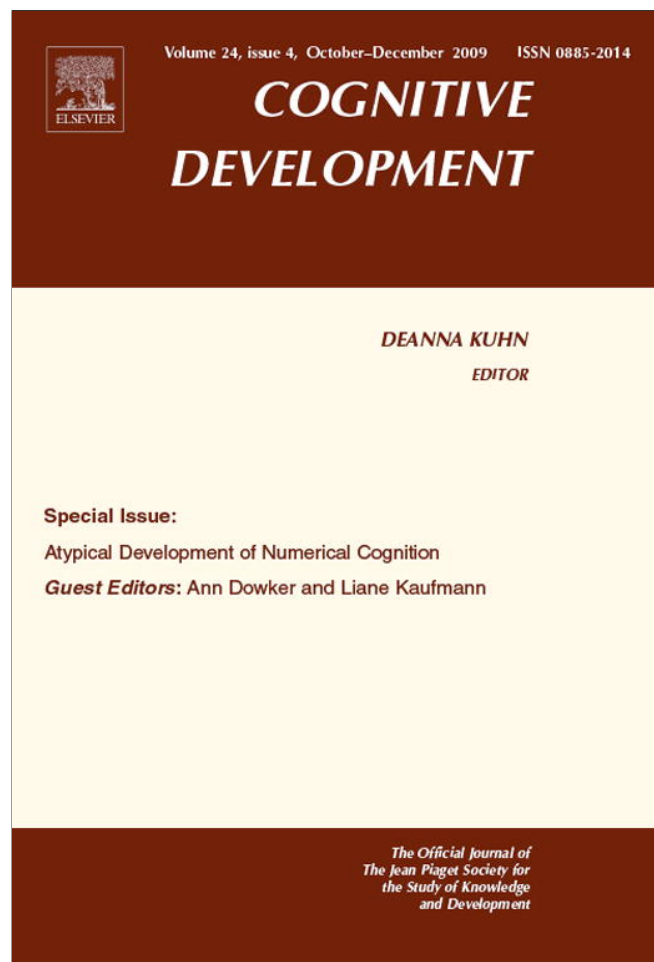


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## Cognitive Development



# Avoiding misinterpretations of Piaget and Vygotsky: Mathematical teaching without learning, learning without teaching, or helpful learning-path teaching?

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### ARTICLE INFO

### ABSTRACT

This article provides an overview of some perspectives about special issues in classroom mathematical teaching and learning that have stemmed from the huge explosion of research in children's mathematical thinking stimulated by Piaget. It concentrates on issues that are particularly important for less-advanced learners and for those who might be having special difficulties in learning mathematics. A major goal of the article is to develop a framework for understanding what effective mathematics teaching and learning is, because doing so is so important for struggling students and for research about them. Piaget's research had a fundamental influence on the on-going tension between understanding and fluency in the classroom, supporting efforts toward increasing understanding. But in some countries, misinterpretations of Piaget led to practices that are counterproductive for children, especially struggling learners. Such misinterpretations are identified and a more balanced approach that also draws on Vygotsky is described—a learning-path developmentally-appropriate learning/teaching approach.

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A major goal of this article is to develop a framework for understanding what effective mathematics teaching and learning is. Doing so is extremely important for students having difficulty learning mathematics. The framework develops out of a major historical tension in mathematics education between understanding and fluency. In the USA this tension has been so intense that it has been termed “the math wars.” It, and variations in the relative emphases on understanding and fluency, has also influenced mathematics education in European countries, although these histories vary in

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the timing and intensity of the tension. Piaget's research had a fundamental influence on this tension, supporting efforts toward increasing understanding. But in some countries (perhaps especially in the USA), misinterpretations of Piaget led to practices that are counterproductive for children and especially for struggling learners.

This article focuses on early numerical learning (up through multi-digit addition and subtraction) because that is the focus of most of the research on students with learning difficulties and because it is especially important for such students to gain competence in these early topics. The article necessarily simplifies complex issues as it summarizes some broad themes and a few more particular points. It is not a review of the literature, although some major researchers are mentioned along the way. For more detail and fuller references, see research reviews by Clements and Sarama (2007), Fuson (1992a, 1992b), Verschaffel, Greer, and DeCorte (2007), and the research summarized in the NRC *Adding It Up* report, 2001, and the NRC early childhood math report, 2009.

## 1. The powerful influence of Piaget on research about children's mathematical learning

Many researchers of my generation probably remember the first time they read a book by Piaget on children's mathematical knowledge. In 1965 as a math major at Oberlin College, I was interested in children's math learning. A math professor sent me to look at Piaget's books, and I sat stunned, reading more and more vignettes that revealed how children thought about mathematical ideas. They did not think like adults! And there were learning progressions in their thinking—their ideas *developed* to become adult ideas. And you could invent wonderful tasks (or at least Piaget could) that would reveal how children did think differently and then progressed in their learning. It is difficult to explain to current researchers how revolutionary these ideas were at the time or the pervasive extent of the explosion of creative research they spawned, all aimed at understanding children's mathematical cognition in more detail.

Because of this explosion of research, we know hugely more about children's mathematical thinking than before Piaget's work became widely known. Many of the research studies from mathematics education, developmental psychology, and cognitive development summarized in the two latest Handbooks of Research on Mathematics Education (Grouws, 1992; Lester, 2007) are influenced in some way by Piaget's ideas that children construct their own concepts and that these ideas develop along learning paths.

Later researchers adapted and extended Piaget's theory. Neo-Piagetians like Case cast Piagetian stages into a neater organization through studies that added successive steps in problem situations (Case, 1991; Case & Okamoto, 1996). He and his students also did important educational research in several mathematical domains (e.g., Griffin, 2005; Kalchman & Koedinger, 2005; Moss, 2005). These studies often used *bridging contexts* that were carefully chosen to be clear examples of the class of situations for the mathematical domain that could support student understanding. Vergnaud (2009) developed the framework of conceptual fields that extended Piaget's notion of scheme to include goals, rules to generate activity, concepts- and theorems-in-action, and generative possibilities of inference. Concepts involve a set of situations and a set of linguistic and symbolic representations as well as the operational invariants contained in schemes. These extensions in the conceptual field framework allow the specific characters of various mathematical domains to enter the analysis, as is necessary, and shift the focus from the general logical operations studied by Piaget to specific mathematical operations.

Piaget's theories and the subsequent research about children's understandings were especially important to educators and researchers who were concerned about the lack of mathematical understanding resulting from traditional rote-teaching methods. Far too often the teacher stood at the front and did a problem or two with no or minimal explanation and then the students were to imitate (or there even was no initial demonstration at all). These rote-teaching methods failed to reach many children, especially those less-advanced than their peers and thus all children with mathematical learning difficulties. Such traditional approaches emphasized fluency and not understanding, and the opponents of such approaches might cleverly have termed them *teaching without learning*.

In this context, the idea that children could and did think, and had their own mathematical ideas and approaches, was very refreshing and encouraging. But Piaget's focus on the concrete operational period in children's thinking and on the centrality of children's interactions with objects during this

**Table 1**

Overinterpretations of Piaget that underpin “the math wars”.

<p>Mathematical learning without teaching: an overemphasis on activities with manipulatives For successful learning by children up to age 8 or 9 (and for some who are older), children</p> <p>A. Must manipulate real objects B. Must manipulate real objects not connected to mathematical words or symbols or cultural methods C. Must manipulate real objects not connected to mathematical words or symbols or cultural methods for a really long time and then perhaps only connect briefly to symbols D. Must not be taught any methods at all E. Are in a classroom in which celebration of every student's thinking as an equal contribution or simple turn-taking to explain every approach replaces a prolonged focus on mathematically important ideas in children's problem solving and discussion about such solving. This emphasis is on understanding at the expense of fluency, and its adherents assert that this approach is the only developmentally appropriate approach.</p> <p>Problems as these overinterpretations are enacted in the classroom are: A. The real objects sometimes do not show the math ideas clearly enough, they may be difficult to organize, children can be distracted by them and use them improperly (e.g., throwing them). It can take a long time to understand how to do each activity before the math elements emerge and are able to be focused on. Activities may have too much “activity” and not enough “mathematics” (e.g., covering whole irregular surface with squares and then counting them to learn place value: children can not see the tens and ones, and this takes a long time). B. When objects and symbols are not connected, the formal mathematics does not take on meanings. Also, too often the real objects are used only to produce answers rather than to help children build meanings for actions on the objects that connect to the mathematical symbols. C. There is a very delayed or no real learning path to more-advanced methods and insufficient time to build meanings for symbols. D. It is assumed that any teaching with symbols results in rote learning rather than in understanding, but meaningful teaching approaches are not tried or even assumed to be possible in some extreme cases. For some programs or in some research writing, it is forbidden (or called immoral) for teachers to explain with visual referents research-based methods that children understand and that are in their learning path (e.g., counting on). E. Math Talk may be about turn taking (without deep listening) rather than an instructional conversation focused on mathematical ideas. <i>All of the ways</i> rather than <i>mathematically important and accessible ways</i> are often emphasized.</p> <p>Teaching Without Learning: The other side of the “math war” These overinterpretations of Piaget arose and still arise partly in resistance to “traditional” rote teaching of math in which the teacher stands in front and shows what to do (and does not explain why or link the symbols to any visual referents or other meaning-make sources); then the children try to imitate the teacher and do endless worksheets of symbolic work with no visual referents or other meaning-making supports or even any expectation of meaning-making (understanding). Or worse yet, the teachers just has students open the textbook to page x and start doing the problems with no demonstration at all. This emphasis is on fluency with no emphasis on understanding.</p>
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period led to (or reinforced) an educational theory emphasizing an activity approach in which children spent most of their classroom time interacting with objects. At least in England, Australia, and the USA, this theory was widely adopted for early childhood but also was used in some places up into the primary and even middle grades, although there traditional teaching also remained pervasive. This swing of the educational pendulum emphasized understanding at the expense of fluency and might have been termed by its opponents *mathematical learning without teaching*. Various attributes of this view are listed at the top of Table 1, and problems with each attribute are listed in the middle of the table. To emphasize that the extremes of this approach (the misinterpretations of Piaget) do reflect an understandable but out-of-balance concern with the dangers of traditional teaching, such traditional teaching that downplays or ignores understanding is summarized at the bottom of the table. The misinterpretations of Piaget summarized in the table are particularly problematic for children with difficulties in learning mathematics because these children especially need meaning-making supports for learning and help moving on to more advanced and mathematically effective strategies. It is important for such children that classrooms move beyond these misinterpretations.

## 2. Enter Vygotsky, culturally specific knowledge, and symbolic tools for learning

The explosion of research on early mathematical thinking stimulated by Piaget can be thought of as falling into four levels that follow children from birth to age 8:

- A1: mathematical thinking without language (to about age one),
- A2: understanding and use of number words and socio-cultural tools like counting,
- A3: adding, subtracting, and comparing single-digit numbers,
- A4: adding, subtracting, and comparing multidigit numbers.

The first area is consistent with Piaget's focus on biological, relatively culturally invariant modes of thinking, but the others involve culturally specific knowledge and oral and written symbolic tools for learning and thinking that were the focus of Vygotsky's work (1934/1986, 1978). The research in these areas provides our detailed knowledge about progressions in children's learning that can be supported in homes and in classrooms. Major aspects of these progressions are similar around the world, but important parts of them depend on specifics of the language, the cultural view of mathematics learning, and the learning supports widely available in a particular culture or in its classrooms.

The implication for research on mathematics learning difficulties is that some such difficulties result from cognitive processing difficulties but that others may result from, and most may be exacerbated by, specific cultural issues. It is important for researchers and educators to understand the level of cultural supports in their language, homes, and schools in order to interpret the nature of the various difficulties and powerful ways to overcome them. Examples of helpful supports are provided later for several core mathematical topics.

An example of culturally specific knowledge that is pervasive in early math learning is the extent to which the number words in the language clearly show the structure of numbers up to 100 as groups of tens and of ones (e.g., 52 is 5 tens 2 ones). The Chinese-based named-value system used in many East Asian languages is quite clear, regular, and in the same order as the written numerals: 5678 is said (translated into English) as *five thousand six hundred seven ten eight*. Each larger value is named in a regular fashion. In European languages there may be reversals in words for different values (in English, *fourteen* for 14 where the *four* is said first but written second), no clear words that say the place value, especially for the tens (in English, *teen* and *ty* but not *ten*), and even more complex constructions such as the French number words based on twenty. There is considerable variation across European languages, but these irregularities do result in many European children learning their number word sequences more slowly than do East Asian children (Miller, Smith, Zhu, & Zhang, 1995). Children speaking such languages also as a group use quantities of ten in their computations less easily and much later, if at all. This delay is also related to the lack of learning supports in many classrooms that can help children build and use concepts of ten (Fuson, 1990; Fuson & Kwon, 1991). These research results are for "normal" children, so children with language or sequential processing difficulties may have even more problems. Other mathematical terms may also be confusing or at least not as clear in a given language, such as fractions said using ordinal words (in English  $3/5$  is said as *three-fifths*) rather than the Chinese *of five equal parts (take) three*. Therefore mathematical vocabulary may be an extra complicating factor in learning rather than a potential learning support for the meaning of that vocabulary (Han & Ginsburg, 2001).

Children need additional learning supports to construct the mathematical ideas obfuscated by their language. For example, we used visual referents for the tens and ones (penny-dime strips with ten pennies on one side and a dime on the other in the early classrooms discussed by Hiebert et al. (1997) and math drawings of tens and ones exemplified later by Fuson, Smith, and Lo Cicero (1997)). However, for children who entered first grade not counting or only counting to ten, it still was a major task to learn to count to 100 and relate these words to the groups of tens and ones in the visual supports. We found that using a modified English form of the Chinese number words (25 was said as *two tens five ones*) allowed all children to enter the conversation about place-value meanings and 2-digit addition with regrouping before some of them had fully mastered the English number word sequence to 100. This place-value language could later be used to describe their addition and subtraction methods involving regrouping. Of course, we also use the standard English number words because children have to learn their own cultural words. But the meaningful place-value words allowed all children to build and use multi-unit place-value concepts in adding and subtracting while solidifying their fluency with the irregular English number words. These linguistic issues need to be considered in research about mathematical learning difficulties.



**Table 2**  
Piaget and Vygotsky: learning-path developmentally-appropriate learning-teaching.

Extreme views of learning	Balanced learning-teaching	Extreme views of teaching
Piaget “constructivist” knowledge Meaning-making only via real objects	Integration of Piagetian and Vygotskiiian forms of knowing Meaning-making by connecting visual referents of ideas to cultural symbol systems and using meaningful language and clear mathematical situations (e.g., bridging contexts). Math drawings are particularly useful visual referents that children can make themselves.	Vygotsky’s formal cultural knowledge Meaning-making only via disconnected cultural symbol systems (e.g., learn the “dance of the digits” by observation and imitation)
(only) Child-invented methods	Sharing child-invented, mathematically desirable and accessible methods, and culturally valued methods linked to math drawings or other manageable visual referents	(only) Culturally valued and efficient method
Classroom is child-dictated; is under-structured for some	Teacher (and the math program) structure a learning path for all children that is based on children’s learning progressions	Classroom is teacher-dictated; is over-structured for some
Understanding	Understanding and fluency	Fluency

### 3. Piaget and Vygotsky, learning and teaching, and learning path developmentally appropriate teaching

An outline of an educational framework reducing misinterpretations of Piaget and of Vygotsky appears in Table 2. It shows how integrating components of Piaget and Vygotsky can lead to a more balanced perspective that in turn can lead to more effective learning situations that can benefit all children, but especially those with mathematical learning difficulties.

The first important issue is summarized in the top row of the table. Over-emphases either on learning or on teaching lead to the problems noted earlier in the discussions of tensions between understanding and fluency. A balance of focus on both understanding and fluency is needed, with an understanding of the continual interaction of learning activity and teaching activity. Each teaching activity of a teacher results in some learning activity by each child using her own schemes and schemas (Piaget’s terms for mental knowledge structures). This occurs even in traditional teaching where there is little emphasis on meaning-making by the teacher or the math program. Children always construct their own knowledge and seek patterns. But many more children can construct meaningful connected webs of useful and fluent knowledge if their teacher and math program use the results of the explosion of research about children’s mathematical thinking to weave together the best of Piaget’s and Vygotsky’s theoretical perspectives. These integrated perspectives are outlined in the middle column of Table 2.

The second row in Table 2 stems from Vygotsky’s distinction between two forms of knowledge and of thinking, informal/unscientific and formal/scientific/academic (1934/1986). Children learn the first outside of school and in natural environments (the focus of Piaget’s interest), and they learn the latter in school. To become meaningful, Vygotsky asserted, children must connect these two forms of knowledge, but to date there are few specifics about how to do this. The tension between understanding and fluency can be thought of as an overemphasis on informal/unscientific knowledge not linked to formal/scientific/academic knowledge (learning without teaching) or as an overemphasis on the reverse—on formal/scientific/academic knowledge not linked to informal/unscientific knowledge (teaching without learning).

The third row of the table uses the research of many math educators who have developed visual referents for math concepts (a major theme of such research) and the extensions of Case and Vergnaud mentioned earlier. Mathematical ideas are abstract, so children do need to experience visual referents for the situations or concepts as a way to relate their developing schemes and schemas to cultural symbol systems. But one of the major results of the ten years of classroom research developing my own K-5 math program is that children, even in first grade, can make math drawings for many mathematical

concepts. Math drawings are simplified drawings that capture the important mathematical aspects. For example, children can draw 4 circles and then 3 circles, rather than animals or pictures of animals, for the cultural symbols  $4 + 3$  and then count them to find that  $4 + 3 = 7$ . Place-value drawings of hundreds, tens, and ones can be linked step-by-step to methods of multi-digit addition and subtraction to make the steps meaningful and accurate. Drawings of rectangles that show tens and ones along the edges and hundreds, tens, and ones inside can do the same for multi-digit multiplication and division. These math drawings facilitate explanations and discussions of various methods (Math Talk about) as well as supporting the mathematical thinking of individuals. Math drawings have many advantages over manipulated objects in school classrooms (Fuson, Atler, Roedel, & Zaccariello, 2009).

The fourth row of the table summarizes the fact that traditional and Piagetian teaching differ not only by their relative emphasis on meaning-making (a little and a lot), but also on the source of mathematical solution methods. In traditional teaching, a culturally valued and efficient method is taught (usually separated from attempts to make it meaningful), but such traditional methods often ignore how easy it is for children to make sense of them or to carry out their steps without errors. In Piagetian approaches, the focus is on methods invented by children. When the only methods in the classroom are those invented by children, a long time may be spent on relatively primitive methods invented by children. These invented and traditional methods are often so far apart that it is difficult to bridge them.

There are methods in the middle that are both mathematically desirable (generalizable and efficient enough) and accessible to children at various places along their learning path. Methods for research Levels A3 and A4 (single-digit and multi-digit addition and subtraction) that are particularly useful for children with learning difficulties are examined in the next section. If a classroom provides a meaning-making environment in which math concepts and operations are supported by meaningful situations and visual supports like math drawings, some children can invent methods. These can be elicited from children and then rather quickly one or more mathematically desirable and accessible methods can be shared with children. These all can be compared and contrasted, and any culturally valued efficient methods can also be introduced for discussion if they are not introduced into the classroom by children using them.

The overemphases of the left and the right columns of Table 2 are corrected by this approach because children who invent primitive methods are helped to move beyond them to a mathematically valuable method, and children who have learned a culturally valued efficient method by rote can be helped to understand it. In this way the whole learning path of methods that children do, can, and “are supposed” to learn enter the classroom conversation but in a meaningful way supported by the math drawings and by Math Talk. Children can choose which mathematically desirable method appeals most to them, and they work to become fluent in it as well as to understand and be able to explain it linked to a math drawing. Thus, such a classroom supports both understanding and fluency. These mathematically desirable and accessible methods and their learning supports are especially appropriate for intervention research with children having difficulties in math. Research on diagnosing and identifying such children can also benefit from understanding that children’s difficulties might have been exacerbated by extreme views in the left or right column of Table 2.

The final two rows in Table 2 restate the extreme views in other language. Child-dictated classrooms emphasize understanding, teacher-dictated classrooms emphasize fluency, and a learning path program based on children’s learning progressions supports both understanding and fluency.

#### **4. Mathematically desirable and accessible methods for all children but especially helpful for many children with mathematical learning difficulties**

##### *4.1. Single-digit addition and subtraction*

One of the most robust results within the explosion of research stimulated by Piaget is the identification and detailed working out of a world-wide learning path of methods within single-digit addition and subtraction (Level A3), as detailed in research by Baroody, Carpenter, Cobb, Fuson, Siegler, and Steffe summarized in the overview references cited earlier. This path is summarized in Table 3.





Children everywhere begin with a first level at which each of the three quantities in the addition or subtraction must be represented sequentially (count all or take away). If children count out the two addends themselves, they must use relationships between counting and cardinality shift from the cardinal to the count meaning to count out the known groups, and then make the shift from a counting to a cardinal meaning after counting the total. For subtraction these same shifts are involved, but the child first counts out the total and then counts the known addend to take it away, and then counts the remaining objects to find the unknown addend (how many are left).

At Level 2 the counting words move from being a list matched with outside objects as a child counts them to being an internal representational tool where the number words themselves are the objects that are counted or matched. These number words represent the numbers in the addition or subtraction as children count on to find the total (when adding) or to find the unknown addend (when subtracting). When the number counting on is more than two or three, some method of keeping track of how many are counted on must be used (raising fingers as each word is said, double counting verbally, or using a visual image as one counts).

Some to almost all children then move on to use Level 3 recomposing methods, make-a-ten or doubles. The make-a-ten methods are taught to all first graders in East Asian countries and are learned successfully by most of them (Fuson & Kwon, 1992; Murata, 2004). But these methods are easier for children speaking East Asian languages than those speaking European languages because there is an extra step at the end for the latter children. When recomposing  $8 + 6$  to make  $10 + 4$ , the step from thinking *ten and four* to thinking 14 is much easier in East Asian languages because 14 is said as *ten four* rather than as *fourteen* (or some other European less-explicit word). There also is more cultural pull for this method because the ten in each teen number is clearer in East Asia because of the ten in the language and because groups of five and ten are more visible in the culture.

For many years I designed various methods to help children in the USA learn this method and its three prerequisites:

- (a) knowing for each number its partner to ten (the number that with it makes ten);
- (b) knowing the partners for each single-digit number, and
- (c) knowing immediately  $10 + n$ .

I also designed meaningful methods for moving children to Level 2 counting on. The latter is very successful for almost all first graders, while some children never learn the make-a-ten methods because they have too many steps. Quite a few children when supported to learn them do so when making math drawings for multi-digit addition and subtraction, a context in which they are particularly useful and where the math drawings support these methods (see the discussion of Fig. 1). Therefore in the USA and probably in many countries speaking European languages, counting on for addition and subtraction is the mathematically desirable and accessible method that can be taught to (almost) everyone.

#### 4.2. Subtraction as finding an unknown addend and thus using forward (not backward) methods

In the USA and some European countries, children's invented methods for subtraction tend to be counting down. Counting down methods are considerably less accurate than are counting on methods because counting down is difficult, and the count-cardinal relationships are complicated and result in four different counting down methods, only two of which are correct (Fuson, 1984; Fuson & Willis, 1988). For example, to solve  $14 - 8$ , you can start with 14 or with 13 and you can end on the eighth word said or the ninth word said. The two correct methods are

- (a) 14, 13, 12, 11, 10, 9, 8, 7; that's eight numbers, so the next number tells how many are left, it is 6,
- (b) 13 is one taken away, then 12, 11, 10, 9, 8, 7, 6 is 8 numbers taken away; so 6.

But children imitate their counting on methods, and also do the two incorrect methods, ending with one too many or one too few. Many countries around the world emphasize the relationships between addition and subtraction and teach forward methods like counting on for subtraction:  $14 - 8 = ?$  means

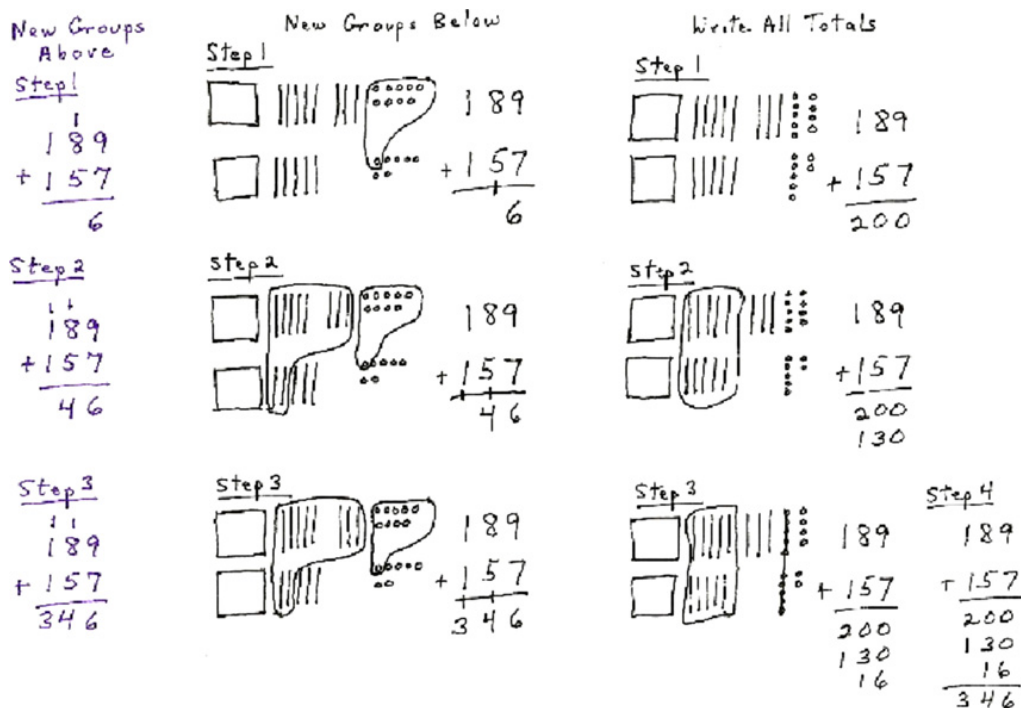


Fig. 1. Linked drawing and numerical steps for addition methods.

$8 + ? = 14$ ? My research over many years has demonstrated how effective such an approach is (Fuson, 1992a, 1992b). Failure to help children move from backward to forward methods for subtraction makes subtraction seem more difficult than it has to be for children. Kamii (1989, 1994), for example, suggests that subtraction is more difficult than multiplication and that it should follow multiplication, but her teaching-learning methods do not include helping children sufficiently to solve subtraction using forward methods.

For both Level 2 and Level 3 methods, the forward methods of subtraction are procedurally easier than are the related addition methods. For counting on, the child must just monitor the word said and stop when s/he hears the total (e.g., 14); then s/he can look at the fingers and see how many were counted on. For addition, the fingers keeping track must be monitored with each count, and the finger patterns for each number (e.g., 6 fingers) must be very well known in order to be recognized in action. For the subtraction make-a-ten method (see Table 3), children are just adding the amount to make ten (e.g., 2 for  $8 + ? = 14$ ) and the number seen in the ones place (e.g., the 4 in 14) to find the unknown addend. So subtraction is actually adding. For addition they must also know the break-apart partners for each number added on (e.g., after finding 2 to make 10 with 8, they have to know that  $2 + ?$  makes 6,  $?$  is 4) and know the teen sums. Hsiu-fei Lee, a graduate student of mine, tested over 100 low-achieving Taiwanese first graders and found that they all learned make-a-ten for subtraction but some still counted on for addition because the addition make-a-ten method was too difficult for them (Duncan et al., 2000).

Because many children diagnosed with learning difficulties in mathematics have problems with single-digit addition and subtraction, it is especially important that they have support to learn the mathematically desirable and accessible Level 2 counting on methods. These are generalizable to all single-digit numbers and can become fast and accurate enough to use in any problem solving or multi-digit calculation. An important focus for helping all children learn to count on for both addition and subtraction is how different learning difficulties affect counting on. Some children might not be able to use fingers to keep track but could use visual images, while others might be able to do the reverse.

The focus of information-processing models that emphasize memorizing and recall of addition and subtraction “facts”, and state or national goals that use this language and emphasize memorization, ignore this huge research literature on children’s learning path for single-digit addition and subtraction

methods. Of course all along the course of learning children do learn to solve particular additions or subtractions very rapidly; they seem memorized and may be memorized. This is normal and helpful, especially for smaller numbers. But mathematics is special because general solution methods do exist in many domains, and this A3 domain is one. The mathematically desirable and accessible counting-on methods are examples of these.

Single-digit multiplications and divisions are a more complex issue. Students do need to learn patterns in how various multiples do or do not fit with the base-ten system (e.g., fives fit and so are easy, and 7s do not and thus are difficult). But there are no general methods that work across all single-digit numbers, so students do need to spend time working on remembering and differentiating various multiplications and divisions.

#### 4.3. Multi-digit addition and subtraction

Many methods invented by children for solving 2-digit and even 3-digit addition and subtraction are extensions of the Level 2 counting-on or counting-down methods for single-digit numbers (e.g., the Open Number Line and other approaches summarized in [Beishuizen, Gravemeijer, & van Lieshout, 1997](#); [Cobb, 1987](#); [Kamii, 1994](#)). However, provision in the classroom of objects showing tens and ones do support methods that combine like places ([Fuson et al., 1997a, 1997b](#); [Hiebert et al., 1997](#); [McClain, Cobb, & Bowers, 1998](#)). We found that the counting skills required by the methods starting with one whole number (counting on or counting back by hundreds, tens, and ones) were difficult for many children and became cumbersome with 3-digit numbers and worse than cumbersome for larger numbers (i.e., they do not generalize readily). We instead found methods that were easy for all children to relate to math drawings of quantities and to explain using quantity language. These methods also related readily to the traditional methods, so these also could be explained and related to math drawings.

Math drawings of hundreds, tens, and ones that can show various multi-digit methods are shown in [Fig. 1](#) along with the current common traditional method (for the USA and some European and East Asian countries) and two mathematically desirable and accessible methods I have used in hundreds of classrooms with positive results from students and teachers (they truly are accessible methods). Of course the quantitative meanings of any math drawings must be developed with students. These are developed by originally drawing on 10 cm × 10 cm grid dots. The ten-sticks are then abbreviated to quick-tens (plain little sticks) that do not show the ones, and the hundreds-boxes are abbreviated from the boxes drawn around ten ten-sticks (and 100 dots) to be just empty boxes that convey the meaning of one hundred things. A vital aspect is that each step in a drawing is linked with each step in the written method. Initially problems are presented in situations or horizontally, but children come to see how it simplifies problem solving to align like places above each other, both in the math drawings and in the numeric method.

In the widespread traditional method we call the New Groups Above method (see the left column of [Fig. 1](#)), any new ten or hundred is written above the problem in its column ready to be added with those multi-units (the new ten with the tens, etc.). But what some students say about this method is that it changes the problem. And it does change the original addition of two numbers. A simple modification of that method is shown in the second column with drawings. This New Groups Below method is conceptually clearer because each number is in its own horizontal space so you can see each of the three quantities (the two addends and the total) without the conceptual confusion of the little 1s at the top. This method also has several other advantages identified by children and by teachers. First, you can see the totals in a column, the 16 ones (see the first row) and the 14 tens (see the second row), more clearly when the 1 is written below close to the 6 and the 4. Second, children can write the teen total (e.g., 16 or 14) in the order in which they usually write it (1 then 6) rather than “write the 6 and carry the one”. When children “carry the 6” it is sometimes because of this writing order and not because of a conceptual error. Third, it is easier to add the three numbers because you add the two numbers you see (e.g., 8 and 5) and then increase that total by 1. With New Groups Above, children tend to forget to add the 1 if they add the two big numbers they see. So teachers tend to urge students to add from the top down. This requires them to add the 1 to the 8, then ignore the 8 they see and remember a 9 they do not see, and add the 9 to the 5. So the New Groups Below method avoids

some complexities of the current traditional method. Teachers have reported that it is much easier for students with mathematical learning difficulties, who may have difficulties with the spatial issues of writing numbers, the spatial or conceptual organization of the three quantities, or the actual adding of single-digit numbers. The math drawings linked to the symbolic method can help with any of these issues.

Children's invented methods tend to move from left to right, as does reading in European languages. The Write All Totals methods (shown in the right column of Fig. 1) is helpful to many children with this preference. It also clearly shows the addition of the quantities in each place by writing out the numbers fully. This is conceptually clearer for many children. This method can also be done from the right.

The drawings use the 5-groups that are used widely in East Asian textbooks. They support children's seeing and using the single-digit make-a-ten methods, so many children do move on to these mental methods after repetitive experiencing with such drawings. As a child would explain for the top left drawing, *I can see that the nine needs one more to make-a-ten, so the seven gives it one more. That leaves six in the seven, so ten and six is sixteen* (writing 16). Even with the 5-groups, children are very creative with their drawings. The second child liked to make-a-ten from the two fives because it looks like a ten-stick. This also is a mental method that works for all problems when both addends are five or more.

Most children stop using the math drawings when they are not needed and the steps with the written numerals have taken on quantitative significance. But for some children, continuing the drawings for a long time is better than not being able to solve problems at all.

These examples show how we can teach children methods that are accessible to them but are also mathematically desirable—generalizable, rapid enough, and showing important mathematical ideas. So modifying the cultural method and supporting meaning making in ways that are simpler than always (and just) using real objects or just using written symbols can help many children with mathematical learning difficulties. Research then can focus on what other supports are necessary beyond these to help overcome specific learning issues particular children might have.

Thus, one important focus of future research on mathematical learning difficulties is to determine the extent to which particular difficulties children have at Levels A2, A3, and A4 really are special learning difficulties of the identified children rather than ones stemming from inadequacies in the learning opportunities that the children experienced. Did their classrooms not provide sufficient learning supports or support movement to a mathematically desirable method? Or did it demand unnecessary performance such as memorized addition–subtraction facts rather than a general counting-on method or require a difficult algorithm rather than a mathematically desirable but accessible one? Geary's (1994) recommendations for learning interventions with children having different types of learning difficulties are consistent with the balanced learning–teaching view in Table 2. Therefore experiencing such approaches in original learning might reduce the number of children experiencing learning difficulties, though they might need more time to master particular topics.

## **5. Insufficient meaning-making among researchers: some issues in the research Levels A1, A2, A3, and A4**

### *5.1. Level A1: the earliest capacities*

Research area A1 with human infants has explored issues of what biological precursors for mathematical knowledge are wired into human infants, primates, birds, and other species capable of participating in research tasks without verbal knowledge. These studies largely center around differentiating very small numbers (1, 2, and 3) and estimates or comparisons of magnitudes larger than these (adult primates seem to have more capacity here than human infants). Methodological difficulties plague the design and interpretation of these studies (e.g., are infants processing number or area or what?). Models of numerical processing are an important focus of this research, but these often seem mere restatements in information-processing or other language of steps that might account for the behavior. And the claims that these early competencies form the bases of later mathematical performance seem under-supported, even analytically.

## 5.2. Meanings of more

The evidence does seem clear that humans (and primates and some birds) are wired to attend to quantitative attributes in the environment. These probably are numerical for very small numbers of whole things and probably are numerical or area or density or some combination of these for judgments of which of two larger groups of things has more. In real life there are fewer things that are exactly the same size than in our modern manufactured lives. Nuts and fruits and seeds and other things to eat vary in area/volume, so a more/less judgment mechanism that confounded or integrated these different mathematical attributes would be helpful rather than problematic. This is also consistent with Piaget's findings of children's initial use of length and density rather than number to make more/less judgments in the conservation of number task. The two rows of objects originally correspond on all of these (the number, length, and densities of the two rows were all equal), but this clever task pits use of number against use of density or length when the objects in one row are moved to be closer together (or farther apart) than those in the other row (Piaget, 1941/1965). Adults and older children (aged 7 or 8) have un-confounded number from length and density and have a sense of logical necessity about their judgments. They can give the Piagetian general arguments about why such movements have not actually made one row more than the other (e.g., reversibility, compensation of length and density, you have not added or subtracted any). They often exhibit the facial or body language message "Are you crazy? Why would I think that moving those things apart would change the number?" even though they did think that two years ago when you interviewed them on the same task.

However, such children have also learned to differentiate meanings of the word *more*. As part of my work in research areas A2 and A3 (Fuson, 1988), we told 4- and 5-year-olds that they would see some rows in which one row would *look like more* but the rows would really be the same and some cases in which one row would *look like more* and really be more. Eighty percent of 4-year-olds and all 5-year-olds chose the longer row as the row that "looks like more" and then spontaneously counted on about two-thirds of the trials and matched on one-third of the rows to decide that the rows "really are the same." Therefore, discussion of both meanings of *more* (the length focus that *looks like more* and the focus on the number of objects as *is really more*) can help clarify what these words mean and what strategies are most appropriate. These children were not yet Piagetian conservers by his logical necessity criterion, but they were able to use language and the cultural competencies of counting and matching to make correct more/less judgments in his difficult situation (see also research summarized in Clements & Sarama, 2007; Fuson, 1988, 1992a, 1992b; Sophian, 1988). This again underscores the importance of language and situations in mathematical learning.

## 5.3. Confusions in the use of the mathematical terms ordering, order relations, and ordinal number

The use of counting to make more/less judgments involves a group of related mathematical concepts and words that are frequently misused by researchers as they confound various mathematical meanings. When asked to count in conservation of number situations, many 4-year-olds can count both rows accurately, remember both count words, change them to cardinal numbers, and find the order relation (more/less) on the cardinal numbers (Fuson, 1988). Children use the number word list to find more/less by generalizing that, for two number words, the word farther out in the list names the set that is more than the other set (Fuson, Richards, & Briars, 1982). This complex task involves related but different mathematical concepts: *ordered* or *an ordering* on the words in the number-word list and on the objects in counting, *cardinal number*, *ordinal number* (the *fifth* block from the start), and *order relation* (on cardinal numbers, 6 is more than 4,  $6 > 4$ ). The number-word list of each language has a linear ordering that must be used and that allows it to become a mental representational tool for comparing (finding more/less) and adding/subtracting. Children establish a linear ordering on objects as they count them, but this ordering is not unique: Any ordering that uses all of the objects once and only once will do. Each counting word is attached to one object via an indicating act like pointing that relates the word said in time to the object located in space. The last word said in counting is thus related by the indicating act to the last object in the ordering used in the counting. That last word must then undergo a conceptual shift from that counting meaning to a cardinal meaning as referring



to all of the objects just counted. A request to decide which of two sets is more (or less) is asking for an *order relation* (< or >) on the *cardinal numbers* of those two sets. This is different from an *ordinal number* that arises in a situation that has a linear ordering (like a line at a ticket office or the list of the runners in order as they cross the finish line). Here one might count to find the ordinal number, but then one must make a shift at the end from the counting number (e.g., the seven said as one points to one's sister in the ticket line) to the ordinal number word that tells in which place in such an ordering a particular object is (*My sister is seventh in line.*). Most languages have the same words for counting and for cardinal numbers but have special words or endings for ordinal numbers. Confusing these mathematical terms obfuscates the diagnosis of difficulties in learning mathematics as well as making comparisons across different research studies more difficult.

#### 5.4. A number line is not an early representation of number for children but a number path and a number list is

The linearly ordered mental number-word list can be symbolized in written numerals in two different forms: as a count model (a number path or a number list) and as a measure model (a number line). These two forms are shown in Fig. 2. The top count-cardinal model (the *number path*) is the form discussed above for the three levels of addition and subtraction methods. This model could also be simply a number list of the written numerals equally spaced. In this model, each number word is taken as a unit to be counted, matched, added, or subtracted. The *number path* form is often used in children's games: the circles or squares around each numeral form a path along which the game tokens move. This is a *count/cardinal* model in which each number word or numeral counts (is paired with) things. The count of things (numerals or squares) uses the *count word reference* shown in the third row of Fig. 2: *This square is where I say five.* The counter then makes a shift to think of that number word *five* as a cardinal number that tells how many objects in all so far, the *cardinal word reference* shown in the fourth row. A *number path* can also have the numbers outside the squares or circles just below, above, or beside them, and the squares or circles can be grouped into fives and tens to show place-value meanings of the cardinal numbers. A *number path* with some shapes designating and thus unitizing each numeral is helpful for games and learning supports. However, the internal number-word list discussed above that children use to carry out Level 2 and Level 3 mental addition and subtraction methods probably has no such extra shapes, may have different orientations (mine is often vertical), and may be primarily auditory rather than visual for some children.

A *number line* (see Fig. 2) is a measure length model like a ruler or a bar graph in which numbers are represented by the *length from zero* along a line segmented into equal lengths. The numerals on a number line tell how many total unit lengths so far. Young children have difficulties with the number line representation because they have difficulty seeing the units—they need to see and count things, and the unit length intervals are not as salient as are the numbers or the interval end-marks. So children often focus on the numbers instead of on the lengths, count the starting 0, and then are off by one in their count. The *Adding It Up* report (NRC, 2001) recognized the difficulties of the number line representation for young children and recommended that its use begin at Grade 2 and not earlier. The *number line* notation of writing the numerals at the end of the lengths is not needed until one wants to show fractional parts of one whole such as one-half. For whole numbers, to clarify that what is being counted is unit lengths, the numerals could be written *under* the lengths. But when one puts points in between the whole numbers, one needs to shift to the *number line* notation where the numerals (whole numbers or fractions) are written at the end of their *total length so far*. Grade 2 is an appropriate time for this.

In early childhood materials and in research, the term *number line* or *mental number line* really means a *number path count* model. Case's (1991) cognitive model uses this term *mental number line* when a *mental number list path* is actually meant (personal communication, 1998). The NRC Committee's report on Early Childhood Math (2009) does make this crucial distinction because it is so important for children to use visual models that reflect their thinking. It addresses why number lines are not appropriate in early childhood, but number paths or number lists are.

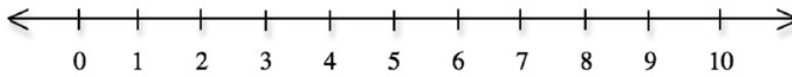
There is a considerable body of research on children's and adult's use of a mental model termed the *analog magnitude system* to estimate large quantities or to say where specified larger numbers



**A Number Path: Counting Things**

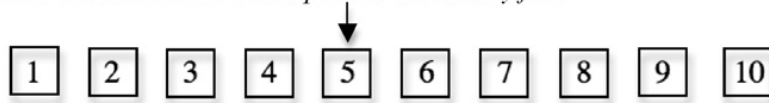


**A Number Line: Counting Length Units**



**Count and Cardinal Word Meanings When Counting Things in a Number Path**

Count word reference: *This square is where I say five.*

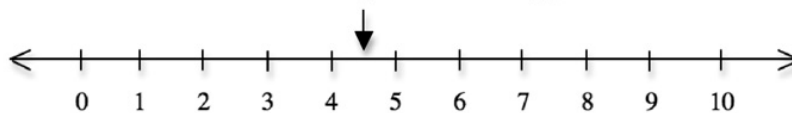


Cardinal word reference: *These are five squares.*

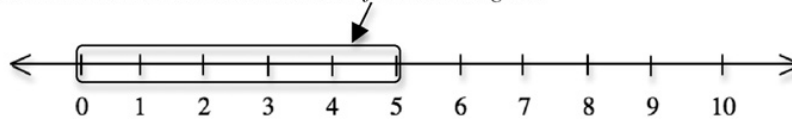


**Count and Measure Word Meanings When Counting Unit Lengths on a Number Line**

Count word reference: *This unit length is where I say five.*



Measure word reference: *These are five unit lengths.*



**Fig. 2.** A number path and a number line.

fall along a number line. However, it is not clear, especially for younger children, whether they are using a mental *number list* or a *number line*. The crucial research issue is the change in the spacing and differentiation of the numbers with age, and this could come either from children's use of a mental *number list/path* or a *number line*. Given children's massive experience with counting and the number list, and the lack of experience of many children with a number line, it seems much more likely that they are using a mental *count list/path* rather than a *measure length/number line*. Confusing these terms, and attributing number line use when it is really a number path can create problems as we try to think about children's conceptions and what is really going on mentally (in both the visual and auditory systems). The use of number lines such as in a ruler or a bar graph scale is an important part of measurement and is appropriate in these contexts. But for cardinal number situations and relations and operations on these cardinal numbers (groups of things), children solve with external groups of

objects and then internalize the groups of objects to form mental count-cardinal models. Therefore number lists/paths are the more appropriate pedagogical models to support number understandings, and they are more appropriate cognitive models for the numerical functioning of young children. When tasks use 2-digit numbers (e.g., to measure the numerical distance effect), it is important to consider that children must build the A4 conceptions of multiunit numbers as involving tens and ones and integrate these conceptions with the patterns in their A3 unitary counting list of numbers from one to one hundred. Forming concepts of length measure, bar graph scales, and fractions using number lines is a later development.

## **6. Interventions for Learning Difficulties in Levels A2, A3, and A4: counting, single-digit, and multi-digit number and addition and subtraction**

Interventions for helping children with difficulties in learning arithmetic are reviewed by Dowker (2005, chap. 12). Several programs target children who are far behind their peers. Some of these focus on children from backgrounds of poverty, who have consistently been found—as a group—to be behind children from middle-class homes (or having educated mothers—measures vary across studies). Klein and Starkey (2004) and Starkey, Klein, and Wakeley (2004) developed the Berkeley Math Readiness Program, which enabled preschool children to show gains in several areas. Clements and Samara's Building Blocks program (Clements & Samara, 2007, 2008) is the best-researched preschool program, and it also clearly articulates research-based learning trajectories that underlie the program (Clements & Sarama, 2009). The program shows significant benefits even during scaling up (the most challenging phase of programs). Griffin, Case, and Siegler (1994) reported positive results for the RightStart program at the Kindergarten level that permitted the 23 children to remain ahead of the control children in first grade. The program and its more recent published version, Griffin's Number Worlds (SRA/McGraw Hill), uses the model of a mental number list (most program materials are actually number paths, as is appropriate, though some are number lines and all are called number lines). Ramani and Siegler (2008) have recently carried out short effective interventions with preschool children playing a game on a number path (squares are drawn around the numbers) counting on 1 or 2 and saying the numbers on which their token moves. My own Math Expressions Kindergarten program (Fuson, 2006, 2009) can function as such an intervention for English-language learners or for children from backgrounds of poverty (or both). Use of the program in half-day kindergartens for 30 min a day as the whole-class math program led to understandings of teen numbers and of word problems with totals to ten equivalent to those of East Asian first graders and above that of American first graders (Fuson, 2009). Because math learning is cumulative, closing the SES gap before entry to first grade can considerably decrease the numbers of first-grade children who will experience learning difficulties stemming from having less knowledge than their classmates.

Two other programs target children identified by teachers as experiencing learning difficulties in the early grades. Mathematics Recovery (Wright, Martland, & Stafford, 2006a; Wright, Martland, Stafford, & Stanger, 2006b; Wright, Stanger, Stafford, & Martl, 2006c; Wright, 2003, 2008) has been used widely in English-speaking countries (Australia where it was developed, the USA, England, Ireland) and is quite successful in enabling children to catch up. It builds upon Steffe's extensive research (Steffe, 1992; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983) on the details of children's progressive use of different kinds of units for counting (a particularly detailed treatment of the three levels identified earlier that are used in most summaries of the research). Teachers work intensively with individual children 30 min a day for 12–14 weeks moving through the progression from where the child enters to counting on and counting back for numbers under 100. Numeracy Recovery (which now has become Catchup in Numeracy) was developed and tested by Dowker (2005, 2008). It is aimed at the lowest performing 15–20% of first graders. It identifies nine target areas, and the research indicates that children have gaps in varying ways. Teachers are released for half a day a week and work with each identified child in their class for 30 min a week for about 30 weeks. This much less intensive approach does enable children to catch up in their own needed areas. Having teachers work with children from their own class enables the teachers to understand and support these children better in the regular class. This program includes more topics than Math Recovery (for example, word problems of each type, arithmetic principles, and

estimation) and concentrates less on moving children through the core counting levels discussed earlier.

My Math Expressions regular classroom math program (Fuson, 2006, 2009) combines many of the features of these two programs in terms of content, although it is intended as the initial classroom learning experience and not as an individualized intervention scheme. We find that most children can close their gaps in Kindergarten, with appropriate instruction, but a few benefit from extra time and targeted attention. Some first graders who enter without the Kindergarten preparation also benefit from extra time on the program learning activities and targeted attention for any gaps. As discussed earlier for the multi-digit methods, the Math Expressions program (Fuson, 2006, 2009) also moves rapidly to mathematically desirable methods that are more accessible and generalize more readily than do multi-digit methods based on counting on and counting back.

Thus we see the need for three aspects of early number learning through about age 8. One is for regular math programs that support children through the research-based learning paths of conceptually based solution methods and that reflect the balanced approach integrating Piaget and Vygotsky summarized in Table 2. A second is for extra supported learning time, preferably early in the year, for children who enter a given grade way-behind in this learning path so that they can catch up. A third is for some weekly help for slower learners to help them consolidate current topics or review and consolidate past topics. Together these steps would greatly increase the number of children gaining grade level understanding and fluency in the early years of school.

There is considerable research seeking to identify types of learning difficulties children have in arithmetic, but less work on interventions to overcome such difficulties. Geary (1994) and Geary and Hoard (2005) identify three subtypes of learning difficulties: procedural, semantic memory, and visuospatial. Butterworth (1999, 2005) summarizes aspects of developmental dyscalculia. Jordan, Hanich, and Uberti (2003) review aspects of children having mathematical difficulties. A developmental delay is a major element in many such cases, and difficulty (slowness, errors) with single-digit addition and subtraction is a frequent occurrence. Use of primitive strategies is sometimes identified as involved, but often with no differentiation between Level 1 counting all or Level 2 counting on. I have found in extensive work in many classrooms, including special education and learning disabilities classrooms, that counting on for addition, and especially counting on for subtraction as a forward method, is fast and accurate enough for almost all children to carry out more-advanced mathematics in later grades. Thus use of counting on for totals between 7 and 18 combined with rapid knowledge of addition and subtraction for smaller totals (with occasional use of counting on as a backup strategy) is really acceptable for less-advanced and struggling students. Forcing rote memorization of additions can interfere with later learning of multiplications.

What would be helpful for future intervention research is ascertaining the special kinds of difficulties particular children might have even in learning counting on for adding and subtracting. Methods of keeping track of how many have been counted on can involve fingers or visual, auditory, or body movement patterns. A child might have difficulties with one of these but not another. It also would be helpful to identify issues particular children have making and using math drawings or other place value representations to support understanding of multi-digit methods of adding and subtracting.

## 7. Conclusions

Piaget and Vygotsky have had a massive effect on our knowledge about children's mathematical learning up to age 8 or so. We now have detailed understanding of children's learning paths for the Levels A2, A3, and A4—counting and cardinal relationships, single-digit addition subtraction, and multi-digit addition and subtraction. We also have some knowledge of cultural and symbolic issues that affect movement along these learning paths. The integration of Piagetian and Vygotskian perspectives on learning and teaching reflected in Table 2 provides a strong basis for mathematical school learning that can move children along known learning paths and support both understanding and fluency. Interventions based on the explosion of research have

been created, and research could productively continue on these. It now seems that research on learning difficulties in number and arithmetic (the most central aspects of early mathematics) could shift from debates about the nature and extent of such difficulties to how to overcome them.

Research on mathematical learning difficulties is also in a particularly good position to help disentangle which kinds of learning difficulties stem from cognitive deficits that impair particular aspects of numerical processing, which kinds have been caused or increased by inadequate learning supports in the home or school, and which parts of solution approaches within learning paths that flow relatively easily for normal children present particular difficulties for which kinds of learning difficulties. Avoiding the misuse of mathematical terms, and of undefined and vague terms like *number sense*, and using precise and mathematically correct terminology, will also help the field progress. Intervention studies of ways to help children overcome learning difficulties is a particularly powerful research approach to disentangle these issues while also contributing to improved learning for these children. More attention to the crucial research areas A3 and A4 that form the foundation for further mathematical success would also be helpful.

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