Developing Mathematical Power in Whole Number Operations

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Although much of the recent research reviewed in this chapter is empirical, much is also conceptual and analytic. Important advances have been made in describing and categorizing real-world situations in each domain, in analyzing attributes of and potential problems with the mathematical notations and words in a domain, and in designing conceptual supports to facilitate learners' understanding in a given domain. Some research involves a mixture of empirical and analytical approaches, such as analyzing advantages and disadvantages of particular algorithms (partly from seeing children use them) or identifying children's errors and the reasoning behind them. The following reviews and summaries of the literature are used extensively in this paper: Baroody and Coslick, 1998; Baroody and Ginsburg, 1986; Bergeron and Herscovics, 1990; Brophy, 1997; Carpenter and Moser, 1984; Cotton, 1995; Davis, 1984; Dixon, Carnine, Kameenui, Simmons, Lee, Wallin, and Chard, 1998; Fuson, 1992a, 1992b; Geary, 1994; Ginsburg, 1984; Greer, 1992; Hiebert, 1986, 1992; Hiebert and Carpenter, 1992; Lampert, 1992; Nesher, 1992; Resnick, 1992; and Siegler, this volume. These include reviews carried out by experts in mathematics education, cognitive science, learning disabilities, special education, educational psychology, and developmental psychology. The reviews in Leinhardt, Putnam, and Hattrup (1992) were written especially for teachers and other educational leaders; they also include analyses of textbook approaches to teaching. To avoid excessive citations, results that are strong, salient, and clear in these reviews and summaries are not cited separately. More specialized results are cited.

Real-World Situations, Problem Solving, and Computation: Continual Intertwining for Sense Making and Computational Fluency

Traditionally in the United States and Canada, computation of whole numbers has been taught first, and then problems using that kind of computation have been presented as applications. This approach has several problems. First, less-advanced students sometimes never reach the application phase, limiting greatly their learning. Second, word problems are usually put at the end of each section or chapter on computation, so sensible students do not read the problems carefully: They just perform the operation they have just practiced on the numbers in the problem. This practice, plus the emphasis on teaching students to focus on key words in problems rather than to build a complete mental model of the problem situation, leads to poor problem solving because students never learn to read and model the problems themselves. Third, seeing problem situations only after learning operations keeps the meanings in the problem situations from becoming linked to those operations. This isolation limits the meaningfulness of the operations and the ability of children to use the operations in a variety of situations.

Research has indicated that beginning with problem situations yields greater problem-solving competence and equal or better computational competence. Children who start with problem situations directly model solutions to these problems. They later move on to more advanced mathematical approaches as they move through levels of solutions and of problem difficulty. Thus, the development of computational fluency and problem solving is intertwined when both develop together with understanding.

For many years, researchers have contrasted conceptual and procedural aspects of learning mathematics. Which should come first has been debated for a long time. Recent research portrays a much more complex relationship between these conceptual and procedural aspects than simply that one precedes the other. Instead, they are continually intertwined and potentially facilitate each other. As a child comes to understand more, the method the child is using becomes more integrated internally and in relation to other methods. As a method becomes more automatic, reflection about some aspect may become more possible, leading to new understanding. These conceptual and procedural interconnections are forged in individuals in individual ways. It may not even be useful to distinguish between these two aspects of learning because doing and understanding are always intertwined in complex ways. Furthermore, different researchers may refer to the same method as a procedure or as a concept, depending upon whether the focus is on carrying out the method or on its conceptual underpinnings. And, in a given classroom at a given time, some students may be carrying out what looks like the same method, but they may well have different degrees of understanding of that method at that time. This is what the helping aspects of classroom teaching are all about—helping all students to relate their methods to their knowledge in ways that give them fluency and flexibility.

The discussion in the rest of this paper does not continue to reiterate that the most effective teaching and learning help students to intertwine doing and understanding mathematics. Instead, this result from the research literature forms the backdrop for what follows.

The types of real-world situations that have been identified for addition and subtraction and for multiplication and division are briefly presented next. Such situations in the form of word problems and real situations brought into the classroom by students can provide contexts within which the operations of addition, subtraction, multiplication, and division can come to take on their whole range of required mathematical meanings. The rest of the paper focuses on developmental progressions in the methods students can come to use for these operations on single-digit and multidigit whole numbers. The vast amount of research on these topics in the past 30 years indicates that substantial changes are needed in classroom teaching and learning of whole number operations.

Types of Real-World Addition, Subtraction, Multiplication, and Division Situations

Researchers from around the world have reached considerable consensus in identifying the types of real-world situations that involve addition or subtraction, although minor variations in terminology exist. One classification of such situations is given in Table 1. Note that for each type of problem, each of the three quantities involved can be the unknown. There is a substantial literature on aspects of problems that increase their difficulty. In general, problem statements, syntax, or sentence orders that do not follow the action in a situation are more difficult than those that do. The language of problems that involve comparing quantities is difficult for children at first, partly because the structure in English lumps together two kinds of information: who has more and how much more. Even kindergarten children can solve many of these problems if they use objects to directly model the situation. Textbooks, however, typically include only the simplest variation of each problem type. In contrast, in the texts of the Soviet Union, problems were given equally across the various types and unknowns, and 40% of the problems in the first-grade books and 60% of the problems in the second-grade books were two-step problems (Stigler, Fuson, Ham, & Kim, 1986).

Insert Table 1 about here; see Table 1 at the end of this paper

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An important distinction needs to be made between a situation representation (an equation or a drawing) and a solution representation. The most powerful problem-solving approach is to understand the situation deeply—draw it or otherwise represent it to oneself. This is the natural method used by young students. But textbooks and teachers influenced by textbooks push students to write solution representations that are not consistent with their view of the situation. Students will write 8 + A = 14 for a problem like "Erica had $8. She babysat last night and now has $14. How much did she earn babysitting?" Textbooks often push students to write 14 – 8, but that is not how most students represent or solve the problem. Allowing students to represent the situation in their own way communicates that the goal of problem solving is to understand the problem deeply. Once they see the goal, students can experience success and move on to more difficult problems.

There is less consensus about whole number multiplication and division situations (see the reviews by Nesher, 1992, and Greer, 1992). These situations are usually addressed in the literature on rational numbers, which focuses on fractions as well as on whole numbers. Whole number situations identified by almost all researchers include grouping situations, multiplicative-comparing situations, and cross-product or combination situations. Grouping situations involve some number of equal groups such as 3 packages each containing 5 apples. For the two corresponding division situations, you may either (a) know the product 15 and how many packages (3) and need to find out how many are in a package, or (b) know the product 15 apples and the size of the packages (5) and need to find out how many packages. The multiplicative comparing situations use the language "*x* times as many as" or the reverse fractional language "1/*x*th as many" as in "Maria had 15 books. She had 5 times as many as Saul. How many books did Saul have?" As with the additive comparing situations, the comparing sentence can be said in two ways (here it can also be said as "Saul had one-fifth as many books as Maria."). The cross-product or combination situations are those in which everything in one group is combined with everything in a second group. Arrays are a good way to show these situations. Familiar examples are problems involving clothes (There are 3 shirts and 2 pants. How many different outfits?) or sundaes (There are 4 kinds of ice cream and 2 kinds of topping. How many different kinds of sundaes?). Area is one kind of cross-product situation.

Experience with these various addition, subtraction, multiplication, and division situations, and with the language involved in them, allows students to build a mathematically adequate understanding of the operations and notations by using them in these situations. The symbol – means more than take away, and *x* means more than repeated addition. Solving and posing problems from a wide range of real-world whole number situations enables students to understand alternative meanings.

Building Fluency With Computational Methods: General Issues

Fluency with computational methods is the heart of what many people in the United States and Canada consider to be the elementary mathematics curriculum. Learning and practicing computational methods is central to many memories of learning in the twentieth century. Twentieth-century mathematics teaching and learning, however, were driven by goals and by theories of learning that are not sufficient for the twenty-first century. We enter the new century with inexpensive machine calculators widely available, computers increasingly appearing in schools and libraries, the World-Wide Web giving access to a huge variety of information, and supercomputers continuing to create demands for new kinds of machine algorithms (general multistep methods). The information age creates for all citizens the need for lifelong learning and for flexible approaches to solving problems. Everyone needs the ability to use calculating machines with understanding.

Clearly the twenty-first century requires a greater focus on a wider range of problem-solving experiences and a reduced focus on learning and practicing by rote a large body of standard calculation methods. How to use the scarce hours of mathematics learning time in schools is a central issue. This decision requires in part a value judgment as to which needs are most important. But new research can also inform our choices. This paper summarizes some results of research that can help in constructing a twenty-first century image of what building fluency with computational methods would entail. Educators and the public are still attempting to reach consensus on what kinds and how much computational fluency are necessary today. Computational fluency is one vital component of developing mathematical power. Other components include understanding the uses of computation and understanding computational methods. This paper reviews how all of these aspects of mathematical power can be supported for students and for teachers. Given that mathematics learning time is a scarce resource, it is useful to know roughly how long it takes various children to reach various levels of computational fluency. Only with such knowledge can we make sensible decisions about how scarce learning time should be allocated for reaching, among all of the worthwhile goals for mathematics learning, computational fluency.

The goal of computational fluency for all has been an elusive goal at least since the 1950s. It is not the case that the United States and Canada have had a successful computational curriculum that is now at risk of being thrown over by "math reform." Research studies, national reports, and international comparisons have for decades identified many aspects of computation in which children's performance was disappointing. These results have sometimes been overshadowed by even worse results for problem solving or applications of calculations, making calculation seem less of an issue than it has consistently been. Many of the calls for school mathematics reform have been at least partially focused on teaching for understanding as a way to eliminate computational errors and thus increase computational performance. For example, on standardized tests U.S. Grade 2 norms for two-digit subtraction requiring borrowing (e.g., 62 – 48) are 38% correct. Many children subtract the smaller from the larger number in each column to get 26 as the answer to 62 – 48. This top-from-bottom error is largely eliminated when children learn to subtract with understanding (e.g., Fuson & Briars, 1990; Fuson, Wearne, et al., 1997; Hiebert et al., 1997). Building on a foundation of understanding can help all students achieve computational fluency.

Several themes characterize much of the research on computational methods over the past 30 years. These themes apply across computational domains (e.g., single-digit addition and subtraction, multidigit multiplication and division). First, I briefly outline these themes. Then, because one of the most central themes is that each computational domain has a great deal of specific domain knowledge, I turn briefly to each domain in turn to indicate these central aspects of domain knowledge. I conclude with a few more general points from the research.

Within each computational domain, individual learners move through progressions of methods from initial, transparent, problem-modeling, concretely represented methods to less transparent, more problem-independent, mathematically sophisticated, symbolic methods. At a given moment, each learner knows and uses a range of methods that may vary according to the numbers in the problem, the problem situation, or other individual and classroom variables. A learner may use different methods even on very similar problems, and any new method competes for a long time with older methods and may not be used consistently. Typical errors can be identified for each domain and for many methods (e.g., the reviews referenced above and Ashlock, 1998), and ways to help students overcome these errors have been designed and studied. A detailed understanding of methods in each domain enables us to identify prerequisite competencies that can be developed in learners to make those methods accessible to all.

The constant cycles of mathematical doing and knowing in a given domain lead to learners' construction of representational tools that are used mentally for finding solutions in that domain. For example, the counting-word list initially is just a list of words used to find how many objects there are in a given group. Children use that list many times for counting, adding, and subtracting. Gradually, the list itself becomes a mathematical object. The words themselves become objects that are counted, added, and subtracted; other objects are not necessary. For students who have opportunities to learn with understanding, the written place-value notation can become a representational tool for multidigit calculations as the digits in various positions are decomposed or composed, and proportional statements can become a representational tool for solving a range of problems involving ratio and proportion.

Learners invent varying methods regardless of whether their classrooms have been focused on teaching for understanding or on rote memorizing of a particular method. In classrooms where there is teaching for understanding, however, a wider range of effective methods is developed. In classrooms in which rote methods are used, learner's inventiveness is often focused on generating many different kinds of errors, most of which are partially correct methods created by a particular misunderstanding. Thus, even in traditional classrooms focused on memorizing standard computational methods, learners are not passive absorbers of knowledge. They build and use their own meaning and doing, and they generalize and reorganize this meaning and doing.

Multidigit addition, subtraction, multiplication, and division solution methods are called algorithms. An algorithm is a general multistep procedure that will produce an answer for a given class of problems. Computers use many different algorithms to solve different kinds of problems. Inventing new algorithms for new kinds of problems is an increasingly important area of applied mathematics. Throughout history and at present around the world, many different algorithms have been invented and taught for multidigit addition, subtraction, multiplication, and division. Different algorithms have been taught at different times in U.S. and Canadian schools. Each algorithm has advantages and disadvantages. Therefore, the decisions to be made about computational fluency concern in part the algorithms that might be supported in classrooms and the bases for selecting those algorithms. This issue is pursued in the discussion of particular mathematical domains as I identify particularly powerful algorithms.

One goal of the following sections is to underscore that it is possible to understand various computational methods. Because such understanding has not been a goal of school mathematics for most of the past century, most adults in charge of making educational decisions have not had an opportunity to understand the standard algorithms or to appreciate the wide variety of algorithms that are possible. Most teachers also have not had that opportunity, and most textbooks do not provide sufficient help with such understanding. This paper aims at demonstrating that such understanding can be facilitated in children for accessible algorithms if so-called quantity supports are used for understanding the meanings of the numbers, notations, and steps in the accessible algorithm. This understanding is not in conflict with developing computational fluency but rather is a foundation for it. Children need supported practice with whatever methods they are using if they are to become more fluent in orchestrating the several steps in any algorithm. Understanding can serve as a continual directive toward correct steps and as a constraint on the many creative calculating errors invented by students taught algorithms by rote. Because all algorithms are not equally accessible to understanding (e.g., many sacrifice comprehensibility so as to save space in writing), I describe at least one algorithm that has been demonstrated to be accessible to a wide range of students. My criteria for such accessible algorithms are that they scaffold the understanding of key steps in the domain, generalize readily to large numbers, have variations that provide for individual differences in thinking, and are procedurally simple to carry out (they require the minimum of computational subskills so that valuable learning time is not required to bring unnecessary subskills to the needed level of accuracy).

Single-Digit Computation: Much More Than "Learning the Facts"

Learning single-digit addition, subtraction, multiplication, and division has for much of a century been characterized in the United States and Canada as "learning math facts." The predominant learning theory was of these facts as rote paired-associate learning in which each pair of numbers was a stimulus (e.g., 7 + 6) and the answer (13) needed to be memorized as the response to that stimulus. "Memorizing the math facts" has been a central focus of the mathematics curriculum, and many pages of textbooks presented these stimuli, to which children were to respond with their memorized response.

This view of how children learn basic single-digit computation was invalidated by one line of research earlier in the century (by Brownell, 1956/1987) and by much research from all over the world during the last thirty years. We now have very robust knowledge of how children in many countries actually learn single-digit addition and subtraction. Below I explore the research for addition in some depth both because it is the largest body of research and to set the scene for understanding calculation in other areas. The research in other areas is then summarized more briefly.

Single-Digit Addition and Subtraction

Single-digit addition. Some research throughout the century (e.g., Brownell, 1956/1987) presented a complex view of children as using a variety of methods. Substantial research now indicates that children do move through an experiential progression of single-digit addition methods (e.g., see reviews in Fuson, 1992a, 1992b; Siegler, this volume). These methods are not ordinarily taught in the United States, Canada, and many other countries. The methods are also invented by adults whose cultures are stimulating new demands for calculating (Saxe, 1982). Analyses of these methods reveal that learners build later methods from earlier methods by chunking, recognizing and eliminating redundancies, using parts instead of entire methods, and using their knowledge of specific numbers. Thus, the methods seem to be accessible to almost all learners by natural and general learning processes. When these more-advanced methods are not supported in the classroom, however, several years separate the earliest and latest users of advanced methods. In contrast, helping children progress through methods can lead all first graders to methods that are efficient enough to use for all later multidigit calculation.

Children's tools for an initial understanding of addition are the counting word list ("one, two, three, four," etc.), the ability to count objects, some indicating act (e.g., pointing, moving objects) that ties words said to objects counted (one at a time), and the so-called count-cardinal knowledge that the last count word said tells how many objects there are in all. Many but not all children in the United States and Canada learn these tools in the preschool years. With these tools, addition can be done orally using concrete situations that are comprehensible to young learners. They count out objects for the first addend, count out objects for the second addend, and then count all of the objects (*count all*). This general counting-all method then becomes abbreviated, internalized, chunked, and abstracted as children become more experienced with it.

The major steps in this worldwide learning progression are shown in Figure 1 (various minor abbreviations and mini-steps are neither shown in Figure 1 nor discussed here). Children notice that they do not have to count the objects for the first addend but that they can start with the number in one addend and count on the objects in the other addend (*count on*). Children count on with objects. They then begin to use the counting words themselves as countable objects and keep track of how many words are counted on by using fingers or auditory patterns (the counting list has become a representational tool). With time, children chunk smaller numbers into larger numbers and use thinking strategies in which they turn an addition they do not know into an addition they do know. In the United States and Canada, this conversion is usually made by using a *double* addition (2 + 2, 3 + 3, etc.). These doubles are learned very quickly. As noted in Figure 1, throughout this learning progression, particular addition combinations move into the category of being rapidly recalled rather than solved in some way. Individuals vary in which sums become recalled readily, though doubles, sums that involve adding 1, and combinations of small numbers are the most readily recalled sums for most children.

Insert Figure 1 about here; see Figure 1 at the end of this paper

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In many other parts of the world, children are taught a general thinking strategy: Make a ten by giving some from one addend to the other addend. Children in the United States or Canada who speak English seldom invent this "make a ten" method. Nor are they taught it in textbooks. But the method is taught in first grade in China, Japan, Korea, and Taiwan (Fuson & Kwon, 1992; Fuson, Stigler, & Bartsch, 1988). This method is facilitated by the number words in these countries: "ten, ten one, ten two, ten three," etc. Many children in these countries also learn numbers and addition using a *ten-frame*: an arrangement of small circles into 2 rows of 5. This pattern emphasizes 6, 7, 8, and 9 as 5 + 1, 5 + 2, 5 + 3, and 5 + 4. Work with this visual pattern enables many children to "see" these small sums under ten as using a five-pattern. For example, some Japanese adults report adding 6 + 3 by thinking or visualizing [(5 + 1) plus 3] = 5 + 4 = 9. This reduction of 6 + 3 to 5 + 1 + 3 = 5 + 4 = 9 is done very rapidly and without effort, as automatically as recall the sum. The ten-frame is also used to teach the make-a-ten method. For example, 8 has 2 missing in the ten-frame, so 8 + 6 requires 2 from the 6 to fill the ten-frame, leaving 4 to make 10 + 4 = 14. By the end of first grade, most children in these Asian countries rapidly use these five-patterns or ten-patterns to add single-digit numbers mentally.

The make-a-ten method is also taught in some European countries. There are three prerequisites that children must learn in order to use the make-a-ten method effectively. They must know what number makes ten with each number up to ten (e.g., 10 = 8 + 2 or 6 + 4 or 7 + 3), be able to break apart a number into any of its two addends (in particular, for numbers over ten, to make ten plus the remainder), and know 10 + *n* (e.g., 10 + 5 = 15). In countries that teach the make-a-ten method, these prerequisites are developed before the method is introduced. Many first and even second graders in the United States do not have these prerequisites consolidated, and they are rarely developed sufficiently in textbooks. Counting on does not help United States or Canadian children to move on to a make-a-ten method for two reasons. English counting words do not signal a change at ten as do the East Asian words "ten, ten one, ten two." And the fingers do not use or show a ten because most children use their fingers to keep track of the words counted on ("8, 9, 10, 11, 12, 13, 14"). This use is in contrast to that of Korean or some Latino children, who make the first number on their fingers and then count on ("1, 2, 3, 4, 5, 6 is 14") as they use 6 more fingers: 2 more to fill 10 fingers and 4 more over ten. This latter way of counting on does support a make-a-ten method.

Textbooks in the United States typically have shown little understanding of children's progression of methods. They moved directly from counting all (e.g., 4 + 3 shows 4 objects and then 3 objects) to using numerals only. Children are then expected to begin to "memorize their facts" (which they cannot do because no answers are given). Children respond by following the experiential trajectory of methods discussed above and summarized in Figure 1. They use their fingers or make little marks to count all, eventually invent counting on, and may go on to invent thinking strategies, especially using doubles. As indicated in Figure 1, particular sums all along the way become recalled sums (the answer is produced rapidly and automatically and without knowledge of a solution method that can be reported). Children are not supported by most textbooks to learn these strategies or to move through this progression of methods. Nor are they typically given visual supports such as a ten-frame for adding by using patterns of numbers. Or if these alternatives are provided, they are often not orchestrated with sufficient practice of subskills or methods to reach mastery.

This lack of fit between what is in textbooks and how children think is exacerbated by other features of the textbook treatment of addition. Compared with other countries, the United States has had a very delayed placement of topics in the elementary school curriculum (Fuson, 1992a; Fuson, Stigler, & Bartsch, 1988). Almost all of first grade has been spent on addition and subtraction below ten. Such simple problems have then also been emphasized and reviewed in Grade 2, resulting overall in doing many more of the easier sums and relatively fewer of the more difficult sums (Hamann & Ashcraft, 1986). Thus, in contrast to East Asian children who are shown in first grade effective methods for solving the difficult sums over ten in visually and conceptually supported ways, many U.S. children have had little opportunity to solve such problems in first grade and have not been supported with any effective methods to do so.

Intervention studies indicate that teaching counting on in a conceptual way makes all single-digit additions accessible to U.S. first graders, including learning-disabled students and English-as-a-second-language students (Fuson & Fuson, 1992; Fuson & Secada, 1986). These results have been replicated in a range of urban and suburban classrooms in English and in Spanish (see reviews in Fuson, 1992a, 1992b, as well as Fuson, Perry, & Kwon, 1994; Fuson, Perry, & Ron, 1996). Children in the United States show numbers on their fingers in various ways, some starting with the index finger, some with the thumb, and some with the little finger. Any of these are effective ways to keep track of the second addend while counting on. With practice, counting on can be done rapidly and accurately enough to be used in multidigit calculations of all kinds. Conversations with many adults reveal the use of counting on by adults in everyday uses where accuracy counts. Counting on is a powerful, general, and rapid-enough method for most purposes.

There is to date very little research with U.S. or Canadian children on using spatial patterns for learning addition with small numbers. This approach might be very powerful, especially for some children who have difficulty with sequential information or who have strong spatial competence. There are a few studies on using ten-frames or other visual supports for the make-a-ten method (Thornton, Jones, & Toohey, 1983), but none on developing the three prerequisites for that method. Some Latino children use finger methods that support the make-a-ten method (Fuson, Perry, & Kwon, 1994; Fuson, Perry, & Ron, 1994) and go on to invent and use it. How accessible the method is to all children is not clear, given the irregularities in the English number words. That it is taught in France and some other European countries with similar irregularities in the words used for the teens suggests that it is worth trying in North America to see how rapid and automatic it could become. It is very useful in multidigit addition because it gives the answer already prepared for regrouping (carrying, trading) as, for example, 1 ten and 4 ones.

The prerequisites for the make-a-ten method are important learning goals themselves, so research would be helpful for understanding how to help students learn them rapidly. More conceptual work focused on the meaning of teens words and written numbers as tens and ones would be especially valuable. Several studies using different methods indicate that for many U.S. children, what you see in a teen number is what you get: They see 15 as a one and a five, not as a ten and a five (see the reviews in Fuson, 1992a, 1992b). Many children speaking European languages have similar difficulties. It is clear, however, that U.S. children can learn the meanings of teens as tens if they are supported to do so (Fuson, Smith, & Lo Cicero, 1997).

Single-digit subtraction. Subtraction follows a progression that is similar to that for addition in its major categories (see Figure 1). Most children in the United States, however, invent counting-down methods that model the taking away of numbers from the total. Counting down is difficult (e.g., try generating an alphabet list backwards from K to C), and it takes a long time for some children to learn. Counting down is also subject to more errors because the cardinal correspondence of the objects in the addends is not clear. Children use two distinct counting-down methods. One starts with the total, and the other starts with the word before the total. Many children create combinations of the two that give an answer that is either one too many or one too few. Many other children make errors when counting backwards or are very slow at it.

In contrast, in Latin America and many countries in Europe, children learn to subtract by counting up from the known addend to the known total. For example, 14 – 8 is solved by counting up from 8 to 14: 8 things I have, so 9, 10, 11, 12, 13, 14, that's 6 more I counted to get to 14. Counting up is even easier than counting on because you only have to listen for the total; you do not have to know and monitor a finger pattern for an addend as you count. So counting up makes subtraction easier than addition.

Intervention studies with U.S. first graders that help them see subtraction situations as taking away the first x objects enables them easily to learn a counting-up-to process for subtraction. All children, including learning-disabled and special education students, were able to learn all single-digit subtraction combinations in first grade. This result was revolutionary for first-grade teachers, who typically see children having much more difficulty with subtraction than with addition. U.S. children invent counting on for situations in which an unknown quantity is added to a known quantity. Many go on later to use counting on in taking-away subtraction situations because counting on is easier (Carpenter & Moser, 1984). But that invention is delayed for many children until second grade, and many never subtract by counting up. Discussing counting up to for taking-away situations in first grade makes it accessible to all children then.

Less research is available about thinking strategies in subtraction than in addition. In East Asia, two methods using ten are taught, and different children use each of these. One is just fast counting on with chunking at ten: 15 – 9 is 9 + 1 (up to ten) + 5 (up to 15), so 9 + 6 = 15. The other goes down over ten: 15 – 9 is 5 down to ten, and 1 more down to 9 is 6.

Summary. The unitary progression of methods used worldwide by children stems from the sequential nature of the list of number words. This list is first used as a counting tool, and then it becomes a representational tool in which the number words themselves are the objects that are counted (Bergeron & Herscovics, 1990; Fuson, 1986b; Steffe, Cobb, & von Glasersfeld, 1998). Counting becomes abbreviated and rapid. Some (or in some settings, many) children then chunk numbers using thinking strategies. These chunking actions turn sums children do not know into sums they do know (often using doubles). During this progression, which may last into third or even into fourth or fifth grade for some children (because they are not helped through the progression), individual children use a range of different methods on different problems. Learning-disabled children and others having difficulty with mathematics do not use methods that differ from this progression. They are just slower than others in the progression (Geary, 1994; Ginsburg & Allardice, 1984; Goldman, Pellegrino, et al., 1988; Kerkman & Siegler, 1993). Counting on can be made accessible to first graders; it makes possible rapid and accurate addition of all single-digit numbers. Single-digit subtraction is usually more difficult than addition is for U.S. children. Learning to think of subtraction as counting up to the known total, as is done in many other countries, makes subtraction as easy as, or easier than, addition. But at present, the counting-up-to method rarely appears in textbooks. Research is needed on using spatial patterns to make sums and differences below ten accessible visually, on supporting thinking strategies and their prerequisites for all children, and on effective organizations of practice and support that will enable all children to progress to rapid and accurate methods of single-digit sums and differences by the end of first grade (at least counting on and counting up to).

Single-Digit Multiplication and Division

There is much less research on single-digit multiplication and division than on addition and subtraction. U.S. students do go through an experiential progression of multiplication methods that is somewhat similar to that for addition (e.g., Anghileri, 1989; Baek, 1998; Mulligan & Mitchelmore, 1997; Steffe, 1994). Eventually they make equal groups and count them all. They learn count-on lists for different products (e.g., counting by 4s gives 4, 8, 12, 16, 20, etc.). They then count up and down these lists, using their fingers to keep track, to find different products. They may use a combination method in which they enter a list at a point they know and then count on by ones to get to the product (e.g., to find 5 × 6, 5 × 5 is 25, and 26, 27, 28, 29, 30 is 5 more). They invent thinking strategies in which they derive related products from products they know.

As with addition and subtraction, many of these methods are developed as individual inventions by children who are not supported by textbooks or instruction. Some very recent textbooks are supporting children's pattern finding and use of count-by lists in multiplication. But most older books used the nineteenth-century model of learning multiplication by memorizing isolated facts using rote associations.

To see the problems with this limited view of rote "memorizing of multiplication facts," consider Figure 2. The figure shows a table of multiplication products using an alphabet analogy: Suppose you were to learn the multiplication combinations for a new counting list (C, D, E, F, G, H, I, J, K). We pose this task so that you will not already know the answers. Look at the table and see all of the interference involved in learning each of these facts separately as a rote response to 2 stimuli. It is a formidable task because the "numbers" all look so similar (they *are* similar, just as 1, 2, 3, 4, etc., are for young children). Look at the table for a while, and think how you might go about this formidable task in another way. See all of the patterns you can find. Do you see a pattern for multiplying by C (look at the top row or left column)? (I give all of these patterns in the next paragraph to give you an opportunity to see some of them for yourself.) Do you see a pattern for multiplying by CL (look at the last column or the bottom row)? Look at the interesting pattern for G. Can you find a pattern for D? The patterns for E and F are more subtle. K has a wonderful pattern (can you explain it?).

Insert Figure 2 about here; see Figure 2 at the end of this paper

Finding and using patterns greatly simplifies the task of learning multiplication combinations. Moreover, such finding and describing of patterns is one of the very essences of mathematics. Thus, approaching multiplication learning as pattern finding both simplifies greatly the task and is a core mathematical approach. (The patterns in the table are as follows: C just copies the number being multiplied to give the same number as the product. Multiplying by CL just adds an L, as D becomes DL, E becomes EL, etc. G products alternate between G and L in the ones place, and the tens place has two Cs, two Ds, two Es, etc. The K pattern has the tens place increasing by one and the ones place decreasing by one; the total of the tens and ones numbers is K. This pattern occurs because as you go over each successive ten in the number list, one more of each group of K has to go to make a whole ten: in English number words and numbers, 2 nines are 2 tens – 2 = 20 – 2 = 18, 3 nines are 3 tens - 3 = 30 – 3 = 27, etc.)

After patterns are identified, children still need much experience to produce count-by lists and individual products rapidly. There is little research about how to accomplish this fluency. There is also little research about how to link the patterns in the numbers to the underlying groupings spatially or conceptually, that is, how to relate all of the different groupings to the 10-groupings in the base-ten multidigit numbers. Nor is there much research about how children learn products in other countries. Informal inquiries of researchers outside the United States suggest some methods that might be pursued. In France, where children learn multiplication effectively (Lemaire & Siegler, 1995), an elaborate yearlong social organization may be used that involves an extra mathematics period and extra help from the teacher for those who need it. In Japan, oral rhythmic chanting of the tables is used; this approach is facilitated by the structure of the Japanese number words. In China, commutativity ($4 \times 7 = 7 \times 4$) is focused upon heavily, reducing the number of products by half (notice the line of symmetry in Figure 2 moving from top left to bottom right). The effectiveness of this approach is reflected in the reaction-time patterns of Chinese adults, which differ from those of U.S. adults (LeFevre & Liu, 1997).

Division combinations can be approached in terms of the related products. For example, 72/9 = ? can be thought of as 9 × ? = 72. Again, research has as yet little to offer. It is not clear whether the division-multiplication relationship can be introduced very early, with quotients learned and practiced at the same time as products, or whether products need to be learned first. How to help children learn and use easily all of the different symbols for division (15/3, 15 ÷ 3, $\frac{15}{3}$, and the reversed $3\overline{)15}$ is also not clear.

The general methods of counting on for addition and counting up for subtraction are readily learned. There are no similar rapid general methods for single-digit multiplication and division. Rather, there is much specific pattern-based knowledge that needs to be orchestrated into accessible and rapid-enough multiplication and division. Research is needed into ways to support such pattern finding and then to organize the necessary follow-up specific learning conceptually, motivationally, and socially in classrooms if this gatekeeper knowledge is to be learned by U.S. children by the end of Grade 3, as it is in other countries in which students' mathematical performance is high.

Traditional learning of addition and multiplication facts creates interference between these two operations (LeFevre, Kulak, & Bisantz, 1991; Lemaire, Barrett, Fayol, & Abdi, 1994; Miller & Paredes, 1990). Thus, when children begin learning multiplication combinations, their addition performance decreases. This phenomenon is a strong reason to encourage learning of general addition and subtraction methods; these methods do not interfere with multiplication or division. Interference between addition and multiplication for combinations involving 0 and 1 is particularly great. Children readily learn the patterns involved in 7 + 0, 7 + 1, 7 × 0, and 7 × 1, but they tend to confuse them. These patterns are complex across operations because although 7 + 0 = 7 × 1, these look maximally dissimilar.

Timed tests have been a controversial part of single-digit computational practice. There is no definitive evidence about their use. Certainly they can be counterproductive and feed math anxiety if they are used in any of the following ways: If they are used before students have conceptual knowledge in a domain that enables them to generate solutions; if they are used in a competitive fashion, so that some students are losers, rather than focusing on monitoring and improving their own individual progress; or if they are used in a nonsupportive environment so that students feel isolated or hopeless. In some situations, students enjoy timed tests as a challenge, and individuals can watch their own progress.

Multidigit Addition and Subtraction

There is considerable research on the ways in which children learn various multidigit addition and subtraction methods, though not nearly as much research as on single-digit addition and subtraction. In single-digit addition and subtraction, the same learning progression occurs in many different countries in spite of not being taught. Multidigit addition and subtraction depend much more on what is taught, and different children even within the same class may follow different learning progressions and use different methods. Multidigit addition and subtraction knowledge seems to consist much more of different pieces that are put together in different orders and in different ways by different children (e.g., Hiebert & Wearne, 1986).

Difficulties with Words and Numbers

As with the teens words, the English number words between 20 and 100 complicate the teaching and learning task for multidigit addition and subtraction. English names the hundreds and thousands regularly, but it does not do so for the tens. For example, 3333 is said “3 thousand 3 hundred thirty 3” not “3 thousand 3 hundred 3 ten 3.” English-speaking children must learn and then use a special sequence of decade words for 20, 30, 40, and so forth. This sequence, like the teens, has irregularities. Furthermore, teens words and decade words sound alike: In a classroom, it is often difficult to hear the difference between “eighteen” and “eighty.” The same numbers 1 through 9 are reused to write how many tens, hundreds, thousands, and so forth. Whether it is 3 tens or 3 hundreds or 3 thousands is shown by the relative position of the 3: how many places to the left of the number farthest to the right is the 3? Relative position is a complex concept. French is even more complex, with its use of 20 as a base in some number words.

The written place-value system is a very efficient system that lets people write very large numbers. But it is very abstract and can be misleading: The digits in every place look the same. To understand the meaning of the digits in the various places, children need experience with some kind of *size quantity supports* (e.g., manipulatives, or objects that they can handle, such as buttons or beads) that show tens to be collections of ten ones and show hundreds to be simultaneously ten tens and one hundred ones, and so on. Various kinds of such supports have been designed and used in teaching the written system of place value. Some support understanding the sizes involved in place values, and some can support understanding patterns in the numbers (e.g., a hundreds grid). Classrooms, however, rarely have enough such supports for children themselves to use them—especially size quantity supports—and many classrooms do not use anything. Such supports are rarely used in multidigit addition and subtraction; when used, they may be used alone at first to get answers without sufficient linking to a written method that is related to the manipulative method.

As a result, many studies indicate that U.S. children do not have or use a quantity understanding of multidigit numbers (see reviews in Fuson, 1990, 1992a, 1992b). Instead, children view numbers as single digits side by side: 827 is functionally "eight two seven" and not 8 groups of 1 hundred, 2 groups of ten, and 7 single ones. Children make many different errors in adding and subtracting multidigit numbers, and many who add or subtract correctly cannot explain how they got their answers.

Teaching for Understanding and Fluency

In contrast, research on instructional programs in the United States, Europe, and South Africa indicate that focusing on understanding multidigit addition and subtraction methods results in much higher levels of correct multidigit methods and produces children who can explain how they got their answers using quantity language (Beishuizen, 1993; Beishuizen, Gravemeijer, & van Lieshout, 1997; Carpenter, Franke, Jacobs, & Fennema, 1998; McClain, Cobb, & Bowers, 1998; Fuson & Briars, 1990; Fuson & Burghardt, 1997, in press; Fuson, Smith, & LoCicero, 1997; Fuson, Wearne, et al., 1997). Characteristics of all of these approaches are that students used some kind of visual quantity support to learn meanings of hundreds, tens, and ones, and these meanings were related to the oral and written numerical methods developed in the classrooms. Many different addition and subtraction methods were developed in these studies, often in the same classrooms (see Fuson, Wearne, et al., 1997, and Fuson & Burghardt, in press, for summaries of many methods). In most of these studies, children invented various methods and described them to each other, but in some studies conceptual supports were used to give meaning to a chosen algorithm. Many studies were intensive studies of children's thinking in one or a few classrooms, but some involved 10 or more classrooms, including one study of all second-grade classrooms in a large urban school district (Fuson & Briars, 1990). In all studies, a strong emphasis was placed on children understanding and explaining their method using quantity terms (e.g., using hundreds, tens, ones or the names of the object supports being used).

Roughly three classes of effective methods can be used for multidigit addition and subtraction, although some methods are mixtures. *Counting list methods* are extensions of the single-digit counting methods. Children initially may count large numbers by ones, but these unitary methods are highly inaccurate and are not effective. All children need to be helped as rapidly as possible to develop prerequisites for methods using tens. In counting-list methods using tens, children count on or count up by tens and by ones. These methods generalize readily to counting on or up by hundreds, but become unwieldy for larger numbers. In *decomposing methods,* children decompose numbers so that they can add or subtract the units that are alike (add tens to tens, ones to ones, hundreds to hundreds, etc.). These methods generalize easily to very large numbers. *Recomposing methods* are like the make-a-ten or doubles methods. The solver changes both numbers by giving some of one number to another number (in adding) or by changing both numbers equivalently to maintain the same difference (in subtracting). These methods are highly useful in special cases such as 398 + 276: the 276 gives 2 to make the 398 into 400, so 400 + 274 is 674. But they do not generalize easily to all numbers, and the addition and subtraction methods may interfere with each other if students (or teachers) do not understand them well enough.

Different kinds of conceptual supports (e.g., manipulatives) have been used successfully in classroom research. Each has its advantages and disadvantages, and each supports some methods more clearly than others (see Fuson & Smith, 1997, for an analysis). Number lines and 100s grids of numbers (10 rows showing 1 to 10, 11 to 20, etc. to 100) support counting-list methods better. These supports do not generalize easily to numbers above 100. Children may use the 100s grid in particular by rote to get answers without really seeing tens on it. Decomposition methods are facilitated by supports that enable the different quantity units to be added and subtracted physically. For example, base-ten blocks show ones, tens (ten attached centimeter cubes), hundreds (a 10 cm by 10 cm flat block), and thousands (a 10 cm by 10 cm by 10 cm large cube). These blocks have been used successfully for children inventing their own methods and for understanding chosen methods.

Because of the expense and management problems posed by objects that are conceptual supports, some studies have also introduced some system of drawing ones, tens, and hundreds (e.g., circles or small dashes for ones, vertical sticks for tens, and squares for hundreds) or of recording on an open number line. Such drawings leave records for a teacher to see after class, and children can draw figures on the board to explain their method. The drawings are also easy to link to the written numbers so that the numbers begin to take on quantity meanings for children.

The function of size quantity supports is to suggest meanings that can be attached to the written numerals and to the steps in the solution method with numbers. Therefore, methods of relating the size quantity supports and the written method through linked actions and through verbal descriptions of the numerical method are crucial. However, in the classroom, supports are often used without recording anything except the answer at the end, and then students are led to use written methods without linking them to the steps taken with the supports. Thus, the written numerals do not necessarily take on the meaning of tens, hundreds, and so forth, and the steps in the numeral method may be thought of as involving only single digits rather than their actual quantity meanings. This development leaves students vulnerable to the many errors they create without the meanings to direct or constrain them. Even for students who initially learn a meaningful method, the appearance of a multidigit number as a collection of single digits may cause errors to creep in. An important step in maintaining the meaning of the steps is to have students occasionally explain their method, using the names for their quantity support (e.g., big cubes, etc., or money).

Solution Methods and Accessible Algorithms

Many different methods of multidigit addition and subtraction are invented by children and are used in different countries. There is not space here to describe all of them or to analyze their respective advantages and disadvantages. I have, however, selected two addition methods and one subtraction method for discussion. These methods are especially clear conceptually, are easy for even less-advanced students to carry out, and are less prone to errors than many other methods are. I also show the addition and the subtraction algorithms that are currently taught most frequently in textbooks in the United States and Canada.

In Figure 3, the algorithm on the top left is the addition method currently appearing in most U.S. textbooks. It starts at the right, in contrast to reading, which starts at the left. Most methods that children invent start at the left, perhaps because they are used to reading from the left and perhaps because number words in English are read starting at the left. The current addition algorithm has two major problems. One is that many children object initially (if they are in a position in which mathematical objections can be voiced) to putting the little 1s above the top number. They say that you are changing the problem. And in fact, this algorithm does change the numbers it is adding, as it proceeds by adding in these carries to the digits in the top number. The second method in the top row of Figure 3 does not change the top number: The new 1 ten is written down in the space for the total on this line (children using base-ten blocks in Fuson & Burghardt, 1993, in press, invented this method so that they did not change the answer as they went). It is also easier to see the total 14 ones when the 1 is written so close to the 4. The second problem with the present U.S. algorithm is that it makes single-digit addition difficult. You must add in the 1 to the top number, remember it even though it is not written, and add that remembered number to the bottom number. If, instead, you add the two numbers you see, you may forget to go up to add on the 1 ten (or 1 hundred). The second method solves this problem: You just add the two numbers you see and then increase that total by 1. This method makes the adding much easier for less-advanced children.

Insert Figure 3 about here; see Figure 3 at the end of this paper

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Both of these methods require that children understand two aspects of multidigit numbers: (1) that they must add like units to each other; and (2) that when they get 10 or more of anything, they must give 1 group of ten of those things to the next left place and record the remaining things. The second understanding has been called “carrying” or “regrouping” or “trading.” This grouping is done after the adding of each kind of unit. The make-a-ten method of single-digit addition described earlier is clearly helpful for such grouping because it makes a number into 1 ten and some ones. Multidigit addition is a useful place to use this make-a-ten method. Unless the structure of teen numbers as 1 ten and some ones is strongly experienced in the classroom, however, children may have trouble knowing how to break a teen number for regrouping. Again, the teen words in English obfuscate the tens, and all calculation is carried out by the solver using number words (even though these words may only be said internally). In one study with base-ten blocks, some first and second graders who were successfully adding 4-digit numbers and explaining their methods still had trouble with the grouping step when they did not use blocks. They knew that each teen word had a ten and some ones; they just did not know how many ones were in a given teen word. Instead, they used their knowledge of written numbers to write their total off to the side: For example, they said, “8 plus 6 is fourteen” and wrote 14, which they then read as 1 ten and 4 ones. Work on teens as 1 ten and *x* ones would have been helpful to these children.

Method B in Figure 3 separates the two major steps in multidigit adding. The total for adding each kind of multiunit is written on a new line, emphasizing that you are adding each kind of multiunit. The carrying-grouping-trading is done as part of the adding of each kind of multiunit: The new 1 ten of the next larger multiunit is simply written in the next-left column. One then does the final step of multidigit adding: Add all of the partial additions to find the total. Method B can be done in either direction (Figure 3 shows the left-to-right version). Because you write out the whole value of each addition (e.g., 500 + 800 = 1300), this method facilitates children's thinking about and explaining of how and what they are adding.

The drawings at the far right can be used with any of the three methods shown in Figure 3 to support understanding of the major components of the methods. The different sizes of the ones, tens, and hundreds in the drawings support children's adding of those like quantities to each other. Ten of a given unit can be encircled to make 1 of the next higher unit (10 ones = 1 ten, 10 tens = 1 hundred, 10 hundreds = 1 thousand). The issue for each algorithm then is how to record the adding of each kind of unit, the making of each 1 new larger unit from 10 of the smaller units, and the adding of the partial additions to make the total. Circling the new ten units can also support the general make-a-ten single-digit methods.

Under the drawing are summarized the two vital elements of using drawings or objects to support understanding of addition methods. First is a long Stage 1 in which the objects or drawings are linked to the steps in the algorithms to give meanings to the numerical notations in those algorithms. A second but crucial Stage 2 then lasts an even longer time (over years) in which students only carry out the numerical algorithm but occasionally explain it using words describing quantity objects or drawings so that meanings stay attached to the steps of the algorithm. Stage 2 is vital because of the single-digit appearance of the written numerals. Numerals do not facilitate correct methods, or inhibit incorrect methods, the way the objects and drawings do, and errors can creep into already understood methods, especially as children learn other solution methods in other domains.

Two subtraction methods are shown in Figure 3. The method on the left is the most widely used current U.S. algorithm. It moves from right to left, and it alternates between the two major subtraction steps: Step 1 is ungrouping (borrowing, trading) to get 10 more of a given unit so that unit can be subtracted (necessary when the top unit is less than the bottom unit), and Step 2 is subtracting after the top number has been ungrouped. The regrouping may be written in different ways (e.g., as a little 1 beside the 4 instead of crossing out the 4 and writing 14 above). The alternating between the two major subtracting steps presents three kinds of difficulties to students. One is initially learning this alternation. Two is then remembering to alternate the steps. The third is that the alternation renders students susceptible to the pervasive subtracting error: subtracting a smaller top number from a larger bottom number (e.g., doing 62 - 15 as 53). When moving left using the current method, a solver sees two numbers in a column while primed to subtract. For example, after ungrouping in 1444 – 568 to get 14 in the rightmost column and subtracting 14 – 6 to get 8, one sees 3 at the top and 6 at the bottom of the next column. Automatically the answer 3 is produced (6 – 3 = 3). This answer must be inhibited while one thinks about the direction of subtracting and asks whether the top number is larger than the bottom (i.e., asks oneself whether regrouping or borrowing is necessary).

The accessible subtraction method shown in the bottom middle of Figure 3 separates the two steps used in the current method. First, a student asks the ungrouping (borrowing) question for every column, in any direction. The goal is to rewrite the whole top number so that every top digit is larger than the bottom digit. This rewriting makes the conceptual goal clearer: You are rearranging the units in the top number so that they are available for subtracting like units. It also prevents the ubiquitous top-from-bottom error because you fix everything (ungrouping if necessary) before doing any subtracting. Doing the fixing in any direction allows children to think in their own way. The second major step is then to subtract the digits in every column, which also can be done in any order.

The drawing at the bottom right of Figure 3 shows how a size drawing or size objects can support the two aspects of multidigit subtracting. There are not enough ones, or tens, or hundreds to do the needed subtracting, so 1 larger unit is opened up to make 10 of the needed units. The subtraction can be done from this 10, facilitating the "take from ten" single-digit subtraction method. Or students can count up to find the difference in the written number problem.

The irregular structure of the English words between twenty and ninety-nine continues to present problems in multidigit problems because all single-digit and multidigit calculation is done using the words as oral intermediaries for the written numbers, and these words do not show the tens in the numbers. Using English forms of the regular East Asian words ("1 ten 4 ones" for 14) along with the ordinary English number words has been reported to be helpful (Fuson, Smith, & Lo Cicero, 1997). This approach permits children to generalize single-digit methods meaningfully. For example, for 48 + 36, students can use their single-digit knowledge and think, "4 tens + 3 tens is 7 tens," rather than having to think "forty plus thirty is ?" or use only single-digit language ("four plus three is seven"), thus ignoring the values of the numbers.

Textbook and Curricular Issues

U.S. textbooks have several problematic features that complicate children's learning of multidigit addition and subtraction methods. The grade placement of topics is delayed compared to that of other countries (Fuson, Stigler, & Bartsch, 1988), and problems have one more digit each year so that this topic continues into Grade 5 or even Grade 6. In contrast, multidigit addition and subtraction for large numbers are completed in some countries by Grade 3. In the first grade in the United States, two-digit addition and subtraction problems with no regrouping (carrying or borrowing) are given, but no problems requiring regrouping are given until almost a year later, in second grade. Problems with no regrouping set children up for making the most common errors, especially subtracting the smaller digit from the larger even when the larger digit is on the bottom (e.g., 72 – 38 = 46). This error is one major reason that on standardized tests only 38% of U.S. second graders are accurate on problems such as 72 – 38. Accessibility studies indicate that first graders can solve two-digit addition problems with trading if they can use drawings or quantity supports (Fuson, Smith, & Lo Cicero, 1997; Carpenter et al., 1998). Because knowing when to make 1 new ten is an excellent use of place-value knowledge, such problems can be thought of as consolidating place-value ideas, not just as doing addition. Giving children from the beginning subtraction problems that require regrouping would help them understand the general nature of two-digit subtraction. This topic might well be delayed until second grade because children find two-digit subtraction much more difficult than addition. But second graders learning with quantity supports and with a focus on understanding their methods can have high levels of success.

Textbooks or approaches characterized as "reform" may have different shortcomings than those of traditional textbooks. No study using a reform approach or focused on teaching for understanding has reported children doing worse on multidigit computation (or single-digit computation) than children using traditional textbooks. But a couple of studies have reported some children using unitary count-all multidigit strategies as late as Grades 3 and 4, suggesting insufficient attention to helping all children learn prerequisite counting and quantity understanding for effective methods using tens. A 5-year longitudinal study following 20 classes of children using a reform textbook *Everyday Mathematics* (EM) suggests other issues that need to be considered if U.S. children's multidigit performance is to improve above that of standard textbooks. Overall, achievement results were very positive (Carroll, Fuson, & Drueck, 2000; Carroll & Fuson, 1999): At every grade level, children who used *Everyday Mathematics* outperformed comparison groups using traditional U.S. textbooks on a wide range of topics. The only exception was in single-digit and multidigit addition and subtraction problems, in which EM children's performance was the same as that of comparison children using standard textbooks.

A focus group of teachers and researchers identified several attributes of the EM curriculum or its use in classrooms that seemed to be sources of the lower-than-desired multidigit performance. I summarize these here because they indicate issues that may need to be addressed as classrooms move to teaching for understanding. They are relatively easy to avoid and are being addressed in current EM revisions. Although there was an emphasis in most EM classrooms on using alternative methods and explaining them, children were not using quantity supports except for the 100s grid. The grid was usually used as a counting tool without tens and ones being explicit on it (the first addend was identified as the square containing 38 rather than 38 being the 3 rows of ten squares and the 8 squares in the fourth row, and counting was done by rote with the vertical ten-jump rarely justified or explained). Most explanations of methods were verbal only, so that less-advanced children had difficulty following the explanations. A few teachers did write numbers on the board as children explained, but writing numbers for all problems and also using quantity referents of some kinds (e.g., drawings on the board) would have made the explanations more accessible to all children.

No meaningful treatment of the standard algorithms was included in the lessons or in most classrooms. Some students inevitably brought the standard algorithms from home, and teachers did not know how to help children explain them meaningfully. Furthermore, because of test pressures, some teachers taught standard algorithms right before standardized tests but without meaning for the algorithms. A difficult subtraction method (recomposing both numbers) was included in lessons, but its meaning was given insufficient scaffolding. Some teachers and children then confused it with the addition recomposition method, leading to errors (in the addition method, one addend is increased and the other addend is decreased by the same amount; in the subtraction method, both numbers must be increased or decreased by the same amount to maintain the difference).

Many of the multidigit lessons used contexts to give real-world meanings to the numbers (e.g., temperature). But the focus in many lessons was heavily on the context and insufficiently on the multidigit processes. There was an insufficient focus on all children learning prerequisites for effective methods. EM children did better on multidigit combinations in word problem situations than in vertical columns. Their success on word problems is noteworthy, but the errors on combinations written in column form indicate insufficient strength of place-value quantity meanings in the face of the single-digit appearance of the numbers. EM did introduce the addition method on the right in Figure 3, and many children in some classrooms did use it and explain it effectively.

This review suggests some central features for effective reform and traditional texts. Any algorithms that are included need to be accessible to children and to teachers, and support needs to be provided so that the algorithms are learned with understanding. The research-based accessible methods in Figure 3 were included here to indicate algorithms that are more accessible than those presently appearing in most U.S. textbooks. Further, children need to use quantity supports in initial experiences with multidigit solving and multidigit algorithms so that these can be learned with meaning. Finally, students and teachers need to use referents when discussing methods so that everyone can follow the discussion. Drawing quantities can be helpful in such discussions.

Conclusion

Recent research clearly indicates that nontraditional approaches can help children come to carry out, understand, and explain methods of multidigit addition and subtraction rather than only carry out a method. This higher level of performance can also be accomplished at earlier grades than those at which, at present, only answers are expected. Features of classrooms engendering this higher level of performance are as follows: an emphasis on understanding and explaining methods; initial use by children of quantity supports or drawings that show the different sizes of ones, tens, and hundreds in order to give meanings to methods with numbers; and sufficient time and support for children to develop meanings for methods with numbers and for prerequisite understandings (these may be developed alongside the development of methods) and to negotiate and become more skilled with the complexities of multistep multidigit methods.

The research is not yet clear about which quantity supports, which multidigit methods, or which details of classroom functioning can maximize learning for all. The most effective approach at present seems to be to make the learning of algorithms more mathematical by considering it an important arena of mathematical pattern finding and invention that will use and contribute to robust understandings of the place-value system of written numeration. Meaningful discussion of various standard algorithms brought into the classroom from children's homes (e.g., the subtraction algorithm widely used in Latin America and Europe, see Ron, 1998) has an important role. Seeking to discover why each algorithm works provides excellent mathematical investigations. It also seems to be important to share accessible methods with less-advanced children so that they have a method they understand and can use. The instructional focus, however, should be on their understanding and explaining, not just on rote use. All three of the accessible methods in Figure 3 were invented by children but have also been shared with and learned meaningfully by many children. There may well be other methods not yet discovered (or rediscovered) that are even more powerful. Comparing methods to see how they take care of the crucial issues of the domain facilitates reflection by everyone on the underlying conceptual and notational issues of that domain. This focus seems much more appropriate than others in the twenty-first century, where new machine algorithms will be needed and new technology will require many people to learn complex multistep algorithmic processes. If the focus is accompanied by a continual focus on testing and teaching accessible methods as well as on fostering invention, all children should be able to learn and explain a multidigit addition and subtraction method as well as carry it out accurately.

Multidigit Multiplication and Division

There is much less research on children's understandings of multidigit multiplication and division than on the operations already discussed. Some sample teaching lessons have been published (e.g., Lampert, 1986, 1992). Teaching alternative methods for accomplishing these operations has been explored (e.g., Carroll & Porter, 1998). A preliminary learning progression of multidigit methods has been reported for third- to fifth-grade classrooms in which children's invention of algorithms was fostered (Baek, 1998). These methods moved from (a) direct modeling with objects or drawings (by ones and by tens and ones), to (b) written methods involving repeatedly adding (sometimes by repeated doubling, a surprisingly effective method used historically), to (c) partitioning methods. The partitioning methods ranged from partitioning with various partitions using numbers other than ten, partitioning one number into tens and ones, and partitioning both numbers into tens and ones.

Current and Accessible Methods

The multiplication and division algorithms currently most prevalent are complex embedded methods that are not easy to understand or to carry out (see the left-most methods in Figure 4). They demand high levels of skill in multiplying a multidigit number by a single-digit number within complex embedded formats in which multiplying and adding alternate. In these algorithms, the meaning and scaffolding of substeps have been sacrificed to using a small amount of paper. Both use aligning methods that keep the steps organized by correct place value without requiring any understanding of what is actually happening with the ones, tens, and hundreds.

Modifications of these methods that clarify the meaning and purpose of each step are given in Figure 4. The separation of the steps in each of these accessible methods also facilitates the linking of each step to the quantities involved. An array drawing is used to show the quantities; arrays are powerful models of multiplication and division. The accessible methods and drawings demonstrate key features in multidigit multiplication and division that students must come to understand and be able to do.

Insert Figure 4 about here; see Figure 4 at the end of this paper.

Accessible Multiplication Methods

For multiplication, an array-size model is shown first. Such a model provides initial support for the crucial understandings of the effects of multiplying by 1, 10, and 100. It also shows clearly how each of the tens and ones numbers in 46 and 68 are multiplied by each other and are then added after all multiplication operations are done. The sizes of the resulting squares or rectangles indicate the sizes of these various products and thus support the key understanding. As one looks across each row in the array, one can see in the top row 10 × 46 as 10 × 40 (4 squares of 100) plus 10 × 6 (6 columns of 10 each). Multiplying by 60 creates 6 such rows of 10 products, so multiplying by 60 is multiplying by 10 and then multiplying by 6. Then one sees 8 rows of 1 × 46 as 1 × 40 and 1 × 6 (8 rows of each). The abbreviated model (shown next in Figure 4) can be drawn to summarize steps in multidigit multiplication. Its separation into tens and ones facilitates the multiplication operations involved.

The accessible multiplication algorithm shown in the top right of Figure 4 is the fullest form with all possible supports. As students come to understand each aspect of multiplication, each of the supports can be dropped, resulting in a streamlined version that is a simple expanded form of the usual U.S. method. Variations of the accessible algorithm have been widely used in research classrooms and in some innovative textbooks. Its key feature is a clear record of each of the four pairs of numbers (40 × 60, 40 × 8, 6 × 60, 6 × 8) that need to be multiplied. The vertical and diagonal marks are a way to record as you go which numbers you have already multiplied. Unlike the current U.S. algorithm, which starts at the right and multiplies units first, the accessible algorithm begins at the left, as students prefer to do. This also has the advantage that the first product written is the largest, which permits all of the smaller products to be aligned easily under it in their correct places. Writing out the factors at the side of each product emphasizes what one is actually doing in each step and permits an easy check. Writing out the separate products for 40 × 60 and 40 × 8 is much easier for students than doing the usual procedure: multiply 40 × 8, write part of the answer down below and part above the problem, multiply 40 × 60, and then add in the number written above the problem. The complex alternation of multiplying and adding in the usual algorithm is not necessary, is a source of errors, and obfuscates what one is actually doing in multidigit multiplying: multiplying each combination of units and adding all of them up (see the abbreviated model). Students who understand and wish to drop steps in the accessible algorithm do so readily, with a result looking like the usual U.S. method except that it has four instead of two products to be added. These four can even be folded into two for those students who wish to do so. Therefore the accessible model permits students to function at their own level of supported understanding and helps them explain what they are doing.

The accessible algorithm also generalizes more readily to algebraic polynomial multiplication than the current U.S. algorithm does. The abbreviated drawing can show, for example, $2x + 3$ across the top and $y + 4$ along the bottom. The model, and students' previous experience with multidigit multiplication, then clarify that one just multiplies each kind of unit in one number with each kind of unit in the other to find the product $2xy + 8x + 3y + 12$.

Multiplying by three-digit numbers is a simple extension of the two-digit version. After a conceptual development of the results of multiplying by 100 (numbers get two places larger so they move left two places), abbreviated drawings can demonstrate the nine combinations of products that need to be found and added. The accessible algorithm for these larger numbers is easy to carry out because it scaffolds the necessary steps. Given the accessibility of calculators, it is not clear how much valuable school learning time should be devoted to such large multiplication problems. But they could easily be introduced in a conceptual fashion that then relates to estimating the product, especially when the largest product is found first, as in the accessible method shown in Figure 4.

Current and Accessible Division Methods

The usual U.S. division algorithm has two aspects that create difficulties for students. First, it requires students to determine exactly the maximum copies of the divisor that can be taken from the dividend. This feature is a source of anxiety because it is often difficult to estimate exactly how many will fit. Students commonly multiply trial products off to the side until they find the exact one. Second, the current algorithm creates no sense of the size of the answers one is writing, and, in fact, one is always multiplying by single digits. In the example in Figure 4, you just write a 6 above the line; there is no sense of 60 because you literally are only multiplying 46 by 6. Thus, it is difficult for students to accumulate experience with estimating the correct order of magnitude of answers in division when they are using the current U.S. algorithm.

The accessible division method shown in Figure 4 facilitates safe underestimating. It builds experience with estimating and later accurate assessment of calculator answers because students multiply by the correct number (e.g., 60, not 6). It is procedurally easy for those still gaining mastery of single-digit multiplication because it permits the use of easy known products. It can be abbreviated to be as brief as the current algorithm for those who can manage the abbreviation. This accessible division algorithm has been used in various innovative materials since at least the 1960s.

The example of the accessible method given first in Figure 4 shows a solution that might be done by a student very early in division learning. Conceptually the drawing and the written algorithm work together to show the meaning of long division: It is like a puzzle in which you take away copies of the divisor (here, 46) until you cannot take away any more copies. You are solving the equation "$46 \times ? = 3129$" using the notion of division as the inverse of multiplication. The drawing shows these copies being added to make the total 3128 as 46×68 (remainder of 1), and the written algorithm subtracts each large copy as you go to keep track of how many more you have still to take away. The drawing can scaffold the 1-digit by 2-digit multiplication necessary at each step: 50×46 is split into $50 \times 40 = 2000$ and $50 \times 6 = 300$ to make 2300. The scaffolding is important because this combination of multiplying and adding is complex for some students. The example shows the student selecting to multiply by 50 because 5s facts are learned easily and accurately (remember the nice repeating pattern for the 5s—the Gs—in the Figure 2 alphabet multiplication table?). The student then sees that he or she can take away another 10 copy of 46, which is simple to do. The student then cleverly uses a product he or she has already found (50×46) to take away 5 copies of 46. Doubling is also easy, though many students would probably have multiplied by 3 at that point. Successive doubling is actually the basis of multiplication and division algorithms that were used historically in Europe. A version of the same problem that might be done by that same student after more experience is given at the right in Figure 4. At this point the student may not need the drawing to scaffold the steps, meanings, or multiplication operations.

Both the accessible algorithms for multiplication and division depend heavily on fluency with multiplication and addition (and in division, with multidigit subtraction). The difficulties many students have in subtraction noticeably affect division, so understanding and fluency in multidigit subtraction is very important. Because students typically range very substantially in the rate at which they have learned all of their multiplications, a few to many students may not have full fluency by the time their class is discussing multidigit multiplication and division. In such cases, it seems advisable to provide such students with a multiplication table that can be used to check their multiplications as they go. This aid will permit them to keep up with the class and learn an algorithm. Furthermore, each verification of or search for a product in the table provides another learning trial for basic multiplication. Of course, providing separate learning opportunities for multiplication combinations with which the student is not yet fluent would also be helpful.

How Much Consolidation Time?

How much valuable school mathematics time should be spent on multidigit multiplication and division is a question whose answer probably will need to be continually revised during the twenty-first century. New goals will arise to compete with these domains, as they have already. At present, it does seem worthwhile to spend some time on conceptual and accessible approaches that facilitate students' understanding of how multidigit multiplication and division can be built from key concepts of place value and basic multiplication combinations. During that time students could also be bringing to mastery those combinations. Drilling for long periods on problems involving large numbers seems a goal more appropriate to the twentieth than to the twenty-first century.

General Issues in Achieving Computational Fluency

Curricular Issues

The U.S. curriculum has been characterized as "underachieving" and recently characterized in the TIMSS international study as "a mile wide and an inch deep" (McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1989; McKnight & Schmidt, 1998; Peak, 1996). Countries whose students score high in international studies select vital grade level topics and devote enough time so that students can gain initial understanding and mastery. In the United States, no teacher and no grade level are responsible for a given topic. Topics such as multidigit computations are distributed over several years, doing one digit larger each year. Large amounts of time are devoted at the beginning of each year and each new topic to teach what was not learned or was learned incorrectly in the year before. It is much easier, however, to help students build initial correct computational methods than to correct errors. For example, second graders using base-ten blocks for initial learning of multidigit addition and subtraction explained answers and achieved high levels of accuracy that were maintained over time (Fuson, 1986a; Fuson & Briars, 1990). Older students who had been making subtraction errors for years did learn in one session with base-ten blocks to correct their errors, but many later regressed to their old errors (Resnick & Omanson, 1987). Carefully designed practice, help during learning, and other aspects described above and below are important for computational fluency. But the most severe problem at this point is helping students learn in a timely fashion any correct generalizable method that they understand. Such initial learning must be deep and accurate. Only with understanding can interference from later similar notations and methods be reduced.

Helpful Instructional Phases

What features of classrooms can contribute to computational fluency? A recent review of the literature contrasts the many studies that found an experimental instructional method superior to a traditional control method (Dixon et al., 1998). The less-effective traditional methods involved two phases: A teacher presentation of some topic (with students observing passively) followed by independent student practice of that topic, with or without teacher monitoring, giving feedback, and so forth. Superior learning was achieved by effective methods that had three phases. First, teachers initially involved students in the introduction of the topic through explanations, questions, and discussion; students were active learners whose initial knowledge about a domain was elicited. Second was a long period in which students were helped to move from teacher-regulated to self-regulated solution processes. Teachers structured a significant period of help that was gradually phased out. This help was given in different ways: by scaffolded problems and visual or other supports, by peers, and by the teacher or aides. During this sustained helping period, students received feedback on their performance, got corrective help so that they did not practice errors, and received (and often gave) explanations. The third phase of effective instruction was a brief assessment of students' ability to apply knowledge to untaught problems (so-called near transfer) in which students worked independently. Such independent work might then be distributed over time. Other relevant results from studies that Dixon et al. reviewed were that strategy instruction of various kinds was superior to not giving such instruction, working fewer problems in depth was more effective than working more problems quickly, writing as well as solving problems was helpful, and solving concept examples sequenced for generalization and discrimination was helpful.

The implications for computational fluency of all of these results are that all students had sustained supported time to learn a given domain deeply and accurately. Such deep sustained accurate learning over time is necessary for complex domains requiring multistep solution methods. Students need to learn the central principles of a domain (e.g., in multidigit addition and subtraction, that you add or subtract like multiunits), learn the overall shape of a given method, learn in detail the steps of the method, and weave this developing knowledge together so that it operates fluidly and accurately. This is true whether the students invent the method or learn it from other students or from the teacher. Practice is important, but effective practice is supported by monitoring and help that are focused on doing and on understanding. In contrast, *drill and practice* frequently carries the connotation of rote practice, has little sense of monitoring or feedback, and no connotation of helping or of visual, conceptual, psychological, or motivational support for learning throughout the practicing phase.

At present, not enough is known about effective ways to orchestrate the helping period to deliver feedback and help to students as they need it, especially in classrooms where students are using different methods. Given the heterogeneity of most classrooms, such orchestration is difficult. Designing and testing effective helping methods and effective ways to give feedback on answers are a vital area for further research. It is also important to ascertain how to facilitate peer helping, given that peer helping or cooperative learning frequently but not always results in better learning. Some methods used in other school subjects, such as jigsaw methods in which each group member is given different knowledge to contribute, have been difficult to use in mathematics.

A textbook issue that at present interferes with the more effective three-phase method (and even with effective teacher presentation of topics in the less-effective two-phase traditional approach) is the common misuse of art (photographs, drawings, cartoons, etc.) in U.S., mathematics textbooks. In many other countries, the art is designed to support conceptual thinking. In the United States, art frequently distracts from conceptual understanding because it is irrelevant or overwhelmingly busy.

Helping Diverse Learners

A related review of literature concerning school success of diverse learners (Kameenui & Carnine, 1998) identified six crucial aspects of teaching and of learning materials: structuring around big ideas, teaching conspicuous strategies, priming background knowledge, using mediated scaffolding (e.g., peer tutoring, giving feedback about thinking, providing visual supports that provide cues for correct methods), using strategic integration (integration into complex applications to provide distributed practice in more complex situations), and designing judicious review. Diverse learners are those who may experience difficulties in learning because of low-income backgrounds, speaking English as a second language (or not at all), or other reasons.

The first aspect, structuring around big ideas, is absolutely necessary to obtain sufficient time so that students, especially diverse learners, can learn deeply the core concepts of that grade level. It is the antithesis of the present "mile wide and inch deep" U.S. curriculum. This issue must be resolved if diverse learners are to obtain computational fluency. The next three aspects specify aspects of the initial active learning phase and the helping phase in the three-phase effective teaching model outlined above.

Using strategic integration and designing judicious review are aspects of computational fluency that follow deep and effective initial learning in a domain. Strategic integration of various computational methods into moderately complex problems increases problem-solving competence by increasing the range of situations in which students use that computational method. It also provides for distributed practice of the method, one of the most effective kinds of practice.

Judicious review is defined as being plentiful, distributed, cumulative, and appropriately varied. It follows initial deep learning. Distributed and monitored practice requires working one or two examples occasionally, with immediate help for wrong answers. This practice is important even after successful meaningful learning because the nonsupportive or misleading mathematical words or notations in many domains continually suggest wrong methods (e.g., *adding* the top and the bottom numbers when adding fractions). Furthermore, many computational domains are similar, and learning new domains creates interference with old domains (e.g., you do *multiply* the tops and bottoms of fractions). Therefore, after deep and successful initial learning, distributed practice of a couple of problems of a given kind can check whether errors are creeping in. Frequently, helping students correct their methods is as simple as suggesting that they remember original supports. For example, as some errors crept into multidigit methods learned with base-ten blocks, asking students to "think about the blocks" was sufficient for them to correct their own errors in subtracting with zeroes in the top number (Fuson, 1986a).

The research of Knapp and associates (Knapp, 1995; Zucker, 1995) on attributes of successful high-poverty classrooms underscores these results. They found that a balance between conceptual understanding and skills practice resulted in higher computational and problem-solving performance by lower-achieving and higher-achieving students. Successful teachers supported conceptual understanding by focusing students on alternative solution methods (not just answers), elicited thinking and discussion about solution methods, used multiple representations and real-life situations to facilitate meaning-making, and modeled ways to probe meaning of mathematical problems or methods. These teachers also provided a "healthy dose" of skills practice.

Individual Differences

As in other school subjects, substantial social-class and ethnicity differences in achievement exist in mathematics (e.g., Ginsburg & Russell, 1981; Secada, 1992). Kerkman and Siegler (1993) found that low-income children had less practice in solving problems and that they executed strategies less well. Strategy instruction and monitored practice were therefore recommended for such students. Individual differences as early as first grade cut across gender and income levels to differentiate children into what Siegler (1988) has termed good students, not-so-good students, and perfectionists. Roughly half of the not-so-good students went on to be identified as having mathematical disabilities by fourth grade versus none of the other groups. On single-digit addition tasks, these students were characterized by use of more primitive methods and by more production of errors on problems on which they could have used (but did not use) more accurate but effortful strategies (e.g., counting with their fingers). Thus, these students were producing incorrect answers more often, thereby creating responses that competed with their experiences of correct answers. Siegler's model for the learning of single-digit addition emphasizes the importance of avoiding generating errors because these interfere with remembering the correct answer. Thus, the importance of feedback and immediate help is underscored. Perfectionists and good students had similar long-term outcomes, but the perfectionists were much more likely to use slower and effortful methods even on simpler problems than were the good students. This finding emphasizes that methods of practice should facilitate individuals understanding their own growth and progress rather than lead to the comparing of individuals. Practice should also be varied so that sometimes speed is important but, at other times, the use of a method in a complex situation is important. An overemphasis on either could lead to rigidity rather than computational fluency.

There has been less work on mathematics disability than on reading disability, especially with younger children. Different kinds of mathematics disability have been identified. Geary's (1994) review identifies four types and recommends different kinds of learning supports for each kind. Students with semantic memory disabilities have difficulty with verbal, and especially phonetic, memory, but many have normal visuospatial skills. These students have great difficulty memorizing basic computations because these rely on a phonetic code. Therefore instructional supports that use visual rather than phonetic cues and teaching strategies for basic calculations are recommended for these students. Students with procedural deficits use less-advanced methods than their peers. Though many eventually catch up, this long period of using primitive methods may be detrimental. Such children do not seem to invent more-advanced methods as readily as their peers do. Therefore, conceptually based strategy instruction that helps these children use and understand more-advanced strategies such as counting on can be helpful. Students with visuospatial disabilities have difficulties with concepts that use spatial representations, such as place value. Research is not clear about the developmental prognosis of such children, but suggested methods of remediation are to support visual processing with extra cues. Because directionality is a special problem with such students, the accessible methods described in this paper that can be carried out in either direction might be especially helpful for such students. Difficulties with mathematical problem solving that go beyond arithmetic deficits also characterize some students. Supports for problem solving such as drawing the problem situation that were discussed in an earlier section of this paper are suggested as useful for these students. Technology may also help provide complex problem-solving situations that are nevertheless accessible to students with disabilities in mathematics (Goldman, Hasselman, & the Cognition and Technology Group at Vanderbilt, in press).

Although one might think that students identified as learning disabled in mathematics might need special learning situations, the recommendations for all types of disabilities in mathematics summarized by Geary (1994, p. 285), one of the most prominent researchers in that field, sound like a summary of the results in this chapter. Thus, the kinds of teaching recommended by research for helping students in general to computational fluency may especially help those students with mathematical disabilities. The use of accessible methods also may be especially helpful to these students because they tend to be behind and discouraged, and accessible methods are learned and understood more quickly and easily.

Teaching to Prepare Students for Rational Numbers

Teaching and learning with whole numbers can lay an adequate foundation for later work with rational numbers, including decimal and ordinary fractions. Or it can make such work more difficult. At the present time, students make many errors in decimal and ordinary fractions because they incorrectly generalize concepts from whole numbers (e.g., Hiebert & Wearne, 1986; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989). Approaching all domains with a focus on the meanings for the notations, and with explicit consideration of what does and what does not generalize, could improve student competence in these advanced domains. Deep understanding of place value and of the regular ten-for-one trades to the left as numbers get larger can facilitate understanding decimal fractions as regular one-for-ten trades to the right, as quantities get smaller. Understanding multidigit addition and subtraction as adding or subtracting like quantities (ones to ones, tens to tens, etc.) can facilitate the related understanding that adding or subtracting decimal fractions or regular fractions must also involve adding or subtracting like quantities (for decimal fractions such as tenths to tenths or hundredths to hundredths, and for fractions such as fourths to fourths or thirds to thirds). Deeply understanding quantities for fractions and decimal fractions is necessary in order to overcome meanings suggested by whole number notation (e.g., that 0.25 > 0.3 because 25 > 3). Similarly, it is important for students to reassess the whole number notions that "multiplication makes larger" and "division makes smaller" when they multiply and divide fractions. If whole number knowing and doing has been a sense-making process intertwined with problem solving and explaining one's thinking, it will be easier for students to make the necessary extensions and adjustments to their whole number knowledge as they enter these more advanced domains.

Conclusion: Achieving Mathematical Power in Whole Number Operations

The reform approach as outlined in the first round of the NCTM standards documents stimulated much action in the United States and Canada and contributed to a broader view of mathematics learning and teaching. As is inevitable with such documents, however, the approach was also sometimes misunderstood and distorted in ways that are counterproductive to good mathematics teaching and learning. The following have sometimes been thought by some people to characterize "reform math teaching": extensive unfocused, meandering discussion; mathematical content restricted to children's current knowledge and interests; real-world contexts or activities in which the mathematical content is not clear or is so complex that little mathematical learning occurs; teachers who give no information of any kind including standard mathematical vocabulary or notation; and prolonged periods of "invention" of solution methods in which children struggling with mathematics use very primitive methods rather than building prerequisite knowledge for more-advanced methods or being helped to learn such methods.

The *Principles and Standards for School Mathematics* attempts to clarify and correct such misunderstanding. The new document instead recommends, and the research literature supports, ambitious mathematical goals, teacher-led and monitored discussion that focuses on central mathematical ideas, teachers explaining and clarifying as well as children explaining and clarifying, using and building on children's knowledge but extending that knowledge in mathematically important ways, and using carefully chosen real-world contexts as well as carefully designed pedagogical learning supports (e.g., selected manipulatives or drawings) to facilitate meaning-building by all children. Teachers have vital roles both in helping children build initial understanding and in supporting them to achieve computational fluency and mathematical power. There is considerably less research on such productive teacher roles than on student understanding, errors, and methods (but see, e.g., Hiebert et al., 1997; Fraivillig, Murphy, & Fuson, 1999; Fuson & Burghardt, in press; Simon, 1995; Stipek, Salmon, Givvin, Kazemi, Saxe, & MacGyvers, 1998). As this body of research grows, teachers will have more detailed guides to developing mathematical power in all of their students.

Meanwhile, the research that does exist provides substantial direction for improving the students' mathematical power. All students require a constant intertwining of understanding and doing, of building meaning, problem solving, and computing. Learning "the basic facts" is important. But research indicates that, in addition and subtraction, children around the world progress from simple methods with objects through a progression of more rapid methods. Children can be helped to progress through these methods to powerful and rapid general methods. Multiplication and division involve different patterns for different numbers, and students also progress through a learning path of more rapid methods. Many different algorithms (general methods) exist for solving multidigit addition, subtraction, multiplication, and division. Research and analysis have identified some that are both easy to understand and to carry out. These all relate to the methods commonly taught now in the United States and Canada, but are conceptually more powerful or easier to carry out. Students in the United States and Canada can learn to understand and explain computational methods if these methods are approached as sense-making endeavors. Practice is important, as is learning prerequisite knowledge that facilitates more advanced methods. Problem solving can be used from the beginning to provide meaning for computations and then can be continually intertwined as both methods and problem solving become consolidated.

The new research-based view of achieving computational fluency is a more complex and connected view than is the past linear view of count, memorize facts, solve problems, learn algorithms, and then solve problems with them. However, a new, more complex view is necessary to achieve the new, more complex goals of mathematics learning and teaching necessary for the twenty-first century. A new kind of computational fluency is needed for the challenges and changes Americans and Canadians will face during the coming 100 years.

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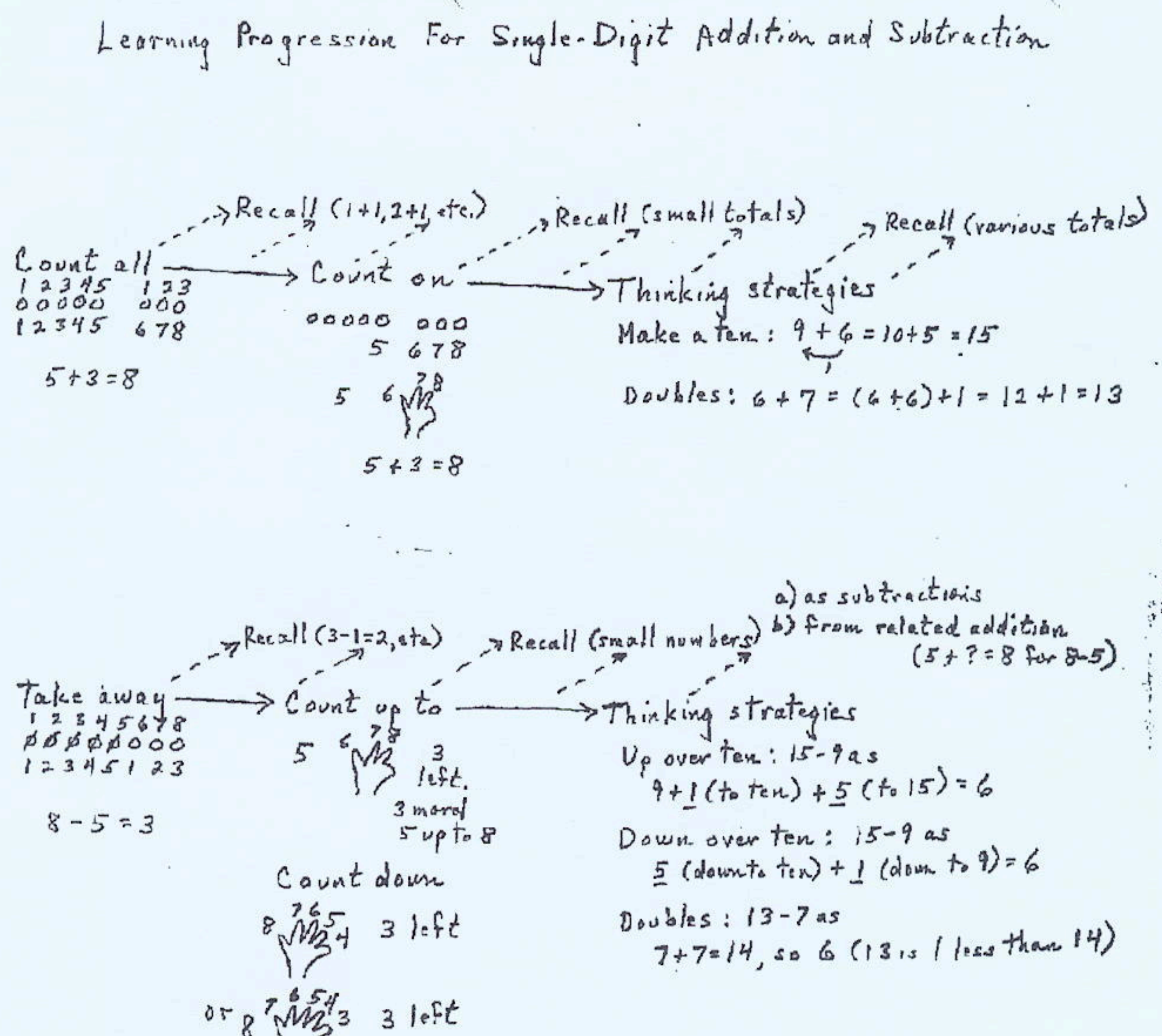
Table 1: Types of Addition and Subtraction Situations Given as Word Problems: Change Add To/Take From, Put Together/Separate, and Compare

	Change Add To and Change Take From	
Change-Add-To unknown result	Change-Add-To unknown change	Change-Add-To unknown start
Miguel had 4 dollars. Giovanni paid Miguel 3 dollars for a milk carton. How many dollars does Miguel have now?	Eliany had 5 packets of ten candies and 7 loose ones and went to the store and bought some more candy. Now she has 8 packets of ten candies and 6 loose ones. How much candy did Eliany buy?	Pablo had some pencils and bought 9 more. Now Pablo has 16 pencils. How many pencils did Pablo have to start with?
Change-Take-From unknown result	Change-Take-From unknown change	Change-Take-From unknown start
Doridalia had 32 dollars. She went to the store and paid 13 dollars for some crayons. How many dollars does Doridalia have now?	Aunt Pat had 11 ears of corn. Then the children ate some of them. Now Aunt Pat has 6 ears of corn. How many ears of corn did the children eat?	Mitzi went to Roberto's store and bought 2 packets of ten peanuts and 8 loose ones from him. Now Roberto has 4 packets of ten peanuts and 7 loose ones. How many peanuts were there in Roberto's store before Mitzi bought her peanuts?
	Put Together and Take Apart	
Put Together unknown total	Put Together unknown part	Put Together unknown part
Mario bought 3 packets of ten colored pencils and 5 loose ones. Edwin bought 2 packets of ten colored pencils and 9 loose ones. How many pencils did they buy altogether?	Ed has 15 kittens and puppies. 7 of them are puppies. How many of them are kittens?	Isabel has a flower shop. In her shop there are 13 roses and some carnations. Altogether there are 37 flowers in Isabel's shop. How many carnations does Isabel have in her shop?
Take Apart unknown total	Take Apart unknown part	Take Apart unknown part
Dad picked some flowers. He put 7 in the red vase and 9 in the blue vase. How many flowers did he pick?	Rachna picked 42 apples at the tree farm. She put them in a bag for her grandmother and a bag for her mother. There were 28 in the bag for her mother. How many were in the bag for her grandmother?	Farmer Brown's sheep walk back and forth between two fields. He counted 6 sheep in one field. He has 14 sheep. How many sheep should be in the other field?
	Compare More and Compare Less/Fewer	
Compare unknown difference	Compare consistent (with more) Compare inconsistent (with less)	Compare inconsistent
Tom has 8 stamps. Sue has 13 stamps. How many more stamps does Sue have than Tom? How many fewer stamps does Tom have than Sue?	My friend and I went to the store to buy notebooks. My friend paid $.64 more than I did. If I paid $1.68, how much did my friend pay? I paid $.64 less than my friend.	Rodrigo has 16 books. Rodrigo has 7 more books than Aki has. How many books does Aki have? Aki has 7 fewer books than Rodrigo.

Note: Compare situations have an additive or subtractive character depening on the language in the comparing sentence. “More” suuggests addition, while “less” suggests subtraction. These suggestions are even stronger in languages such as Spanish, where the same words are used for “more” and “add” and for “less/fewer” and “subtract.” The difficult compare problems have insonsistent language: the word in the question suggests the operation opposite to that required to solve the problem. The comparing sentence can always be said in two ways, one uusing “more” and one using “fewer/less” (these are underlined above. Thuus one can change a difficult inconsistent problem into a simpler consistent problem by changing the question. Other language can be used to make the comparison: “How many books does Aki have to get to have as many as Rodrigo?” or “If Rodrigo and Aki match their books, how many extra will there be?”



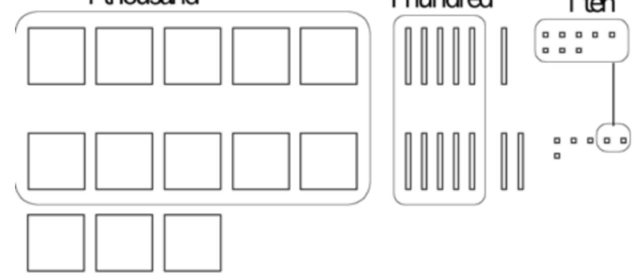
Learning Progression For Single-Digit Addition and Subtraction
Recall (1+1, 2+1, etc.)
Recall (small totals)
Recall (various totals)
Count all
1 2 3 4 5 1 2 3
0 0 0 0 0 0 0 0
1 2 3 4 5 6 7 8
5+3=8
Count on
0 0 0 0 0 0 0 0
5 6 7 8
5 6 7 8
5+3=8
Thinking strategies
Make a ten: 9+6=10+5=15
Doubles: 6+7=(6+6)+1=12+1=13
Recall (3-1=2, etc.)
Recall (small numbers)
a) as subtractions
b) from related addition
(5+?=8 for 8-5)
Take away
1 2 3 4 5 6 7 8
1 2 3 4 5 1 2 3
8-5=3
Count up to
5 6 7 8
3 left.
3 more
5 up to 8
Count down
8 7 6 5 4 3 left
or 8 7 6 5 4 3 3 left
Thinking strategies
Up over ten: 15-9 as
9+1 (to ten) + 5 (to 15) = 6
Down over ten: 15-9 as
5 (down to ten) + 1 (down to 9) = 6
Doubles: 13-7 as
7+7=14, so 6 (13 is 1 less than 14)

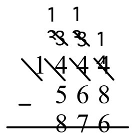
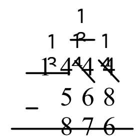
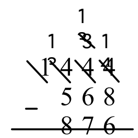
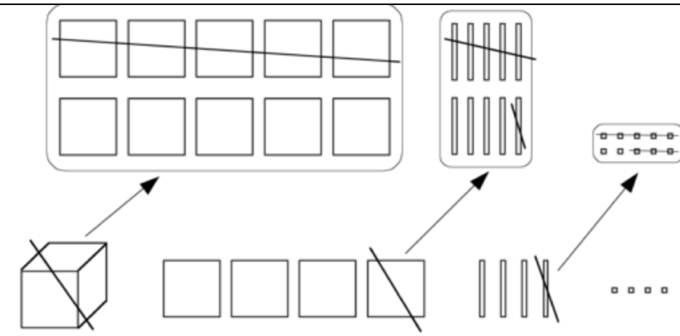
Figure 1

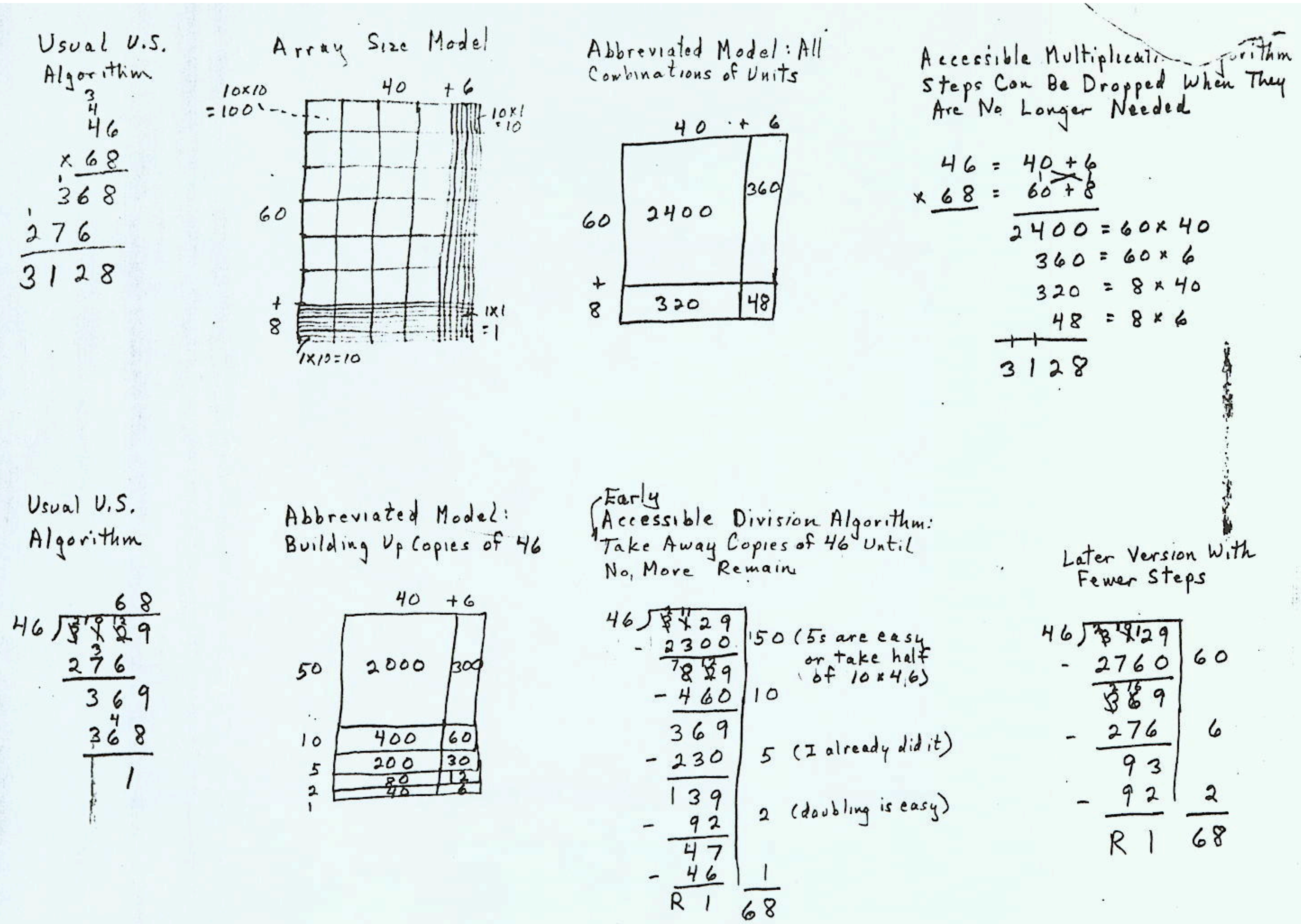
Figure 2: A Multiplication Table for Numbers A Through CL

	C	D	E	F	G	H	I	J	K	CL
C	C	D	E	F	G	H	I	J	K	CL
D	D	F	H	J	CL	CD	CF	CH	CJ	DL
E	E	H	K	CD	CG	CJ	DC	DF	DI	EL
F	F	J	CD	CH	DL	DF	DJ	ED	EH	FL
G	G	CL	CG	DL	DG	EL	EG	FL	FG	GL
H	H	CD	FJ	DF	EL	EH	FD	FJ	GF	HL
I	I	CF	DC	DJ	EG	FD	FK	GH	HE	IL
J	J	CH	DF	ED	FL	FJ	GH	HF	ID	JL
K	K	CJ	DI	EH	FG	GF	HE	ID	JC	KL
CL	CL	DL	EL	FL	GL	HL	IL	JL	KL	CLL

Figure 3. Multidigit Addition and Subtraction

Typical U.S. Algorithms	Accessible Generalizable Methods		Drawings to Show Quantities
	Method A: New Groups Below	**Method B: See Place Values**	
11 568 +876 1444	568 +876 1 4 → 568 +876 1 1 44 → 568 +876 1 1 1444	568 +876 1300 130 14 1444	
move right to left add ones, carry 1 to above left; add tens, carry 1 to above left usually add carry to top number, remember that number while adding it to bottom number	move right to left 1 new group goes below in answer space, keeping total together add 2 numbers you see, then increase that number by 1 to add the new group	can be done in either direction add each kind of unit first, then add those totals	Stage 1: Sustained linking of quantities to written algorithm to build understanding of quantity meanings Stage 2: Only do numerical algorithm but occasionally explain using quantity words (thousands, hundreds, tens)

 move right to left alternate ungrouping and subtracting	**Ungroup Everything First (As Necessary) Then Subtract Everywhere**   left-to-right ungrouping right-to-left ungrouping Do all ungrouping, in any order, until every top number is larger than the bottom number. Then subtract each	



Multidigit Multiplication and Division

Figure 4