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Teaching as Assisting Individual Constructive Paths Within an Interdependent Class Learning Zone: Japanese First Graders Learning to Add Using 10

Aki Murata
Stanford University

Karen Fuson
Northwestern University

The framework of Tharp and Gallimore (1988) was adapted to form a ZPD (Zone of Proximal Development) Model of Mathematical Proficiency that identifies two interacting kinds of learning activities: instructional conversations that assist understanding and practice that develops fluency. A Class Learning Path was conceptualized as a classroom path that includes a small number of different learning paths followed by students, and it permits a teacher to provide assistance to students at their own levels. A case study illustrates this model by describing how one teacher in a Japanese Grade 1 classroom assisted student learning of addition with teen totals by valuing students' informal knowledge and individual approaches, bridging the distance between their existing knowledge and the new culturally valued method, and giving carefully structured practice. The teacher decreased assistance over time but increased it for transitions to new problem types and for students who needed it. Students interacted, influenced/supported one another, and moved forward along their own learning paths within the Class Learning Path.

Key words: Addition and subtraction, Elementary K–8, Learning, Number sense, Social and cultural issues, Teaching, Vygotsky, Whole numbers

National reports summarizing research describe a new view of teaching mathematics that builds competence in culturally valued knowledge by relating such

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knowledge to what students already know (e.g., Bransford, Brown, & Cocking, 1999; Fuson, Kalchman, & Bransford, 2005) and that balances conceptual understanding and procedural fluency (and also develops productive disposition, strategic competence, and adaptive reasoning) to develop mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001). Implementing these views in the classroom requires new classroom teaching models. This article presents an overview of a perspective on teaching literacy that reflects the views of Tharp and Gallimore (1988). We adapt this perspective to teaching mathematics by identifying some new theoretical constructs and describe a new classroom teaching model, the ZPD Mathematical Proficiency Model.

The framework of Tharp and Gallimore (1988) specifies aspects of teaching as they relate to the Zone of Proximal Development (ZPD). It was developed in extensive classroom work with children from backgrounds of poverty and non-native speakers of English, and it carries an equity perspective. We apply it to the case study in mathematics so that we can be specific about the mathematical cultural tools used to support student learning, see student mathematical thinking in action within this model, and check how the theoretical aspects of teaching literacy extend to mathematics. We offer our ZPD Mathematical Proficiency Model to help "bridge the individual, the social, and the mathematical" (Thames & Ball, 2004, p. 432, in their review of Kieren, Forman, & Sfard, 2002). Our bridging of the individual and the social follows others who have emphasized that the constructivist/individual sense-making perspective and the sociocultural-situative-enculturation-communicational perspective can be viewed as complementary because individuals within a classroom learning community (including the teacher) are continually interpreting and adapting as they participate in the learning-teaching community (e.g., Cobb, 1994; 1996; Sfard, 1998). The bridging to the mathematical involves identifying aspects within the ZPD model that are specific to teaching and learning mathematics as well as those aspects that apply across subject domains.

We wrote the article with a Bakhtinian awareness (Bakhtin, 1986; Clark & Holquist, 1984) of how present in our thinking were the voices and words of so many others: the children and teachers in our classroom research, our readings of and conversations with many other articulations of theoretical perspectives on mathematics teaching and learning or of related perspectives (e.g., Cobb & Bauersfeld, 1995; Cobb, Yackel, & McClain, 2000; Fuson, 1979a, 1979b; Kieren, Forman, & Sfard, 2002; Lee & Smagorinsky, 2000; Pirie & Keiren, 1994; Wertsch, 1985), and our own continuing dialogue with one another.

TEACHING AS ASSISTING

Tharp and Gallimore (1988) articulate a Vygotskiian (Vygotsky, 1978, 1987) view of teaching and learning in which "teaching can be said to occur when assistance is offered at points in the Zone of Proximal Development (ZPD) at which performance requires assistance" (Tharp & Gallimore, 1988, p. 41). The ZPD is defined as the distance between the child's actual developmental level and his or her potential development under the guidance of or in collaboration with a more-experienced partner. Thus, teaching is a shared social activity. Tharp and Gallimore (1988) identify six different means of assisting performance: modeling, managing, giving feedback, instructing, questioning, and cognitive structuring. They summarize different activity settings that support literacy teaching and learning via collaborative interaction, intersubjectivity, and assisted performance. One such activity setting, instructional conversation, is particularly powerful in creating intersubjectivity, i.e., understanding what others are thinking and meaning by creating taken-as-shared meanings (Cobb & Yackel, 1996). Instructional conversations also enable assistance to be offered by more-experienced peers as well as by the teacher. In an instructional conversation, listeners and speakers make mutual adaptations by using the body language, intonation, and other means used by participants in ordinary conversations. However, the focus is on helping all participants in the conversation—verbal and nonverbal participants and the assister—learn more about the target instructional goals.

In this article, we articulate our extension of Tharp and Gallimore's (1988) model to mathematics and exemplify it with a case study. The case study reports how different means of assistance come together to support student learning as well as how individual learning paths relate to the Classroom Learning Zone and interact with each other in a Japanese Grade 1 classroom. Many of the aspects of the model were used implicitly during the intensive study by the first author of Japanese classrooms (the case study was taken from this research) and in a 10-year project in classrooms by the second author (the Children's Math Worlds project). This project focused on articulating for different mathematics topics, student errors, solution methods, and visual and contextual means of assistance that would enable teachers to support mathematical proficiency using a Class Learning Zone. This earlier perspective was described as using Piagetian notions of learning and Vygotskiian notions of teaching. The model emerged as it was progressively modified interactively in dialogue with data from our classroom research; with each other; and with helpful reviewers, the editor, and other readers of earlier drafts of this article.

We will work in this article with a specification of Tharp and Gallimore's definition of teaching: "Teaching occurs when *responsive* assistance is offered *by more capable others* at points at which performance *and understanding* require assistance" (italics are our extensions). This teaching may be about the mathematical instructional goals, about social goals for helping students become effective members of their culture within and outside of school, or about social norms concerning how the classroom itself will function to accomplish the mathematical and social goals. *Responsive* was added to emphasize the close listening and openness required by any assister. Because most school settings have many students with only one teacher, the inclusion of *by more capable others* emphasizes how much students can learn from their more-experienced peers. Such helping also assists the more-capable peers in learning more because they must take the view of another and think more deeply about the topic at hand. We expanded *performance* to *performance and understanding* to clarify that we mean the conceptual underpinnings of performance as well as the performance.[1]

In our model, assistance is offered only when it is needed; decreasing amounts of assistance are needed as students progress through their own learning path in any given topic. Tharp and Gallimore identified four stages in an individual's moving through the ZPD for a given performance goal (see the top of Figure 1): Stage I is assistance provided by more capable others, Stage II is assistance provided by the self (as the means of assistance of others are internalized into speech-for-self), Stage III is internalization-automatization-fossilization, and Stage IV is de-automatization with recursion through the stages as performance that was once mastered slips away over time. This often occurs as backing up one stage at a time until performance can be recovered and involves processes such as *folding back* described by Pirie and Kieren (1994). Decreasing assistance over time is part of *responsive assistance*, a term that emphasizes the need for creating intersubjectivity between the assister and the assistee and for giving assistance adapted to the assistee. This underscores Vygotsky's view of learning as a constructive activity by the learner so that the *internalization* process across these stages does not involve rote copying of behavior. Tharp and Gallimore suggest *guided reinvention through mutual participation* as capturing this sense of a Vygotskiian learner's individual activity set within an interdependent cultural activity setting.

Such Vygotskiian teaching is well documented in many cultures and in many different activity settings around the world (e.g., Fuson et al., 2000; Rogoff, 1990; Tharp & Gallimore, 1988; Wertsch, 1985). However, it often occurs with one learner and one assister. So how can such an ideal view of teaching possibly work in a classroom with one teacher and as many as 20 or even 35 students? We propose a theoretical construct to help us see how our definition of teaching could be enacted for individual students within the whole-class setting: *a Class Learning Zone within a Class Learning Path*. Key to this notion is our knowledge from research (e.g., Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003; Grouws, 1992; Kilpatrick, Martin, & Schifter, 2003) that for many mathematics topics, there are a few typical errors that stem from partial but incomplete understandings and some other more random errors from momentary lapses of attention or effort. Likewise, there are usually several solution methods, but these are limited in number and vary in their sophistication, generalizability, and ease of understanding. Thus, for any given mathematics topic, there are not 20 or 35 different learning paths or strategies for the teacher to understand and assist. Instead there are usually 3 to 6 strategies with minor variations, and these can be summarized in curricular materials that assist teachers in learning to assist students. Also, visual supports can be developed and shared with teachers to aid them in teaching particular topics. Of course for any mathematics topic or problem, there is always a possibility of new solutions or strategies, and not all can be anticipated, so the Class Learning Path needs to be responsive to such possibilities.

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[1] Tharp and Gallimore did mean both, but specificity about this point seems helpful.

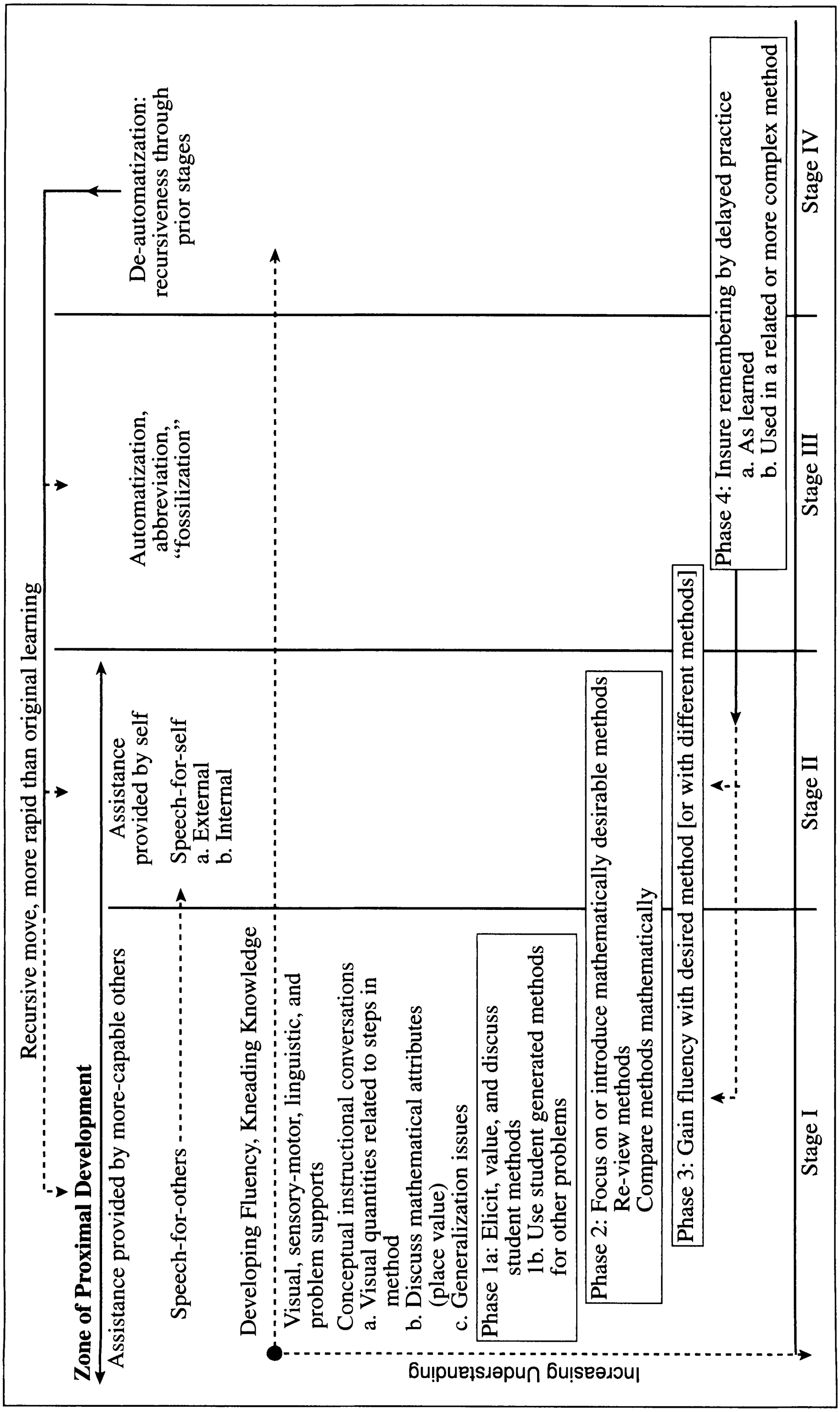


Figure 1. Stages of learning and class learning zone phases in the ZPD mathematical proficiency model.

Thus, when teaching using a Class Learning Zone, a teacher would orient students to the new instructional topic and then elicit from students their methods for solving such problems or for thinking about such contexts. With assistance from teaching materials, the teacher begins moving along a Class Learning Path that will provide assistance to move students forward to a good-enough and culturally valued general solution, with individual students starting from their own initial knowledge. The Class Learning Zone is the day-to-day learning zone within which the teacher organizes assistance to various students. Exceptional students (either extremely advanced or extremely delayed) may fall outside the Class Learning Zone. The former may assist others but may need assistance to do so (or to want to do so). The Class Learning Path is the day-to-day sum of the learning paths of most of the students in the class (reflecting the state of growth in their methods and in their understandings each day), but this falls within manageable groups of related-enough mathematical assistance needs. A few students may not fully master a target solution method, but assistance will continue in subsequent units toward mastery. And students may also continue to use any powerful or general enough method of their own choice. Learning for all students includes increased understanding of how other students solve problems and increased ability to assist other students.

Figure 1 summarizes our ZPD Mathematical Proficiency Model. The model is further specified in Table 1, where four aspects of teaching are described for our case study, and in Figure 2, where the mathematical support tools that assist semiotic cognitive structuring are described. With this model, we attempt to show the process of and the relationships between the development of conceptual understanding and fluency within the Tharp and Gallimore (1988) framework while paying special attention to the ways that this model applied to our case study. From Tharp and Gallimore's work, we moved *internalization* from Stage III to Stage II and specified the movement of speech-for-self from (a) external to (b) internal. This is an important self-regulating developmental cognitive step discussed by Vygotsky (1987).[2] We added *abbreviation* to Stage III because this is an important part of building fluency in many mathematical processes. We also identified four Class Learning Zone Phases in teaching that reflect the general characteristics of the ZPD but that cut across the stages and describe general aspects of teaching mathematics. These phases will be discussed in more detail in the case study.

We identified in the ZPD model two independent but continually interacting aspects of teaching over time: developing understanding and developing fluency. Fluency moves to the right horizontally along Tharp and Gallimore's four stages. Understanding moves down vertically and is central in our first two phases. However, the visual, sensory-motor, and linguistic supports that provide the bases for building understanding and the conceptual discussion that assists understanding can continue at any stage or phase of learning in the ZPD. These cultural tools form the backdrop for all collective and individual functioning within the

[2] See Fuson, 1979a, 1979b, 1980, about this step in mathematics and Fuson, 1979a, for a review of self-regulating speech.

Table 1
Aspects of Teaching for Understanding and Fluency: Examples from the Japanese Class Learning Zone Classroom

Focus on Meaning Supports (Representational and Cultural/Visual Tools) and on Conceptual Discussion	
Visual, linguistic, and sensory-motor representational support for learning steps	Use visual representations (physical objects, drawings, and fingers along with oral explanations) to strengthen students' understanding of crucial steps: • Move objects to show 9 becoming 10 • Circle numbers to make 10 • Draw upside-down "v" to show break-apart pairs • Emphasize the critical conceptual step by using a colored ten in the drawing Help students make connections between different representations
Focus on Individual Mathematical Thinking	
Discuss, value, and assist students' ideas and thinking	Allow students to share ideas and different approaches Ask questions to guide student thinking Maintain students' ownership of ideas (call different methods by students' names, vote for methods after discussing advantages and disadvantages)
Support students' different learning paths	Vary questioning patterns to meet different levels of understanding of individual students and provide modeling and explanation when needed Include less-advanced students in whole-class practice to allow them to experience the whole process rapidly but support their individual solving as necessary with questions and modeling Consider differences among students as strengths, and create situations where they benefit from the differences
Focus on Mathematics	
Support generalization and focus on the mathematics of their learning	Support generalization of problems with the smaller number first: • Introduce and practice problems by their mathematical structure (e.g., $9 + n$ problems, then $8 + n$ problems) to support initial learning • Discuss the similarities and differences of problems according to their mathematical structure (size of first addend) Discuss mathematical aspects of methods (e.g., the new unit of 10 related to place value). Discuss whether to start with the smaller or larger addend
Focus on Assisting All Students to Speech-for-Self, Abbreviation, and Automatization	
Facilitate Fluency	Provide opportunities to practice with decreasing visual and question support Pair students and encourage them to practice using flashcards that mix the problem types (practice with immediate feedback) Send a worksheet packet home that explains to parents what students are learning and asks them to time the students as they finish their homework. This helps the teacher understand the fluency level of each student.

Class Learning Zone; the kinds of learning supports and the mathematical points within the conceptual discussion vary with the mathematical topic. The necessity to identify the learning supports for particular topics is part of our model and our research. Our focus on the separate goals of understanding and of fluency is

Steps of the BAMT method	Step 1 Find that 9 needs 1 more to make 10	Step 2 Separate 4 into 1 and the rest (3)	Step 3 Add 9 and 1 to make 10	Step 4 Add 10 and 3 to make 13
Counter use makes objects change from 9 + 4 to 10 + 3	Count 9 counters and 4 counters, then move 1 from 4 to make a group of 10 with 9.	See 3 left in 4.	Just see and think 9 and 1 is 10 (making of 10 already happened in step 1).	See 10 counters and see 3 counters and think ten-three or count on "ten-one, ten-two, ten-three."
Finger use makes each step visible separately	Open 9 fingers, see 1 more finger is folded to reach 10.	Open 4 fingers, fold 1, and see 3 fingers are still left.	Open 9 fingers, open 1 more, and see 10 fingers, or remember it has been done already with step 1.	Open 10 fingers, say "ten," fold them again, and open 3 more fingers and count-on as they are folded, ["ten-one, ten-two, ten-three"] or know 10 and 3, 13.
Visual representational drawings, make a numerical trace of old and new problems and of steps in the change process visible	9 + 4 / **1** The line under 4 toward the place between 9 and 4 helps students know they need to think of 9's partner to make 10.	9 + 4 / \\ **1 3** Two lines under 4 indicate how the number 4 is separated into two partners.	(9 + 4) **10** / \\ 1) 3 Circling of 9 and 1 shows how two numbers are combined to make 10.	(9 + 4 = 13 10 / \\ 1) 3 Shows 3 is the only number that is not yet a part of 10. So 10 + 3.

Note: The first 2 steps are also facilitated by the linguistic support of the term "partners" for the 2 addends that form the totals. The final step is also facilitated by the Japanese linguistic form of 13 as "ten three." When written on the board and in notebooks, the oval and the 10 in Step 3 were in red to highlight the making of the 10.

Figure 2. Teaching supports and their facilitation of the BAMT steps (example: 9 + 4).

consistent with the positions of the national reports described earlier and with conclusions of Schwartz, Bransford, and Sears (2005), who differentiated innovation (supported by understanding) and efficiency (fluency) in transfer studies and argued that both are important instructional goals and that both are needed to achieve adaptive expertise.

Our adaptations to Tharp and Gallimore's (1988) specific means of teaching assistance will be discussed at the end of our presentation of the case study. As we present case study transcripts, we will identify in brackets means of assistance we see operating in the classroom except that we will omit questioning because it was so frequent and inserting it breaks up the flow of examples. We found in writing the article that sometimes it felt more natural to use the word *support* than *assist*. *Support* seems more omnipresent and implicit, whereas *assistance* seems more from the outside and explicit. We think that these words form a continuum, and we will shift back and forth between them in the article.

JAPANESE TEACHING AND THE CASE STUDY UNIT

Lewis (1995) described how Japanese schools work to develop a sense of belonging among students by meeting their social and emotional needs, and how in return students develop strong motivation to want to do well academically. The Third International Mathematics and Science Study videotape study (National Center for Education Statistics, 2003) presented Grade 8 Japanese mathematics classrooms as places where students' ideas are developed as their contributions to classroom discussions build on and support one another, and teachers' questions carefully guide them to productive interactions. Teachers typically do not give evaluative feedback on students' answers but rather focus their effort on accepting and/or clarifying particular students' ideas while having the whole class evaluate those ideas. In such a learning environment, every student is expected to participate in helping others learn, and this interdependent learning strengthens the relationships between members of the classroom.

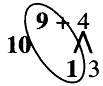
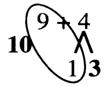
Our case study examines these processes at work in a Grade 1 classroom. We overview the teaching means of assistance that one Japanese teacher provided as students learned a culturally valued mathematics concept, how he changed the levels of support for the class and for individual students as the unit progressed, and how he used teaching supports and tools. We also see how students assisted other students within the Class Learning Zone.

The goal of the unit chosen for our case study was to learn to add numbers with totals in the teens. This unit was chosen because it involves learning a complex multistep method, the Break-Apart-to-Make-Ten (BAMT) method, which is specified in the Japanese National Course of Study. The complexity of this method pushes our concept of a Class Learning Zone because this method is demanding for less-advanced students. Teachers and students' parents had also learned this method when they were students. Students' understanding of and fluency with the BAMT method are viewed as important for their future learning of multidigit addition and subtraction in the curriculum because it helps them make sense of and use the values of 10-ness in the number system and it is a general addition method useful in multidigit addition, where students will be moving the new group of 10 (of whatever units) to the next left column. This method also prepares for related methods for subtractions.

Prior to this particular curricular unit, the Grade 1 students had explored the concepts of numerals and counting, decomposition using break-apart partners of numbers less than and equal to 10 (e.g., $6 = 5 + 1 = 4 + 2 = 3 + 3$), addition and subtraction of numbers with totals less than 10, the number structure for teen numbers as 10 + another number, addition and subtraction of teen numbers using the 10 structure (e.g., $10 + 2 = 12$, $18 - 8 = 10$), and addition and subtraction with three addends using 10s (e.g., $4 + 6 + 3 = 10 + 3 = 13$, $15 - 5 - 9 = 10 - 9 = 1$). Students needed all of these prior understandings of mathematics as they learned to add numbers using the BAMT method (except the subtraction knowledge, which they would use in the following unit on subtraction using tens).

The top 3 rows of Figure 3 show steps in the BAMT method using an example of 9 + 4 and a representational drawing taken from the Japanese teacher's manual (Tokyo Publishing, 2000) and used in the case-study classroom. In this example, 9 + 4 becomes 10 and 3, which is 13 (ten-three). For step 1, as a student focuses on the first addend, 9, he or she finds that 9 needs 1 more to make 10. For this, the student is using his or her prior knowledge of 10 partners, that 9 and 1 make 10 together, but in the form of an unknown addend ($9 + n = 10$). For step 2, the student separates the other addend, 4, into two numbers, the "1" that will be combined with 9 to make 10 and the rest of 4. This step also requires the student to find an unknown addend ($4 = 1 + n$), which is cognitively more demanding than finding a total. Step 3 begins the final addition of the new problem 10 + 3, as 9 and 1 are added to make 10. This step was already done as step 1 but now is done again as part of adding the 3 numbers 9 + 1 + 3. For step 4, a student adds the 10 and the other number 3 left from the breaking apart of 4 to get the total of 13. For this step, students use their prior knowledge of the structure of teen numbers, that 10 and another number make a teen number ($10 + 3 = 13$). The BAMT method supports student thinking of numbers in terms of 10 by reframing the total of two numbers as 10 + another number (e.g., 9 + 4 becomes 10 and 3). This method is facilitated by the Japanese number words above 10: 11 is *ten one*, 12 is *ten two*, 13 is *ten three*, etc., so the final step is linguistically simpler in Japanese than in English.

It may seem restricting to prioritize one method (the BAMT method) over others when teaching young children because their prior experiences and mathematics backgrounds are likely to vary. However, as will be discussed later, focusing on the BAMT method was a gradual and natural process in this case-study classroom, and students used other methods (or partial methods) when they felt it would help them solve the problem. (For example, less-advanced students often used fingers to count for particular steps when they felt it necessary). Also the method was not "given" to students—it was elicited from students at an early stage of their learning to foster a sense of ownership. The approach was what Hiebert et al. (1997) called

Goals	Make 9 into 10 with part ofthe other addend	Find how many more still to add to 10	In new problem, 10 + 3, make 10	In new problem, 10 + 3, find total
Steps	Step 1 Find that 9 needs 1 more to make 10	Step 2 Separate 4 into 1 and the rest (3)	Step 3 Add 9 and 1 to make 10	Step 4 Add 10 and 3 to make 13
Visual representational support in textbooks and on the board	**9** + 4 / 1	9 + **4** /\\ **1 3**		
Level A Support: Steps 1–4	"9 and what number make 10?" [Teacher points to 9.]	Teacher draws sticks to elicit break-apart partners for 4. "What two numbers are you separating 4 into [to make 10]?"	"What do 9 and 1 make?" [Teacher circles 9 and 1, writes 10 next to the circle.]	"What do 10 and 3 make?" [Teacher points to numbers 10 and 3, says "ten and three" to make connections to the total "ten-three."]
Level B Support: Steps 2–4		Teacher draws sticks to elicit break-apart partners for 4. "What two numbers are you separating 4 into [to make 10]?"	"9 and 1 make …?" [Teacher points to 9 and 1.]	"10 and 3 make …?" [Teacher points to 3.]
Level C Support: Steps 2 and 4		Teacher draws sticks to elicit break-apart partners for 4. "4 is what number and what number?"		"10 and 3 make …?" [Teacher points to 3.]
Level D Support: Step 2 visually		Teacher draws sticks to elicit break-apart partners for 4. No verbal guiding.		
Level E Support: Step 4 visually and with partners		Break-apart partners are filled in from Level D. No verbal guiding.		(No guiding question)

Note: Each level supports fewer steps. Levels D and E often occurred in combination. For Level D, Mr. Otani all but once elicited only step 2 and students typically gave break-apart partners of the addends. Following this step, Level E support occurred when the break-apart partners remained on the board for a visual cue while students stated answers to problems without verbal guiding.

Figure 3. Steps, drawing, and levels of teacher assistance for learning the break-apart-to-make-10 (BAMT) method.

problematizing and sense-making: "The teacher's role includes developing a social community . . . that problematizes mathematics and shares in searching for solutions" (p. 16). The teacher problematized solution methods for addition with teen totals, and the class searched for the easiest/most useful method while individuals continued on their own learning paths toward that method, always in a sense-making environment.

It is also important to note that, according to prior research on children's number understanding in the United States and Europe (e.g., see summary of research in Fuson, 1992), the BAMT method is a conceptually powerful and general method compared to other methods young children are likely to use (e.g., counting-based methods). Very young children typically see a quantity as an aggregate of single units, and thus they need to count when finding a total in an addition situation. By counting each addend and then the total, children experience the units within the whole. As children gain more experience, they come to abbreviate the count of one of the addends and thus only count on from one addend to find the total. For example, for 5 + 3, they would abbreviate the count of the first addend and say, "five," then count on three times, "six, seven, eight," to find the answer 8. After more experiences, children see quantities as made by smaller chunks (e.g., 8 is 5 and 3 or 6 and 2), and become able to separate and combine these chunks freely. This way of thinking was supported in the Japanese mathematics curricular unit on number partners. Research has found that at this stage Asian children who speak Chinese-based languages tend to use 10-based method to add numbers when the total is larger than 10 (Fuson & Kwon, 1992; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Murata, 2004). In contrast, children who speak European languages tend to use known additions, especially doubles (e.g., for 6 + 7, think as 6 + 6 = 12, one more is 13). However, using doubles (even doubles ± 2) is not a general method, nor does this method give the teen number ready for regrouping as one 10 and some ones because it does not highlight the 10- in the process. The BAMT method is helpful in multidigit addition because the 10 that needs to be given to the next left column is already separated from the ones that will stay in the added column. The Japanese teachers' manual motivated the teaching of the BAMT method by discussing difficulties that students may experience as the numbers become bigger if they continue to use counting-based methods, and the teacher also communicated the advantage of the BAMT method compared to other methods for the parents through classroom newsletters.

The BAMT method is a general decomposition method that uses 10 as a base number and thus is especially useful in the future for solving multidigit addition (for giving 10 to the next place to the left). The Japanese course of study (and those in China, Korea, and Taiwan) has chosen to develop this general mathematically powerful method in all students and has increased the number of students who can learn it at this stage in several ways: by carefully developing the conceptual prerequisites in earlier units, providing visual learning supports for students at different levels of concreteness, organizing the problems to facilitate learning and understanding, and allocating a great number of lessons so that most students can develop understanding and fluency. The case study discusses these conceptual prerequisites and also identifies other assistance given by the teacher.

METHOD

Participants and Setting

Twenty-five Grade 1 students and their classroom teacher, Mr. Otani,[3] participated in the study, which was conducted at a full-day Japanese school in a suburb of a midwestern metropolitan city in the Untied States. The school is operated by the Japanese Ministry of Education and closely follows the Japanese National Course of Study. Administrators and teachers are sent directly from Japan through the ministry, and the instructional language is Japanese. The school houses approximately 200 students in first through ninth grades. These students' families typically come to the United States due to the fathers' work, stay for 2 to 5 years, and then return to Japan. Thus, they differ from other ethnic minority groups in the United States who immigrate to stay. This Japanese community puts much effort into preserving the culture in their children's lives as well as maintaining Japanese ways of teaching and learning, because they wish their children to have successful school experiences when they return to Japan.

Japanese schools are more homogeneous than U.S. schools are in terms of students' ethnic backgrounds, family's socioeconomic status, and student academic achievement levels (Rohlen, 1997; Schmidt, McKnight, & Raizen, 1997; Stevenson & Lee, 1997; Stevenson & Stigler, 1992; White, 1987). All schools use textbooks that closely follow the guidelines set by the Ministry of Education. When asked if or how this school differed from schools in Japan, the teachers, parents, and administrators commented that they felt this school was different because it gathered students from different geographical areas across Japan and that the students may be somewhat more alike because of their families' living situations in the United States and the level of fathers' jobs. However, with respect to teaching and learning, they felt that the differences between schools in Japan and this particular school were small.

Mr. Otani had taught upper-elementary and middle school grades in Japan for 8 years prior to coming to the school. In his 1st year, there were two Grade 1 classrooms, and Mr. Otani collaborated with the other experienced Grade 1 teacher. In the 2nd year, he taught the only Grade 1 classroom, which is the case-study classroom of this study. These 2 years were the first time he had taught young students. However, it is common for Japanese teachers to change grade levels yearly. In their first 10 years of teaching, most Japanese elementary teachers teach all six elementary grade levels and sometimes middle school levels (up to Grade 9). In some schools, teachers move along with students over several grade levels; in others, they move independently of students and often skip grades. Because of this

[3] The real name of the teacher is used at his request; student names are pseudonyms.

teaching across grades, Japanese teachers understand children's learning trajectories across grade levels.

To trace different learning trajectories in the classroom, six target students were selected (in consultation with Mr. Otani) to represent a range of performance in an interview conducted at the beginning of the school year and on their performance in the classroom (see Table 2).

Data Collection

The unit consisted of 11 lessons over a 3-week period during the 5th month of the school year. The researcher observed all but Lesson 10. For each observation, careful field notes were taken to record (1) general lesson procedures; (2) teaching-learning activities and their structures; (3) kinds of student participation; (4) students' engagement and reaction; (5) transcripts of whole-class discussions and discussions among peers during independent work (as many as possible of the latter); (6) questions and responses of Mr. Otani and students; (7) the strategies of the six target students while engaging in independent work; (8) use of different teaching supports (e.g., counters, fingers); and (9) conversations of the researcher (first author) with Mr. Otani or with students before, during, and after the class. The lessons were also videotaped to check, support, and supplement the observation field notes.

The target students were interviewed at the beginning of the school year (Month 1), right before the unit of addition with totals in the teens (Month 4), right after the unit (Month 5), and at the end of the school year (Month 11) to trace their addition methods. The interviews were conducted individually in Japanese. The students were asked to solve addition problems (2 + 6, 4 + 4, 6 + 9, 7 + 7 for Months 1, 4, and 5 interviews; 6 + 7 was added to the four questions for the Month 11) and to explain their thinking. The methods the target students used in the classroom as they solved addition problems were recorded. Nontarget students were also interviewed in Months 1, 5, and 11 as described above.

Data Analysis

Data from the observation field notes were analyzed to illustrate how Mr. Otani (1) provided assistance as students learned the steps of the BAMT method, and (2) changed his support levels for individual students and also over time. Field notes were coded for the external problem-solving steps that students took in the whole-class context and individually in independent work as they learned to use the BAMT method for addition, and for the kinds of support Mr. Otani provided for particular steps. Other means of support were also identified. Special attention was paid to the methods students used, difficulties students had, and the adjustments that Mr. Otani made to support students with these difficulties. An extensive data table was then created to contrast classroom activities and the shifting of the levels and steps over the course of the 11 lessons.

Table 2
Target Students' Method Use Change Over the School Year

Name	Class Performance	Beginning of Year (Month 1) Interview		Before the BAMT Unit (Month 4) Interview		After the BAMT Unit (Month 7) Interview		End of Year (Month 11) Interview		
		6 + 9	7 + 7	6 + 9	7 + 7	6 + 9	7 + 7	6 + 9	7 + 7	6 + 7
Shinobu	High	Count on from 9 [b] Show 6 as 5 + 1 using blocks .	Count on [b] Show 7 as 5 + 2 using blocks.	I	Count all [b] (A, T) Used 7 blocks and counted twice.	I	I	5-and-5	5-and-5	BAMT (2 sec) "Took 3 from 6 to make 10, 13."
Yuichiro	Middle	Count on from 9 [f, 1 by 1][i]	BAMT [f] (6 sec) Show 7 fingers, "separate 7 into 3 and 4, add 3 and 7 to make 10, 4 more is 14."	BAMT [f] (I) Show 9 fingers, "6 is 5 and 1, use 1, so 5 is left, 15."	BAMT [f] (4 sec) Show 7 fingers, "3 and 4 is 7, I take 3, 4 is left, 14."	BAMT [s] (7 sec) "for 9, bring 1, then 5 more is 15."	BAMT [s] (4 sec) "for 7, bring 3 more over, then 4 more is 14."	I	BAMT (3 sec) "Took 3, then added 4."	BAMT (2 sec) "Took 3, then added 3."
Kensuke	High	Count on from 9 [f, 1 by 1]	Count on [f, 1 by 1]	UOA (6 + 6)	Count on [f, 1 by 1][ii]	BAMT (I) "Bring 1 to 9, then 10 and 5 is 15."	BAMT (I) "Bring 3 to 7, then 4 is 14."	I	I	(Said, "Took 7 from 10 is 3, add 6 is 9": incorrect answer)
Kiyomi	Middle	Count all (A, A, T) [f, 1 by 1]	Count all (A, A, T) [f, 1 by 1]	Count all (T) [f]	Count all (T) [f, 1 by 1]	BAMT (4 sec) "I made 10 with 9 and 1, then 6 become 5, so 15."	BAMT (5 sec) "I made 10 with 7 and 3, then 4, so 14."	I	I	I

Table 2—*Continued*
Target Students' Method Use Change Over the School Year

Name	Class Perfor-mance	Beginning of Year (Month 1) Interview		Before the BAMT Unit (Month 4) Interview		After the BAMT Unit (Month 7) Interview		End of Year (Month 11) Interview		
		6 + 9	7 + 7	6 + 9	7 + 7	6 + 9	7 + 7	6 + 9	7 + 7	6 + 7
Yukiko	Low	Count all (A, A, T) [f]	Count all (A, A, T) [f]	Count all (A, A, T)	I	BAMT (I) "Gave 1 from 6 to 9, then 5 is left over, so 15."	BAMT (I) "Gave 3 from 7, then 4 and 10 is 14."	BAMT (2 sec) "From 6, took 1 and added 9, then 6 became 5 and 10, so 15."	BAMT (2 sec) "3 from 7, added 7, then left-over 4, and 10, so 14."	BAMT (3 sec) "Added 3 from 6 and 7, then 6 became 3, and 7 became 10, so 13."
Akemi	Low	Count all (A, A, T) [b]	Needed Further Explanation	Count all (A, A, T) [b]	Count on [f, 1 by 1]	Count on from 9 [s]	Count on [s]	Count on from 9 [s]	UOA (7 + 6)	BAMT (5 sec) "I made ten."

Note: Class performance was assessed by Mr. Otani. "Needed Further Explanation" is when students needed explanation of additive situation before they could solve the problem. Count on by fingers 1 by 1 involved unfolding fingers 1 by 1, unless otherwise noted. 5-and-5 is the five-and-five-to-make-ten method. UOA stands for use-other-addition (e.g., for 7 + 7, used 6 + 7 = 13 + 1 = 14). "I" means immediate answer; this may be recall, or very rapid use of BAMT, or of a pattern. For assisting the solution: [f] for fingers, [b] for blocks (base-ten unit and tens blocks were available), and [s] for speech. For count all, the part of the problem actually counted is identified as A (addend) and T (total): A, A, T is the full count all method, but some students did more-advanced abbreviations of it. For use of the BAMT method, the number of seconds taken to give an answer is noted to show progress.

[i] As 1 by 1, but folded 6 fingers.
[ii] As 1 by 1, opened and folded 7 fingers and then opened them as counted on.

After the field-note data were coded, the videotaped data were reviewed to verify the data coded from the field notes. The videotapes were transcribed and annotated for nonverbal aspects of interaction. Transcripts of the videotape and field notes were reviewed by another trained researcher to assess intercoder reliability for the levels of teacher assistance shown in Figure 3. Because the steps and levels were clearly distinguishable, agreement was 100%.

For individual student learning, the different methods and the steps of the BAMT method the six target students used were also analyzed using the framework in Figure 3. Changes in external steps used by the target students are shown in Table 3 (to be discussed in detail later). The researchers who coded the classroom data for the levels of assistance and the steps students took also analyzed the individual student data to assess intercoder reliability. Once again, agreement was 100%.

Field notes and videotape data were also reviewed for the visual and sensory-motor means of support (representational drawings, counters, and fingers) used by Mr. Otani and his students. The student textbook and teacher's manual were also analyzed for the presentation of such means of assistance.

RESULTS

Visual, Linguistic, and Sensory-Motor Means of Assistance

Mr. Otani often used counters, fingers, or visual representational drawings to clarify ideas (cognitive structuring). These teaching supports were tools that assisted students' performance (Murata, 2006a) by facilitating particular steps of the method and by working as an integrated system to assist coordination of the multiple steps into a fluent whole (Murata, 2006b). Figure 2 describes how these various teaching supports assisted student thinking and facilitated each step. Individual students used these supports in different ways, and Mr. Otani suggested the use of particular supports for students as they needed them.

Another learning support was the arrangement of problems within the BAMT unit (Murata, 2006b). Problems with the same addend were introduced and practiced together so that the first step of the BAMT process would be the same. Problems beginning with a 9 were given first because these have the easiest first step (9 just needs 1 to make 10), and problems were then given in increasing difficulty of that first step (increasing distance from 10 so 8, then 7, then 6) followed by mixed addends. Other problem types followed the 9s problem with the same assistance format.

The Teaching Phases and the Means of Assistance

All of the Tharp and Gallimore (1988) means of assistance were used in our case study. Mr. Otani used questioning and cognitive structuring pervasively; he used feeding back, modeling, instructing, and managing to a lesser extent. We also identified a seventh means of assistance that was used frequently that we called

Table 3
Target Students' Externally Supported Steps in Method Over 11 Lessons

	Shinobu	Yuichiro	Kensuke	Kiyomi	Yukiko	Akemi
1					C 1&2&3&4	
2						
3						
4	C 9: 2&4, a C 8: a C	C 9: 2&4, a C 8: 2, a	C 9: 2&4, a C 8: a	C 9: 2&4, a C 8: a	C 9: **1**&2&**3**&4, a C 8: 2, a	C 9: 2&**3**&4, a C 8: a
5	C 9/8: 2, a C 7: a	C 9/8: 2, a C 7: 2, a I 7: 2&4 [s]	C 9/8: 2, a C 7: 2, a I 7: a	C 9/8: 2, a C 7: a I 7: 2&4 [s]	C 9/8: 2, a C 7: 2, a I 7:1&2&3&4 [d]	C 9/8: 2, a C 7: a I 7:1&2&3&4 [d]
6	I 6: a	I 6: 2&4 [d] Mix L+S: 2&4 [d]	I 6: a	I 6:1&2&3&4 [d] I Mix L+S: 2&4 [f, d]	I 6:1&2&3&4 [d]	I 6: 2&4 [d] I Mix L+S: 2&4 [f, d]
7	I S+L: Ct-On (from larger)	I S+L: Ct-On (from larger)	I S+L: Ct-On (from larger)	I S+L: Ct-On (from first)	I S+L: Ct-On (from larger)	I S+L: Ct-On (from first)
8	I Mix: a	I Mix: a	I Mix: mixed method (C-O for # less than 5, BAMT for both numbers 5 or more)	I Mix: 2&4 [d]	I Mix: a (very long time per problem)	I Mix: a (many mistakes)
9	I Mix: a	I Mix: a	I Mix: mixed method	I Mix:2&3&4 [d]	I Mix:2&3&4 [d]	I Mix:2&3&4 [f, d]
10						
11	I Mix: a	I Mix: a	I Mix: mixed method	I Mix: a	I Mix: 2&4 [d]	I Mix: 2&4 [d]

Note: The table shows the steps supported externally, by speech, drawings, and fingers, as students solved problems. C for individual-in-whole-class practice, I is for individual practice. All class responses were at the support level being used by the whole class except when steps are shown in bold. # before colon is the problem type (e.g., 9 means 9+# problems), # after colon is the external BAMT steps used by the students. a means only the answer was stated. Ct-On is count-on. For individual practice, external support for steps are noted as [f]: finger support, [d]: drawing, and [s]: speaking. Almost all answers in individual practice were correct except for Lesson 8 for the individual practice, Akemi made many mistakes. Yukiko, for the same practice, took a very long time solving problems. Because Mr. Otani often asked students who were very slow or making mistakes to use the drawings, it is likely that the change for Yukiko and Akemi from Lesson 8 to Lesson 9 stemmed from Mr. Otani directly or indirectly (perhaps via another student).

engaging and involving. This often accompanied other means of assistance and contributed to a safe and lively classroom environment.

Phase 1: Elicit, value, and discuss student methods

From the very beginning of the first lesson, students' ideas and contributions were used to direct their learning. Students actively shared and discussed different methods and ideas, and all contributions were recognized and appreciated by Mr. Otani and peers. Although Mr. Otani guided and assisted student discussions to create productive interaction, it was very clear that students' ideas were the driving force of the lesson. Students were important mathematical contributors and thinkers in the classroom.

For the initial introduction of the unit, Mr. Otani showed a group of 9 blue magnetic counters and a group of 4 red magnetic counters in a row on the blackboard. Some students immediately shouted out the answer "13!" Mr. Otani then initiated discussion by saying, "Some of you are quick in telling the answer, but who can share with the class your thinking?" Several students raised hands to share their ideas. Sakiko went to the board when called and moved 1 red counter to add to the blue group:

Sakiko: From 4, I add 1 to 9 and make 10.
Mr. Otani: So, the 9 became 10 and 4 became 3?
Sakiko: Then we know 10 and 3 make 13. We learned that before.

Mr. Otani summarized Sakiko's method on the board (modeling, cognitive structuring, and instruction) and continued to ask for other students' contributions to drive the discussion.

Mr. O: Did anyone do this differently? 9 + 4? Nobuhiko?
Nobuhiko: 9 + 4 is . . . at first, 3 and 4 is 7.
Students: What? What are you saying? We don't understand! [Overlapping comments; giving feedback]
Mr. O: Will you say it again, Nobuhiko? [Managing]
Nobuhiko: I took 3 from 9 . . . [Moves 3 counters from 9]
Mr. O: I took 3 from 9?
Nobuhiko: Add that 3 and 4 . . . [Put 3 and 4 counters together]
Mr. O: So, 9 is . . .
Nobuhiko: 9 became 6. Separate 9 into 3 and 6.
Mr. O: OK. [Giving feedback]
Students: Oh, that's what he meant. I know it now. [Overlapping comments; giving feedback]
Nobuhiko: Then, 7 . . . I mean . . .
Mr. O: 7?
Nobuhiko: Became 7. 4 and 3 became 7, so 7 and 6 is 13. [Points to the counters]
Mr. O: 13. 7 and 6 make 13. You like 7 + 6, don't you?
Students: Nobuhiko has remembered that way for a long time!
Mr. O: Is that so? [Friendly laughter]
Students: Yeah, he always does this way. This is Nobuhiko's secret method!

Students worked to understand and evaluate Nobuhiko's solution method. His seemingly incomprehensible idea at the beginning ("at first 3 and 4 is 7") was gradually clarified with guiding questions by Mr. Otani. As other students made sense of his thinking, they accepted his method as a valid mathematical approach. As Nobuhiko worked to articulate his idea to his classmates, he explained one step at a time; careful reflection on his own thinking helped develop mutual understanding in the community. Other students actively evaluated the quality of Nobuhiko's explanation by demanding that he clearly describe the process of his thinking. In these ways, Mr. Otani and his students negotiated and established shared understanding of what a good mathematical solution and explanation of such a solution should be. The fact that this method is an untypical method (it was not listed in the Teacher's Manual) emphasizes how a Class Learning Zone can involve idiosyncratic student thinking and not just typical methods.

In their study of the development of mathematical practice, Bowers, Cobb, and McClain (1999) described how "clarity" came to be an important criterion for a good mathematical explanation in the Grade 2 classroom they studied. Clear explanation was also valued by Mr. Otani's classroom community. As was illustrated in Nobuhiko's example above, students often voiced their difficulty in understanding certain ideas by loudly saying, "I don't understand!" and thus stopping the discussion. Mr. Otani encouraged students to express their difficulty, too, by saying, "It is OK not to understand, but it is not OK not trying to understand." He stopped the class whenever students identified their difficulty and then gave additional explanation and support for understanding. Thus, it was an agreed classroom norm that clarification should be demanded by everyone, and that students should voice their confusion whenever necessary.

After Nobuhiko's contribution in this first lesson, Tadashi volunteered to show the class how he counted unitarily from 1 to 13 to get the answer. Following his contribution, two other students shared their counting methods: Tetsuhiro counted by 2s and Tomoko counted by 3s. Although counting was not the official topic of the lesson, Mr. Otani allowed students time to explain, and he then summarized for the whole class how counting by 1s, 2s, and 3s were different from one another (cognitive structuring).

After demonstrating counting-on, Mr. Otani called on Koichi, one of the most-advanced students in the classroom. He came to the board and did a variation of the make-a-10 method in which the 9 was first separated into groups of 5 and 4, and then another 5 was made by separating the group of 4 counters into 1 and 3, and the 1 was put with the 4 in the 9 to make the second 5. Thus, the 10 method was shown by 2 groups of 5 and the 3. Koichi used two embedded groups in his solution: the subgroups of 5 emphasized for the numbers 6, 7, 8, 9, and 10, and the group of 10 in 13. Seeing numbers 6 through 10 as 5 plus some more was supported in the previous unit on decomposition of numbers, in which students saw such numbers as a row of 5 circles with 1, 2, 3, 4 or 5 circles below to show 6, 7, 8, 9, and 10.

Koichi: I made groups of 5 and 5.

Mr. O: I wonder what is different about this method.

Students: He did 5 + 5 + 3!
Mr. O: Yes, how did you think of this, Koichi?
Koichi: I thought this way. 5 and 5 is 10, so 3 more is 13.

At this point, Mr. Otani summarized students' different approaches. The BAMT method was called the "Sakiko method," changing into 7 + 6 was called the "Nobuhiko method," 5 + 5 + 3 was called the "Koichi method," and the different counting methods and other recomposition methods that used different break-apart pairs of addends were also named after the students who shared those particular methods. The Sakiko method was the primary method to be taught in this unit according to the curriculum, but Mr. Otani spent approximately equal amounts of time explaining each method at this point, asking students questions to support their understanding of each of them.

In this first lesson of the unit, Mr. Otani encouraged students to share their ideas based on their spontaneous thinking and prior knowledge and allowed room for diverse methods. The sharing process at the beginning of the unit provided opportunities for students to re-view previously learned concepts, demonstrate their competence, and set the stage for future exploration. Mr. Otani carefully directed student discussion to focus on the process of solving the problem, and that provided opportunities both for the students who already knew the answer and for those who were experiencing such a complex problem for the first time. The students' thinking and methods were the driving force of the discussion, and they spent time developing clear explanations to show their thinking using the visual representational supports on the board (colored counters and representational drawings).

As the bell rang announcing the end of the period, Mr. Otani quickly asked, "But which one do you think is the easiest to understand? Which one is the most useful?" Students raised hands to vote for methods. The majority voted for the Sakiko method (BAMT), but several students voted for other methods. Mr. Otani stated that most students had voted for the Sakiko method and that they would continue learning the following day.

Phase 2: Focus on the BAMT method

Phase 2 in Mr. Otani's class involved three different kinds of instructional conversations: re-viewing different methods, comparing the methods mathematically and voting on the easiest method, and discussing place-value related to the BAMT method. Each of these will be summarized briefly.

Re-viewing. Figure 4 shows by lesson the types of problems and the kinds of teaching activities. The second and third lessons began with a whole-class instructional conversation re-viewing the different methods students had identified in the first lesson. The fourth and fifth lessons began with re-views of the BAMT method for problems with the first number as 9 (which has 1 as its partner to 10). The sixth lesson began with re-views of the ten partners for 9, 8, and 7 as a way to introduce the BAMT method for problems with the first number 6. These re-views

	Activities	Support						
		A	B	C	D	E	No	V
1	1. Whole-class exploration of different methods for 9 + 4							
	2. Voting for the easiest method [IC]							
2	1. Whole-class re-view of methods (9 + 4) [IC]							
	2. Voting for the easiest method [IC]							
	3. Discussion of place-value and the BAMT method [IC]							
	4. Whole-class intro for 9+ *n*							
	a. Step 1 for the set of 6 problems (discussion of 9's partner to make 10 [IC])	As						
	b. Step 2 for the set of 6 problems	As						
	c. Step 3 for the set of 6 problems	As						
	d. Step 4 for the set of 6 problems	As						
	5. Voting for the easiest method [IC]							
3	1. Whole-class re-view of methods (9 + 4) [IC]							
	2. Whole-class practice of 9 + *n* (3 problems) steps 1 – 4	Ap						
	3. Individual practice, 9 + *n* (4 problems)							V
	4. Individual-in-whole-class practice of 9 + *n* (problems from 3)							
	a. Step 1 for the set of 4 problems	As						
	b. Step 2 for the set of 4 problems	As						
	c. Step 3 for the set of 4 problems	As						
	d. Step 4 for the set of 4 problems	As						
	5. Discussion of 9's partner to make 10 [IC]							
	6. Voting for the easiest method [IC]							
4	1. Whole-class re-view of the BAMT method, 9 + 3, steps 1–4 [IC]	Ap						
	2. Whole-class re-view of 9 + *n* (6 problems) steps 2, 4 [IC]			Cp				
	3. Individuals-in-whole-class review of 9 + *n* (problems from 2) steps 2 and 4			Cp				
	4. Individuals-in-whole-class practice of 9 + *n* (problems from 2), BA partners written on the board (other things erased), oral answers, 6 problems					Ep		
	5. Individual-in-whole-class practice of 9 + *n* (problems from 2), BA erased, oral answers				Dp			
	6. Whole-class intro for 8 + *n* (8 + 3), steps 2 – 4 [IC]		Bp					
	7. Individuals-in-whole-class practice, 8 + *n* (7 problems)							
	a. Step 2 only, with break-apart sticks				Ds			
	b. BA written on the board, oral answers, teacher points to random problems					Ep		
5	1. Whole-class re-view of 9 + 5 and 8 + 6, steps 2–4 [IC]		Bp					
	2. Individual-in-whole-class practice of 15 mixed 9 + *n*, 8 + *n*							
	a. Step 2 only, with break-apart sticks				Ds			
	b. BA written on the board (other things erased), oral answers, teacher points to random problem					Ep		

Figure 4. Levels of support over 11 lessons.

	Activities	Support						
		A	B	C	D	E	No	V
5	3. Individual-in-whole-class intro of 7 + *n* (6 problems) [IC]							
	a. Step 2 only, with break-apart sticks				Ds			
	b. BA written on the board (other things erased), oral answers					Ep		
	4. Whole-class say answers to 7 + *n*, with BA partners written					Ep		
	5. Individual practice of 7 + *n* (4 problems)							V
6	1. Whole-class intro of 6 + 5 by discussing ten partner for 6, re-views of ten partners for 9, 8, 7 [IC]							
	2. Individual practice of 6 + *n* (5 problems)							V
	3. Individual-in-whole-class report of 6 + *n* (problems from 2), class gives feedback, "It is OK!"						No	
	4. Individual practice for 16 mixed 6 + *n*, 7 + *n*, 8 + *n*, and 9 + *n*							V
7	1. Individual-in-whole-class report answers on problems solved in Lesson 6; teacher writes equation and answer as it is shared, class gives feedback, "It is OK!"						No	
	2. Whole-class intro for smaller + larger (4 + 8, equation and answer only) [IC]							
	3. Individual practice of 12 smaller + larger; teacher notices that many students are counting on, so shifts to 4							V
	4. Whole-class discussion on smaller + larger, 2 + 9, steps 1–4, solved from 9 and from 2 [IC]	Ap						
8	1. Individual practice of 11 smaller + larger problems							V
	2. Individual-in-whole-class report answers on problems just solved (as in Lesson 7, 1 above)						No	
	3. Individual practice of two word problems							V
	4. Individual-in-whole-class report on problems just solved (disagreement on quantifiers)						No	
9	1. Individual practice on eight mixed problems							V
	2. Individual-in-whole-class report answers on problems just solved (as in Lesson 7, 1 above)						No	
	3. Individual practice on 8 mixed problems							V
10	Like Lesson 9 (no observation)							V
11	1. Whole-class report answers on eight mixed problems solved in previous class (as in Lesson 7, 1 above)						No	
	2. Individual practice on six mixed problems							V
	3. Individual-in-whole-class report answers on problems just solved (as in Lesson 7, 1 above)						No	

Note: Support always involved drawing on the board and sometimes (especially for individuals) also involved fingers or counters. The support identified is standard support for the class. Some individuals might have received more support. The small "s" or "p" placed after a support level letter stand for "steps" and "problems," respectively. For example, As means Level A support for a step, Ap means Level A support for solving the whole problem. No means no support. V means varied support with students (for individual practice). [IC] means "instructional conversation." BA means break-apart.

Figure 4. Continued

not only helped refresh students' memories but also connected their previous knowledge to the new topic (cognitive structuring). When a new first number was introduced, it was typically introduced as an extension of solving the problems with a more familiar number. For example, in introducing $7 + n$ addition problems in Lesson 5, Mr. Otani first supported students with $9 + n$ and $8 + n$ problems on the board by asking individual students to solve them. He then wrote $7 + n$ problems on a small portable blackboard, placed it right next to the $8 + n$ problems, and continued the previous questioning pattern (cognitive structuring using Level D support and then Level E support). Placing different types of problems side by side highlighted the similarities between the different problem sets. From Day 2 to Day 4, Mr. Otani shifted the conceptual emphasis of the reviews from the first step ("9 needs how many more to become a group of 10?") to the general pattern in the second step for all problems beginning with 9 ("So, we separate the other number to always get 1 as we did,") to a short-cut way of thinking about these problems ("When we try to solve these problems quickly in our heads, all we have to know is the left-over number after separating the number into 1 and another number").

We write "re-view" with a hyphen to convey the substance of these sessions. In Japanese schools, students are instructed to "re-view and pre-view" at home what they study at school each day. As a nightly study routine, students go over the materials they have studied that day as well as study the materials for the following day. This re-view and pre-view perspective is widely shared in the culture, and parents, when they were students, spent their evenings in the same manner. In re-viewing materials, students typically read over the relevant parts of their textbooks (their textbooks are very small and carried back and forth between school and home daily), look at their notebooks, and re-view their mistakes (from very early on in Grade 1 they learn how to take good notes in special notebooks with grid paper), and do additional homework assignments. After the re-view process, students proceed to preview the materials for the following day by reading the next section of the textbooks. Students typically receive much family support and encouragement at home for this re-view and pre-view studying time.

Comparing methods: Student voting. In the first three lessons, after students discussed different methods guided by Mr. Otani's questions, he asked students to vote for the method they thought was the easiest/most useful to use in this addition situation. When the students voted at the end of the first lesson, Sakiko's method (BAMT method) received the most votes, and the other methods each received a few votes. Although this divided students into different camps, there was no word, tone of voice, or observable body language to indicate competition between the groups or teacher pressure on any one to use or vote for a particular method. Norms had already been established in the classroom that different student methods were valued, and students felt comfortable expressing their feelings. This voting for the easiest method then continued in subsequent lessons. By the end of the third lesson, all students agreed that the BAMT method is the easiest method in this addition situation of totals in the teens. The process of voting helped individual students express their choices. Students gradually made decisions for their learning as they came to understand conceptually the generality and effectiveness of the BAMT method in a nonpressured environment.

In discussing methods, Mr. Otani used the terms "easiest" and "most useful." He used both terms for the first voting, and for the subsequent voting, used only the term "easiest." In Japanese, "easy" can be said in two different ways: *kantan* (簡単) and *yasashii* (優しい). *Kantan* means simple, easy to follow, lacking difficulty. It is often used to describe ease in using appliances or in finishing tasks. *Yasashii*, on the other hand, means accessible, friendly, and gentle. It is often used to describe a person's (or living thing's) character. It not only means accessibility to the person but the person's natural willingness to be connected. Mr. Otani used the terms *kantan* and *yasashii* interchangeably throughout the unit. Thus, when he and the students discussed the easiest method, they were talking about how the method was friendly to the students, easily connected to student thinking, and appealed to their natural approaches, as well as how simple it was to use.

Relating the BAMT method to the structure of the written numerals. Students continued to discuss reasons why the BAMT method was the easiest. Near the end of the third lesson, and guided by Mr. Otani's questions, they explored the base-10 number structure and the importance of "10s" in the number system (cognitive structuring). Mr. Otani wrote the numbers 0 to 19 on the board in two rows (0 to 9, 10 to 19, with the teen numbers on the second row under the first row of 0 to 9) and asked students if they noticed any pattern.

Mr. O:	This may be something you had known even before you started first grade. Let's read this together [he points to the numerals as students read them].
Students:	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.
Mr. O:	Does anyone see a pattern?
Hiroshi:	10 is 1 and 0, 11 is 1 and 1.
Mr. O:	What do you mean, Hiroshi?
Hiroshi:	10 is under 0 and is [written as] 1 and 0. 11 is under 1 and is [written as] 1 and 1.
Mr. O:	Is everyone OK with this? It seems like Hiroshi noticed the way we write numbers?
Makiko:	We only use 9 numbers to write numbers, like 1, 2, 3, 4, 5, 6, 7, 8, 9, like that.
Students:	[Overlapping comments] What is she saying? We don't understand!
Mr. O:	I think what Makiko is saying is that we use those same numbers repeatedly to write different numbers. Let's see how we do this. 1, 2, 3 . . . [Points to numbers on the top row, 0 to 9, as students read along]
Students:	1, 2, 3, 4, 5, 6, 7, 8, 9 . . . 10? [Seeing the numeral 9] 9? 10? [Discussion among themselves, some students seem confused]
Mr. O:	Look up here. OK? We may take this for granted, but we have 10 numerals, 10 numbers, and all the numbers are written with these 10 numerals. So, after 9, the next number is . . .
Students:	10!
Mr. O:	10, because after 9, we don't have a new numeral. And, we think of 10 as a chunk, and put that chunk in the tens place, so making 10 as we add numbers make sense here.

Phase 3: Gaining Fluency With the BAMT Method

When the word "practice" is used in Japanese classrooms, it conveys a meaning slightly different from that in English. The Japanese word "practice" is written as a combination of two Chinese characters

(練習).

The first character means "kneading" and the second character means "learning." Together, the word represents the meaning of kneading different ideas and experiences together to learn. Such kneading was observed in all three of the different modes of practice identified: whole-class practice, individuals in whole-class practice, and independent practice (see Figure 4). Students brought different ideas, experiences, and approaches to learning the BAMT method, and the differences were "kneaded" through various practice forms to support each student's learning as well as to establish a common understanding base in the classroom.

For the whole-class practice, Mr. Otani typically stood in front of the class with a set of problems written on the board. As he asked questions to support students to take a specific step in the BAMT method, he pointed to that part of the problem on the board, and students answered the questions together out loud. Mr. Otani often pointed to the questions on the board in order (e.g., from left to right, from top to bottom), but he sometimes pointed to the questions randomly so students could not think ahead. Sometimes, his questions assisted all of the steps to solve one problem and the same questions were asked to solve the next problem (see problem support in Figure 4). At other times, he asked questions for one particular step for all the problems on the board (see step support in Figure 4), then moved on to the next step for all of the problems. Step support happened in Lessons 2 and 3 when students were learning the steps of the BAMT method for the first time, then in Lessons 4 and 5 to assist their learning of step 2 (the most challenging step) for the first addend 8 (Lesson 4) and then addend 7 (Lesson 5) by combining Level D step support and Level E problem support. Students were encouraged to speak loudly for all whole-class procedures, and they shared enthusiasm and energy as they answered.

Individuals-in-whole-class practice on Lessons 3 through 6 followed the whole-class re-views or practice. With a set of problems written on the board, Mr. Otani continued to ask questions to support certain steps in the BAMT method, but students took turns answering the questions individually. Students usually answered by their seating order (e.g., starting from the student who sat at the front row of the rightmost side of the room to the student in the back row, then to the students in the next column, etc.). As with the whole-class practice, Mr. Otani typically pointed to the questions on the board in order, but sometimes changed orders or pointed randomly so that students could not practice their problem ahead of their turn.

The most distinctive difference for this individual-in-whole-class practice is that after one student answered the question, the student always asked the whole class, "Is it OK?" The whole class answered by shouting together, "It is OK!" if they agreed with the student, or "It is not OK!" if they did not. When there was disagreement, Mr. Otani guided the discussion among students to identify and resolve the disagreement.

For independent practice, students worked at their seats solving problems independently. Often, they worked on assigned problems from textbook pages, but once they finished them, they worked on a packet of worksheets Mr. Otani had prepared or a set of calculation cards (small flash cards that are put together by a ring). When individual students had difficulties, students who sat close by spontaneously helped them. Except for testing situations, students often helped one another. Mr. Otani usually walked around the room to monitor student progress during the independent practice time and provided extra assistance when needed. Students practiced independently during all but Lessons 1, 2, and 4.

Individuals-in-whole-class reporting of answers occurred in Lessons 7, 8, 9, and 11 after individual practice in which students had worked problems in their notebook. This gave students a reason to work hard on the problems, as students took turns reading answers from their notebooks. All students circled incorrect problems in red in the notebooks. If the student said an incorrect answer (if he or she received an "it is NOT OK!" response), Mr. Otani assisted that student in solving that problem by following each step to make sure the student understood. After the practice, students corrected all incorrect problems by writing all of the steps using the BAMT drawing in their notebooks. They then turned in their notebook for Mr. Otani to check.

The interactions among these different modes of practice supported student learning in different ways. The whole-class practice provided a fun and safe group-learning environment where students shouted answers together. Individuals-in-whole-class practice offered opportunities for individual students to show their developing fluency with the method and get whole-class feedback. Individual practice allowed students to focus on areas in which they needed more work and also created a foundation before the whole-class sharing of individual-in-whole-class reporting answers.

All of the practice had continual conceptual links. Visual, linguistic, and sensory-motor supports were always present or available. Even during practice sessions, Mr. Otani provided opportunities for students to think about the process by asking why they were making 10 or why students broke apart the addend in a certain way. The curricular chunking of problems by the same larger number created a new opportunity to think about the reason for making 10 each time students moved to a new number. Thus, fluency and understanding were intertwined and practice involved everyone "kneading their knowledge."

For some students, their fluency resulted in an abbreviated BAMT method made by combining steps 1 and 3 as the initial step and then taking steps 2 and 4. In this case, the 10 made in step 1 just remains in the background mentally as the other partner of the second number is found, and the 10 is then added to that partner number. As fluency develops, the whole process becomes almost automatic, and students take the steps so quickly that they may not be conscious of the steps they take to solve problems. With increasing fluency, some students also solved problems of particular types by using the pattern of the partner to ten reducing the ones by that number (e.g., 9 + 4 is 10 + 3, 1 less than 4).

Phase 4: Delayed Practice

Delayed practice (phase 4) is important to move students from the Stage III automatization and abbreviation stage to robust remembering through the use of Stage IV de-automatization (see Figure 1). This was provided after the unit in the re-view section of the textbook where concepts students learned previously are revisited and practiced. Here, students re-view independently in familiar practice contexts. The BAMT method was also used in a related or more complex method in a subsequent unit of subtraction using 10 and in multidigit addition in Grade 2.

Shifts in Teacher Levels of Assistance

The steps involved in the BAMT method were not difficult when they were taken one step at a time because each had been learned in previous units. However, many students experienced difficulty coordinating the steps into a fluent whole. Initially, Mr. Otani supported each step by questions (see Level A support in Figure 3). He then dropped support one step at a time, dropping the easier steps 1 and 3 first and keeping support for the most difficult step 2 at the final level (visual only). However, he always increased the levels of support for students who needed it.

Figure 4 shows how this full support decreased over time through the levels B through E and varied with the kinds of problems. On Days 4, 5, and 6, assistance decreased as the class continued to practice a given type of problem but increased when they began a new type of problem. From Day 4 through Day 7, the initial level of assistance at the beginning of the day decreased.

Sometimes more-advanced students spontaneously modeled for the class a BAMT method with fewer steps than the steps Mr. Otani was supporting in the Class Learning Zone. For example, when students in the whole class practice were experiencing Level C support (steps 2 and 4), Sachiko stood up to solve the problem 9 + 5:

Mr. O: [Points to the problem 9 + 5]

Sachiko: [Sachiko starts talking before Mr. Otani can ask guiding questions.] 10 and 4 is 14.

Mr. O: OK, OK, what did you do first?

Sachiko: Separated 5 into 1 and 4, then 10 and 4 is 14, is it OK?

Students: It is OK!

Here Sachiko was only doing step 4, but Mr. Otani elicited from her steps 2 and 4.

Different practice types and different levels of teacher assistance interacted to support student learning. When decreasing the level of support for a set of problems, Mr. Otani sometimes erased parts of the visual representational drawing of the problems already written on the board from the previous round of practice (e.g., erasing circles and "10" before shifting to Level D support) but continued to have students work on the same set of problems. This repetitiveness and continuity using the same set of problems especially helped lower-achieving students by providing them a sense of ease. In Lessons 4 and 5, Mr. Otani used, as a part of individual-in-whole-class practice, Levels D and E support to emphasize step 2 of the process for the new numbers 8 and then 7. By having students take only step 2 first and leaving the break-apart partners written on board (Level D support), students could focus on finding the final total of 10 + n with the Level E support that followed.

On Day 7, Mr. Otani introduced a new class of problems where the smaller number was the first addend (e.g., 2 + 9 instead of 9 + 2). In the individual practice, many students solved these problems by counting on and did not use the BAMT method. When Mr. Otani realized this, he initiated an instructional conversation shifting back to Level A support to discuss BAMT solutions for 2 + 9 (making a 10 with the second addend), and he related the solutions of 2 + 9 and 9 + 2 to each other. He drew on the board full representational drawings for 2 + 9 (where the 2 was broken into 1 and 1 to make a 10 with 9) and then for 9 + 2 (where the same partners of 1 and 1 were shown under 2 but now on the right). He then guided student discussion of these solutions by using two groups of 2 and 9 counters and asking students, "Can we move counters like this and make 10 on this side [for 1 + 1 + 9 = 11]?" and then for the counters 9 and 2, "Can we move counters to make 10 to make 11 this way [for 9 + 1 + 1 = 11]?" He then wrote 2 + 9 and 9 + 2 on top of each other and led a discussion by questioning to help students analyze which of these was easier and to see the similarities between the new situation and the larger-plus-smaller-addend addition situations they had been solving using the BAMT method. Most students quickly went back to use the BAMT method, and most started with the larger number even if it was the second addend.

Mr. Otani's questions also shifted through levels to become more abstract and informal. His questions at the beginning of the unit were explicit directives (e.g., "What number do 1 and 9 make together?"). As the unit progressed, he was more likely to state the same question as a process in action (e.g., "9 and 1 is . . .?"), or sometimes he only pointed to the numerals on the board as an implied nonverbal question (see Figure 3). His language shifts modeled speech-for-self that students could use externally and then internally to assist themselves.

Individual Student Paths of Understanding and Internalization

Students began the year with a wide range of knowledge about addition with teen numbers (Murata, 2004). In interviews on 6 + 9 and 7 + 7, students needed additional assistance to get started with 16% of the problems, count-based methods were used on 60% of the problems (36% count all, and 24% count on), the BAMT method was used on only 3%, and remembering/recalling the answers on 22% of the problems (the same percentage for both problems). In the interviews of the six target students immediately before the BAMT unit, they showed a wide range of method uses (see Table 2).

Table 3 shows how these six target students gradually but at different times internalized steps in the BAMT method. Data for the individual method uses were recorded during individual practice time ("I" on the table) as well as when individual students showed methods in individual-in-whole-class practice ("C" on the table). Students' beginning points as well as ending points varied in the unit. They used various supports (drawing, finger, speech) to assist the steps they needed to externalize the steps of the BAMT and gradually dropped some of the steps as the process became more automatized. The processes and the rates of the automatizations varied among students, and by the end, four students had internalized all steps when they were observed working alone, and two students were still saying steps 2 and 4 aloud. In the last 4 days of the unit, all problems were solved by the target students using the BAMT method except for Kensuke, who, after Day 7, consistently used the count-on method for problems that had an addend smaller than 5 (e.g., 8 + 3) and used the BAMT method for problems with both addends larger than 5 (e.g., 8 + 6).

Although by the end of the unit all students in the class could do the BAMT method alone (several students still did some steps out loud), on six problems in individual interviews just after the unit, four students (in the whole class) counted on rather than use the BAMT method. Four other students used transitional methods they invented to bridge their understanding from counting on to BAMT (Murata, 2004). Two students counted on to 10 and then chunked the "10 + *n*" to say 13 (e.g., 7 + 5 was 7, 8, 9, 10 + 2 left in 5 was ten-two), while two other students chunked the first number (knew the 10-partner) but counted on to find "10 + *n*" (e.g., 7 + 5 was 7, 10, 11, 12). These transitional methods were never discussed in class, so they reflected independent abbreviations of counting on by these students. In the end-of-the-year interview, these transitional students used the BAMT method or stated answers immediately. In the end-of-the-year interview, six students on a total of seven problems used a related addition (6 + 6 was used to solve 6 + 7 and 6 + 9, 7 + 7 was used to solve 6 + 7, and 6 + 7 was used to solve 7 + 7). One student on 2 problems used a different recomposition method based on 10 (e.g., 6 + 9: the 5 inside 6 and the 5 inside 9 made 10, and the 1 and 4 make 5, so 15). One student used the count-on method for 6 + 9, three students for 6 + 7, and two students on 7 + 7. Thus, even though the emphasis of the unit was the BAMT method, students still made their own transitions and chose their own methods in an interview situation and later in the year moved on to construct and use other conceptually advanced methods. The four students still using counting on in the end-of-the-year interview were students who had counted all or needed assistance to add in the interview at the beginning of the year. Thus, all students in the class did move within the Class Learning Path to more-advanced methods.

The detailed data in the learning trajectories of the target students provide even more understanding of the overall Class Learning Path and variations within it. Levels of the Learning Path from the beginning of the year to end of the year moved from (0) needing extra assistance to understand what addition is to (1) adding by counting all to (2) adding by counting on to (3) use of the BAMT method. Some students also gave immediate answers to problems at various points of the year. It is striking that students did not do this more often on 7 + 7 than on 6 + 9, as students in the United States often do (Fuson, 1992), and occasionally use a different known addition.

The more-advanced target students demonstrated further abbreviation and generalization of numerical patterns within the BAMT method. Yuichiro used the BAMT method from the beginning of the school year, but his use of the method changed over time. At the beginning, he used his fingers to guide his solution with the BAMT method. As the school year progressed, he gradually internalized the steps. At the end of the year, he quickly stated the answers to the interview problems and then explained his solution using the BAMT steps. In Lesson 6, during independent practice time, he communicated with the first author of this article that he was solving problems by using number patterns: for example, because the 10-partner of 9 is 1, he knew the answer to a 9 + # problem would be 10 and 1 less than the second addend. In a similar manner, the answer to an 8 + # problem would be 10 and 2 less than the second addend.

Shinobu was one of the most-advanced students in the class. During independent practice time, she effectively utilized her time after having finished the tasks (she was often the first one to finish) by engaging herself with the calculation cards and supplemental worksheets. In her case, as the BAMT method was automatized at the early part of the unit, she quickly moved to give immediate answers to the teen addition problems for the latter part of the BAMT unit. But she again used decomposition methods (BAMT and 5-and-5-to-make-10 methods) at the end of the school year, perhaps because she did not continue to practice addition so much.

The more-advanced students often helped their classmates during independent practice as it was a norm in the classroom. They also gave a glimpse to other students of more-advanced BAMT solutions during individual-in-whole-class practice by not giving all of the steps as other students did (often purposefully facilitated by Mr. Otani). Working with more-advanced students encouraged all students, as the advanced students received approval and respect and the less-advanced students became motivated to try harder.

Learning was an interactive process in the classroom; the students and Mr. Otani continuously assessed and changed the levels of assistance to make progress toward their goals (Murata, 2006c). These continuing instructional conversations and interactions between Mr. Otani and students were the foundation for students' learning in the classroom. Mr. Otani allowed space and made adjustments to meet different levels of students' understanding, but students also made adjustments to keep the learning going in the classroom. Some students elicited extra assistance by hesitating and giving body language cues or by making mistakes. Faster students assisted slower students by waiting patiently while they answered, giving extra help, and including everyone's efforts in the class by giving their energetic, "It is OK!" feedback. Mr. Otani and students were aware of individual differences, and they worked together interdependently to assist each other's learning. Productive relationships in the classroom were important to everyone in the learning community, and they adjusted their individual goals to maintain meaningful interactions.

RE-VIEW: TEACHING FOR UNDERSTANDING AND FLUENCY

Our case study gave life to the ZPD Model of Mathematical Proficiency. It illustrated how one teacher assisted student learning by valuing students' informal knowledge and innate approaches, allowing students time and opportunities to explore different ideas, helping bridge the distance between their existing knowledge and the new method, and giving time for students to practice and gain fluency with a newly learned method. The focus on meaning supports and conceptual discussion on individual mathematical thinking, on aspects of the mathematics, and on a learning path helped students develop understanding of the overall method, coordinate steps in the multistep method, and develop or move toward fluency. Mr. Otani assisted students of different fluency levels to work together and helped individuals to move forward within their own learning path. The visual and verbal question teaching supports were internalized by students gradually as they used them to provide self-assistance to coordinate or carry out particular steps of the method.

Mr. Otani assisted community and individual interaction for everyone's learning including his own as the class developed shared understandings. The whole-class context appeared enjoyable for students as well as effective for their learning as it worked to meet social and emotional needs while students assisted their own and each other's learning. The class moved within a Class Learning Path of increasing understanding, internalization, and abbreviation of steps accompanied by decreased support. Individual students moved forward in their own individual learning paths within the Class Learning Zone. Mr. Otani assisted less-advanced students by changing the supporting levels, and other students shifted their own levels accordingly to make the learning flow in the classroom. The use of consistent visual representational supports kept the community together as it helped to reduce the differences between individual students during whole-class and individual practice.

Student invention of transitional methods to combine counting with the BAMT method showed how individual students were supplementing the classroom assistance by their own abbreviations and assisted steps. When students who were ahead of or behind the Class Learning Zone had a chance to share their learning in the individuals-in-whole-class practice (or independent practice with peers who sat close by), this sharing reminded other students that there were different ways of learning happening in the classroom. Students were always willing and eager to support and adjust their own levels to the ones whose learning paths were different from their own. The emphasis on relationship and "sameness" in Japanese culture helps create an environment in which students understand difference as a norm but changeable characteristic, thus they try to be like one another. Helping one another is a part of their identities, and it is well supported in various classroom rituals and activities. The U.S. emphasis on individuality and uniqueness may work counter to creating such an expectation. However, we know of similar examples in U.S. schools where teachers work diligently to create a collaborative learning environment (Ball, 1993; Cobb, Yackel, & McClain, 2000; Hiebert et al., 1997; Lampert, 1990, 2001).

Our ZPD Model of Mathematical Proficiency indicates how a Class Learning Path can encompass both conceptual understanding and procedural fluency by moving through four phases in which levels of support decrease and types of classroom activities change. The Japanese notion of practice as *kneading learning* seems to be a helpful reminder of this possibility to forestall the common U.S. stance of emphasizing one at the expense of the other. The use of the individuals-in-whole-class structure seems helpful in providing feedback to the teacher concerning individual performance while providing varied levels of support to the answering student as well as others who are just listening. The whole-class and individual activity structures enabled Mr. Otani to adjust support to individual Learning Zones within the Whole Class Learning Zone as he moved through the four phases to increasing fluency with decreasing support.

Understanding the range of mathematical topics to which our model generalizes readily awaits future research. It would seem to apply well to those mathematical topics that involve multi-step solution methods. The model suggests that teachers (and curricular materials) need to address the four sections of Table 1 for any given topic. It is necessary to outline expected and desired student solution methods, give specific visual, sensory-motor, linguistic, and problem supports for these methods, identify topics for instructional conversations, and plan ways to assist students with particular steps using these supports to facilitate initial and ongoing understanding and fluency. The Class Learning Path should use all of the Tharp and Gallimore (1988) means of assistance (questioning, cognitive structuring, feeding back, modeling, instructing, managing) and our seventh means *engaging and involving*. The coherent learning supports can provide cognitive structuring as can chunking problems in ways that help students build conceptual understanding, generalization, and fluency. Over time, questioning will be used for fewer steps and move to a form that models self-regulating external and then internal speech, and cognitive structuring will involve increasing abbreviation and internalization of steps. The teacher's willingness to listen can also ensure that the inevitable unexpected occurrences will be recognized and supported.

It takes a lot of time to become proficient in a complex multistep method. Therefore, it is vital that any such methods in state or national standards or learning materials be carefully chosen to be of central mathematical importance and be accessible (*easy/friendly*) to students, as are the methods discussed in *Adding It Up* (Kilpatrick et al., 2003). The BAMT method is certainly mathematically important, but it is not as accessible to students speaking languages in which the teen number words are not said in the form *ten one, ten two,* etc. The final step from 10 + 5 to *fifteen* is more complex than is the step from 10 + 5 to *ten five*.[4] We have found in

[4] This step does involve a conceptual construction, as indicated by the data in a study concerning Chinese-speaking kindergarten children by Ho and Fuson (1998).

the Children's Math Worlds project that even though we carefully build all of the prerequisites for the BAMT method in kindergarten and Grade 1, many students stick with counting on (which is rapid, accurate, and general for them) until the BAMT method becomes useful in multidigit addition. The overall multidigit context provides the cognitive structuring about the need for a new group of 10, and students begin to chunk counting on into steps 2 and 3 of the BAMT method.

The Japanese curriculum is coherent, with sustained time on fewer topics that build over the year. This enabled Mr. Otani to begin the BAMT unit knowing that all students had some to considerable proficiency in all of the prerequisites for the BAMT method. In contrast, most U.S. teaching/learning materials reflect a *mile wide inch deep* U.S. pattern that results from the repetitive and unrealistic numbers of state standards at each grade level (Schmidt et al., 1997). A crucial need in the United States is for building reasonable, possible, and coherent state standards that will permit teachers to assist students in reaching mathematical proficiency. Until this happens, teachers and schools must choose some core topics in each grade level that build coherently across the grades and develop mathematical proficiency in these. Other topics must be treated in the best way possible to address the pressures of tests. If no such coherent choices are made within a school or district, deep learning that combines understanding and fluency to develop mathematical proficiency will remain out of reach for too many students.

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Authors

Aki Murata, Stanford University School of Education, 520 Galvez Mall, Stanford, CA 94305; akimura@stanford.edu

Karen Fuson, Northwestern University (Professor Emeritus), 290 Old Hill Road, Fallbrook, CA 92028; fuson@northwestern.edu