A Design for Ratio and Proportion Instruction

DOR ABRAHAMSON AND CHRISTIAN CIGAN

DOR ABRAHAMSON, *abrador@northwestern.edu, is a graduate student in the learning sciences department of the School of Education and Social Policy at Northwestern University. He applies theoretical frameworks from cognitive science to the design and research of curricular units set in collaborative learning environments. The work reported here was part of the Children's Math Worlds project, directed by Karen Fuson.* CHRIS CIGAN, *CCigan@hotmail.com, is a fifth-grade teacher at Orrington Elementary School in Evanston, Illinois, who is interested in teaching reform mathematics in a student-centered classroom.*

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THIS ARTICLE DESCRIBES A METHOD for teaching ratio and proportion in a fifth-grade classroom. Our unit design creates and follows a learning path that begins with revisiting multiplication as repeated addition. It then explores patterns in the multiplication table and moves to the concept of rate as a single column out of the multiplication table. Next, students discuss ratio as two linked rate columns, or a ratio table, then look at two rows out of a ratio table, or a proportion quartet. As a fifth-grade introduction to ratio and proportion, this unit focuses on integer cases, that is, situations in which the question mark in $a:b = c:?$ is an integer. This emphasis fosters multiplicative reasoning in a developmentally appropriate numerical environment.

This framework shows the solution to problems similar to the following:

Every day, Robin and Tim each save at a constant rate. If, on a certain day, Robin has \$6 and Tim has \$10, then how much will Tim have when Robin has \$21?

	Robin 3	Tim 5	
2	6	10	2
7	21	35	7
	3	5	

(d) Proportion Quartet (PQ)
factored outside

PQ as "mini-MT"

PQ as selected RT rows (here, rows 2 and 7)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

(a) Multiplication Table (MT)

Day 1	3	5
0	0	0
1	3	5
2	6	10
3	9	15
4	12	20
5	15	25
6	18	30
7	21	35
8	24	40
9	27	45
10	30	50

Every day, Robin puts \$3 in her cat bank, and Tim puts \$5 in his doggy bank.

(c) Ratio Table (RT)

Rate as single MT column

RT as two simultaneous rate columns

Every day, Robin puts \$3 in her cat bank.

1	3
0	0
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24
9	27
10	30

(b) Rate Table

Fig. 1 Designed framework for conceptualizing ratio and proportion. From left and counterclockwise: (a) the multiplication table (MT), (b) rate table, (c) ratio table (RT), and (d) proportion quartet (PQ). Products and cells of a specific example problem on the top left are enhanced here for illustration.

Introduction to the Unit

THE UNIT DESIGN STARTS WITH THE SIMPLE ideas of addition and multiplication. This introduction strongly connects the more complex ideas of ratio and proportion to the mathematical concepts and operations that fifth-grade students already know and builds on these skills. Over the course of the unit, students are encouraged to draw on and discuss their personal intuitive knowledge, as well as their mathematical knowledge, by relating real-world experiences and mathematical formats. The objective of the unit is to ground students' concepts of ratio and proportion reciprocally in their interpretations of real-world experiences (story situations) and symbolic mathematical representations (tables, numbers, and notation). A reciprocal interpretation of situations and representations, we believe, fosters robust conceptual tools for addressing and solving ratio and proportion problems, whether purely numerical or set in the context of word problems.

The design of the curricular unit evolved from our concern with students' frequently incorrect additive reasoning in ratio and proportion. Such reasoning leads to mistaken assumptions; for example, "The ratio 3:5 = 6:8 because 3 + 2 = 5 and 6 + 2 = 8." In the design of our unit, we sought to link ratio and proportion situations in the world to analysis of the multiplication table to foster students' multiplicative reasoning. Initially, students focused on repeated-addition situations to fill in ratio tables and develop an understanding of the multiplication table; they then moved to multiplicative solution strategies to solve ratio and proportion word problems. This article first presents an overview of our unit, then discusses the results from a fifth-grade classroom.

The four related tables we use in this unit—the multiplication table, the rate table, the ratio table, and the proportion quartet—are shown in **figure 1.** The following paragraphs highlight how we introduce and discuss each table with students, using proportion story situations to form a coherent and integrated understanding.

Exploring Patterns in the Multiplication Table

WE BEGIN OUR UNIT BY REQUESTING THAT STUDENTS "find interesting patterns or facts in the multiplication table." Students discover patterns that they have not seen before and mark them in color, as shown in **figure 2.** The multiplication table is rich in patterns of repetition of rows and columns or diagonals, where sets of numbers are repeated, but translated in position in the table (see **fig. 3**). The simplest version of this pattern is "twin" rows and columns; for example, the 3 row (3, 6, 9, . . . , 30) has a twin in the 3 column (3, 6, 9, . . . , 30). Our students found many more patterns, some of which are less obvious, such as the table's symmetry across the diagonal of squares (1, 4, 9, . . . , 100). Other patterns use rectangular arrays of cells made by the 1 cell as a top left corner and any other cell, such as 45, as a bottom right corner. The number in the bottom right corner tells how many cells are inside the array; this discovery led to a discussion of areas of rectangles.

After this exploration, we focused on the simplest "pattern" of the table, the one that we expect all students to know, but, we found, the one that many did not understand very well. In every row or column of the multiplication table, the first value dictates the "count-

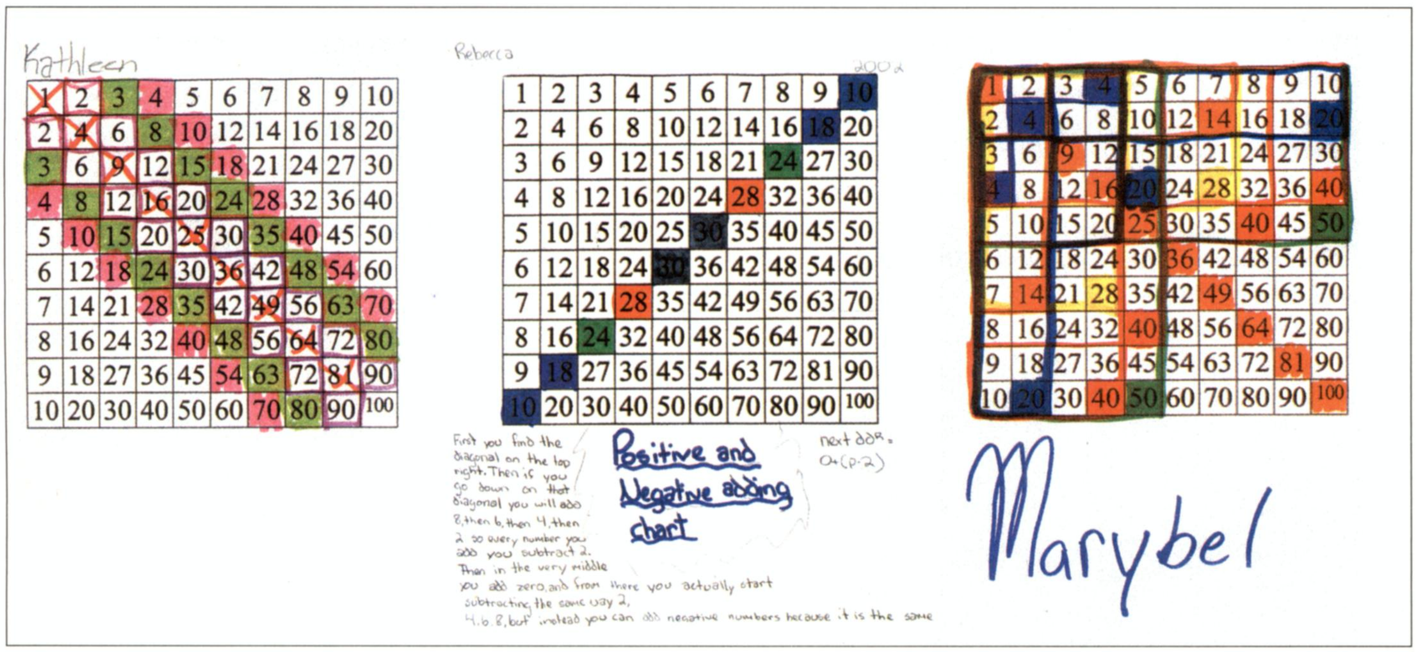

Fig. 2 Students' pattern discoveries in the multiplication table

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
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5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
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5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Fig. 3 Examples of spatial-numerical patterns in the multiplication table

by" value used in skip counting; for example, in the 3 column, we count by threes. Students could easily reproduce the 3 column by reciting and writing the skip count by threes, but the moment we asked them to break the skip count down into a sequence of separate addition operations, students confused the addend and the running total. This confusion impeded students' consolidation of their understanding of multiplication as repeated addition. Our didactic solution was to have students interpret the 3 column as a situated rate narrative.

Seeing One Column as a Situation Rate

TO HELP STUDENTS UNDERSTAND MULTIPLICATION as repeated addition, we gave them the following problem:

> Robin's cat bank is empty. Starting from today, Robin will put \$3 a day into her cat bank. On day 7, how much will she have in her cat bank?

Robin's decision to save \$3 per day is an example of any real-world situation in which some quantity begins from 0, then accumulates by a constant increase linked to a fixed unit of time. (Note that in our work with fifth graders, we occasionally added a 0 to the multiplication tables to tie them to the situation beginnings, although most of the time, we chose not to deviate from the standard format of the multiplication table, which does not have a 0. Thus, we avoided the confusing issue of division by 0.) Accumulation situations, such as the one in the problem, can be thought of as rate situations with a running total, and the process of accumulation over time can be represented by a column in the multiplication table, such as 3, 6, 9, and so on, as shown in **figures 1** and **3.** The numbers in this column, interpreted as a running total, could just as well represent accumulations in other rate situations, such as a squirrel collecting 3 acorns per week; a plant growing up from zero, or ground level, at the rate of 3 centimeters per month; a beard growing

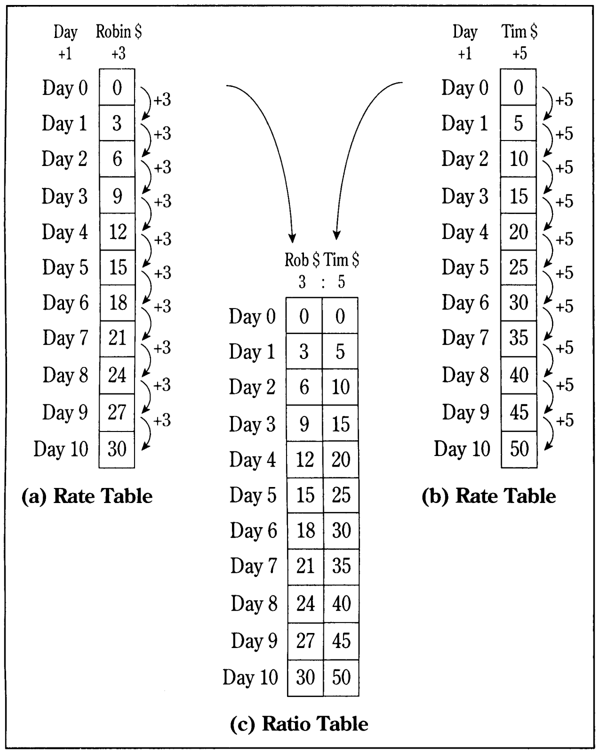

Fig. 4 A ratio illustrated as two rates "stepping in time together"

down from a chin at the rate of 3 inches per year; a person running across a track from a start line at the rate of 3 yards per second; and so on. We used students' language to relate their skip counting to the situations. We referred to the constant increment of the rate situation as the "adding number" or "growing number" and each of the successive values as the "total," the sum after each increment.

Despite the apparent simplicity of interpreting the rate table as a constant-increment situation, students often confused the constant x increase and the cumulative total. In their words, students confused the "growing number," or "what you put"; that is, +3, +3, +3, . . . , and "what you get"; that is, 3, 6, 9, ... For instance, students who attempted to interpret the 3 column in relation to Robin's story said, "On the first day, Robin put in $3, and on the second day, she put in $6." Understanding this distinction is pivotal for safely making the transition from addition to the idea of multiplication as repeated addition. Simple rate situations can provide a bridging context for understanding the notion of adding by a constant value.

Using other rate stories, we worked with students until they could fill in a situated rate table. We found that initially, writing in the increment operations (as shown in the repeated +3 marks in the diagram in **fig. 4a**), then, later, omitting these notations and leaving them as implicit in the rate context helped scaffold students' abilities to interpret and fill in the column in terms of the rate story.

The next step was to move from additive to multiplicative understanding of situations and the multiplication table. The numbers in the 1 column increase in parallel with the numbers in any other column; in other words, they count up using skip counting by 1 and indicate how many amounts of $3 Robin has accumulated, or how many additions of 3 we have made in the 3 column. In that sense, the numbers in the 1 column are the multipliers of 3, and each one of them indicates a number in the 3 column, its multiple of 3, that is in the corresponding row. For instance, we can think of 6 in the 1 column as indicating 18, its multiple of 3 in the corresponding row of the 3 column, "We needed six additions of 3 to get to 18."

For the factors, we focused on the growing number at the top of the column, such as 3, as in $3 per day 1, and on the multiplier in the left-hand column, such as 6, as in day 6; for the product, we focused on the total accumulated quantity, that is, $3 \times 6 = 18$. Students practiced this multiplicative interpretation of the rate table by solving partially filled rate tables. Students reconstructed tables with a missing top number (the constant increment) by dividing a product (e.g., 21) by its day factor (7). Understanding these accumulation situations and rate tables as columns from the multiplication table was essential. Less advanced students initially used a multiplication table for this type of problem solving, thereby increasing their ability to make connections between that mathematical format and proportion problems.

Ratio: Two Linked Rate Situations

IN PROBLEM SITUATIONS, WE LINK TWO UNIT rates into a single ratio. In other words, we think of $3 per 1 day and $5 per 1 day as "stepping in time together" to become $3 per $5. We represent this connection as two linked rate tables that become a single ratio table (see **fig. 4**). The following problem illustrates this type of situation:

> Every day, Robin puts $3 in her cat bank, and Tim puts $5 in his dog bank. When Robin has $21 in her cat bank, how much will Tim have in his dog bank?

Students' conceptual challenge in understanding ratio is that although the increases in each column are constant (3 and 5 in this instance), the difference between the two totals at every increment, or day, is not constant. For example, compare 3:5, which has a difference of 2, and 6:10, which has a difference of 4. A "same difference" misconception demonstrates what we meant earlier by *additive reasoning.* This misconception is so robust that a good method of assessing where your students are in their understanding of ratio and proportion is by evaluating whether they are still confused about this issue.

Robin and Tim begin saving money on the same day. Each child saves the same amount every day, but the amounts are different for each child. After some days, Robin has \$6 and Tim has \$10. On a later day, Robin has \$21. How much does Tim have?

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
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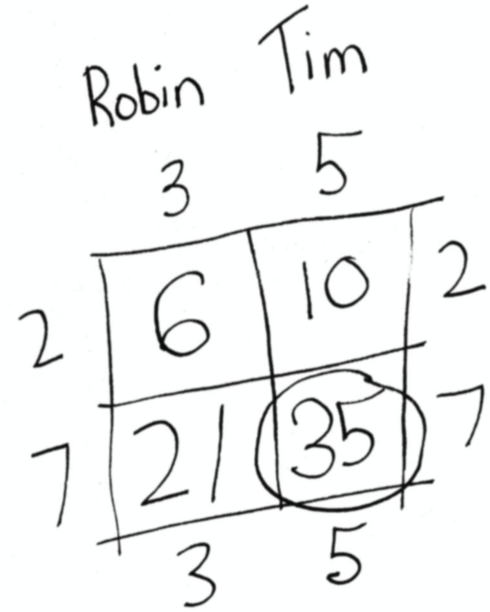


Fig. 5 Proportion-quartet solution format for missing-value ratio and proportion word problems; coming from the multiplication table

We found that the most effective types of demonstration are visual and physical experiences. For example, to represent two flowers growing at different rates, we had students place their hands flat on their desks, then lift them simultaneously, their left hands, 2 centimeters, and their right hands, 3 centimeters. If a student lifted her hands with a fixed distance between them, we challenged her by requesting that she "ungrow" down toward the desk. Once the student's lower hand reached the desk and the other hand was still in the air, we asked how that situation could be possible, given that she had begun the demonstration with both hands on the table. Other examples included a slow-motion running competition across the classroom, with the teacher racing at a rate of 2 yards per second against a student whose rate was 3 yards per second. Students were intrigued and delighted to realize that the distance between the "runners" grew from second to second. Their surprise indicated to us that this type of demonstration is essential for students to gain number sense for ratio and proportion.

A student is presenting a ratio as two rate tables in the multiplication table.

We created illustrations for students' favorite proportion stories, such as the growing plants or the running competition, and glued them to a poster. Throughout the unit, students referred to this poster to make sense of (to "situate") numerical or word problems. These meaningful contexts assisted students in making sense of the proportion quartet, which is merely two rows of a ratio table.

Using Proportion Quartets to Solve Proportion Word Problems Meaningfully

WE INTRODUCED PROPORTION QUARTETS USING the following problem:

Robin and Tim begin saving money on the same day. Each child saves the same amount every day, but the amounts are different for each child. After some days, Robin has \$6 and Tim has \$10. On a later day, Robin has \$21. How much does Tim have?

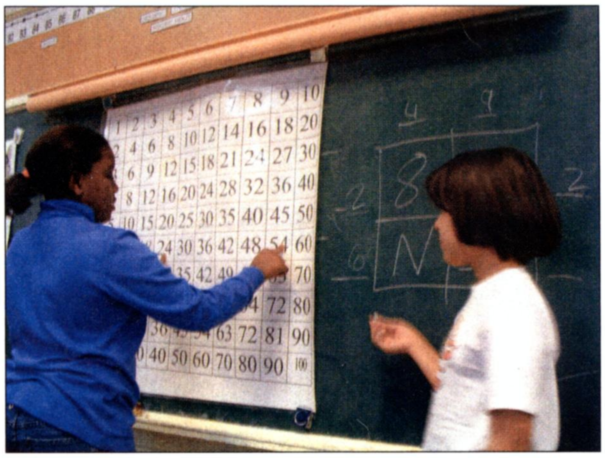
A proportion quartet captures two "time frames" (two rows) in a ratio table, such as day 2 and day 7 in the proportion story of Robin and Tim (see **figs. 1** and **5**). To solve a missing-value proportion quartet additively, we would need to reconstruct the whole ratio table and use the constant increases in each column. Such a task is time-consuming. A much easier task is to focus just on the values in the problem in the proportion quartet and factor these values as interpreted in the setting of the whole ratio table or multiplication table. As students explained their solutions to this problem, they used a large classroom multiplication table to link the additive increases in the ratio and multiplication table to their proportion-quartet solutions.

Of course, we cannot always rely on having a multiplication table nearby whenever we solve a ratio and proportion problem. We must also have a method that enables us to generalize our technique to fractions, very large numbers, or algebra. We do not need to refer to the multiplication table visually, however, but only mentally.

Joan used exactly 15 cans of paint to paint 18 chairs. How many chairs can she paint with 20 cans?

Use **figure 6** to follow our explanation of constructing a proportion quintet or, even better, use paper and pencil and go through the process yourself. Start by creating a 2-by-2 table, and label it according to the variable quantities in the word problem. In using this process in the classroom, students must name the columns and rows according to the specific situation described in the word problem, such as "Cans" for the left column and "Chairs" for the right column, to avoid incorrect entry of data from the word problem or incorrect interpretation of the solution. We remind students of this necessity by telling them to "Label the table!" This step also offers a good opportunity to prepare students for middle school work on data sets, where they will be required to create and manage complex charts. Once we label the two columns, we are committed to maintaining this order in both the top and bottom rows of the proportion quartet.

Next, we enter the three given values of the word problem; in this example, 15 cans and 18 chairs in the top row and 20 cans on the bottom row. We now have a proportion quartet in which the top row has two known values and the bottom row has a known value and an unknown value. If we think of the proportion quartet as an abbreviated multiplication



Students are discussing proportion quartets in the multiplication table.

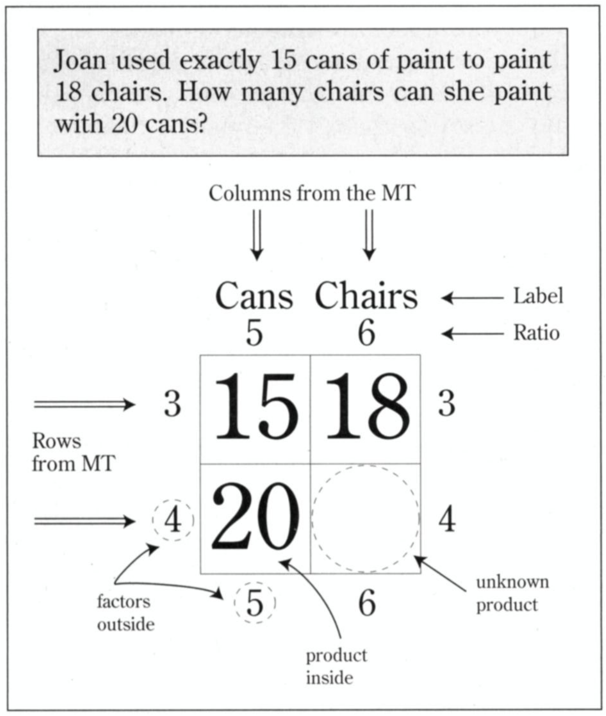
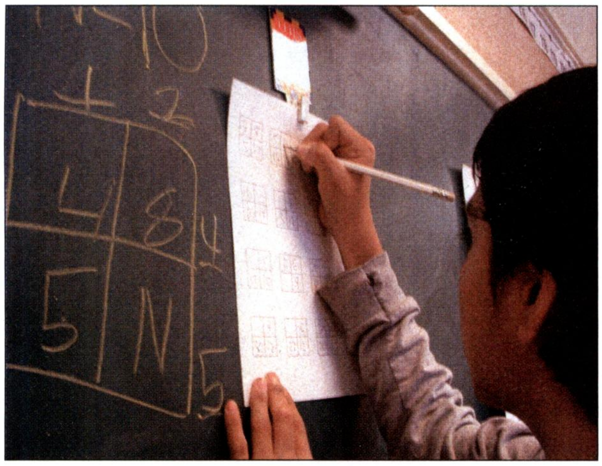


Fig. 6 The proportion-quartet solution strategy for ratio and proportion word problems

table, with factors and products, then finding the unknown value is the same as asking, "What is the row and what is the column of this unknown value?" This analogy is valid because the row and column, for instance, 4 and 6, respectively, are the factors of the unknown product. The numerical solution process now becomes a puzzle that students find accessible and even enjoyable to solve because they can see the puzzle as four cells from the multiplication table, which affords multiple solution methods.

To find which rows and columns of the multiplication table our proportion quartet is using, look at any row or column in the proportion quartet that has two numbers and think of which row or column



A student is solving a proportion quartet.

in the multiplication table it might be in. When we find the row number or column number, such as 3 as the row number of the top row or 5 as the column number of the left column, we write it on both sides of the proportion quartet. As the proportion-quartet puzzle is solved by continuing this process, each small square will have its factors, that is, its rows and columns in the multiplication table, right outside it (see **fig. 6**). Eventually, we find the two factors of the unknown product and multiply them to get the product. The solution is completed by writing a response describing the answer, such as "With 20 cans, Joan can paint 24 chairs."

Conclusion

WE HAVE FOUND THE PROPORTION-QUARTET solution format easily accessible to our fifth graders. They were highly engaged in computing unknown values in proportion-quartet practice sheets out of the context of word problems. Not only is the proportion quartet an effective solution format for ratio and proportion word problems, but it also affords ample practice of multiplication and division in an engaging way.

This approach was originally designed by a university-based research team and was informed by the literature of mathematics cognition and education (see, for example, Vergnaud [1983]; Confrey [1995]). The approach was then further developed through collaboration between a designer-researcher and a fifth-grade teacher of a socioeconomically and ethnically diverse class. The unit was cotaught in the fall of 2001 by both authors over ten 1-hour periods. We evaluated our students' understanding of ratio and proportion by testing them on items taken from previous studies (NAEP 1990; TIMSS 1995; Kaput and West 1994; Stigler, Lee, and Stevenson 1990; Vanhille and Baroody 2002). Our students did well on the posttest, with a mean of 74 percent correct compared with a mean of 53 percent correct achieved by students in the other studies who ranged from one to seven grades ahead of our students.

Students' discussion in class indicated that they could solve ratio and proportion problems meaningfully. Most students could articulate their solution methods and their criteria for interpreting specific word problems as situations of ratio and proportion and for choosing solution strategies accordingly. For example, students would reject a situation as an example of a ratio and proportion problem if they could not assume confidently, on the basis of the given information, that each of two described rates of change was constant, that these rates began simultaneously, and that they were linked by a common time unit. An example of a word problem that students rejected as a situation of ratio and proportion was the following:

> Bob and Joe are brothers who were born exactly two years apart. When Bob was 3 years old, Joe was 5. When Bob was 6, how old was Joe?

Students' struggle to articulate their understanding led to fascinating and unanticipated class discussion of different types of ratio and proportion situations and their different predictive powers. For instance, the fact that Joan needs 5 cans of paint for every 6 chairs is a fixed rule, but if Juanita usually scores 5 of every 6 penalty basketball shots, we cannot be absolutely sure that she will do so in tonight's game.

Another unexpected finding was that over the course of the ten days during which our unit was taught, we witnessed an improvement, in both accuracy and speed, in students' use of multiplication and division facts. For example, we observed that students resorted less frequently to the classroom poster of the multiplication table in their individual work on proportion quartets. This bonus of our approach suggests that this unit could be a good kickoff for grade 5 because it reviews multiplication and division, which are fundamental arithmetic operations for students at this level. Further, the concepts of equivalent fractions (e.g., $3/4 = 6/8$) and percent (e.g., $3/4 = 75\%$) can be readily introduced or revisited by working with the multiplication table and proportion quartets. We also found the proportion quartet to be a promising format for supporting students' discovery of cross multiplication; if students write a proportion quartet with its rows and columns inside the squares instead of outside them, they can see that the left diagonal and the right diagonal contain the same four numbers. Finally, although our unit could stand alone, the broad applications of the proportion quartet to all the domains of rational numbers allow this unit to be readily integrated into a unit on fractions. For example, we witnessed students' spontaneous use of the multiplication table for expanding and reducing fractions.

This unit has been well received by teachers in our district. As one experienced teacher exclaimed at a presentation, "I've been teaching mathematics for thirty-five years, and I thought that the multiplication table was just . . . the multiplication table!" Instead, this well-known tool can be used to support students' meaningful transition to multiplicative solutions for ratio and proportion problems. In later grades, expansion of the multiplication table to decimals and fractions between the whole numbers can support meaningful solutions to noninteger ratio and proportion problems.

References

Confrey, Jere. "Student Voice in Examining 'Splitting' as an Approach to Ratio, Proportions and Fractions." In *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education*, edited by Silvio L. Meira and David Carraher, Vol. 1, pp. 3–29. Recife, Brazil: Universidade Federal de Pernambuco, 1995.

International Study Center, Lynch School of Education, Boston College. *Third International Mathematics and Science Study (TIMSS 1995)*. "Study Instruments and Procedures." timss.bc.edu/timss1995i/t95_study.html (November 21, 2001).

Kaput, James J., and Mary M. West. "Missing-Value Proportional Reasoning Problems: Factors Affecting Informal Reasoning Patterns." In *The Development of Multiplicative Reasoning in the Learning of Mathematics,* edited by Guershon Harel and Jere Confrey, pp. 237–87. Albany, N.Y.: State University of New York, 1994.

National Center for Education Statistics. "NAEP Questions." *National Assessment of Educational Progress, 1990: The Nation's Report Card.* nces.ed.gov/nationsreportcard/itmrls/pickone.asp (November 21, 2001).

Stigler, James W., Shinying Lee, and Harold W. Stevenson. *Mathematical Knowledge.* Reston, Va.: National Council of Teachers of Mathematics, 1990.

Vanhille, Lee S., and Arthur J. Baroody. "Fraction Instruction That Fosters Multiplicative Reasoning." In *Making Sense of Fractions, Ratios, and Proportions,* 2002 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Bonnie H. Litwiller, pp. 224–36. Reston, Va.: NCTM, 2002.

Vergnaud, Gerard. "Multiplicative Structures." In *Acquisition of Mathematical Concepts and Processes*, edited by Richard Lesh and Marsha Landau, pp. 127–74. New York: Academic Press, 1983. □