Multiplication Strategies and the Appropriation of Computational Resources

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This article proposes a taxonomy of strategies for single-digit multiplication, then uses it to elucidate the nature of the learning tasks involved in multiplication. In preceding work, it has generally been assumed that much of children's strategy development is driven by changes in their general conceptual capabilities relating to number. In contrast, we argue that, during the period in which single-digit multiplication is the focus of explicit classroom attention, changes in strategy use are primarily driven by the learning of *number-specific computational resources*. For this reason, we categorize multiplication strategies based upon the number-specific resources that are employed in their execution. To support our conclusions, we draw from a corpus of interviews with third-grade students that were conducted before, during, and after instruction in multiplication.

Key words: Children's strategies; Cognitive theory; Elementary, K–8; Learning; Multiplication, division

A great deal has been written about the development of children's strategies for adding single-digit numbers. Researchers have largely agreed on types of strategies, and there has been some convergence on the terminology for describing these types. For example, using the terminology in the handbook article by Fuson (1992), the earliest adding strategies use a *count-all* procedure: the child starts by directly representing and counting each of the addends, and then counts all the items in the representation, starting at 1 and proceeding to the total. The next strategy to appear is *count-on*: the child starts at one addend, and then counts on from there through the rest of the objects to find the total.

The state of research on single-digit multiplication differs greatly from that on addition. Although there is a growing body of research (e.g., Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997), researchers still differ greatly on the strategies described as well as in the terminology used. Thus, we believe the time is right to attempt to forge consensus on a taxonomy of strategies for multiplication. Building on the work of other researchers, as well as on our own data and analyses, we propose such a taxonomy and use it to elucidate the nature of the learning tasks involved in multiplication. Furthermore, we use this taxonomy to describe how children's use of multiplication strategies changes as a result of growth and instruction. To meet these goals, we have adopted a two-pronged approach. First, we draw together and synthesize the work that has been done by other researchers in an

attempt to build on this existing work and resolve any conflicts, apparent or real. Second, we have supplemented this prior work with analyses of our own data corpus in order to illustrate our points and support our larger argument.

Although we synthesize and build on prior research, there are some fundamental differences between our stance and those adopted in preceding work on both addition and multiplication strategies. In preceding work, it has generally been assumed that much of children's early strategy development is driven by changes in their *general conceptual capabilities* relating to number. For the case of single-digit addition, we believe that many of the important developments in strategy use really are driven by such changes. However, we believe that there are other mechanisms that may sometimes dominate. In particular, for single-digit multiplication, we will argue that during the period in which multiplication is the focus of explicit classroom attention, changes in strategy use are primarily driven by the learning of *number-specific computational resources*. Stated simply, students acquire a great deal of knowledge about specific numbers—such as 4, 12, and 32—and this knowledge allows the use of new strategies or the use of old strategies in new contexts. For this reason, many of the central issues associated with the learning of single-digit multiplication are very different from those associated with addition.

This stance has significant implications for the manner in which we can expect to achieve our stated goals. Most dramatically, this stance moderates the extent to which we can expect to develop a simple and universal account of learning progressions in multiplication strategies. Strategy use by individuals, in a particular circumstance, will be very sensitive to the number-specific resources available, which are in turn sensitive to details of instruction. Thus, while a major goal of this article is to outline broad features of a taxonomy and learning progression, we must also comment on how we expect the learning of computational strategies to vary across cultural and instructional contexts. In addition, there are some important cases in which the scheme we propose breaks down. This is particularly true as students' expertise increases. As students learn, they develop an increasingly rich network of knowledge about specific numbers. In essence, their number-specific resources merge and, because of this merging, it does not make sense to speak of students using one strategy or another. This does not, we believe, diminish the usefulness of our taxonomic scheme. But it does have important implications for the criteria that should be employed as we attempt to forge consensus on a particular taxonomy.

THE LAY OF THE LAND: AN OVERVIEW OF PRIOR RESEARCH

Four major threads characterize much of the range of research pertaining to the learning of single-digit multiplication: research concerned with (1) *semantic types* (models of situations), (2) *intuitive models*, (3) *solution procedures* (computational strategies), and (4) models of *retrieval*. Research on *semantic types* is concerned with categorizing the situations described in word problems according to how they are schematized prior to solution (e.g., Greer, 1992; Kouba, 1989;

Marshall, 1995; Nesher, 1988; Reed, 1999). These categories of types include *equal grouping*, *rate*, and *Cartesian product*. Some of this work is ambitious in the extent to which the authors attempt to provide an integrated account. For example, Vergnaud (1988) places his analysis within a larger framework that is concerned with "multiplicative structures." And Greer (1992) proposed a synthesis of semantic types—which he calls "models of situations"—for both multiplication and division.

Research on *intuitive models,* although closely related to the discussion of semantic types, has generally been treated separately. The discussion of intuitive models can be traced back to a seminal article by Fischbein and colleagues (Fischbein, Deri, Nello, & Marino, 1985).[1] In this article, the authors hypothesize that "Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious and primitive intuitive model" (p. 4). Solving a problem involving two numbers is mediated by this model. Furthermore, in the case of multiplication, this intuitive model is hypothesized to be "repeated addition." The precise relationship between these first two research threads—semantic types and intuitive models—is somewhat subtle and is not of central interest in the present article. However, note that intuitive models are generally assumed to cut across semantic types (e.g., Mulligan & Mitchelmore, 1997). In fact, in the original work by Fischbein et al., it was argued that a single intuitive model underlies *all* understanding of multiplication.

The third thread in single-digit multiplication research pertains to what have been variously called *solution procedures, solution strategies,* and *computational strategies*. Analyses of computational strategies are concerned with describing the sequence of operations that a student performs in order to get from the given numbers to the product. In the research literature, discussions of computational strategies are typically combined with discussions of one of the first two threads (e.g., Kouba, 1989; Mulligan & Mitchelmore, 1997).

The final thread of research focuses on the nature and development of *retrieval*. Typically, one of the goals of instruction in single-digit multiplication is to help students develop the ability to quickly state the product of two given operands. Some researchers have been concerned with building detailed cognitive models of this ability and how it develops (e.g., Baroody, 1999; Cooney, Swanson, & Ladd, 1988; LeFevre & Liu, 1997; Lemaire & Siegler, 1995; Siegler, 1988). Research of this sort usually has some concern with computational strategies, but when categorizing strategies, a simple binary split between retrieval and nonretrieval strategies is often made.

Several authors have tried to address multiple threads simultaneously (particularly the first three threads). Some of this work is rather ambitious, attempting to paint a broad and encompassing picture of the development of multiplicative

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[1] Note that Fischbein et al. (1985) are not only concerned with *single-digit* multiplication. They include, for example, tasks involving decimal numbers. The thread of research, following from this seminal article, that also considers these more advanced tasks (Bell, Greer, Grimison, & Mangan, 1989; Greer, 1992; Schwartz, 1988), will not be considered in this article.

thinking (e.g., Confrey, 1994, 1998; Steffe, 1992; Steffe & Cobb, 1998; Vergnaud, 1988). In contrast, the present article is restricted almost entirely to the third thread, which concerns computational strategies. Furthermore, we will primarily be concerned with the development of these strategies as it occurs during the time when single-digit multiplication is directly addressed in school-based instruction, although our full taxonomy will encompass preinstruction strategies as well.[2]

To be clear, when we speak of a *computational strategy,* we refer to *patterns in computational activity, viewed at a certain level of abstraction.* This is in contrast to an alternative stance that views strategies as *knowledge (cognitive structures) possessed by individuals.* Computational strategies, as we speak of them, are not knowledge; rather, a computational strategy is a pattern in computational activity—a pattern in the steps taken toward producing a numerical result. Sometimes, in our view, there is a simple relationship between a specific computational strategy and knowledge possessed by an individual student, but this need not always be the case.

THE MECHANISMS THAT DRIVE EARLY LEARNING PROGRESSIONS IN ADDITION AND MULTIPLICATION

The purpose of this article is to propose a consensus taxonomy for multiplication strategies and to discuss student learning progressions through this taxonomy. However, prior to proposing a taxonomy, we must address the question of whether it is even possible to develop such a consensus taxonomy. Of course, any learning progression is somewhat dependent on the nature of instruction. Nevertheless, some learning progressions that we discover in mathematics learning may be strongly constrained by factors that are largely outside of our control, such as the inherent structure of the mathematics, the knowledge that students bring to their learning, nearly universal attributes of children's experience, and the more global developmental unfolding of cognitive capabilities. We read prior research as saying that learning progressions in single-digit addition learning are, to a certain extent, of this more constrained sort. In contrast, we believe that the learning progression in single-digit multiplication is less strongly constrained by factors that will be largely independent of context. The entire weight of this article will be needed to fully argue for this claim. Our core point is that the degree of invariance that we can expect in any learning progression will depend on the nature of the mechanisms that drive development in students' strategy use. In the subsections that follow, we discuss, first, the mechanisms that drive strategy development in single-digit addition. Then we describe our hypotheses concerning mechanisms in multiplication.

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[2] Some researchers, particularly those focusing on retrieval, primarily study the development of multiplication skills during and after formal instruction in multiplication. In contrast, other researchers have looked at the development of multiplicative thinking prior to any formal instruction. This, of course, leads to dramatic differences in what is seen, particularly with regard to students' use of invented strategies.

Mechanisms That Drive Strategy Development in Addition

For the case of addition, we begin with the account presented in Fuson (1992), wherein is described a developmental progression with three levels in "conceptual structures for addition and subtraction" (p. 250). The three levels are as follows:

1. *Perceptual unit items*. Children must present addition or subtraction situations to themselves using objects or perceptual unit items.
2. *Embedded integration*. All three quantities involved—the two addends and the sum—can be simultaneously represented by embedding entities for the addends within the sum.
3. *Ideal unit items*. The addends are not embedded within the sum, but are outside and can be compared to the sum. Numbers become units that comprise numerical triads—two known addends and a known sum. This permits recomposition of the addends so that a problem can be transformed into an easier sum of different addends.

In this account, development in computational strategies[3] is seen to happen in conjunction with these changes in fundamental conceptual structures. Students at the first developmental level perform addition by directly modeling the problem with items of some type. First, they count out items for each of the addends, then they count all of the items, starting at 1 and proceeding to the total. In contrast, students at level two are capable of using a *count-on* procedure. They can start at the first addend and then count on the second addend to find the total. Finally, at level three, students can use procedures that involve recomposing the addends, so that a problem can be transformed such that a known number triad can be employed.

Built into this account are some particular assumptions about the mechanisms that drive the development of students' addition strategies. At the larger level, the development is seen as driven by changes in children's ability to conceptualize the quantity relationships that are at the heart of the addition task. It is also implied that this development in conceptual structures is invariant across a wide range of cultural and classroom contexts, and thus that the development in computational strategies is also largely invariant. To the extent that these assumptions are correct, they imply that we should expect substantial invariance in the strategies that we observe across contexts. Furthermore, because strategy change is linked to fundamental conceptual development, it implies that, when this development occurs, we should expect relatively rapid, across-the-board changes in strategy use. For example, when a student reaches the embedded integration level, we can expect that the student will, in a relatively short time period, apply the count-on strategy across a wide range of numbers.

There are some exceptions to this generalization, however. The new strategies associated with the third level of development are called *known fact* and *derived*

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[3] Fuson (1992) uses the phrase "solution procedure" rather than the phrase "computational strategy."

fact (Fuson, 1992). These strategies are based on known number triads—triads that include two given addends and their associated sum. Thus, at least during the latter phase, we must expect strategy development to happen in a more piecemeal way, with the known fact strategy being applied to some numbers and not others. The use of strategies such as *count on* may also depend, to some extent, on some number-specific knowledge.

Our analysis is presented schematically in Figure 1. In this figure, we divide the cognitive resources of individuals into three broad categories. The first category shows changes in fundamental conceptual structures; this is what changes as the students move from perceptual unit items to embedded integration and then to ideal unit items. The developments here essentially correspond to changes in basic capabilities for representing the relationships among quantities in an addition task.

As we began to suggest above, not all of our knowledge of mathematics is of this general sort. All of us who have learned mathematics also know a great deal about specific numbers. We may know, for example, that 13 is prime and that 12 is made

	Cognitive resources			General solution methods (computational strategies)
	Conceptual structures	Number-specific resources	Explicit knowledge of solution methods	
Early	Perceptual unit items	Some remembered triads 1 2 3 4 ... 1 2 3 4 5 2 3 4 5 6 3 4 5 6 7 4 5 6 5 6 7 ...	Specific object, drawing, and finger techniques	Count-all	Learned sum where number-specific resources (triads) permit
	Embedded integration	More remembered triads 1 2 3 4 ... 1 2 3 4 5 2 3 4 5 6 3 4 5 6 7 4 5 6 7 8 5 6 7 ...	New specific object, drawing, and finger techniques	Count-on	Learned sum where number-specific resources (triads) permit
Later	Ideal unit items	Many remembered triads 1 2 3 4 ... 1 2 3 4 5 2 3 4 5 6 3 4 5 6 7 4 5 6 7 8 5 6 7 8 9 ...	New specific object, drawing, and finger techniques	Derived fact: Re-composing to use a known triad	Learned sum where number-specific resources (triads) permit

Figure 1. Mechanisms that drive the development of strategies in single-digit addition

up of three 4's. The second category in Figure 1—number-specific resources—is intended to capture knowledge of this sort. As shown in the figure, knowledge of specific addition triads may develop in parallel with other learning. During the earliest phases, students may know rapidly that $1 + 1 = 2$ and that $2 + 2 = 4$. A little later, this may expand to include triads of the form $X + 1 = Y$ as well as doubles, such as $6 + 6 = 12$. Finally, students may eventually know many addition triads.

The third category of cognitive resources in Figure 1 lists explicit knowledge of solution methods. We expect that, in addition to knowledge that supports solution strategies, students have knowledge of a specific sequence of steps that can be used to solve problems. Given the framework in Figure 1, we can restate our analysis of mechanisms that drive strategy development in addition: The development of addition strategies is strongly linked to changes in the first category of knowledge resource—fundamental conceptual structures—and these changes drive across-the-board changes in strategy use. Secondarily, the development of number-specific knowledge supports some strategy change, particularly at later phases. These changes, however, are specific to the triads that are learned.

Mechanisms That Drive Strategy Development in Multiplication

Some researchers have proposed accounts for single-digit multiplication strategies that are modeled on the accounts for single-digit addition. In some cases, the connection is made explicitly. For example, Anghileri (1989), when discussing the early progression from what she calls unitary counting to rhythmic counting, states: "The development from *unitary counting* to *rhythmic counting* in groups for multiplication relates to the development in children's strategies for adding from the *counting all* procedure to the *counting on* procedure" (p. 374). In her account, the "transition from one stage to the next is marked by the child's ability to recognize that the single word that ends the first count represents the totality of that group" (pp. 374–375). Related abilities to move between counting and cardinal meanings are also seen as key in the development of single-digit addition capabilities (Fuson, 1988, 1992).

In another category of hypothesis, some researchers have linked the progression through single-digit multiplication strategies to underlying conceptual changes of a fundamentally different sort than those described in the addition literature. For example, Mulligan and Mitchelmore's (1997) account is built on Fischbein et al.'s (1985) notion of *intuitive models*. Following Fischbein and citing Kouba (1989), they use the term "intuitive model" to refer to "an internal mental structure that children impose on multiplicative situations" (p. 312) across a range of semantic structures. As shown in Table 1, Mulligan and Mitchelmore associate each of these intuitive models with one or more computational strategies.

Although the preceding perspectives differ markedly, they share an important characteristic with the addition research: They link many of the most important changes in strategy use to changes in an underlying conceptualization. In contrast, we believe that, over the time span during which multiplication is usually taught, the most important changes are not driven primarily by changes in how students

conceptualize quantity; rather, these changes are driven by relatively incremental changes to number-specific computational resources.

Table 1
Intuitive Models and Computational Strategies from Mulligan and Mitchelmore (1997)

Intuitive model	Computational strategy
Direct counting	Unitary counting
Repeated addition	Rhythmic counting
	Skip counting
	Repeated adding
	Additive doubling
Multiplicative operation	Known multiplicative fact
	Derived multiplicative fact

Our account of mechanisms is presented schematically in Figure 2. The most important difference between Figure 2 and Figure 1 is that all of the entries in the "conceptual structures" column are at the top. Our experience is that most students enter formal instruction in multiplication with the addition conceptual structures in place. This includes the single-digit conceptual structures discussed above, as well as some understanding of the meaning of two-digit numbers. Furthermore, by the time of formal instruction, students already possess the fundamental conceptual capabilities required for conceptualizing multiplication. Indeed, it has been documented that, as early as kindergarten, children can solve simple multiplication problems (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993).

There are also important differences in the number-specific resources columns of Figure 2 and Figure 1. As highlighted in Figure 2, the learning of *patterns* is particularly important in multiplication and may be an explicit focus of instruction. Also important is a type of number-specific computational resource that we will refer to as a *count-by sequence.* As part of instruction in single-digit multiplication, students often learn to count rapidly by the integers 2 through 10. For example, a student may learn to say the 4 count-by sequence: 4, 8, 12, 16, 20, 24, etc. Also, as in single-digit addition, students learn number triads (e.g., $4 \times 4 = 16$). This is a more important task in multiplication, however, because without knowledge of some multiplication triads, multiplication computations can be time-consuming and onerous. In contrast, if a student does not recall a particular addition triad, it can often be recomputed in comparatively less time.

CONTEXT, DATA SOURCES, AND RESEARCH METHODS

No single piece of evidence can support a broad stance of the sort laid out in the preceding sections. Instead, we intend this stance to be supported by the overall coherence of our view, as well as by its consistency with and ability to explain a wide range of data. Our argument in this article makes use of multiple, converging

	Cognitive resources			General solution methods (compu-tational strategies)
	Conceptual structures		**Number-specific resources**	**Explicit knowledge of solution methods**	
Early	All addition conceptual structures (ideal unit items) Meaning of two-digit numbers	Multiplication as repeated, same quantity Multiplication as repeated, same *embedded* quantity	Number-specific resources from addition. Build on and use knowledge of doubles addition ($n + n$) for multipli-cation × 2.	Specific object, drawing, and finger techniques	Count-all Simple additive calculations
	All con-ceptual structures carry forward	All con-ceptual structures carry forward	*Each count-by sequence:* 2 4 6 8 10 12 ... 3 6 9 12 15 18 ... 4 8 12 16 20 24 ... 5 10 15 20 25 30 ... 6 12 18 24 30 36 ... ... *Patterns:* $N \times 0 = 0$ $N \times 1 = N$ $N \times 5 = ?5$ $N \times 10 = N0$ 9's patterns *Each multiplication triad:*	Methods for using count-by sequences Solving 9's with hands	Count-by* Pattern-based* Learned product* More complex additive calculation*

(triad table, inset:)

	2	3	4	5	6 ...
2	4	6	8	10	12 ...
3	6	9	12	15	18 ...
4	8	12	16	20	24 ...
5	10	15	20	25	30 ...
6	12	18	24	30	36 ...
...					

Hybrid

| **Later** | | | Rich network of number-specific knowledge, particularly for small numbers | | Increasingly based on rich network of knowledge-specific resources. Strate-gies not distin-guished for small numbers. |

*Note: These solution methods do not appear for all numbers at once. They appear by number as number-specific resources are acquired.

Figure 2. Mechanisms that drive the development of strategies in single-digit multiplication

lines of argumentation and evidence. First, we believe that much of the argument for our view can be made without much in the way of specific supporting data. When one considers the possibility that much of the relevant knowledge underlying single-digit multiplication is number-specific, it becomes clear that there is a prima

facie case to be made for this position. The preceding sections have attempted to make some progress in this manner.

Second, we rely heavily on the work of other researchers. In what follows, we will be systematic and explicit in connecting our analyses to the evidence and argumentation of earlier publications. Third, and finally, we draw on our own empirical work. Our data collection efforts were conducted as part of the Children's Math Worlds Project (CMW). This ongoing project combines the design of curricular materials and professional development for teachers with a range of more traditional research activities. In the most recent phase of this work, which has spanned approximately 3 years, we have been developing and studying full-year curricula for third- and fourth-grade mathematics.

As part of this phase of our project, we conducted 230 interviews with students and completed intensive observations of classrooms. The relevant portions of our interview data were digitized, transcribed, and coded in terms of computational strategies used. We also tested the reliability of our scheme. In Appendix A, we describe our data collection and analysis efforts in more detail. This includes more discussion of the tasks and interviewing techniques employed, as well as quantitative results. The quantitative results presented in Appendix A are intended only to give the reader a sense of the relative frequency of particular strategies *within our data corpus*. Because of the particular character of our data corpus, the specific frequencies we obtained are not directly comparable with those found by earlier studies that sampled all multiplication combinations uniformly (refer to Appendix A for more discussion of this point). As discussed above, we will use our own data selectively, to illustrate and add force to our arguments.

TOWARD CONSENSUS: THE CANONICAL STRATEGIES

In the case of single-digit addition, our understanding of developing conceptual structures provides us with a natural way to categorize students' computational strategies. Although the situation in multiplication is somewhat different than that for addition, we adopt a similar approach, using our understanding of the mechanisms that drive strategy change in order to devise a taxonomic scheme for computational strategies for multiplication. We associate classes of strategies with the type of number-specific computational resources that underpin those strategies. To be more specific, we describe a set of *canonical strategies* that are associated with a particular pattern of use of one or more of these types of number-specific resources. Other strategies will then be understood as variations on those canonical types.

One type of variation involves what we call *hybrids*. Hybrids are combinations of the canonical types in that a student employs more than one computational resource. There is also within-category variation. For example, as we will see, some types of computational strategies require a student to keep track of quantities that change as a computation proceeds. This can pose a particular challenge, and additional resources need to be brought to bear. In particular, we will see that students make use of multiple representational modes and techniques for employing these

modes. Figure 3 gives an overview of our canonical strategies together with some of the more common varieties that we have observed.

Two tables play an important role in this section. Since one of our main goals is to build on prior literature and help work toward consensus, it is critical that we continue to be systematic in making connections to existing literature. Figure 4 and Table 2 together provide an overview of relevant literature. Figure 4 summarizes the taxonomic schemes from the most relevant articles, with a comparison to our own scheme. Table 2 provides some additional details concerning the work described in these articles.

To select the articles described in Figure 4 and Table 2 we employed several criteria. First, we restricted our attention to articles that included some explicit categorization of the range of computational strategies employed by students on single-digit (integer only) tasks. This criterion rules out, for the present purposes, a substantial fraction of the important research on multiplication. For example, this criterion rules out many of the articles, mentioned above, that are primarily concerned with semantic types and intuitive models (Bell et al., 1989; Fischbein et al., 1985; Greer, 1992; Marshall, 1995; Nesher, 1988; Reed, 1999; Schwartz, 1988; Vergnaud, 1982, 1988). Also ruled out is work that is primarily concerned with understanding the early conceptual bases of multiplicative thinking, particularly as it develops in very young children, prior to any formal instruction in multiplication. (Clark & Kamii, 1996; Confrey, 1994, 1998; Steffe, 1992; Steffe & Cobb, 1998).

Among the articles that remain after this first cut, there is another important distinction to be made. There is a substantial body of literature that is primarily concerned with how people come to produce responses to multiplication tasks rapidly (e.g., Ashcraft, 1992; Baroody, 1997, 1999; Campbell, 1994; Campbell & Graham, 1985; Cooney et al., 1988; Koshmider & Ashcraft, 1991; LeFevre et al., 1996; LeFevre & Liu, 1997; Lemaire, Barrett, Fayol, & Abdi, 1994; Lemaire, Fayol, & Abdi, 1991; Lemaire & Siegler, 1995; Siegler, 1988; Siegler & Shipley, 1995; Stazyk, Ashcraft, & Hamann, 1982; Steel & Funnell, 2001). A number of features characterize this research:

- Latency (reaction time) data are collected.
- No word problems are used; only straight number tasks of the form $m \times n$ appear.
- Subjects do not use objects or external representations.
- Subjects are expected to solve individual tasks in a very short amount of time.
- Subjects are sometimes, although not always, adults.
- Research is concerned with validating computer models of various sorts, particularly models based on associative networks that connect operands with products.

Authors of many of these retrieval-focused articles essentially combine computational strategies into two large categories, *retrieval* and *other*. Because the categorization of computational strategies is so coarse in most of this research, it is not particularly relevant to the current endeavor. However, there are a few articles from
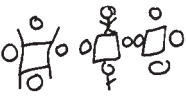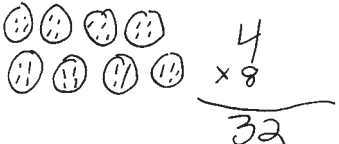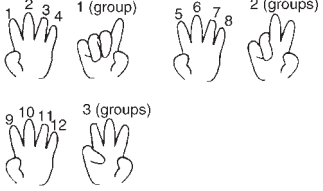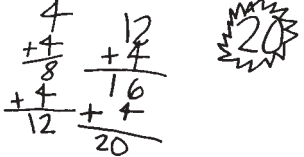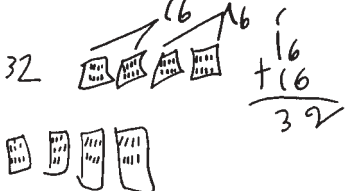
Count-all	**New computational resources:** None	
	Key observational characteristics: All values represented between one and the total	
	Sample varieties	
Count after drawing— semi-situational drawing		Student draws entire figure, separately counting out each group. Then counts from one to total.
Count after drawing— "math drawing"		Same as above, but using what, in CMW, are called "math drawings." These drawings are less direct in representing elements of the situation.
Count-all with fingers		Student counts aloud from one to total, using fingers as an aid. "One, two, three, four, – one. Five, six, seven, eight, – two. Nine, ten, eleven, twelve – three."
Rhythmic counting with fingers		Counting aloud with verbal emphasis at each multiple of the group size, while fingers keep track of the number of groups: "one, two, **three,** four, five, **six,** seven, eight, **nine,** ten, eleven, **twelve.**"

Additive calcu-lation	**New computational resources:** None	
	Key characteristics: Not all values are represented between one and the total. Student may write out component addition tasks using standard notations.	
Repeated addition		The problem is transformed into a sequence of addition problems, adding on the group size repeatedly.
Collapse groups and add		The student adds groups, usually in pairs, then adds the resulting sums, usually using multicolumn techniques. (In some cases, the resulting sums may be collapsed by pairs again before they are added using multicolumn techniques.)

Figure 3. (Continued on the next page)
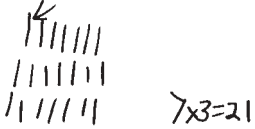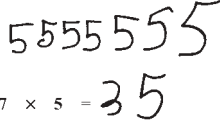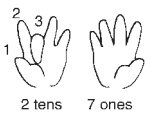
Count-by	**New computational resources:** Count-by sequences for each number: *n, 2n, 3n, 4n,* etc. **Key characteristics:** Values are represented in the regular sequence corresponding to the count-by sequence.	
Count-by with full drawing	$7 \times 3 = 21$	Makes a full drawing, as for a count-all strategy, but finds the product by employing a count-by sequence, pointing to each group in the drawing: "7, 14, 21."
Count-by with written groups	5555555 $7 \times 5 = 35$	Writes a numeral for each group, then says the count-by sequence while pointing to each numeral.
Count-by using fingers		Says the count-by sequence aloud, keeping track of the number of groups on his/her fingers: "4, 8, 12, 16, 24." Students may begin with thumb, index finger, or pinky.

Pattern-based	**New computational resources:** A number of specific patterns, such as $N \times 1 = N$, $N \times 0 = 0$, and a number of patterns and techniques for 9's. Understanding 10's patterns may involve new place-value knowledge. **Key characteristics:** Solutions are generally very rapid. One of the 9's techniques involves a particular type of use of fingers.	
0's rule, 1's rule, 10's rule	No visible use of fingers or drawing.	Response is rapid with no visible computation: "One times seven is seven."
9's finger technique	2 tens 7 ones	To multiply $9 \times N$, the student holds up both hands, and puts down the *N*th finger, counting from the left. The tens digit of the result is given by the number of fingers to the left of the finger that was put down, and the ones digit is given by the number of fingers to the right.

Learned products	**New computational resources:** Learned associations of pairs of factors with their products **Key characteristics:** Solutions are generally very rapid. No verbalization except for the result.	
Learned products	No visible use of fingers or drawing.	Response is rapid with no visible computation: "7 times 6 is 42."

Figure 3. (Continued on the next page)
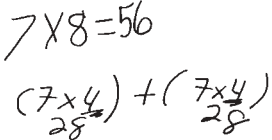
Hybrids	**New computational resources:** Employs combinations of the new resources mentioned above and previously existing computational resources. **Key characteristics:** For each part of the hybrid computation, the observational characteristics are those associated with the substrategy, as described in this table.	
Count-by + count-all	"6, 12, 18, 24, 30, 36"	Uses a count-by sequence to get partway to the total, then count-all to reach the total. "6, 12, 18, 24, 30, 36, 37, 38, 39, 40, 41, 42."
Learned product + count-all	"6 times 6 is 36"	Starts with a learned product below the total, then count-all (in one of the above ways) to reach the total. "6 times 6 is 36. 37, 38, 39, 40, 41, 42."
Learned product + additive calculation	(There may be written components to show the addition.)	"6 times 7 is 42, plus 7 is 49, plus 7 is 56."
Split factor + learned product + additive calculation	$7 \times 8 = 56$ $(7 \times 4) + (7 \times 4)$ $28 \qquad 28$	One of the factors is split, so that the problem is decomposed into two parts. The two subproducts are found using a learned product strategy. These are then added with an additive calculation.

Figure 3. Canonical strategies and sample variants

this genre that do open up the "other" category, at least partly. The four bottommost articles in Figure 4 are of this sort. The articles by LeFevre et al. (1996) and Lemaire and Siegler (Lemaire & Siegler, 1995; Siegler, 1988) have a fairly differentiated taxonomy and are thus unusual in this field. Cooney et al. (1988) has a less-differentiated taxonomy and was selected to be representative of similar research in this genre. The remaining top three articles listed in Figure 4 are the ones that are closest in concern to the current work in that they all present schemes that substantially open up the "other" category. They will thus be given the most emphasis.

   As described in Table 2, the studies listed in Figure 4 differed substantially in the populations studied, the tasks employed, and the data collected. For example, the top three articles looked only at students solving word problems, whereas the others looked only at students solving straight numerical tasks. In some instances, students were given manipulatives or pencil and paper to use (e.g., Anghileri, 1989), whereas in others they were not provided with any external aids (Cooney et al., 1988). Finally, the ages of the subjects studied differed. Lemaire and Siegler (1995) looked only at French second graders, Anghileri (1989) looked at students

	Count-all	Additive calculation	Count-by	Pattern-based	Learned products	Hybrid
Mulligan & Mitchelmore, 1997	Unitary counting (direct counting) / Rhythmic counting	Repeated addition / Repeated adding, additive doubling	Skip counting		Multiplicative calculation / Known multiplication fact	Derived multiplication fact
Kouba, 1989	Direct representation (only use of physical objects)	Additive	Transitional counting		Recalled number facts (includes derived facts)	
Anghileri, 1989	Unitary counting / Rhythmic counting (Unary)		Number pattern		Known fact (binary)	
Lefevre et al., 1996		Repeated addition	Number series	9's rule / 0's and 1's rules	Retrieval	Derived fact
Lemaire & Seigler, 1995; Siegler, 1988	Counting-set-of-objects / Repeated addition			(Rapid responses coded as retrieval)	Retrieval | Writing problem	
Cooney et al., 1988	Counting (not clear if this also includes count-all)			Rules (0 and 1) / Other	Memory retrieval	Derived fact

Figure 4. Overview of strategy taxonomies from selected articles

ages 4 through 12, and LeFevre et al. (1996) looked at adults. For these reasons, we must expect significant differences in the types of strategies reported. In the subsections that follow, we now present our own framework, making detailed comparisons to prior research where appropriate.

Table 2
More Details on the Articles Listed in Figure 4

	Data and methods	Other notes
Mulligan & Mitchelmore, 1997	Tracked 70 girls through grades 2 and 3. At time of first, second, and third interviews, students had no instruction in multiplication. Students received no instruction in word problems.	The framework was structured as a developmental progression in three intuitive models, each of which was associated with one or more computational strategies. The intuitive models were (1) direct counting, (2) repeated addition, (3) multiplicative operation.
Kouba, 1989	Studied first, second, and third graders. All tasks were equal group word problems.	Semantic factors in the word problem were the main concern. Strategies were classified by "degree of abstraction," as well as by "use of physical objects." The strategies given in Figure 4 correspond to Kouba's classification by degree of abstraction. Classification by use of physical objects employed three categories: (1) use as representations of individual elements, (2) use as tallies or repeated references, (3) no use.
Anghileri, 1989	Studied students ages 4–12. In all tasks, students were given physical objects. They employed 6 tasks, one for each of their semantic types.	Multiple categorization schemes for computational strategies were employed in the article. Some of these schemes are coarse, others more fine-grained. Figure 4 reports Anghileri's more fine-grained taxonomy. Anghileri reports a developmental progression that, in some ways, mirrors the development in single-digit addition.
LeFevre et al., 1996	Studied undergraduates age 18–45 years. Subjects given either 5 or 10 seconds to solve, verbally, all tasks of the form $m \times n$. Latency data were collected.	
Lemaire & Seigler, 1995	Longitudinal study of French second graders. Students were interviewed at three times and were given all tasks of the form $m \times n$.	Their central interest is in tracking strategy change—where and how strategies are used—particularly with respect to the use of retrieval.
Siegler, 1988	Studied third-grade students. Tasks were primarily of the form $m \times n$.	Concerned with testing predictions of the "distribution of associations" model, particularly with respect to the use of backup strategies.
Cooney et al, 1988	Studied 10 third and 10 fourth graders. Tasks included 100 problems of the form $m \times n$. They obtained latency data and also followed up some tasks with interviews. For timed tasks, students were not permitted to use paper and pencil.	

COUNT-ALL

The first two classes of strategies in Figure 3 are based on resources that are, in general, already in place at the time of instruction in multiplication. In the first strategy, *count-all,* a student can be seen counting from 1 to the product as they perform the computation. Example 1 describes an incident in which a student, Danny,[4] was presented with the task of finding the total number of children, given that 4 children are seated at each of 3 tables. He solved this problem by first drawing a picture, and then counting all of the children he had drawn.

Across individuals of a wide range of capabilities, count-all strategies can be the most time-consuming and most difficult to enact correctly when the operands are large. Enacting a count-all computation requires that three separate counts are coordinated. For illustration, consider the task of multiplying 3×4. One way to do this is to count to the total made by counting to 4, three times. This requires that we enact and coordinate the three counting sequences shown in Figure 5: (1) We need to count from 1 to 3 to keep track of the number of groups; (2) we need to count from 1 to 4 three times, to keep track of where we are within each group; and (3) we need to count from 1 to 12, thus keeping track of the running total.[5]

		1				2					3	Count of the number of groups
1	2	3	4	1	2	3	4	1	2	3	4	Count of entities in a group
1	2	3	4	5	6	7	8	9	10	11	12	Count of total

Figure 5. The three coordinated counting sequences for multiplying 3×4

Varieties of Count-all

Because of the need to coordinate the three separate counts in count-all, many techniques are employed, and supplementary resources are brought to bear. For this reason, this category of strategies is the largest and most varied part of our taxonomy. Figure 3 lists four varieties of count-all strategies, and these represent just a sample of the diversity that exists. Across all these varieties, a central issue is how external media are used to support the computation. We thus divide the discussion that follows according to the primary medium employed.

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[4] All student names are pseudonyms.
[5] We are aware that an account of this sort hides much conceptual nuance, including some conceptual leaps that may be difficult for younger students.

Count-all—paper-based. The use of paper as an external medium has dramatic effects on the ability of a student to manage a count-all computation. For example, because Danny made a drawing as a first step to multiplying 3×4 (see Example 1), this essentially allowed him to enact the three counts sequentially, rather than simultaneously. First he drew the tables, counting 1 to 3; then he drew the children, counting from 1 to 4 three separate times; finally, he counted from 1 to 12, pointing to each of the children he had drawn. One feature of Danny's drawing merits particular attention: Substantial features of the situation described in the word problem are reflected in the drawing. It is for this reason that, in Figure 3, we refer to this variant of count-all as *count after drawing—semisituational.* When students make situational drawings of this sort, they are making use of drawing techniques that are most likely learned outside of mathematics instruction.

Example 1: Danny, preinterview

Task: There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?

Description: Initially, Danny was unsure how to proceed. Following the suggestion of the interviewer, he drew the situation. When the interviewer asked, "So, how many children are there altogether?" he counted quietly without pointing, but his head moved and he nodded a bit, as if in the direction of each drawn child.
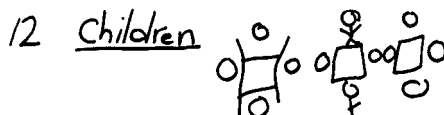
In contrast to Danny's solution in Example 1, some of the students we observed used techniques for making simplified drawings. During the preinterview, Hector made relatively pictorial drawings of the sort shown in Example 2. However, during later interviews, as illustrated in Example 3, Hector used a drawing technique involving boxes and marks. The use of such abstracted drawings has a number of benefits, not least of which is that it can greatly reduce the amount of time necessary to make a drawing. Indeed, in our own data, we rarely saw students make strongly situational drawings after the preinterview. Abstracted diagrams of the sort seen in Example 3 appeared frequently in the CMW classrooms we observed. We refer to this variety of count-all as *count after drawing—math drawing*, as the CMW curriculum uses this term to describe these types of simplified mathematical drawings, which were strongly emphasized.

Count-all—finger-based. An alternative medium that can be used to support the count-all strategy is the medium that consists of the student's *fingers*. Example 4 presents an instance of count-all using fingers. In this example, Sam multiplied 3×4

Example 2: Hector, preinterview

Task: There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?

Description: Hector initially struggled trying to count on his fingers. He kept losing his place. Eventually, the interviewer suggested he draw a picture. Hector drew the diagram. He counted under his breath and said that 18 was the answer. When asked to count aloud, he counted by 2's to 12: "2, 4, etc."
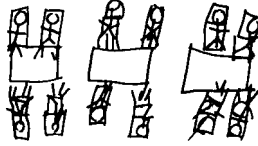
Example 3: Hector, midpoint 1

Task: Martin bought 4 boxes of pencils. There were 8 pencils in each box. How many pencils did Martin buy?

Description: Hector made the diagram, then counted each tally mark from 1 to the total.

by repeatedly putting up three fingers, one at a time, on his left hand. Notice that, unlike the examples that made use of drawings, the three counts in Example 4 were enacted simultaneously. Sam's use of his fingers helped to make this possible; he used his fingers to enact the within-group count, whereas the total count was kept verbally. It is interesting to note that the count corresponding to the number of groups was not enacted in any visible manner. There are six examples in our digital database in which a student uses fingers in the execution of a count-all strategy. In all these examples, the computation was distributed over media in the same manner as in Example 4; fingers were used to enact the within-group count, whereas the total count was kept verbally, and there was no visible counting of the number of groups.

Count-all—rhythmic counting. Figure 3 includes one additional variety of count-all, called *rhythmic counting.* In *rhythmic counting*, the student counts from 1 to the total, saying every value along the way, just as in all count-all variants. However, as they count, the student emphasizes each value that is associated with the completion of a group. So a student multiplying 3×4 might say, "One two three four, five six seven eight, nine ten eleven twelve."

Example 4: Sam, preinterview

Task: There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?

Description: He counted "1, 2, 3" putting up three fingers, one at a time, on his left hand. Then he said "4, 5, 6" again putting up three fingers. Then he continued in the same way up to 12.
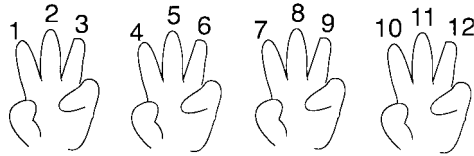
A particular variant of rhythmic counting, *rhythmic counting with fingers,* is shown in Figure 3. In this variant, the number-of-groups count is kept on the fingers, but the within-group count is only carried by the rhythm of emphasis as the total count is said aloud. This is most feasible when the group size is small. Although *rhythmic counting* did not appear in our interview corpus, it is included here because, as we discuss below, *rhythmic counting* figures prominently in the schemes of some other researchers.

Count-all in the Research Literature

There are some differences in how count-all-like strategies have been treated among the articles listed in Figure 4. First, among the three bottom rows of "retrieval-focused" researchers, count-all strategies are given much less attention. LeFevre et al. (1996) report no observation of count-all, likely because their subjects were adults. Cooney et al. (1988) do mention observing count-all, but they include it within a more encompassing "counting" category that places count-all together with what we call *additive calculation* and *count-by.* Lemaire and Siegler (1995) have a similar encompassing category that they call "repeated addition." Siegler (1988), however, splits out one type of strategy that he calls "counting-sets–of-objects." This is a subset of our count-all category that includes only the very specific case in which tally marks are written on a sheet of paper and counted.

Each of the top three articles in Figure 4 has a category that is close to our count-all class of strategies. Kouba (1989) uses the name "direct representation." Both Anghileri (1989) and Mulligan and Mitchelmore (1997) call their respective category "unitary counting," and each explicitly splits out *rhythmic counting*. Anghileri (1989), in particular, ascribes an important role to *rhythmic counting*. In her learning progression, it plays a role that is analogous to the role played by count-on in the addition literature.

As stated above, we saw *rhythmic counting* only rarely in CMW classrooms and not at all in interviews. It is possible that *rhythmic counting* appeared in our classrooms in just a brief transitional stage and that we missed this brief appearance. However, we believe that *rhythmic counting* really was rare in CMW classrooms, perhaps because it did not receive formal attention. This is evidence that we must be careful about ascribing any sort of universal importance to types of strategies that may depend very sensitively on the details of instruction.

ADDITIVE CALCULATION

Because students have prior learning experiences relating to addition, they have existing resources that can provide the basis of strategies that are less time-consuming and easier to enact than count-all strategies. We call these strategies that are based on addition-related techniques *additive calculations.* An instance of this strategy is described in Example 5, wherein Ellen multiplies 3×4 by first adding $4 + 4$ to get 8, and then $8 + 4$ to get 12. This episode has features that clearly distinguish it from episodes of count-all. In Ellen's computation, not every value between 1 and 12 was represented; instead, the computation jumped from 4 to 8 to 12. Furthermore, Ellen's written work made explicit use of addition notations.

Example 5: Ellen, preinterview

Task: There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?

Description: Ellen added two 4's to get 8, and then added an additional 4 to get 12.

Varieties of Additive Calculation

As with count-all, there is some diversity within the additive calculation category. Example 5 is an instance of the subtype that, in Figure 3, we refer to as *repeated addition.* In *repeated addition*, the student performs sequential additions, each time adding the group size onto the current value of the total.

Example 6 and Example 7 show instances of a more advanced variety of additive calculation that we call *collapse groups and add.* In both of these examples, the students are working on a problem that asks them to determine how many pencils are contained in 4 boxes, each of which contain 8 pencils. Their solutions follow a similar pattern. They begin by adding pairs of 8s to get two 16s, which are then

added using multicolumn addition techniques. These computations still have the characteristics that distinguish additive calculations from count-all: not all values between 1 and the total are represented, and standard arithmetic notations appear. However, the pattern of represented quantities is somewhat different than what occurs in the *repeated addition* subtype. If Harry or Jeremy were using *repeated addition*, we would expect to see the pattern 8, 16, 24, 32 in their computations. However, in these examples, we instead see two 16s produced and then combined. Furthermore, 24 did not appear at all in these solutions.

Example 6: Harry, midpoint 1

Task: Martin bought 4 boxes of pencils. There were 8 pencils in each box. How many pencils did Martin buy?

Description: Harry drew the diagram, and labeled it with the two 16's. Then he wrote and solved the multicolumn addition problem shown.
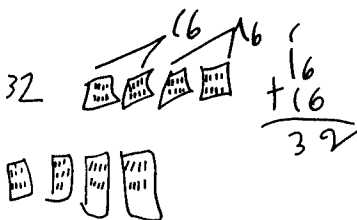
Example 7: Jeremy, midpoint 1

Task: Martin bought 4 boxes of pencils. There were 8 pencils in each box. How many pencils did Martin buy?

Description: Jeremy began by writing four 8's in a row. Then, after a brief pause, he wrote two 16's in multicolumn format. He then proceeded to do the multicolumn addition.

Additive Calculations in the Research Literature

In all but one (Anghileri, 1989) of the articles in Figure 4, a category analogous to our additive calculation category was either explicit or the authors could plausibly have intended similar strategies to be included in one of their categories. In the case of articles listed in the bottom two rows in Figure 4 (Cooney et al., 1988; Lemaire & Siegler, 1995; Siegler, 1988), additive calculations are included (at least, implicitly) as part of the authors' "other" categories. The remaining three studies in Figure 4 (Kouba, 1989; LeFevre et al., 1996; Mulligan & Mitchelmore, 1997) have categories that are in close alignment with our additive calculation strategy. The alignment of Mulligan and Mitchelmore (1997) seems to be the best. They have categories called "repeated adding" and "additive doubling," which may line up with our two subtypes, *repeated addition* and *collapse groups and add*, although they use the phrase "repeated addition" to refer to an intuitive model, not a computational strategy. Kouba (1989), in contrast, has a category called "additive," but this seems to only include our *repeated addition* subtype of additive calculation.

COUNT-BY

When instruction in multiplication begins, students begin the extended task of learning the various number-specific computational resources that can support more efficient and accurate strategies. As discussed earlier, one important and prevalent collection of resources is the *count-by sequences*; students learn to say sequences such as "6, 12, 18, 24, . . ." and "9, 18, 27, 36, . . ." These sequences make possible the class of computational strategies that we refer to simply as *count-by* strategies. In Example 8, we describe an episode in which a student used COUNT-BY to multiply 8×4; she counted by 4's to 32, putting up a finger on each hand to keep track of the number of groups.

As in count-all, it is helpful to think of the enactment of count-by as requiring the coordination of multiple counting sequences. In the case of count-by, only two

Example 8: Linda, postinterview

Task: 8×4

Description: Linda counted by 4's to 32. She said: "4, 8, 12, etc.," putting up a finger as she said each number. She used only her left hand, so she had to reuse some fingers.
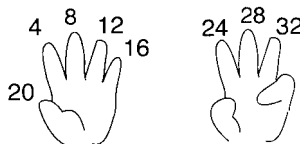
counting sequences must be coordinated, a reduction that greatly reduces the difficulty of accurately enacting count-by strategies. The tradeoff is that a count-by sequence must be learned for each number. This is depicted in Figure 6 for the case of 8×4 (as in Figure 5, the sort of description given in this figure hides important conceptual nuances).

4	8	12	16	20	24	28	32	Count of total
1	2	3	4	5	6	7	8	Number-of-groups count

Figure 6. The two sequences to be coordinated for multiplying 6×5

Varieties of Count-by

In one variant of count-by that we refer to as *count-by using fingers*, students kept track of the number of iterations on their fingers, and spoke the running total aloud. This was illustrated in Example 8. Example 9 illustrates a closely related variant, *count-by with written groups*, in which a sheet of paper is used instead of fingers to keep track of the number of groups that have been counted.

Example 9: Jeremy, preinterview
Task: 7×5
Description: Jeremy counted by 5's, pointing to each of the 5's that he had written.

5 5 5 5 5 5 5

7 × 5 = 35

Count-by in the Research Literature

Among the top four articles in Figure 4, there is substantial consensus in how the count-by strategy is treated. Mulligan and Mitchelmore (1997) have an equivalent category that they call "skip counting." Anghileri (1989) calls her equivalent category "number pattern," and LeFevre et al. call their category "number series." Kouba's (1989) "transitional counting" is very similar to our count-by category, but it seems to include the case where a student uses count-by to get partway to the final

result, then employs count-all to complete the computation. As discussed below, we would treat this latter case as a hybrid.

The situation in the retrieval-focused articles is, of course, somewhat different. Cooney and colleagues (Cooney et al., 1988) include count-by within their large "counting" category. Similarly, it is likely that Lemaire and Siegler (1995) would include count-by in their "repeated-addition" category; however, no explicit mention of skip-counting-like strategies appears in either Lemaire and Siegler or Siegler (1988).

PATTERN-BASED

Patterns of various sorts, such as $N \times 1 = N$ and 9's patterns, are number-specific resources that are often learned in parallel with the count-by sequences (refer to Figure 2). A selection of these patterns is associated with the first pattern subtypes listed in Figure 3: the *0's pattern*, *1's pattern*, and *10's pattern.* These three subtypes allow students to produce certain results rapidly and without visible work. Because these pattern-based strategies are associated with very rapid responses by students, they may be hard, in practice, to distinguish from learned product strategies. Nonetheless, we believe that it makes sense to treat these strategies as part of a separate category (from learned product) because they are based on a very different sort of number-specific resource. These patterns have a broader range of application than, for example, a single number triad. For multiplication by 1 there is just a single pattern to see and learn; it is not necessary for the student to learn a separate rule for each pair of multiplicands.

Beyond the 0's, 1's, and 10's patterns, there are other patterns that students may learn and that may support them in multiplication computations. The 9's products are particularly rich with useful patterns, and recognition of these patterns can reduce the difficulty of multiplication tasks involving 9. In CMW, students first consider 9's patterns based on thinking of 9 as $10 - 1$. For example, for the product 6×9, they first flash 10 fingers 6 times, then fold down 6 fingers from the last 10, leaving 4 ones. Then they raise 5 fingers to show the 5 tens, thus showing 5 tens with one hand and 4 ones with the other.

After working through all of the related $10 - 1$ patterns, students summarize these using the finger shortcut shown in Examples 10 and 11. This shortcut works as follows: If a student wants to multiply $9 \times N$, then the student holds up both hands and puts down their Nth finger, counting from the left. The tens digit of the result is then given by the number of fingers to the left of the finger that was put down, and the ones digit is given by the number of fingers to the right (this works because $9 \times N = 10 \times N - N$).

Pattern-Based Strategies in the Research Literature

For the most part, pattern-based strategies were not treated as a separate category by the researchers listed in Figure 4. In many cases, it is likely that researchers

Example 10: Jeremy, postinterview

Task: 9×6

Description: He held up his hands in front of him, palm up. Then he bent the pinky of his right hand down quickly and for just a moment. Then he said, "54."

Example 11: Charlie, postinterview

Task: 9×3

Description: When asked the question, he looked down at his hands for just a moment, then said the answer, 27. He then explained as follows, holding up his hands to demonstrate:

C: I did my nines trick, you go 1, 2, 3. Then you look at it. . . . And then there's 2 and then there's 7.
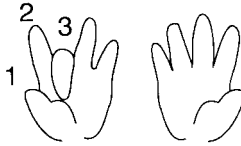
intended these strategies to be included in a learned product-like category. For example, Lemaire and Siegler (1995) and Siegler (1988) explicitly state that they treat all instances in which there is no overt behavior by the student as belonging to their "retrieval" category. There are two exceptions in Figure 4, however. LeFevre et al. (1996) have a special category called "9's rule" and Cooney and colleagues (Cooney et al., 1988) treat pattern-based strategies involving 0 and 1 as a separate case, worthy of its own category.

LEARNED PRODUCT

Our last primary category of strategy, *learned product,* is associated with a large collection of number-specific resources: the multiplication triads. The learning of these multiplication triads typically demands a large amount of student time and effort; the resources are acquired bit by bit, with some triads being learned earlier than others. Example 12 contains a brief episode involving this strategy. In that episode, Jenna quickly writes the product, saying "I just know the answer."

Example 12: Jenna, midpoint

Task: 7×5

Description: Jenna read the problem and then just wrote the answer. When prompted, she explained her solution as follows:

J: I know what this is.

I: How did you get 35? Can you tell me how?

J: I just know that answer.

I: Did you just memorize it?

J: Yeah.

Learned Products in the Research Literature

All the article listed in Figure 4 have a category that is closely related to our learned product category. The top three articles all give their categories a name using "fact" in the title: Mulligan and Mitchelmore (1997) have "known multiplication fact," Kouba (1989) has "recalled number fact," and Anghileri (1989) has "known fact." Where these researchers differ is in how they treat what Mulligan and Mitchelmore (1997) call "derived multiplication facts." In this strategy, the student begins with a known "fact," and then adds or subtracts, in some manner, to derive a solution for the current problem. Mulligan and Mitchelmore treat this as a separate category of strategy, but Kouba (1989) includes this in her "recalled number facts" category. As we will discuss below, we treat these "derived multiplication facts" as hybrid strategies.

The articles in the remaining three rows all have categories that are, once again, quite close to our own, and all employ names with the word "retrieval." The only mild exception is that Lemaire and Siegler (1995) split our learned product category into two parts, one that they call "retrieval" and a smaller category they call "writing problem." This strategy differs from retrieval only in that the student writes out the two multiplicands (e.g., 8×4) and then gives the answer orally, rather than just answering orally. In our own scheme, this type of difference would be treated as a within-category variance associated with differing uses of media.

We conclude this section with a comment concerning our choice of the term *learned product* for this category. We feel that none of the terms that have been employed in the literature for this category—terms that include "fact" or "retrieval"—do justice to this category. These terms all suggest rote lookup, as if from a mental table, and we believe that this is overly simplistic, even when responses are given extremely rapidly by students. For example, it certainly misses much to say that $2 \times 3 = 6$ and $6 \times 10 = 60$ are "just memorized." Each of these multiplication triads will have a different experiential basis, contributing to a unique learning history. Students' understanding of $2 \times 3 = 6$, for example, may be rooted in experiences of visual patterns (⬤⬤⬤ ⬤⬤⬤) or in prior learning of addi-

tion. We will say more about this issue below in our discussion of "Where the computational resources merge."

HYBRIDS

Hybrid strategies are based on combinations of the strategies above. In principle, there is a moderately large number of possible ways that existing strategies can be composed to form hybrid strategies. However, as discussed below, we observed only some of these possibilities.

Varieties of Hybrids

The most common hybrids we observed used count-by or learned product techniques to get partway to the result, and then used count-all or additive calculation to get the rest of the way. Two episodes involving hybrid strategies are described in Example 13 and Example 14. Example 13 is an instance of *learned product + count-all.* In that episode, Jenna multiplied 7×6 by starting from $6 \times 6 = 36$, and then counting from 37 to 42 on her fingers. In Example 14, Jeanne also multiplied 7×6 by starting from $6 \times 6 = 36$, but she added on the last multiple of 6 using additive resources. We would thus describe this as an instance of *learned product + additive calculation*.

Example 13: Jenna, midpoint 2

Task: 7×6

Description: Jenna said, "36," and then counted from 37 to 42 on her fingers. She explained, "I know that $6 \times 6 = 36$ so I added 6 more on my fingers."
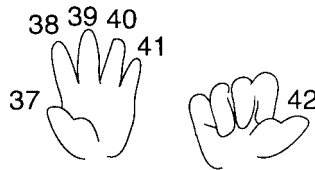
Example 14: Jeanne, postinterview

Task: John had 3 crayons. He decided that he wanted some more crayons, so he went to the store and bought 7 boxes of crayons. There were 6 crayons in each box. How many crayons did John have altogether?

Description: Jeanne has 45 written as her answer. When asked to explain how she got this answer, she stated that 7×6 is 42, and you add 3 more to get 45. The interviewer then asked how she knows that seven 6's are 42. Jeanne said, "Because 6 time 6 is 36 and plus another 6 is 42."

We also observed hybrid strategies in which students combined strategies in ways other than using one strategy to get partway to the result, then another to get the rest of the way. As an example, Figure 3 includes a variety of hybrid that we call *split factor + learned product + additive calculation.* In this type of strategy, students partitioned one of the two multiplicands into two parts, computed the product for each of these parts, and then added the resulting products together. This is how Jane multiplied 7×8 in Example 15. She used retrieval to multiply 7×4, obtaining 28, then she added 28 to 28 to get 56.

Example 15: Jane, postinterview

Task: The walls of the rooms were covered in beautiful tiles. There were 7 rows and 8 columns of tile on each wall. How many tiles were there in all?

Description: Jane explained that she had memorized that 7×4 is 28, and she added 28 and 28 to get 56 (apparently using multicolumn addition done in her head).

$$7 \times 8 = 56$$

$$(7 \times 4) + (7 \times 4)$$
$$\quad\; 28 \qquad\qquad 28$$

Hybrids in the Research Literature

Although none of the articles listed in Figure 4 included a general discussion of hybrids, some specific hybrids did appear. In some cases, these were just mentioned in passing and included in a larger category; in other cases, they were treated as a separate category. The most common hybrid to appear in the literature is what has been referred to as "derived facts." As we discussed earlier, some researchers treated derived facts as part of a learned-product-like category (Kouba, 1989), whereas others had a separate category (Cooney et al., 1988; LeFevre et al., 1996; Mulligan & Mitchelmore, 1997).

Where the Computational Resources Merge

In presenting our category scheme, we described our categories as separate, each based on a specific type of computational resource. However, as we have suggested, this assumption of the separability of computational resources becomes increasingly problematic as students progress. This observation has direct implications for our category scheme; it means that, as students learn more, individual instances of computational behavior no longer belong solely to one of our categories.

The merging of number-specific computational resources is evident even very early in the learning process, particularly when small numbers are involved. For illustration, consider the episode in Example 16. In this episode, Cayla was given the task of multiplying 2×3, and she responded, relatively quickly, with an answer of 6. Then, when prompted to explain, she explained that "you could add three plus three." The point is that it is unclear how to categorize an episode of this sort in terms of our taxonomy. The initial answer was produced very quickly, which suggests that this is an episode of learned product. However, in her explanation, Cayla said that the answer could be found by *adding* $3 + 3$. Indeed, it is not implausible that an answer could be produced quite quickly using this latter strategy. In our view, the appropriate way to understand episodes of this sort is that, in the territory of small numbers, the various computational resources have already become integrated for Cayla, so it simply does not make sense to differentiate among these possibilities.

Example 16: Cayla, preinterview

Task: $2 \times 3 =$

Description: Cayla had $3 + 3 = 6$ written on her paper. When prompted to explain, she said that her previous year's teacher taught her how to do problems of this sort: "Last year my teacher, he.... In my class we had third and second and he taught both grades the same stuff. He said two times three you could add $3 + 3$ two times and the answer would be six. That's how he taught us."

This type of integration becomes more pervasive later in the instructional cycle. One way that this was manifested in our observations was that, following the posing of a problem, there would be a pause of moderate length (a few seconds) before the student stated an answer. For illustration, consider Examples 17 and 18. In the first example, Shanta was given the task of multiplying 4×7. After a pause of about 3 seconds, she stated the answer, 28. When asked how she found the answer, she explained that she counted by 4's. Similarly, when given the same problem, Jane also paused for a few seconds before giving a response. She explained that she had found the result by adding 14 and 14. Once again, episodes of this sort pose a challenge for categorization. The answers are produced moderately quickly, after about 3 seconds in each case. This suggests that learned product may be an appropriate coding. However, the students reported more extended cognitive activity. For example, Shanta stated that she found the answer by counting by 4's.

The difference between interpretations here is a slim one. As students progress to expertise, there may not be much difference between counting by 4's very quickly and retrieving a result. The particular strategies also may become abbreviated so that when initiated, they also stimulate a learned product, which then may or may not be verified by completing the strategy. Similar points have been made elsewhere in the research literature. For example, Ter Heege (1985) stated that

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Example 17: Shanta, postinterview

Task: $4 \times 7 =$

Description: When asked the question, she paused for between 2 and 3 seconds then said, "28." The interviewer then asked her how she got that answer.

I: How did you figure out your answer was 28.

F: 4×7? Well I counted, like, by 4's.

Example 18: Jane, postinterview

Task: $4 \times 7 =$

Description: When asked the question, Jane paused for about 3 seconds and then said "28." During the pause, her lips move slightly as if saying something to herself. The interviewer then asked her how she got that answer.

E: How did you know 4×7 was 28?

K: Well I did 14 plus 14.

E: And you do that in your head?

K: Uh-huh.

---

students can become so skilled "that the border between 'figure out' and 'know by heart' seems to blur" (p. 386). Similarly, Baroody (1997) argued against a clean distinction between retrieval-related resources and other resources that underlie multiplication. He argued that "the representation of basic number combinations is not a distinct aspect of long-term memory but an integral aspect of the structured framework for general arithmetic knowledge" (p. 6).

This observation concerning the merging of resources is important, not only because it suggests some theoretical limitations of the categorization scheme presented in this article, but also because it has quite general implications for how we must understand the nature of "basic skills," such as the ability to "recall" multiplication facts. We must not assume that the end products of learning are memorized count-by sequences or straightforwardly internalized versions of the multiplication table. Although it may occasionally be productive to understand instruction as directed at helping students to acquire these relatively well-defined cultural tools, we must be careful not to presume that there is little complexity, individuality, or variability in the end products of understanding. As was mentioned earlier, the terminology used by many to discuss this learning task ("memorizing multiplication facts") oversimplifies this task and thus may mis-direct learning activity.

DIMENSIONS OF VARIABILITY

In the preceding section, we presented a taxonomic scheme of computational strategies that have been reported by other researchers and that we observed in our

own work. With this taxonomic scheme now available, we use it to overview how strategy use by students varies across contexts and how it changes over time with instruction.

Learning Progressions in Broad Sweep

Throughout this article, we have hypothesized that the learning of number-specific computational resources is the primary driver of strategy change in single-digit multiplication. This hypothesis leads us to have some specific expectations regarding dimensions of variability in strategy use. First, for a given individual, we do not expect across-the-board development in strategy use. Instead, at any given time, the strategy that a child uses will depend on the values of the operands. Second, because the learning of number-specific resources is very sensitive to instructional emphasis, we are led to expect significant variation in learning progressions across classroom contexts.

Nonetheless, there are some generalizations to be made, both within our project and across the literature. The accounts in the research literature are, in broad outline, what one would expect; researchers describe a general movement from the left side to the right side of Figure 4. Students begin with strategies in the vicinity of count-all and progress toward increasing use of learned-product-like strategies. For example, Kouba (1989), in her interviews of students in first though third grades, saw a progression from what she called "direct representation" to "recalled number facts." Similarly, Mulligan and Mitchelmore (1997) documented a steady progression from their "unitary counting" strategy through "repeated addition" to "multiplicative calculation." Although we did not set out to study systematically how the frequency of strategy use changed over time, to the extent that we can draw conclusions, our own data seem to be consistent with this broad developmental outline (refer to Figure 8 in Appendix A).

Variation in Strategy Use Across Problem Contexts

Having looked at the broad sweep of changes in strategy use, we now look at variability in strategy use at a given time during the instructional sequence. Research from the retrieval-focused genre has documented, in a quantitative manner, some broad measures of the variability of strategy use by individuals. It has been shown that children use diverse strategies throughout their learning period and that adults continue to use multiple strategies (Brownell & Carper, 1943; Jerman, 1970; LeFevre et al., 1996). Lemaire and Siegler (1995), using their coarse-grained taxonomic scheme (see Figure 4), found that the number of strategies employed by French second graders began at 3.1, increased to 3.7, and then decreased to 2.4. Similarly, Anghileri (1989) found that only 7 out of the 90 students studied employed the same strategy to solve all of the six tasks that she administered. Furthermore, 78% of the remaining students who solved all six of her tasks used at least three different strategies. Looking at adults, LeFevre et al. (1996) found, in two separate experiments, that subjects used nonretrieval strategies on 17% and 32% of trials.

Although our conditions were different than those in the studies noted above, we found variability on a similar order of magnitude. During a given interview, individual students tended to use multiple strategies. Even during the final interview, they employed an average of 3.0 different canonical strategy types. Although values of this sort are very sensitive to the granularity of the coding scheme (and to difficulties in coding of the sort mentioned above), they can give a sense of the variability in individual student's use of strategies.

Variation in Strategy Use With Operand Values

Next we discuss how strategy use varies depending on the numbers that appear in a task. To begin, the retrieval-focused literature has identified a number of "structural features" that seem to have importance for student solutions in single-digit multiplication:

1. *Problem-size effect*. Tasks with smaller operands are easier for students and are more likely to be solved by learned product (e.g., Campbell & Graham, 1985; LeFevre et al., 1996; Miller, Perlmutter, & Keating, 1984).
2. *The ties effect*. Tasks involving ties (i.e., tasks in which both multiplicands are the same, as in 6×6) are easier for students than would be expected given the problem-size effect and are more likely to be solved by learned product (e.g., Campbell & Graham, 1985; LeFevre et al., 1996; Miller et al., 1984).
3. *5-operand advantage*. Tasks with 5 as an operand are easier for students than one would expect given their problem-size and are more likely to be solved by learned product (e.g., Campbell, 1994; Campbell & Graham, 1985; LeFevre et al., 1996; Miller et al., 1984).

Some of the studies listed in Figure 4 attempted to map out, in terms of their own category schemes, how strategy use depends on structural features of this sort. Lemaire and Siegler (1995) did this for a strict version of the problem-size effect. In their analysis, they divided the problems given into 4 categories: easy (product < 8), relatively easy (product 9–18), relatively hard (20–36) and hard (36–81). At the time of their first interview session, these situations existed:

- Retrieval dominated for the easiest problems.
- Repeated addition and retrieval dominated for the relatively easy problems.
- Repeated addition and "I don't know" were common for the relatively hard problems.
- "I don't know" was most common on the hardest problems.

In contrast, by the third interview session, retrieval was the most common strategy across all the categories of problems. Steel and Funnell (2001) documented a similar pattern in the use of retrieval. When our data are broken out in this manner, we also see differences in strategy use on "easy" and "hard" problems during both the pre- and postinterviews (see Figures 9 and 10 in Appendix A).

However, an analysis that looks only at product size misses many of the important details concerning the dependence of strategy use on multiplicand values, including the possibility that students may tend to use different strategies on ties and 5-operand tasks. Here, the existing literature is somewhat sparse; there have not been systematic attempts to map, in detail, how children's strategy use varies across all multiplicand values. However, LeFevre et al. (1996) present relevant data for adults. In brief they found that—

- for tasks involving 0 or 1 as operands, the only methods reported were "retrieval" and "rule."
- "repeated addition" was used primarily for problems with 2 as an operand.
- "number series" (count-by) was used primarily on problems with 3 or 5 as an operand.
- the majority of uses of "derived fact" were on problems with a product greater than 40.

These brief results suggest more complexity in how strategy use depends on operand than is suggested by the structural features listed above, and we must expect more complex dependence in children, particularly during the time that the students are engaged in the learning of new computational resources.

Issues of Universality Revisited

Figure 7 draws together a rough map of learning progressions that describes what we saw in our own classrooms and that is consistent with what we know from prior research (in contrast to Figure 2, Figure 7 attempts to capture more of the nonlinearity of the learning progression). How universal is the progression described in Figure 7? Any answer to this question must necessarily be speculative. Nonetheless, in this section, we draw together our best guesses concerning how this learning progression may depend on the particularities of classroom and cultural contexts.

We have argued that any learning progression in multiplication strategy use will depend, in a sensitive manner, on the nature of instruction. Nevertheless, there are some reasons to expect rather substantial uniformities across instructional contexts. This is true, first, because there are a number of constraints on strategy development that operate across many contexts in which multiplication is learned. For example, there are some broad cognitive constraints, such as the size of working memory, that strongly constrain the range of computational strategies that are feasible, and there are significant uniformities in the notational systems used across classrooms and context. Furthermore, across classrooms and cultural contexts, there are uniformities in instructional approach that go beyond these constraints. For example, these constraints do not *require* the teaching of multiplication triads, yet multiplication triads are the focus of instructional attention across a wide range of classroom and cultural contexts.

With these thoughts in mind, Table 3 draws together some of our best guesses about the universality of our scheme and learning progressions. For each of the

	Count-all	Additive calculation	Count-by	Pattern-based	Learned products	Hybrids
Entry	Addition conceptual structures in place. Allows solving of multiplication tasks with count-all.	Some capabilities in place. Represent and solve using addition resources: $m \times g = g + g + g$	Some sequences (2, 5, 10) known as a means of fast counting.	Patterns for 0, 1, 5, 10 known or induced early	Some small multiplication triads known early	
	Standardized drawing techniques are standardized finger techniques	Some techniques may be developed and used repeatedly	Increasing knowledge of count-by sequences	Increasing knowledge of patterns, 9's in particular	Gradual learning of multiplication triads; doubles, smaller numbers, learned earlier	Use of multiplication triads and count-by sequences with count-all and additive calculations
		Use as a differentiable strategy fades for smaller numbers				
Later	May continue to use on word problems beyond use on number problems	Use as a differentiable strategy fades	Use as a differentiable strategy fades		6's, 7's, and 8's learned latest	

Computational resources merge into rich knowledge of multiplicative structure of integers ≤ 81. Strategies become difficult to differentiate, particularly for smaller multiplicands.

Figure 7. A rough map of the learning progression

strategy types, this table describes what we believe will be nearly universal across cultures and context, what we believe will be somewhat culturally dependent, and what we believe will depend strongly on features that are likely to vary across classroom contexts. As stated in Table 3, we expect the use of count-all and additive calculation strategies to be fairly universal, as long as students have had prior instruc-

Table 3
Conclusions About the Universality of Our Taxonomy and Progression

Strategy	Nearly universal	Culturally dependent	Strongly context dependence
Count-all	When given the opportunity, students who have had prior instruction in addition will invent some varieties.	Drawing conventions will affect types of drawings. There will be variation across cultures in conventional ways that fingers are used for counting.	Students may use very specific drawing and finger counting techniques that are practiced in the classroom or learned in some homes.
Additive calculation	When given the opportunity, students who have had prior instruction in addition will invent some varieties.	There will be variability across cultures associated with cultural variation in addition notations and techniques	Students may learn and practice particular techniques in some classroom contexts.
Count-by	Some sequences may be discovered and practiced independently by students, but use will be limited without formal classroom attention. The base system used has implications for which sequences are easier to learn. (10's, 5's, and 9's have simple patterns.)	The learning of count-by sequences or tables may be more frequent in some cultures. The structure of number words affects which count-by sequences are easier to learn.	Use is strongly dependent on classroom learning and individual practice of the count-by sequences. The order in which count-by sequences are learned, and relationships taught between them, may vary across instructional contexts.
Pattern-based	Induction of 0 and 1-based patterns by students should be universal. 5 and 10 patterns should be universal across cultures that employ base 10.	Explicit instruction in certain rules or patterns may be more traditional and frequent in some cultures.	The learning of some pattern-based strategies will be strongly dependent on explicit classroom attention. For example, 9's pattern-based strategies may be much more likely to appear if they are an explicit focus in the classroom.
Learned product	Students may incidentally learn some pair-product associations, but use will be limited without formal classroom attention or out-of-school experience. The base system used has implications for which pair-product associations are easier to learn. Thus, there are some learning sequences that are universally more sensible than others.	The learning of pair-product associations may be more or less traditional in some cultures.	Learning of number triads is strongly dependent on classroom learning and practice. The sequence in which pair-product associations are learned may vary across instructional contexts.

tion in addition and they are given the opportunity to employ these strategies. However, we do expect significant cultural and classroom variation in the specific subtechniques that are employed.

In contrast, the use of count-by and learned product strategies is somewhat more dependent on features of the classroom context. Students may induce some count-by sequences on their own or they may learn them in the context of addition instruction (e.g., the sequences for 2 and 5). But the learning of other count-by sequences (or tables) likely requires explicit classroom attention, and this classroom attention may be more or less conventional across cultural contexts. Similarly, students may learn multiplication triads for some small numbers on their own, but broad learning of multiplication triads requires substantial effort. Where students *are* taught count-by sequences and number triads, there are constraints that make some instructional sequences more sensible than others. For example, because we employ a base-10 system, the 5 count-by sequence can be learned comparatively rapidly; thus, it is sensible to teach this sequence early in the instructional cycle.

The story for pattern-based strategies is mixed. The pattern-based strategies involving 0 and 1 should be induced, nearly universally, by students; the associated patterns do not even depend on the use of base 10. Similarly, we expect the use of pattern-based strategies with 5's and 10's to be fairly universal across cultures that employ base 10. The use of other pattern-based strategies may depend more sensitively on details of instruction, even though the patterns themselves are essentially determined by our use of base 10. For example, it is less likely that students will recognize patterns in multiples of 9's if these are not addressed instructionally.

Finally, the use of learned product by students, across a wide range of multiplicands, requires explicit instructional attention. As stated above, students may learn some multiplication triads on their own, but broad learning of multiplication triads is likely to be dependent on substantial instructional focus. Thus, broadly speaking, classrooms and cultures that mobilize organized and sustained efforts for such learning will be more successful.

SUMMARY AND CONCLUSION

The purpose of this article was to attempt to work toward consensus on a taxonomy of strategies for single-digit multiplication. Our goal was to present a scheme that is fully fleshed out and that leaves little room for misunderstanding. In addition, we wanted to give the reader a sense for the range of variability. For these reasons, we presented and discussed numerous examples, and we frequently returned to the research literature to make explicit comparisons.

There are some important respects in which the stance we adopted differs from that of previous work. Our taxonomy of strategies is determined by an understanding of the mechanisms that govern strategy development. For the particular case of single-digit multiplication, we contended that the primary mechanism is the incre-

mental appropriation, by students, of number-specific computational resources. This stance had major implications for our larger task of understanding the development of multiplication strategies. We were led to expect operand-dependence of strategy use at all stages of learning, even into adulthood. Our stance implied that strategies, as well as learning progressions through strategies, are sensitively dependent on certain details of instruction. Additionally, the boundaries between categories become increasingly fuzzy over time, because computational resources cease to be separable.

As we discussed earlier, no single piece of evidence can support a broad stance of this sort. Instead, we intend this stance to be supported by the overall coherence of this view, as well as by its consistency with and ability to explain a wide range of data. Here, we summarize the arguments and evidence that can be drawn from the presentation in the earlier parts of this article.

First, much of our argument was made without specific supporting data. We believe it is manifestly clear that at least some of the relevant knowledge is number-specific. Some classes of strategies are, by their very nature, specific to operand value. For example, some of the pattern-based strategies only work when 9 is an operand. Similarly, some variants of strategies, such as certain finger-counting techniques, only work for a small range of operand values. Further adding to this prima facie case is the fact that, in order for some of our classes of strategies to work, a child must acquire supporting number-specific knowledge. For example, the count-by strategy requires that the student acquire the count-by sequences for each operand. And learned product requires that specific number triads are learned. No across-the-board conceptual development can obviate the need for the learning of these number-specific computational resources.

Following from these observations, a prima facie case can also be made for some amount of sensitivity to instruction. Without explicit instructional attention, it is unlikely that children would learn most of the single-digit number triads or that they would learn the count-by sequences. Thus, the appearance of these strategies likely requires this instructional attention.

In addition to this prima facie case for our view, there is also a variety of empirical support spread throughout this article. This support includes the following observations:

- *Dependence of strategy use on operand value*. We presented evidence that the strategy employed depends on operand values.
- *Variability in strategy use persists through instruction and into adulthood.* Following instruction, students continue to use several strategies. Still more dramatically, adults continue to use different strategies, depending on operand values.
- *Diversity of strategy variants*. The sheer diversity and nature of the strategies observed constitute evidence for our stance. It is not simply the case that the *selection* of strategies depends on operand values. As we stated just above, we observed strategy variants that developed for particular tasks, with particular

operand values. This suggests a richer texture to the learning than could be explained by an across-the-board conceptual shift.

- *Instructional sensitivity*. We stated above that a prima facie case can be made for instructional dependence. We also believe that we have seen the beginnings of empirical evidence for this dependence. In particular, there were idiosyncrasies in our data corpus that can be plausibly tied to features of CMW. This included differences in strategy use between our observations and those of other researchers.
- *The merging of computational resources*. We presented several examples that we argued were intrinsically ambiguous—they could not be placed within any of the sort of categories of computational strategies that are discussed in the literature. Within our framework, this particular variety of ambiguity was expected, since strategy use depends on a moderately large number of interrelated number-specific resources.

Again, no single category of evidence is the linchpin in the argument for our view. Rather, the support comes from the broad consistency of our stance with these observations.

Instructional Implications

There is a temptation for us, as researchers, to want to discover universal progressions in learning that are driven by deep changes in conceptual structure. The very nature of mathematics makes it seem particularly suited to such an approach and, to be sure, there are cases in which these discoveries are there to be found. However, there are parts of mathematics learning that, although important and complex, are driven by more incremental mechanisms. We have argued that this is true for the learning of single-digit multiplication.

We must be careful, however, in how we understand the instructional implications of this claim. We have essentially argued that the development of strategy use in multiplication is not driven by central conceptual developments. But it would be a mistake to take this as implying that the learning of multiplication need only be based on repeated practice with isolated facts. Our claim suggests that we stake out a middle ground. Students require help to acquire number-specific computational resources, but these resources must not be thought of as consisting of a collection of isolated "facts." This point was emphasized, above, when we discussed the increased merging of computational resources with learning. Based on the observations reported there, we argued that it is not appropriate to think of the end products of learning as a straightforwardly internalized version of the multiplication table, consisting of individual and separate cells.

Taken as a whole, this suggests that we cannot give up on practice of a certain sort; students must have experience working with specific operands. But it also suggests that this practice will have its greatest effect when "facts" are not treated in isolation, and when practice on number triads follows, and is continually linked

to, meaningful examination of patterns and strategies. Practice must be done in such a way that it helps students become familiar with, and continues to support student understanding of, the patterns and structure across computational resources, so that each child can form a rich network of number-specific resources.

These conclusions point to the need for some specific varieties of future work in order to improve pedagogy in this arena. We must work to determine what computational resources students should acquire, and how they can best acquire them. This suggests relatively focused questions, such as which patterns are powerful enough that they deserve instructional effort. But we may also consider some more radical restructuring of instruction. For example, in recent iterations of CMW, we have explored the possibility of teaching multiplication and division together. Our belief is that this approach can help students get a handle on the rich network of multiplicative structure of the integers less than or equal to 81. In our view, this is the sort of pedagogical direction suggested by the analysis presented in this article.

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APPENDIX A

Additional Detail on the Empirical Work

In this appendix, we provide details concerning the context and scope of our empirical work. We briefly describe the larger project of which this was a part, and we overview our data collection and analysis techniques.

Context: The Children's Math Worlds Project (CMW)

Our data collection efforts were conducted as part of one phase of the Children's Math Worlds Project (CMW), during which we worked to develop full-year curricula for third- and fourth-grade mathematics. The topics covered in these curricula include the usual grade-level topics, such as single-digit multiplication and division, as well as some topics that are not usually addressed until later years. There is a strong emphasis on fostering classroom discourse around mathematics; students are encouraged to develop and share their own strategies, and drawn representations are particularly valued. However, this focus on discourse is not done at the expense of grade-level mastery. Patterns for all factors are discussed, and students learn fundamental strategies such as count-by.

Data Collection

During this phase of the CMW project, we have worked closely with a number of classrooms and engaged in a range of data-collection activities, including teacher interviews, student interviews, written assessments, and frequent classroom observations. In this article, we have drawn examples from the interviews we conducted, all of which were videotaped. However, our conclusions here were greatly informed by the full range of our experience in classrooms, particularly by the detailed ethnographic work by members of our team.

Table 4 is an overview of the interview data collection on which we drew in our study of single-digit strategies for multiplication. As described in this table, we conducted our first interviews near the end of Year 1 of this phase of the project. Thirty-seven interviews were conducted with students in two classrooms. With regard to our study of computational strategies, these first interviews allowed us to refine our interviewing strategies.

During Year 2, we engaged in our most extensive and systematic data collection related to single-digit computational strategies. We interviewed third-grade CMW students at the start of the year, at two midpoints during a unit on multiplication, and then after the multiplication unit. Finally, during Year 3, we interviewed a selection of fourth graders at the start of the year. For the purposes of the current work, these last interviews allowed us to test the reliability and generalizability of categories we had developed from our analysis of the previous year's data.

Two classrooms from the Year 2 data corpus provided a sufficient range of examples to illustrate our classification, and our examples were all drawn from these class-

Table 4
Overview of Student Interview Data Collection Relevant to Single-digit Multiplication Strategies

Year views	Grade	When?	No. of classes	No. of inte
1	3	End of year	1	22
	4	End of year	1	15
2	3	Start of year	4	69
	3	Midpoint 1	2	8
	3	Midpoint 2	2	7
	3	After instruction	3	45
3	4	Start of year	3	64

rooms. Teachers TD and NQ[6] taught in public schools that differed dramatically in the makeup of their student bodies. TD's classroom was in a suburban school that was 55% White and 31% Asian, with the remainder being Black and Hispanic. Furthermore, 14% of students were from low-income families, and 14% were reported as having limited English proficiency. NQ's classroom was in an urban school with a student population that was 53% Black, 43% Hispanic, 2% White, and 1% Asian. Ninety-two percent of students were reported as low income, and 30% as possessing limited English proficiency.

Table 5 provides a breakdown of the interviews conducted with students in TD's and NQ's classrooms. We attempted to interview all the students in both classes at the start of the year and after the multiplication unit. However, NQ declined to have us conduct postinterviews because of time pressures at the school. A selected group of students in each classroom were also interviewed at two midpoints during the multiplication unit.

Table 5
Interviews Conducted in the Classrooms of TD and NQ

	Pre	Midpoint 1	Midpoint 2	Post	Total
NQ	22	4	3	0	29
TD	12	4	4	15	35
Total	34	8	7	15	64

Interviewing Tasks and Techniques

In the interviews mentioned above, our goals with respect to computational strategies were intentionally broad. As much as possible, we wanted to be able to see the full range of diversity of computational strategies. Furthermore, we wanted to understand as much as possible about how each individual strategy worked; for example,

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[6] TD and NQ are abbreviations of pseudonyms, not of the teachers' true names.

we wanted to see what types of drawings students made and how they made these drawings, and we wanted to see how students used their fingers in computations.

This interest in mapping the "flora and fauna" of student strategies placed difficult demands on our interview data collection. Ideally, we would have given all of the single-digit multiplication combinations to every student, on each type of word problem, and at many points of time. However, pragmatically, it was not possible for us to cross all possible number combinations with all types of word problems. Some studies with adults administer all possible number combinations to individual subjects, but they do not employ word problems, and they give subjects a maximum time limit on each question of perhaps 5 or 10 seconds (e.g., LeFevre et al., 1996). In contrast, because we were interested in understanding, in full detail, how subjects execute strategies, including very laborious strategies, we often needed to give students many minutes to solve a problem. This was particularly an issue during early interviews, when students used less efficient strategies.

The situation was complicated still further by the fact that our one-on-one interview time had to serve multiple needs. We had other research concerns operating in parallel, including research concerned with semantic types. Furthermore, because this work was conducted during an early iteration in the design of our curricular materials, we used our interviews to help us understand student difficulties, so that we could refine our ongoing instruction.

The result of these multiple desires and constraints is that it was not possible for us to cover all possible dimensions during every interview. Instead, we used our evolving understanding to sample widely across the range of phenomena of interest. The result is that some particular types of quantitative analysis were not accessible to us. For example, although it was possible for us to compute the frequency with which students employed certain strategies, these frequencies are not directly comparable with those found by earlier studies that sampled all multiplication combinations uniformly. However, our method of broad sampling has put us in a position to understand, in qualitative detail, the range of computational strategies employed by students. In our view, this is precisely what is needed at the current time. As we have discussed, there is already a substantial body of experimental studies in which tens or even hundreds of subjects solve all the single-digit multiplication combinations. But, in order to get this coverage, these studies suppress detail, and they do not look at word problems (e.g., LeFevre et al., 1996; LeFevre & Morris, 1999; Lemaire et al., 1991; Lemaire & Siegler, 1995; Siegler, 1988).

Furthermore, there are some reasons that CMW classrooms provide a particularly appropriate context for this work. Although CMW works hard to help all students develop efficient strategies, student strategies are certainly not suppressed. Indeed, CMW places a premium on students being able to communicate their strategies to others. Moreover, if we accept that multiplication strategies are highly dependent on instruction, it makes sense to study strategies in instructional contexts that we believe are promising.

Given our desire to sample broadly, the tasks employed in our interviews were quite varied. They included multiplication word problems as well as tasks in which

students were simply asked to find the product of two numbers. We do not present a comprehensive list of all the tasks we employed. Instead, each example episode in the article is accompanied by the task.

Our interviewers adopted a technique in which increasing support was given to students until the student was able to solve the task. In tasks concerned with multiplication strategies, the interviewer always attempted first to elicit a solution without guidance. If the student struggled, the interviewer would then provide increasing support. Eventually, if necessary, the interaction would become strongly tutorial in character, so that we could study the learning of multiplication strategies. Except where explicitly noted, the examples presented in this paper all describe student solutions that were produced without tutorial guidance by the interviewer.

Analysis Methods

Our first categories were formed by our early classroom experiences and Year 1 interviews as well as by looking at existing research. With these initial categories in mind, we engaged in systematic and intensive analysis of the Year 2 data corpus, during which we coded and recoded the relevant problem-solving episodes. This recoding proceeded until the team had reached convergence on an analysis scheme. This effort was greatly facilitated by the creation of a digital database. The videotapes were first digitized and stored on a centralized server. Then, as we viewed and coded the video, we created a database with indices into the digitized video files. In all, this database contained 397 episodes of students solving problems, ranging in duration from 5 seconds to 15 minutes. Of these episodes, 291 were multiplication problems (the remaining 107 required some division). This digital system allowed for rapid comparison across the 291 instances of computational behavior, and it facilitated convergence on a set of categories.

Because our presentation in this article does not rely on the detailed results (in the form of frequencies) of our coding efforts, we do not report on our analysis

Table 6
Summary of Results of Coding of Year 2 Corpus

	Count-all	Repeated-addition	Count-by	Pattern-based	Learned product	Hybrid	No solution	Total
TD								
Pre	6	1	14		5	7	26	59
Midpoint 1	3	3	2		2	1	1	12
Midpoint 2	3	1				1	4	9
NQ								
Pre	2	6	7		1	1	9	26
Midpoint 1	1	2	3		1	3	2	12
Midpoint 2	2	6	1		1	2		12
Post	5	6	21	7	94	9	19	161
Total	22	25	48	7	104	24	61	291

procedures in any greater detail. From the point of view of this work, the purpose of the somewhat systematic analysis was primarily to ensure that we were attentive to the range of phenomena in our data corpus. Furthermore, as discussed in the body of the article, we believe that coding becomes increasingly difficult as students move toward expertise, because of a real merging of the strategies. Nonetheless, in order to give the reader some sense for the range of variability in our corpus, the overall results of our coding of the Year 2 data are summarized in Table 6. These results are also shown graphically in Figure 8.[7] In addition, Figures 9 and 10 show the pre- and postinterview data broken out by product size (see pages 394 and 395).
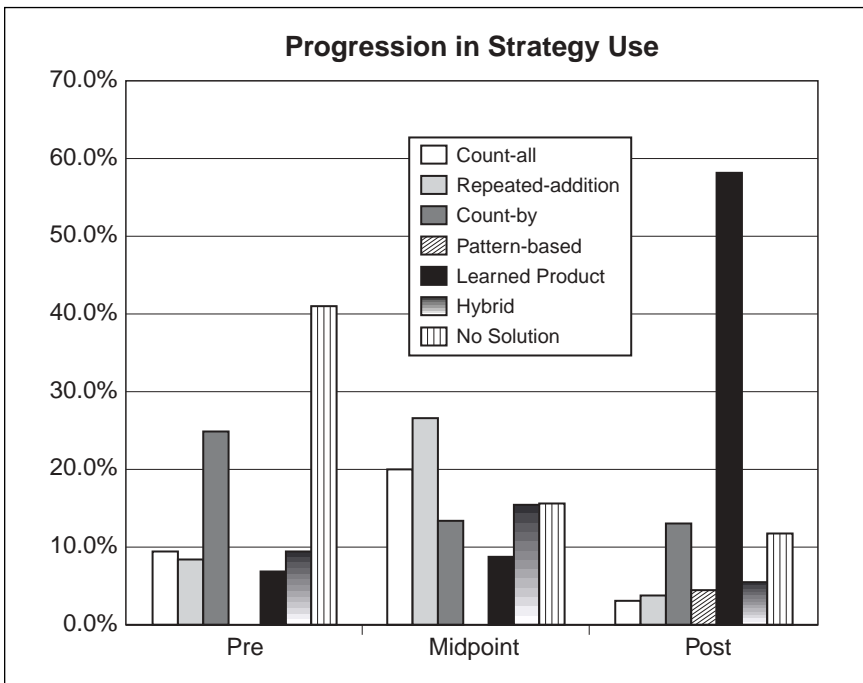

Figure 8. Percentage use of strategies before, during, and after instruction

[7] In this chart, we have combined the results of the two midpoint sessions because the number of focus interviews was small in comparison to the pre- and postinterviews. In addition, it should be kept in mind that the postinterviews were only conducted in one of these two classes. Also, the relative prevalence of skip counting during the preinterview was due, in large measure, to the inclusion of tasks in which the number 5 was one of the multiplicands.
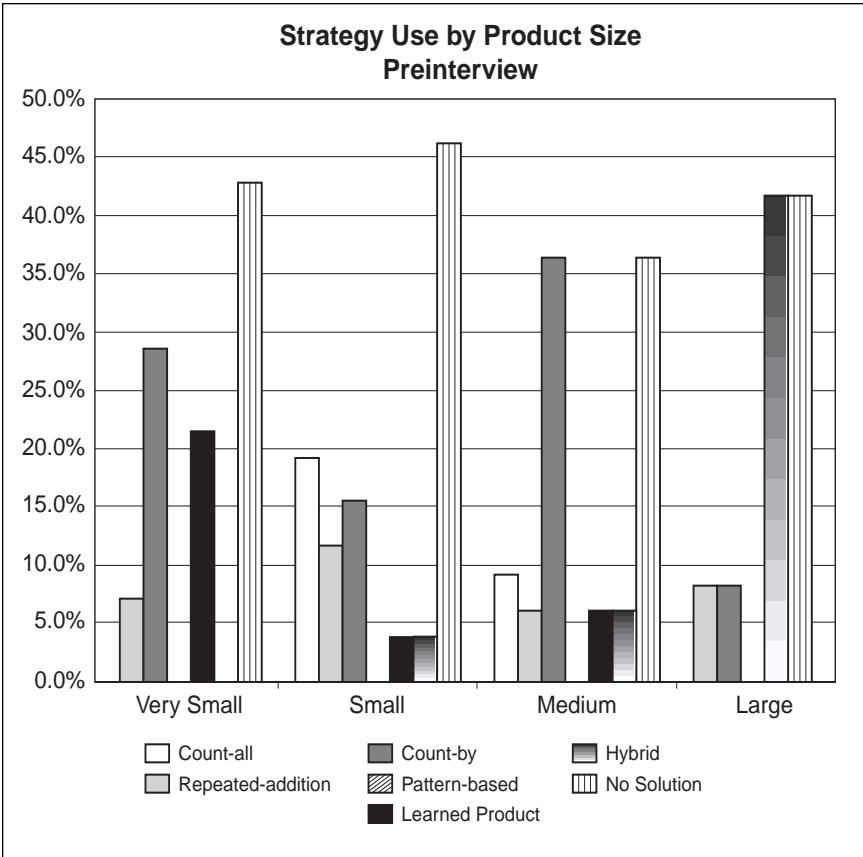
Figure 9. Strategy by product size for the preinterview

The Year 3 data were used as a test of reliability of the scheme developed during Year 2. The coding was done by two undergraduates who were trained to code using our category scheme. In all, these undergraduates coded 380 instances of single-digit multiplication in the Year 3 data. After the first coding pass, the two coders disagreed on 47 of the 380 instances (12.4%). Of these 47 disagreements, 15 were easily resolvable by the two coders because of an error by one coder. This result is acceptable given the inherent complexity of the data and given our position that coding will be intrinsically difficult in some cases. Indeed, our inspection of the remaining 32 disagreements revealed that these episodes were quite ambiguous and difficult to code with any confidence. Of these 32 instances, 25 involved hybrid codes, which are inherently more difficult.
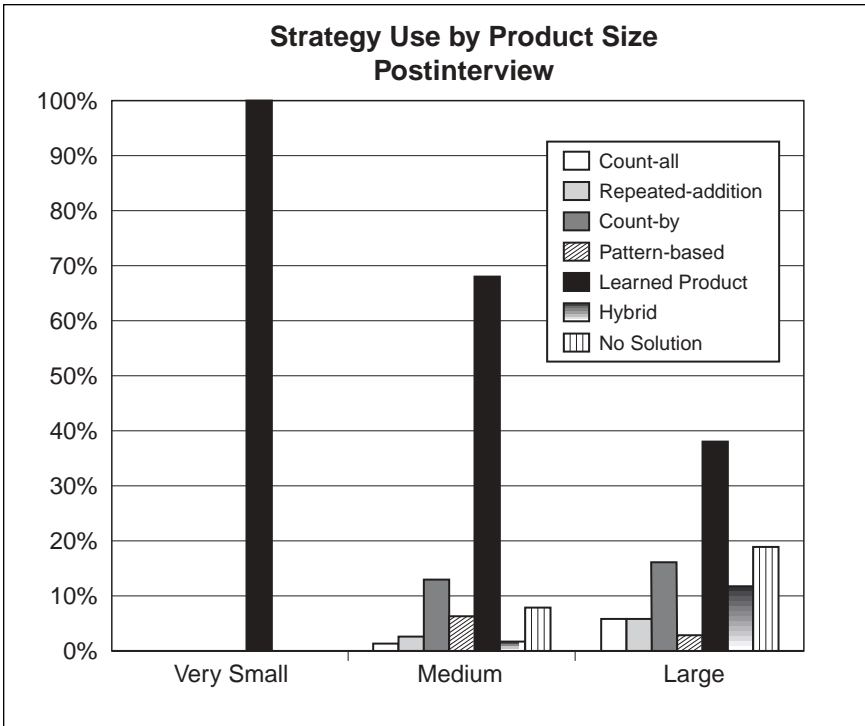
Figure 10. Strategy by product size for the postinterview