The CCSS OA Learning Progression: Coherent Visual Models that Support Problem Solving and the Mathematical Practices

Professor Karen C. Fuson

Northwestern University

For more details about the CCSS-M and visual supports, please see the series of visual with audio Teaching Progressions I have made for various math domains. These can be found at karenfusonmath.com

Learning Path Teaching-Learning:
Differentiating within Whole-Class Instruction by Using
the Math Talk Community

Bridging for teachers and students
by coherent learning supports

Learning
Path

Phase 3: Compact methods for **fluency**

Math Sense-Making   Math Drawings
Math Structure   Math Explaining
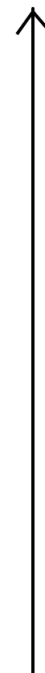
Phase 2: Research-based mathematically-desirable and
accessible methods in the middle for
understanding and growing fluency

Math Sense-Making   Math Drawings
Math Structure   Math Explaining

Phase 1: Students' methods elicited for **understanding**
but move rapidly to Phase 2

Common Core Mathematical Practices
Used in a Math Talk Community

Math Sense-Making: Make sense and use appropriate precision	Math Drawings: Model and use tools
1 Make sense of problems and perservere in solving them. 6 Attend to precision.	4 Model with mathematics. 5 Use appropriate tools strategically.
Math Structure: See structure and generalize	Math Explaining: Reason, explain, and question
7 Look for and make use of structure. 8 Look for and express regularity in repeated reasoning.	2 Reason abstractly and quantitatively. 3 Construct viable arguments and critique the reasoning of others.

Figure 2

The Math Practices in action

A teacher asks every day:

Did I do math sense-making about math structure using math drawings to support math explaining?

Can I do some part of this better tomorrow?

Learning paths within and across grades for

- situations (problem types) that give meanings for operations
- single-digit computation (+- and x ÷)

Students represent using drawings/diagrams and/or equations, then solve.

Students understand and apply properties of operations and the relationship between addition/subtraction and multiplication/division).

What is new in OA?

a) Solve problems with all 3 unknowns.

Each situation can have 3 unknowns.

This creates a learning path of difficulty from Kindergarten to Grade 1 to Grade 2.

b) Show the situation with a math drawing or diagram.
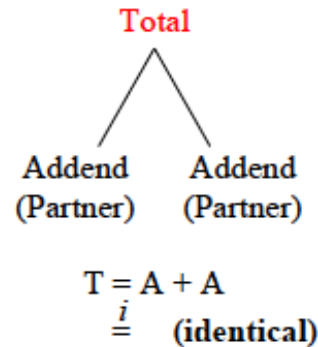
The 6 Situations

**Add To
Take From**

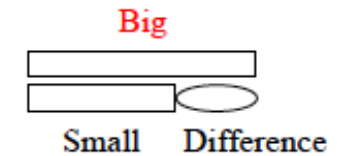Start + Change = **Result**
Start − Change = Result

⟶
(becomes)

**Put Together/
Take Apart**
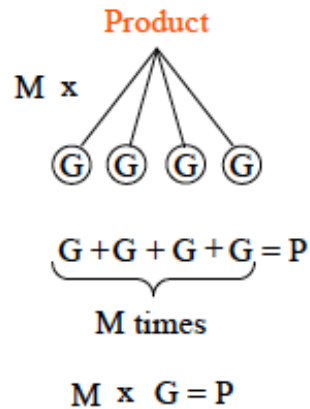
Total

Addend   Addend
(Partner)   (Partner)

$$T = A + A$$
$$\overset{i}{=} \quad \text{(identical)}$$

**Additive
Comparison**

Big

Small   Difference

Small + Difference = Big
Big − Difference = Small
Big − Small = Difference
$$\overset{n}{=} \quad \text{(same number)}$$

Equal Groups

Product

M x

Ⓖ Ⓖ Ⓖ Ⓖ

$$\underbrace{G + G + G + G}_{M \text{ times}} = P$$

M x G = P

⟶
(becomes)

Rectangular Everything Times Everything

Array

Factor
○○○○○
Factor ○ **Product**
○

(Long Division
Format)

Factor
Factor | **Product** |

(table)

$$\overset{i}{=} \quad \text{(identical)}$$

Area

Factor
├┼┼┼┼┤
Factor ╎ **Product**

Factor

F x F = P
P ÷ F = F
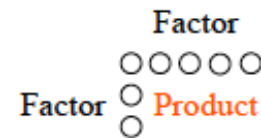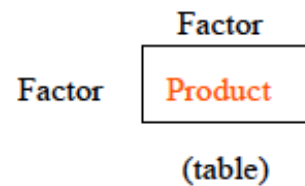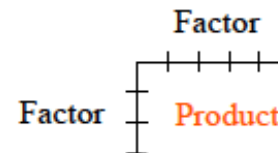
**Multiplicative
Comparison**

Big = 3 x Small

Big | 5 | 5 | 5 |
Small | 5 |
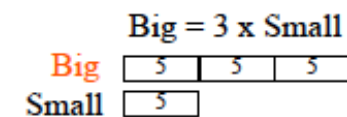
$$\text{Small} = \frac{1}{3} \text{ x Big}$$

Big ÷ 3 = Small

$$\overset{n}{=} \quad \text{(same number)}$$

CCSS Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each

Problem Difficulty Learning Path:

K is dark grey.
G1 is grey.
G2 is white.

Table 2. Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first two? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	"How many more?" version. Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? "How many fewer?" version. Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	"More" version suggests operation. Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? "Fewer" version suggests wrong operation. Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	"Fewer" version suggests operation. Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? "More" version suggests wrong operation. Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Special Difficulties with Compare Language

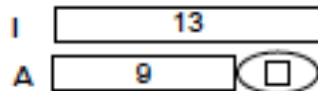## Compare Situations

Difference Unknown

Ali has 9 balloons.
Lisa has 13 balloons.

How Many More?

How many more balloons does Lisa have than Ali?

How Many Fewer?

How many fewer balloons does Ali have than Lisa?
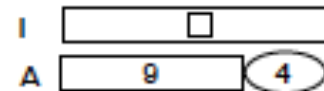

Bigger Amount Unknown

Ali has 9 balloons.

Leading Language

Lisa has 4 more than Ali.

Misleading Language

He has 4 fewer than Lisa.

How many balloons does Lisa have?
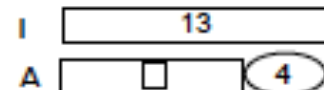

Smaller Amount Unknown

Lisa has 13 balloons.

Leading Language

Ali has 4 fewer than Lisa.

Misleading Language

She has 4 more than Ali.

How many balloons does Ali have?

Represent the Situation
OA: Operations and Algebraic Thinking

Grade 1 and Grade 2 subtypes involve algebraic thinking:

Represent the situation with a drawing, diagram, and/or an equation.

Then decide how to solve for the answer.

Situation Equations vs. Solution Equations

A situation equation shows the situation.

$$\square - 7 = 5 \qquad\qquad 189 + \square = 346 \qquad\qquad \square - 27 = 82$$

A solution equation shows the solution operation.

$$7 + 5 = \square \qquad\qquad 346 - 189 = \square \qquad\qquad 82 + 27 = \square$$

Yolanda has a box of golf balls.
Eddie took 7 of them.
Now Yolanda has 5 left.
How many golf balls did Yolanda
have in the beginning?

Did I do math sense-making
about math structure
using math drawings
to support math explaining?
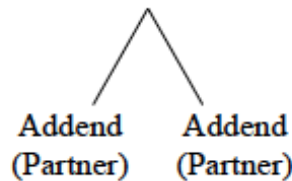
The 6 Situations

Add To Take From	Put Together/ Take Apart	Additive Comparison
K	K	Gr1

Add To / Take From (K)

Start + Change = Result
Start − Change = Result

⟶
(becomes)

Put Together/ Take Apart (K)

Total

Addend (Partner)   Addend (Partner)

$T = A + A$
$\overset{i}{=}$ (identical)

Additive Comparison (Gr1)

Big

Small   Difference

Small + Difference = Big
Big − Difference = Small
Big − Small = Difference
$\overset{n}{=}$ (same number)

Gr3	Rectangular Everything Times Everything	Gr3	Gr4
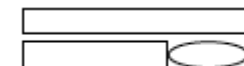
Yolanda has a box of golf balls. Eddie took 7 of them. Now Yolanda has 5 left. How many golf balls did Yolanda have in the beginning?

The key to solving story problems is understanding the situation. Students' equations often show the situation rather than the solution. Students drawings should be labeled to show which numbers or objects show which parts of the story situation.

Beginning [12] golf balls.
label

YOl E

[12] − 7 = 5

Total

Y in Beginning

[12]

[12] golf balls
label

7

5

I put the golf balls back together.

7 E
+ 5 Yolanda
12

in all
[12] golf ball
label

In the summer Jana trimmed 346 bushes.
Lisa trimmed 189 bushes.
How many fewer bushes did Lisa trim than Jana?

Did I do math sense-making about math structure using math drawings to support math explaining?
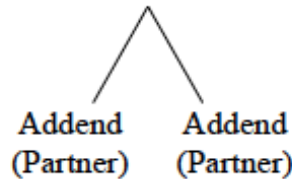
The 6 Situations

K	**K**	**Gr1**

Add To
Take From

Start + Change = Result
Start − Change = Result

→
(becomes)

Put Together/
Take Apart

Total

Addend   Addend
(Partner)   (Partner)

$T = A + A$
i
$=$ (identical)

Additive
Comparison

Big

Small   Difference

Small + Difference = Big
Big − Difference = Small
Big − Small = Difference

n
$=$ (same number)

Rectangular Everything Times Everything
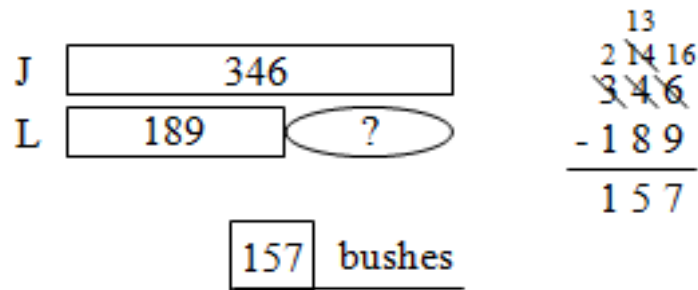
Gr3	**Gr3**	**Gr4**

Grade 3 Solution Approaches to an Additive Comparison Problem

In the summer Jana trimmed 346 bushes. Lisa trimmed 189 bushes. How many fewer bushes did Lisa trim than Jana?
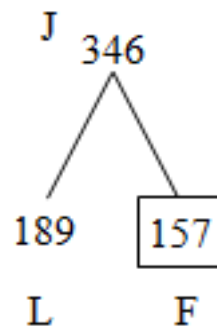
Comparison Bar Drawing of Quantities

J | 346 |

L | 189 | ? |

$$\begin{array}{r} 13 \\ 2\,\cancel{14}\,16 \\ \cancel{3}\cancel{4}\cancel{6} \\ -\ 1\ 8\ 9 \\ \hline 1\ 5\ 7 \end{array}$$

| 157 | bushes

Situation Equation

L d J

$189 + \boxed{157} = 346$

$189 + \ 11 = 200$

$200 + 146 = 346$
$$\overline{}$$
157 | 157 | bushes

Numerical Relationships Shown in Math Mountain

J 346

189 | 157 |

L   F

$$\begin{array}{r} 13 \\ 2\,\cancel{3}\,16 \\ \cancel{3}\cancel{4}\cancel{6} \\ -\ 1\ 8\ 9 \\ \hline 1\ 5\ 7 \end{array}$$

| 157 | bushes

Solution Equation

Jana   Lisa   fewer

$346 - 189 = \boxed{157}$

$$\begin{array}{r} 13 \\ 2\,\cancel{14}\,16 \\ \cancel{3}\cancel{4}\cancel{6} \\ -\ 1\ 8\ 9 \\ \hline 1\ 5\ 7 \end{array}$$

| 157 | bushes

**Some bunnies were sitting on the grass.
27 more bunnies hopped there.
Then there were 64 bunnies.
How many bunnies were on the grass before?**

Did I do math sense-making
about math structure
using math drawings
to support math explaining?
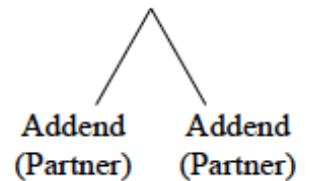

The 6 Situations

K

**Add To
Take From**

Start + Change = Result
Start − Change = Result

⟶
(becomes)

K

**Put Together/
Take Apart**

Total

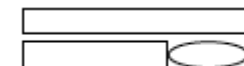Addend   Addend
(Partner)   (Partner)

$T \underset{i}{=} A + A$
(identical)

Gr1

**Additive
Comparison**

Big

Small   Difference

Small + Difference = Big
Big − Difference = Small
Big − Small = Difference

$\underset{n}{=}$ (same number)

Gr3   Rectangular Everything Times Everything   Gr3   Gr4

The Problem Solving Process

Part A: Understand and represent: Conceptualize bottom up from the situation

Part B: Re-represent and solve: Use related problem types, representations, properties, and /or relationships between + - or x÷

A1. Understand the problem situation

Mathematize (and Storyize)

A2. Represent the problem situation in a drawing/diagram and/or an equation

Then focus on the question and:

B1. Re-represent to find the unknown

Do the solution actions

B2. Write the answer and check that it makes sense

Districts Record Students Explaining These Key Milestones with Drawings and Share with Parents

Kindergarten: Ten in teens   Subtraction WP (e.g., 9 – 5)

G1: 2-d addition with new groups   Unknown addend WP (8 + ? = 14)

G2: 3-d subtraction (e.g., 163 – 89)   Start unknown WP (e.g., ? – 6 = 8)

G3: 3-d addition (e.g., 387 + 259)   3-d subtraction (e.g., 802 – 356)

**   with no drawing (fluency level) but use place value words for explaining**

G4: 2-d x 2-d (e.g., 37 x 65)   3-d ÷ 1-d with remainder (e.g., 293 ÷ 8)

G5: 3/4 + 2/5   3/4 x 2/5

G6: 3/4 ÷ 2/5   division with decimals (e.g., 1.984 ÷ 0.32)