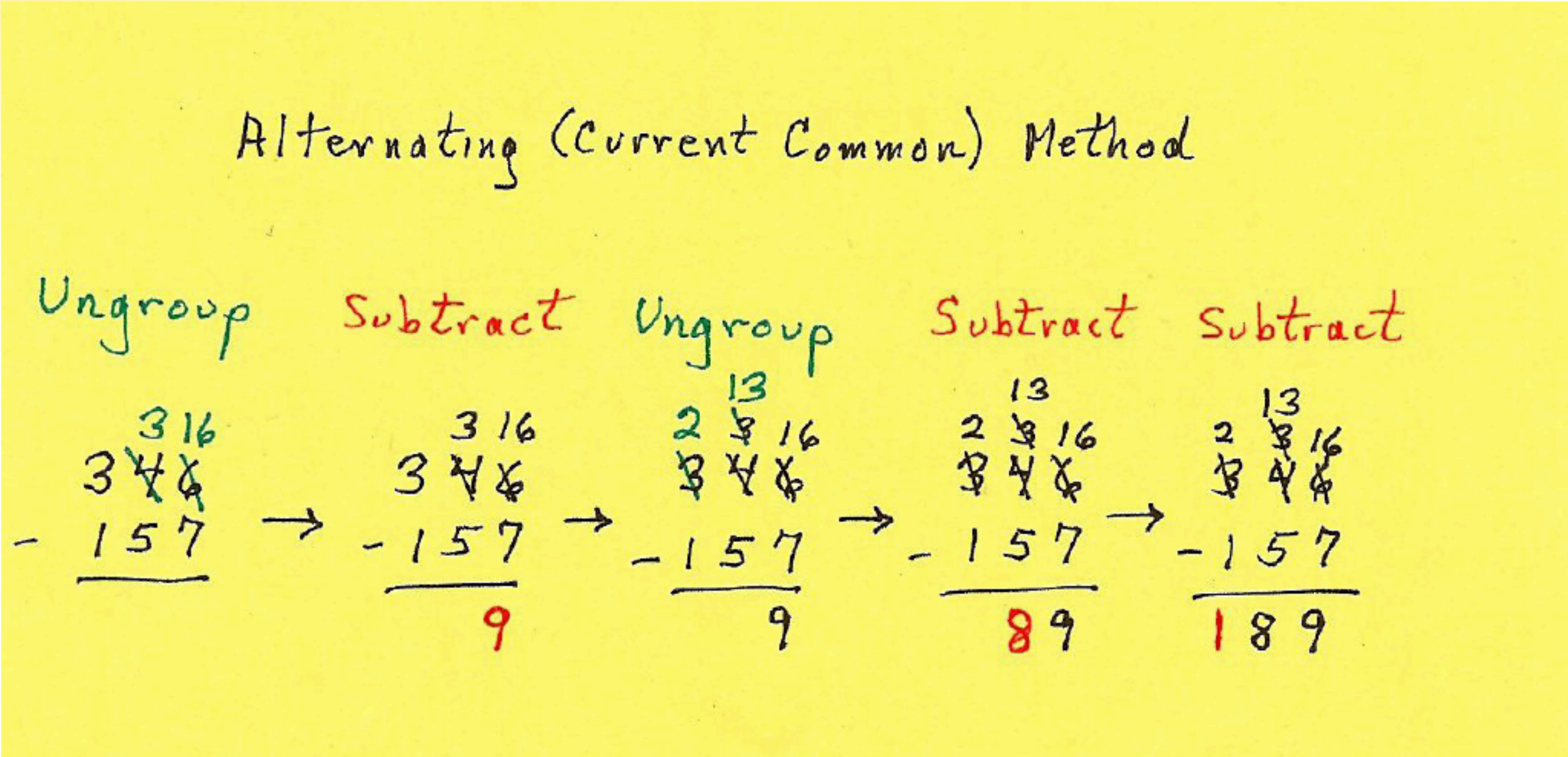
$$\begin{array}{r} 57 \\ +\ 35 \\ \hline 80 \\ 12 \\ \hline 92 \end{array}$$

G2 Multidigit Subtraction Common Error

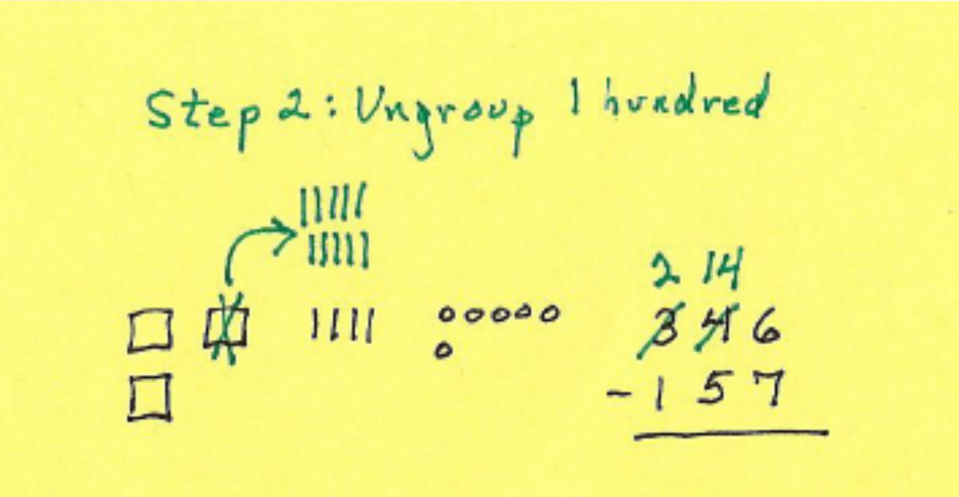
$$\begin{array}{r} 83 \\ -\ 57 \\ \hline 34 \end{array}$$

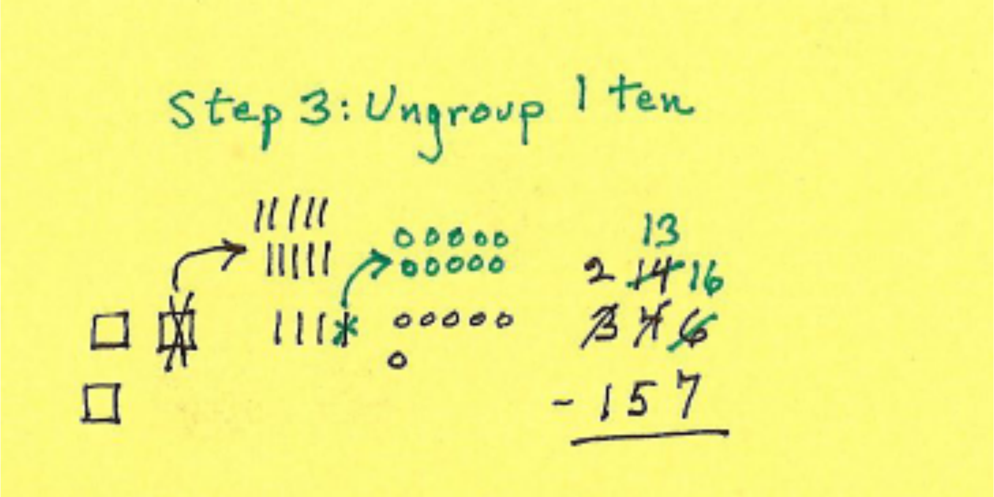
$$\begin{array}{r} 346 \\ -157 \\ \hline 211 \end{array}$$

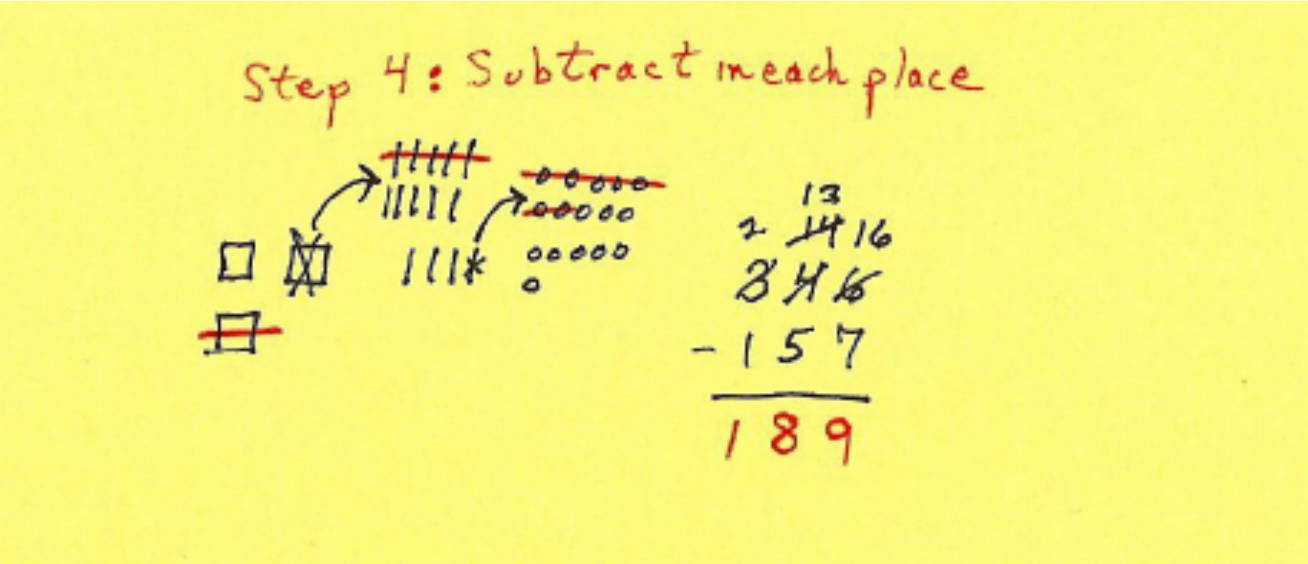
G2 The Alternating Method



G2 Ungroup First Method Within 1000

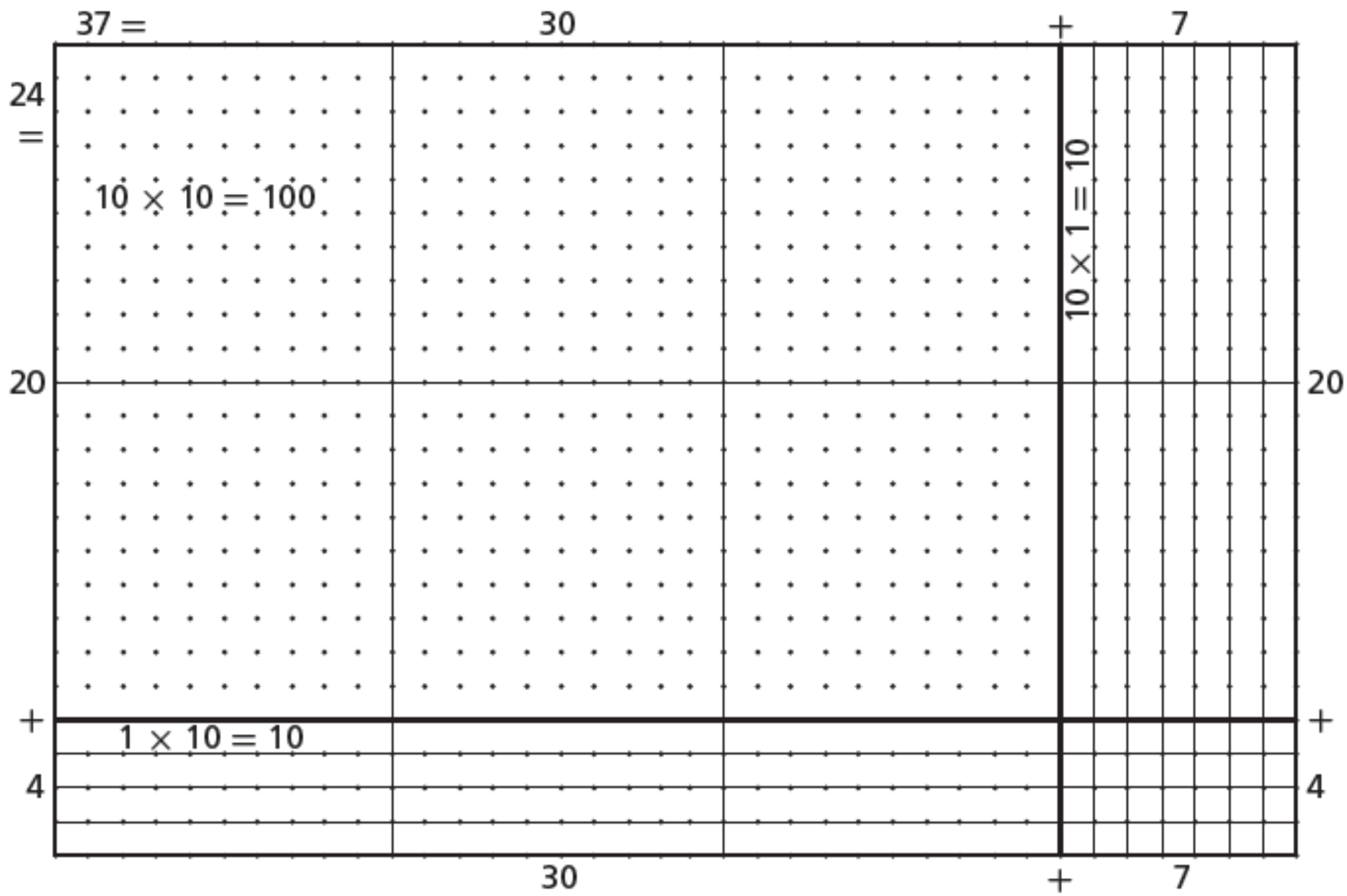






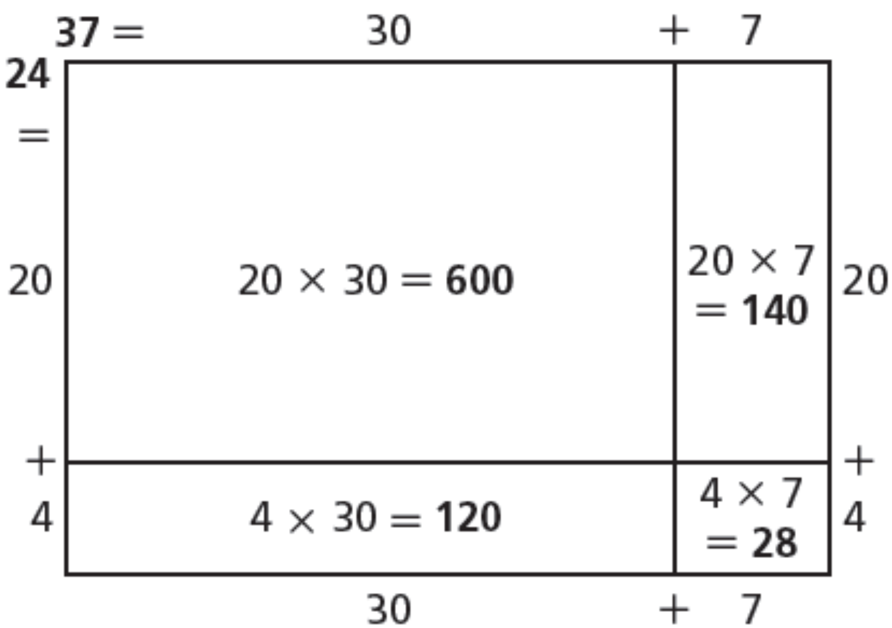
G4 2-Digit x 2-Digit Drawing

A coin-collecting book holds 24 coins on a page. There are 37 pages in the book. How many coins can the book hold? The models below all show the solution to 24 × 37.



G4 2-Digit x 2-Digit Sketches

Area Model Sketch



Place Value Sections Method

$$
\begin{aligned}
20 \times 30 &= \mathbf{600} \\
20 \times 7 &= \mathbf{140} \\
4 \times 30 &= \mathbf{120} \\
4 \times 7 &= \mathbf{28} \\
\hline
&\mathbf{888}
\end{aligned}
$$

G4 Discuss 2-Digit x 2-Digit Methods

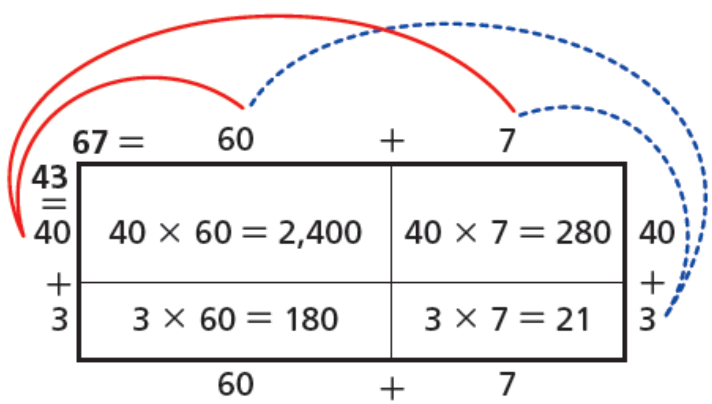
Place Value Sections Method

67 =	60	+ 7
43 = 40	40 × 60 = 2,400	40 × 7 = 280
+ 3	3 × 60 = 180	3 × 7 = 21
	60	+ 7

$$\begin{aligned} 40 \times 60 &= 2{,}400 \\ 40 \times 7 &= \phantom{2{,}}280 \\ 3 \times 60 &= \phantom{2{,}}180 \\ 3 \times 7 &= +\ \phantom{2{,}}21 \\ \hline & \ 2{,}881 \end{aligned}$$

Expanded Notation Method

Expanded Notation Method



$$\begin{aligned} 67 &= 60 + 7 \\ \times\ 43 &= 40 + 3 \\ \hline 40 \times 60 &= 2{,}400 \\ 40 \times 7 &= \phantom{2{,}}280 \\ 3 \times 60 &= \phantom{2{,}}180 \\ 3 \times 7 &= \phantom{2{,}2}21 \\ \hline & \ 2{,}881 \end{aligned}$$

G4 Discuss Algebraic Notation Method

Algebraic Notation Method

67 = 60 + 7

43 =	60	7	
40	$40 \times 60 = 2{,}400$	$40 \times 7 = 280$	40
+			+
3	$3 \times 60 = 180$	$3 \times 7 = 21$	3

60 + 7

$$43 \cdot 67 = (40 + 3) \cdot (60 + 7)$$

$$= 2{,}400 + 280 + 180 + 21$$

$$= 2{,}881$$

G4 Discuss More Compact Methods

Discuss how the recording methods below show the partial products in different ways.

Show partial products

67	
× 43	
21	3 × 7
180	3 × 6 tens
280	4 tens × 7
+ 2,400	4 tens × 6 tens
2,881	

Show new groups

67
× 43
1 2
81
2 2
+ 480
2,881

Alternate multiplying and adding steps

1-Row New Groups Above Method

Where is the place value error here?

Step 1	Step 2	Step 3	Step 4	Step 5
		2	2	2
2	2	2	2	2
67	67	67	67	67
× 43	× 43	× 43	× 43	× 43
1	201	201	201	201
		8	268	+ 268
				2,881

Step 3: 40 x 7 = 280, but the 2 in the 200 is written above the tens column.

Alternate multiplying and adding steps

1-Row New Groups Below Method

Step 1	Step 2	Step 3	Step 4	Step 5
67	67	67	67	67
× 43	× 43	× 43	× 43	× 43
2	2	2 2	2 2	2 2
1	201	201	201	201
		8	268	+ 268
				2,881

Here in Step 3 the 2 in the 200 is written correctly in the hundreds column.

G5 Fluency with Multiplication

Expanded Notation

$$\begin{array}{rcl} 43 & = & 40 + 3 \\ \times\ 67 & = & 60 + 7 \\ \hline 60 \times 40 & = & 2400 \\ 60 \times 3 & = & 180 \\ 7 \times 40 & = & 280 \\ 7 \times 3 & = & 21 \\ \hline & & 2881 \end{array}$$

Partial Products

$$\begin{array}{r} 43 \\ \times\ 67 \\ \hline 2400 \\ 180 \\ 280 \\ 21 \\ \hline 2881 \end{array}$$

Drop the helping steps to achieve fluency.

What to emphasize and where to intervene as needed

Grades K to 2 are more ambitious than some/many earlier state standards:

K: The ten in teen numbers

G1: + within 100 with composing a new ten; ok if many children still use math drawings; no subtraction without decomposing a ten

G2: a) +- total ≤100 with composing and decomposing a ten; use math drawings initially, but fluency requires no math drawings

b) +- totals 101 to 1,000 with math drawings; vital get mastery by most so that G3 can focus on x÷; intervene with as many as possible to get G2 mastery

Grades 3 to 6 are less ambitious than some/many earlier state standards:

G3: Fluency for G2 problem sizes so can focus on x÷ [intervene for x÷ all year]

G4 and G5: a) x only up to 1-digit x 2-, 3-, 4-digits and 2-digits x 2-digits; not need mastery of 1-row methods for multiplication [so have time for fractions]

b) division has only the related unknown factor problems; 1-digit divisors G4 and 2-digit divisors G5; fluency G6

Districts Record Students Explaining These Key Milestones with Drawings and Share with Parents

Kindergarten: Ten in teens Subtraction WP (e.g., 9 – 5)

G1: 2-d addition with new groups Unknown addend WP (8 + ? = 14)

G2: 3-d subtraction (e.g., 163 – 89) Start unknown WP (e.g., ? – 6 = 8)

G3: 3-d addition (e.g., 387 + 259) 3-d subtraction (e.g., 802 – 356)

with no drawing (fluency level) but use place value words for explaining

G4: 2-d x 2-d (e.g., 37 x 65) 3-d ÷ 1-d with remainder (e.g., 293 ÷ 8)

G5: 3/4 + 2/5 3/4 x 2/5

G6: 3/4 ÷ 2/5 division with decimals (e.g., 1.984 ÷ 0.32)

The Computation Learning Path

Any method that is taught or used must have a learning path resting on visual models and on students explaining the reasoning used using place value.

Methods are elicited from students and discussed, but good variations of writing the standard algorithm are introduced early on so that all students can experience them.

Steps in written methods are related to steps in visual models.

Experiencing and discussing variations in writing a method is important mathematically.

Students stop making drawings when they do not need them. Fluency means solving without a drawing.

Students drop steps of helping step methods when they can move to a short written variation of the standard algorithm for fluency.