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CHAPTER 10

MULTIDIGIT ADDITION AND SUBTRACTION METHODS INVENTED IN SMALL GROUPS AND TEACHER SUPPORT OF PROBLEM SOLVING AND REFLECTION

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A strong national effort is currently under way in the United States to reform school mathematics instruction, so that children become autonomous problem solvers (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 2000). One recommended change is encouraging students to invent their own solution procedures for problems or calculation tasks. This is important for fostering their mathematical power or proficiency, which involves an understanding of mathematics, the thoughtful and fluent use of procedures, the capacity to engage in mathematical inquiry, and a positive disposition toward learning and applying mathematics. (For discussion of how, see Ambrose, Baek, & Carpenter, chap. 11, this volume; Baroody, chap. 1, this volume; Baroody, with Coslick, 1998.)

There is presently, however, an insufficient research base for understanding the range of solution procedures children can invent in a socially and cognitively supportive environment or of what supporting roles teachers should play in promoting such inventions. Most previous work on children's invented multidigit addition and subtraction procedures was limited to two-digit numbers (for reviews, see Beishuizen, Gravemeijer, & van Lieshout, 1997; Fuson, 1990; Fuson & Smith, 1997; Fuson, Wearne, et al., 1997; Labinowicz, 1985). Unfortunately, many of these procedures do not generalize well to larger numbers or depend on special solutions for the decades.

In this chapter, we report on an investigation into the mathematical thinking of high-achieving second graders as they worked in groups, learning to add and subtract horizontal four-digit symbolic expressions using base-ten blocks and written marks. Our focus is on conceptual learning of multidigit concepts and methods and on how to support it. We begin the chapter by describing the study. Next, we discuss the educational implications of its results. The most important is that children can invent efficient written procedures, some of which are conceptually or procedurally superior to the standard school-taught algorithms. We end the chapter with some general conclusions about the teaching and learning of multidigit addition and subtraction. Our key conclusions are: (a) Teaching tools such as using cooperative learning groups and manipulatives (e.g., base-ten blocks) must be used carefully and thoughtfully to promote conceptual learning. In regard to the former, teachers must actively help children learn how to use small-group discussions as a problem-solving resource. In regard to using manipulatives, it is particularly important that children be encouraged to consider each step in their concrete block procedure when inventing (or attempting to understand a written procedure) and connect each

Some of the methods children invented in this study were used in the later classroom research of the Children's Math Worlds Research Project in which children made drawings of hundreds, tens, and ones instead of using base-ten blocks.

concrete step with the corresponding step in the written procedure. (b) In contrast to the rigid use of algorithms learned by rote, conceptual learning promotes adaptive expertise and flexibility.

The Study

In this section, we discuss, in turn, the theoretical perspective, methods, and results of the study.

THEORETICAL PERSPECTIVE

The theoretical perspective for this chapter is the analysis of children's conceptual structures for understanding English number words, multidigit written marks, and multidigit addition and subtraction described in Fuson (1990), extended in Fuson, Wearne, et al. (1997) and in Fuson, Smith, and Lo Cicero (1997), and summarized in Fig. 10.1. Children's construction of the connected network of conceptual structures identified in those studies was supported in the present study by the use of base-ten blocks, which can represent units of one, ten, hundred, and thousand that can be added, subtracted, and traded (units are represented by a "cube" 1 cm x 1 cm x 1 cm; tens, by a "long" 10 cm x 1 cm x 1 cm; hundreds, by a "flat" 10 cm x 10 cm x 1 cm; and thousands, by a "large cube" 10 cm x 10 cm x 10 cm).[1] These units can be described in block words and English number words and can be recorded as multidigit written marks. The term "written marks" is used to remind the reader that the meaning of multidigit numerals at any moment for any viewer of those numerals may differ from the meanings for those numerals used by the expert readers of this article.

Fuson (1990) identified two different unitary conceptions commonly used by children in the United States. One is a total unitary conception (e.g., viewing the numeral 26 as a pile of "twenty-six" units); the other, a concatenated single-digit conception (e.g., viewing 26 as a unitary 2 next to a unitary 6). A third common conception is a sequence-tens conception (e.g., viewing 26 as "ten, twenty, twenty-one, . . . twenty-six"). A productive meaning of multidigit numerals that is used less often by U.S. children is a multiunit conception in which each digit represents the number of ones, tens, hundreds, thousands, and so forth. Although many children can name these digit positions, only some use these quantities in their conceptions of multidigit numbers, and fewer yet apply them to their addition and subtraction methods. Base-ten blocks were found to be helpful in facilitating children's constructions of multiunit conceptions used in understanding traditional algorithms (Fuson, 1986a; Fuson & Briars, 1990). This study extended the use of base-ten blocks to supporting children's invention of multidigit addition and subtraction methods.

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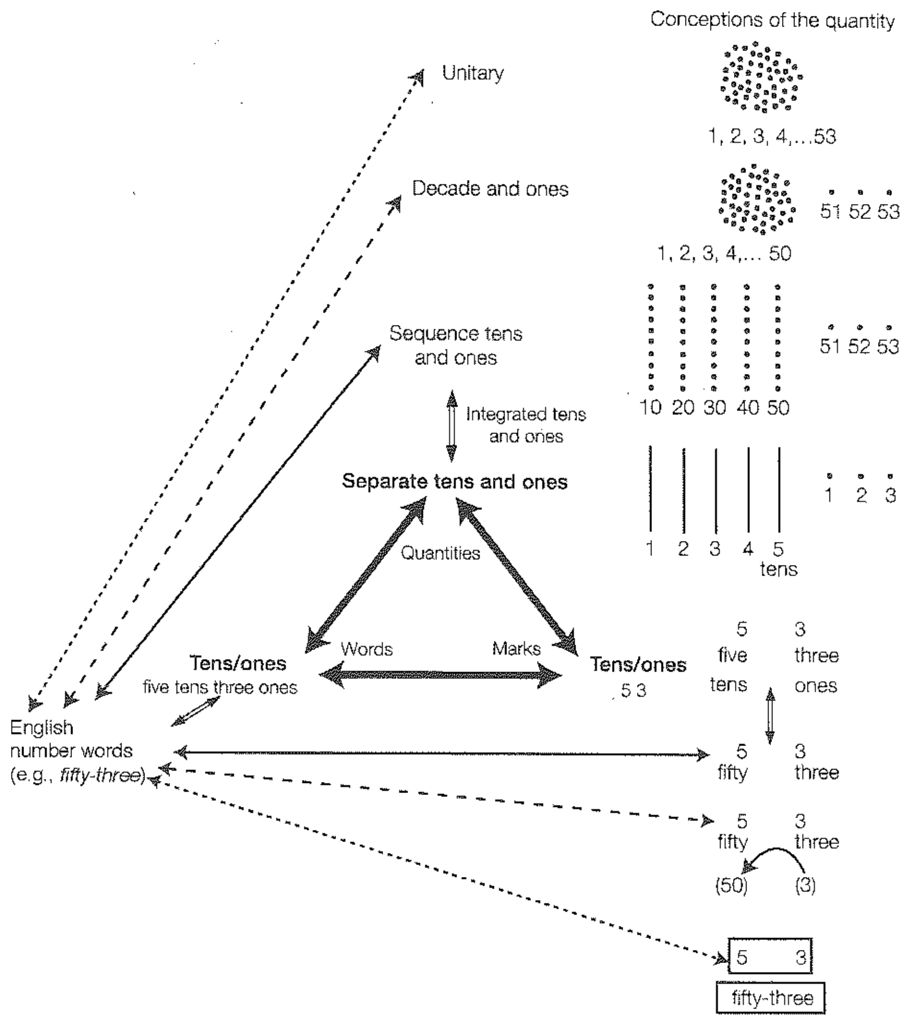
[1] Base-ten blocks are one useful manipulative for fostering grouping and place-value concepts and skills. A model that requires children to create (compose) their own group of ten from units or decompose a group of ten into units is somewhat more concrete than that with base-ten blocks, in which the units of the ten are already "glued" together. For a more complete discussion of grouping and place-value models, see Baroody (1990) or Baroody, with Coslick (1998).

Figure 10.1: Children's Conceptual Structures for Understanding English Number Words (Fuson, Smith, & Lo Cicero, 1997)



METHOD

Participants

The participants were the 26 students in the highest achieving class of three second-grade classes in a public school in a small city with a heterogeneous population. The study was done in the fall before they had received any formal multidigit addition or subtraction instruction. In first grade, about a third of the children had used base-ten blocks to model place value but not multidigit addition or subtraction. Written pretests of multidigit addition and subtraction were administered and followed by pretest interviews to assess place-value knowledge and conceptual understanding of multidigit addition and subtraction. The students were assigned to high-, medium-, or low-achievement groups on the basis of their pretest scores; there were two groups at each level. Because gender has been found to affect interaction in small groups (Kimball, 1989; Peterson & Fennema, 1985), the groups were also balanced as much as possible by gender. All but two groups contained 2 girls and 2 boys; both remaining groups had 2 boys and 3 girls. On the pretest, children ranged from solving no addition items correctly (6 children) to solving all three vertical (two-, three-, and four-digit items) and both horizontal (two- and four-digit) items correctly (4 children). Participants showed a similar range in place-value knowledge and conceptual explanations for two-digit and four-digit trading and alignment of uneven items (those with addends that had a different number of digits).

Training

One whole-class session on working together in groups preceded the small-group sessions. The latter consisted of reading a story that posed an interesting problem to be solved by the children in a group. While working on the problem, participants were guided to use the following four skills: conciseness, listening, reflecting on what others have said, and being sure that everyone gets a chance to contribute. This was done according to a plan laid out in "Epstein's Four Stage Rocket," a task found in Cohen (1984). Then, during the study, to help distribute students' power in their groups, official "leader" and "checker" roles rotated daily among the children in each group.

Three groups (one at each achievement level) then worked in large stair-landing spaces outside the classroom for 14 consecutive group sessions. Meanwhile, the teacher continued with other topics with the remaining students in the classroom. The same training was then undertaken with the remaining three groups, while the first three groups worked in the classroom. Class periods were 40 min in length, but effective learning time was only about 30 min because of set-up and clean-up time.

For each group, a supervising adult guided the initial 2- to 3-day experience with the base-ten blocks. This ensured the children chose names for each kind of block, knew the English names for the blocks, understood the relations between block arrays and the usual four-digit marks, and understood the ten-for-one trades for next larger blocks. Versions of the English words that regularized the words for the tens place (e.g., "four thousand five hundred *six ten* seven") were also used and practiced during this time. These words are modeled after the regular number words used in China, Japan, Korea, and Taiwan (e.g., Fuson & Kwon, 1992; Menninger, 1958/1969). These words potentially support children's construction of, and easy reference to, multiunit conceptual structures. The first three groups used digit cards to show their marks values for blocks and for addition and subtraction. These digit cards were index cards with a numeral written on each one; they were stored in a small box by numeral value. Because these were very time consuming to find and arrange (and because children often did not link blocks and digit cards), the second three groups wrote their block values and their addition and subtraction problems on a "magic pad." This was a large pad on which actions with the blocks had to be recorded as soon as the actions were done (children were to "beep" if such recordings were not done). Recording was done with a marker so that a permanent record could be obtained and analyzed. In both sessions, children also wrote on individual papers after the first several days of addition or subtraction.

In the addition phase, children were given horizontally written four-digit (and sometimes a three-digit plus four-digit) addition expressions to solve with the blocks and written multidigit marks. They were asked to link their block and mark solutions by recording everything they did with the blocks with the digit cards (first three groups) or the magic pad (second three groups). After a group had devised at least one correct addition method with the blocks and the marks, and most group members could carry out this method, that group moved on to subtraction. Subtraction was also presented as horizontally written four-digit (or three- and four-digit) numerical subtractions. The first additions required a trade from the thousands to the hundreds and a trade from the tens to the ones but not a trade from hundreds to tens. Later problems involved all three trades and various combinations of just two trades. The two high-achieving groups reached the last and most difficult task, subtractions with zeros in the top number. Groups spent from 5 to 8 days on addition and from 3 to 8 days on subtraction. The school allowed us only 14 sessions with each group. Therefore, the middle and low groups left addition before their lowest members fully understood it, and no group had sufficient time for every member to understand and be able to explain subtraction thoroughly, as judged by behavior in the group sessions.

An adult oversaw the videotaping of each group, took notes, and intervened when children's behavior became too rowdy or when the group became stuck on a mistaken procedure for a long time. (The criterion for a long time for the second three groups was one whole period; this time was sometimes longer for the first three groups.) Adults intervened minimally to provide children the opportunity to resolve conflicts and solve problems creatively and to approximate the environment of a real classroom in which a teacher would be able to observe and help any one group only a fraction of a period.

Data Collection

The videotapes of the group sessions were transcribed to provide verbatim records of mathematical statements made by children or adults, summaries of off-task discussions, descriptions of actions by the children, and descriptions of emotional states and interactions. Another transcriber checked each transcript.

Separate drawings keyed to the transcripts were made of all movement of blocks and writing of multidigit marks with digit cards, on magic pads, and on individual student papers (originals of the latter two were used for this summary).

RESULTS

In this section, the results regarding children's construction during the preaddition phase of relations among English number words, base-ten blocks, and written multidigit addition are briefly summarized to provide a context for the addition and subtraction results given next (for a full report, see Fuson, Fraivillig, & Burghardt, 1992). Next, the invention of multidigit addition with blocks and marks is briefly summarized across groups (for a fuller report of group interactions, see Burghardt, 1992, and Fuson & Burghardt, 1993). Incorrect addition methods and then correct ones are described and discussed. Subtraction is treated similarly (for a fuller report of subtraction group interactions, see Fuson & Burghardt, 1997). The supporting roles teachers played are mentioned where relevant throughout this section (and summarized in the next section on educational implications).

Summary of the Preaddition Phase

The names of the blocks chosen by each group related to perceptual features (mostly shape and size); food names were common. Children easily found the ten-for-one equivalencies between the little cubes and the longs, the longs and the flats, and the flats and the big cubes. They made few errors with respect to these equivalencies throughout the preaddition phase. Many verbal statements of the children about equivalencies were quite abbreviated and not very helpful to other group members ("ten of those"). Children were accurate in using English words and block words. They learned the regularized (East Asian based) tens words readily, though individuals varied in the extent to which they spontaneously used them in subsequent discussions. Adults did not support the use of these regularized tens words, because they were trying to intervene as little as possible. Children easily established relations among blocks, English words, and written marks. The need for zero arose in all groups and was successfully resolved. Some groups grappled with the issue of how to write block arrays that had more than 10 of one kind of block; the adult leader of other groups suggested that children only make block arrays with less than 10 of each kind of block. Some children regularized the English words even beyond the regular tens forms. They added the unit word *ones* so that every unit would be named. Such children also used zero with a multiunit or unit name, rather than omit that multiunit as English does (e.g., "three hundred zero tens four ones" instead of "three hundred four"). The rapidity of learning and low level of errors in this phase were partly due to the presence in most groups of a child who had used the blocks in first grade.

Addition

Overview of Block and Mark Multidigit Addition. In spite of the adults' initial instructions to link the blocks with written marks during addition, children in five of the six groups often functioned in two separate worlds—the blocks world and the written-marks world (see Thompson, 1992, for a similar result). Children quickly devised correct addition methods with the blocks (see Fuson et al., 1992, for more details). The collectible multiunits in the blocks were sufficient in most cases to direct correct block methods and constrain incorrect methods. These visually salient multiunits suggested solutions to the following two major components of multidigit addition: (a) what to add to what (the items were written horizontally and so did not cue adding like places) and (b) what to do when you have more than nine of a given unit. Every group immediately added like blocks to each other. They all eventually arranged the blocks for the second addend so that like blocks were below each other. Some groups pushed together blocks to make the answer, whereas other groups used extra blocks to make the answer in a third row below the two addends (simulating a procedure with written numbers). All groups added the blocks from left to right. Groups solved the second component (having more than nine of a given multiunit) in various ways. All, though, traded 10 of one kind of block for 1 of the next larger unit or put the 10 blocks in the next column to the left (often arranged spatially so that these blocks looked like one of the next larger block). Where and when they traded or put blocks varied, and the variations led to different written marks methods (described later). Children also often solved the single-digit addition of a unit mentally or with fingers rather than counting the blocks.

Necessary preaddition skills (counting blocks, copying the expression, writing digits) rarely caused difficulty, except that some magic-pad records were quite messy and difficult to read. However, children sometimes played with blocks from a model, thus changing the quantities added. Also, blocks used to represent addends occasionally merged with the bank of currently unused blocks. Such unintended block changes sometimes were responsible for long, unproductive digressions.

In the independent-marks world (marks methods not connected to blocks methods), different incorrect methods were used, some for several days. (Examples will be described next with the incorrect addition methods.) All but one of these methods involved a conception of written marks as concatenated single digits. In this conception, no multiunit values (e.g., tens, hundreds) were attached to the different marks' positions, so no information (other than, for some children, partial memory of two-digit addition instruction) was available to direct or constrain the placement of marks.

All of these incorrect block and mark methods were corrected by the groups when the adults insisted that the participants link the blocks and written marks. This linking enabled children to use the multiunit value of the blocks to identify erroneous methods and direct a correct trade.

Children's descriptions or explanations that used block words or multiunit names often facilitated correcting erroneous methods. Initially, though, participants ordinarily carried out addition with blocks or marks with little or no discussion or explanation. Consequently, it is likely that the different group members were not always aware of all of the important aspects of each other's activities. Furthermore, spontaneous descriptions or explanations often omitted block names or multiunit names. This frequently led to confused discourse (e.g., "ten," if said alone, could refer to one ten, ten ones, or ten of another multiunit) and to the arousal of concatenated single-digit conceptions in the listener. Imprecise language used to describe block actions also sometimes suggested wrong marks methods. Therefore, the adult leaders asked children to explain methods using block or multiunit names. This resulted in a more complete and accurate explanations by children and seemed to help the other members of a group.

Children rarely discussed issues of mathematical importance, even though they arose at least implicitly. For example, having more than nine of a given multiunit is only a problem when writing the answer in written marks. Specifically, if more than nine is written in a given column, this will make the whole multidigit number one digit too large. In contrast, noncanonical (more than nine of a given multiunit) block arrays or English number word forms are not ambiguous (we even say *fifteen hundred* as an acceptable alternative for *one thousand five hundred*). Only one group, however, said clearly why writing two digits in a given column is problematic. This group had wrestled with this issue in the preaddition phase when writing noncanonical block arrays. For example, when recording three flats, four longs, and 29 units as 3429, they said, "How come when Dh writes it on the magic pad it looks like *three thousand*?" (instead of three hundred . . .).

Consider a second example of the common failure to discuss mathematical issues. Several groups had controversies (some extended over days) about whether to add from the left or from the right. Without adult support, no group decided to compare these methods. Instead, the majority or a dominant member made an arbitrary decision. When the adult in one group suggested comparing the two methods and discussing them, the children did carry out both methods and engage in a good discussion. They decided that it was better to add from the right "because it was faster." (In that direction, they did not have to cross out their answers so often to increase them by the new traded multiunit.) Teacher support to identify and discuss such issues of mathematical importance seems to be necessary, at least until children are experienced in doing so.

The path of an individual group through multidigit addition was influenced by the extent to which the dominant and more socially skilled individuals in the group possessed and used conceptual multiunit knowledge they had learned before the study (see Fuson & Burghardt, 1993, and Fuson et al., 1992, for group-case studies of multidigit addition). Although the leader role was effective in that every child in the leader role talked more when being the leader than on preceding days, natural leaders, or at least dominant group members, also emerged for most groups. In some groups, a dominant group member imposed an incorrect method. In such cases, it was necessary for the adult to support less dominant group members by asking them to explain their view about these incorrect methods or about alternatives they were trying to propose. These children were unable or unwilling to confront the dominant member without such adult support.

Incorrect Addition Methods. Seven incorrect addition methods used with written marks are shown in Fig. 10.2. All of the methods are variations of the standard addition algorithm in the United States. In the standard U.S. method, one begins adding from the right, and any two-digit sum results in a small 1 being written above the next column to the left. Many students had been shown this multidigit addition procedure in first grade or by parents or siblings (facts

Figure 10.2: Incorrect Invented Multidigit Addition Methods

*A) When adding left to right, write 1 above next column to the right

```
   1 1
 2 8 3 4
+1 9 6 3
 3 7 0 8
```

B) Write all 1s above the far-left column

```
 1
 1
 1
 1 8 3 6
+3 5 9 7
 6 3 2 3
```

C) Write one 1 above far-left column; can't have more than one 1, so only write one 1

```
 1
 1 8 3 6
+3 5 9 7
 4 3 2 3
```

D) Write the 1 in the column that was added

```
   1 1 1
 1 8 3 6
+3 5 9 7
 3 3 2 3
```

E) Write 1 below the column, write right digit of the teen sum above the next column

```
4   3
 1 5 2 7
+1 9 4 6
 6 1 9 1
```

F) Keep 9 (because "you can't have ten in a column"); write the rest above the next column to the left

```
                       1
                       1
                       1
 4 6 4                 1
 1 2 4 8         1 2 4 8
+2 5 7 5        +2 5 7 5
 7 9 9 9               9
```

G) Each two-digit sum has tens and ones. Put the ten in the tens column and the ones in the ones column [or in its own column for hundreds]

```
                              1
     1             1 5        1 5
 3 9 6 7  -->  3 9 6 7  -->  3 9 6 7
+1 5 8 5      +1 5 8 5      +1 5 8 5
       2           1 2       4 4 1 2
```

Note. All methods except those with an asterisk were done from right to left.

that emerged during the study). However, errors arose because they did not understand the standard algorithm and either misremembered this procedure (remembered it incompletely or inaccurately) or misgeneralized it. Sometimes versions of "teacher (or sister) rules" were used to justify a method.

The first five incorrect methods (Methods A to E in Fig. 10.2) involved writing the 1 somewhere other than above the next column to the left. Children's discourse about these methods involved concatenated single-digit language; they spoke about writing 1s someplace, not about adding multiunit quantities. The last two incorrect methods (Methods F and G) involved other types of invented procedures.

- *Method A.* This method involved adding left to right and writing 1 above the next column to the right. Method A arose in several groups because it modeled their left-to-right addition with blocks (which all groups did). However, because the addition with marks tended to be carried out independently from the addition with blocks, children did not use the block multiunit values to direct their mark trading. When adding 8 flats and 9 flats, for example, children always put 10 of the flats with the big cubes or traded them for a big cube, which was put with the existing big cubes. In both cases, they put the newly formed "group of a thousand" to the left of extant thousand blocks. In contrast, with no perceptual multiunits to direct the marks procedure, some children interpreted the standard procedure as "write the little one in the column *you are going to add next*" (the next column to the right), rather than as the column to the left (the column 10 times larger).
- *Methods B and C.* Both of these methods involved interpreting the standard two-digit algorithm as "write the 1 above the *far-left* column," rather than as "write the 1 above the *next* column to the left." Whereas Method B involved writing multiple 1s above the far-left column, Method C involved writing a single 1. Note that the latter also involved a second rule about adding ("You can't have more than one 1 in a column."). VanLehn (1986) considered a related subtraction error (always borrow left). As VanLehn discussed for the subtraction version of this error, when learned without multiunit quantity meanings, there is nothing about the standard two-digit addition algorithm that permits a child to distinguish these alternatives. From two-digit problems one cannot tell whether one puts the 1 above the next-left or far-left column. Similarly, with respect to Method A, one cannot tell from watching the standard algorithm being carried out whether the 1 is written above the next column you are adding or the column to the left. Methods B and C were used by several students until they were requested by the adult to link block and written mark methods. These incorrect procedures then immediately disappeared.
- *Method D.* This method involved writing the 1 above the column that was added. Unlike the common early multidigit error "vanishing the one" (Fuson & Briars, 1990), in which children do not know what to do with the 1 and so ignore it altogether (vanish it), Method D reflects some knowledge of the standard algorithm. Unfortunately, Method D results in the same answer as "vanishing the one" because the 1 is written above a column that has already been totaled and is, thus, never included in the sum.
- *Method E.* In this method, the places of writing the numbers in a two-digit partial sum are reversed. That is, the 1 in a partial sum such as 13 is written below the column, and the ones digit (the 3) is written above the next column to

the left. This reversal error might occur more often if it were not for the fact that the addition of two numbers always results in a tens digit of 1. The pattern of seeing only 1s above a column is highly salient, and so many children may adopt this procedure even though they do not understand it. Methods D and E were infrequent.

- *Method F.* Imposed by a dominant girl, this method persisted for several days in one of the two groups in which it arose. The dominating girl quashed the frequent questions raised by all group members (including herself) by restating the following partially remembered teacher rules: "You can't have ten in a column" and "You can't have more than nine in a column." In one group, this rule came from an older sibling, as well as from a teacher. The restatements in both groups transformed this rule into *"you must have 9 (keep the 9)"* and then the rest were put above the next column to the left, an imitation of the standard algorithm. The amount over nine was written with a stack of 1s in one group to conform to the standard algorithm's writing of 1s. In one group the amount written or put above the next column to the left was sometimes the amount above 10, but 9 was still written below the column "because you can't write 10 there." In the group that most persistently used it, application of the method seemed to be facilitated or prompted, by the block-trading method and the language the girls in the group used to describe it. Everyone in this group almost always used concatenated single-digit language (no block or multiunit words). The girls, who were the leaders in this group, would find the sum of a given multiunit mentally or with fingers. For example, when adding pancakes/hundreds, they would say, "six plus seven is thirteen." They then counted 3 of these 13 pancakes to indicate the number of hundreds and, without comment, simply replaced the other 10 pancakes with a big cube. The girls not only failed to connect discarding 10 units (here, 10 pancakes) to adding a one new multiunit (here, a big cube), they described the carrying process as "take away" (e.g., taking three from the total 13 pancakes). As a result, for a long time, the boys did not realize 10 blocks were traded for a larger block. Indeed, the whole group continued to describe this trading as "take away *x*" (the number of ones in the teen sum).
- *Method G.* This method was invented and pushed by a dominant child who had strong understanding of two-digit numbers as tens and ones. However, for two-digit totals of tens and hundreds, he did not simultaneously consider the multiunit values (e.g., the sum of 1 plus 6 + 8 in the tens place was interpreted as 15, not 15 tens). His two-digit tens-and-ones conception of the written marks directed a modification of the standard algorithm in which the tens in any column sum were to be put with the tens, and the ones were to be put with the ones. Most of the children in the group found his basic argument to be sensible (put the tens with the tens and the ones with the ones), and their conceptions of two-digit numbers as tens and ones were strong enough to support this method. Other children in the group offered somewhat different variations of this procedure, and although the group spent several days using Method G or a similar method, the children seemed uncomfortable with these methods. In particular, when the sums of the hundreds or thousands columns were greater than 9, the ones digits for these columns were sometimes written in these columns to provide answers there, rather than in the ones place as their procedure demanded. The adult finally asked the group to link block and digit-card solutions and describe their methods in block words. The children immediately saw why their

method did not work: "You have fifteen *long legs* or fifteen tens, not just fifteen, so it is one iceberg and five long legs [one hundred and five tens], not one long leg and five little guys [one ten and five ones]." Method G arose again, however, on the following day when they were using English words and not block words. The adult supported a child who objected to the group's return to this method but whose arguments were neither being accepted by the dominant member nor listened to by others. Once her arguments were heard, everyone, including its inventor, rejected Method G.

Correct Invented Addition Methods. The standard U.S. algorithm and the six correct methods invented by the groups are shown in Fig. 10.3. Because every group contained at least one member who knew the standard U.S. algorithm for two-digit numbers, this method strongly influenced the other correct methods used by the groups. The groups linked all of these correct methods to a parallel block solution. In each case, the written form captured what was done with the blocks. Methods I through N were each done by one to three groups.

Figure 10.3: Correct Invented Multidigit Addition Methods

H) Standard U.S. algorithm; add right to left; put the 1 above;

```
   1 1 1
   2 6 8 5
 + 1 9 6 7
   -------
   4 6 5 2
```

I) Write 1 within total location

```
   2 6 8 5
 + 1 9 6 7
   -------
   1 1 1
   4 6 5 2
```

J) Add 1 into top addend

```
   3 7 9
   2 6 8 5   (2, 6, 8 crossed out)
 + 1 9 6 7
   -------
   4 6 5 2
```

*K) Mentally add in 1

```
   2 6 8 5
 + 1 9 6 7
   -------
   4 6 5 2
```

*L) Add left to right; write 1 above and fix answers as go

```
   1 1 1
   2 6 8 5
 + 1 9 6 7
   -------
   3 5 4 2   (3, 5, 4 crossed out)
   4 6 5
```

M) Add all multiunits first; fix total to standard numbers

```
   1 1 1
   2 6 8 5
 + 1 9 6 7
   ---------
   3 15 14 12            1  1  1
   4 5             or    3 15 14 12
   4 6 4                   x  x  x
   4 6 5 2               4 6  5  2
```

N) Write problem in multiunit values

```
   1   1   1
   2000600805
 + 1000900607
   ----------
   4000600502
```

Note. All methods except those with an asterisk were done from right to left. Method K was done in both direction.

• *Method H.* In the blocks version of the standard U.S. algorithm (Method H), whenever the total of a given kind of blocks was 10 or more, 10 of the blocks were traded for one of the next larger multiunit. This new block was then put above the blocks of that multiunit for the first addend, or 10 of the blocks were put there, often laid in the form of one of the new multiunit blocks. Thus, the blocks showed the meaning of the carried 1 in the standard algorithm as one of the next multiunit, which was made from 10 of the multiunit to the right. Children in all groups eventually described these block trades or block puts using block or multiunit names so that the trading involved was clear, though several groups did not do so spontaneously and needed the suggestion of the adult that they do so. In the traditional method, that 1 is mentally added to the top number, and then that number is added to the given second addend. One could, however, add the two given numbers in any order (e.g., add smaller to larger) and then add the 1 to that number. However, in several informal surveys (by the first author) of university students from all parts of the United States, almost all of them added in the right-to-left order as they were taught to do. Those who added in a different order said that they had invented their procedure. Such adding is quite difficult because you have to keep a new number in mind and add it even though it is not written there.

• *Methods I through N.* All of the invented methods (Methods I through N) are superior in some way to the standard algorithm, though some also have limitations that the standard method does not have. They all stemmed in some way from doing multidigit addition with the blocks. Some methods solved difficulties some children had with the standard algorithm, though none of them were direct responses to such difficulties.

Method I is superior to the standard U.S. algorithm conceptually and procedurally. In the standard algorithm, one adds part of the sum into the top addend (i.e., adds part of the answer back into the "problem"). By doing this, one continually changes the problem as one solves it. Some of the children in our study were confused when other children did this standard U.S. method with blocks (traded or put a block into a top addend pile of blocks), saying that people were changing the original problem. A couple of children also objected to this aspect of the standard written algorithm for the same reason. In Method I, each addend and the total have their own horizontal space; the total is always below the line. Thus, one is truly *regrouping the total* (moving part of it over to await adding it with that multiunit's total).

Method I is also procedurally easier than the standard U.S. method because it puts fewer demands on working memory, particularly for children who must compute single-digit sums. In adding the tens digits of 2,685 + 1,967, for instance, Method I involves adding 8 and 6 and then the resulting partial sum 14 and 1, whereas the standard algorithm involves adding 1 and 8 and then the partial sum 9 and 6. Note that, with the former, the addends of the more difficult computation (8 + 6) are visually present, whereas with the latter one addend (9) is not. Furthermore, if the more difficult computation must be redone (e.g., because of a perceived error) or is delayed (e.g., by a difficulty with the computational process or a distraction), children using Method I have a readily available record of the original item, whereas those who use the standard procedure must either compute 1 + 8 = 9 again or try to remember this partial sum. Note also that the more difficult computation in the case of the former (8 + 6) is slightly less difficult than that of the former (9 + 6). Perhaps more critically is when the more difficult computation in each case is done. For Method I, a child must store the partial sum for 8 + 6 (14) in working memory only briefly, because the next step is simply incrementing it by one, a relatively automatic process (Baroody, 1987). With the standard algorithm, a child must store the partial sum of 1 + 8 (9) in working memory and then tackle the more difficult computation of 9 + 6. Method I also makes it easier, particularly for developmentally less advanced children, to add two digits in any order (i.e., count- or add-on from larger) with less risk of forgetting to add the 1.

Method J is similar conceptually to the standard U.S. algorithm, but it makes the addition within a column easier. As with a blocks model, when the newly traded block is actually added into the pile, the traded 1 is added into the top digit, and this total is written above the original top digit. With Method J, children do not have to remember partial sums but simply add the two numbers they see. This method was taught to children in the Fuson (1986a) study to simplify the single-digit additions with base-ten blocks.

Children in two groups also added with blocks in a method related to Methods I and J. This method was never recorded in marks, so it is not listed in Fig. 10.3. In this method the traded block was put with the second addend blocks. This could be recorded by increasing the bottom number by one. This recording is as easy to add as is Method J. It might, however, be messier because there is not much room to write the increased number. Alternatively, the traded block could be written as a 1 below the second addend. Then, it would be close procedurally to the simple Method I (the two visible numbers would be added and the total increased by one), but not as robust conceptually because the trade is still being put into one of the addends.

Method K was used in two types of situations. In one group, it was a version of the standard U.S. algorithm in which the trades were not recorded. This method may have evolved in this group partly because the children would take away the blocks and the digit cards representing the addends and replace them with blocks and digit cards showing the answer. Although they did trades with blocks, they immediately took away the digit cards for that column and put in the answer card and never developed a method for recording their block trades with the digit cards. On individual papers, three of the four children did method K, adding right to left as in the standard algorithm and adding in the trade mentally. The fourth child used the standard algorithm, though he had not done so in the pretest.

Method K done with erasures also is a written version of the left-to-right blocks methods used in most groups. When a column addition required a "carry" to the previously computed column to the left, children erased the latter's original column total and then wrote the new total. Repeated use of this method might lead children to anticipate trades and record only the final sum for each column. This latter procedure is identical to the common European left-to-right look-ahead method: Add a column and then look ahead at the next-right column to decide if its sum is 10 or more; if so, the total of the first column is increased by 1 and recorded.

Method L was the other way in which children represented their left-to-right block method with marks. This method was similar to Method K, but using cross-outs instead of erasures and with the carried (traded or "put") 1 written above the appropriate column. Occasionally, children also wrote hybrids of these two methods, like Method K but using cross-outs instead of erasures or, like Method L, using erasures instead of cross-outs. When adding left to right, writing the small 1s above each column to show the traded or put block probably arose from children's exposure to the standard U.S. algorithm. Because the column was already totaled, increasing the total by one was simple, so there was really little need to record the 1 anywhere. In the standard algorithm, it is more important to record the 1 lest one forget while adding that column that a 1 had been traded or put from the previous column's sum. In Method L, it also would make more sense to record the block-trade or -put total (below the line) because the addends for a given column have already been used. As with Method I, this method is simpler than the standard U.S. method, because the 1 is added at the end (the column has been totaled and the ones digit in the total is written by the time you find out from the next-right column that a 1 is added). If the 1 were written with the total, Method L would be conceptually superior to the standard algorithm in the same way as Method I is. More important, both Methods K and L capture the strong tendency of children to move from left to right in adding the blocks and in other methods invented for two-digit addition (e.g., Kamii, 1985, 1989). Adding from the left is also beneficial because it supports children's competence with estimation by encouraging them to think about the largest digits first.

Method M was the result of an extended collaborative group effort in which a marks method arose from addition with the blocks without interference from the standard algorithm. This group worked well together, and they formed and used the social norms of including and explaining to all group members. They were the only group that continually linked block and marks methods. Their method separates multidigit addition into its two component processes. First, one adds like multiunits. The group did so by pushing the blocks of a particular size together and then recording them. Adding like multiunits can be done in any order, and these children did add and record the blocks addition in any order. The second component was to fix the answer into its standard written form. This was done by trading 10 of any block for one of the next-larger blocks and recording this trade. This also can be done in any order, and children did trade and fix in various orders. This method is conceptually robust, because it clearly shows the components of multidigit addition. It is also procedurally simple; one does only one kind of simple step (adding or trading/fixing) at a time and does not alternate between two kinds of steps. Writing unfixed answers and not fixing them is a common early error of children doing multidigit addition (e.g., Fuson & Briars, 1990). However, these children understood that writing two-digit sums was only the first step in their adding method and that they had to change the sum into a standard written form so that others could understand it. This group had had an extended discussion in the preaddition phase about how to "fix" noncanonical block arrays, so no one objected to their unfixed sums when they first made them (see Fuson et al., 1992).

The group did need adult support in order to move from the marks method as a way to record blocks addition to being able to use marks without adding blocks. First, they could not always reflect on their written fixing method, because their mark recordings were sometimes extremely messy, with the sums written all over the page. The adult asked them to make all future recordings neater and then showed them examples of their neat unfixed and fixed answers. Second, she asked them how they could fix a marks answer without doing the block trading. The children responded with many descriptions of block trades without doing such trades, so eventually everyone could fix the sum without actually trading blocks, if they thought about the individual block trades. In response to repeated requests to think of a way to fix the marks without thinking through all of the block trades, two children invented a method in which they wrote a small 1 above the right-hand digit to the left and wrote an x below that fixed 1 (see the answer to the right for Method M). After each 1 in the sum had an x below it (i.e., after all trades had been made), the sum could be fixed in one step by adding in the traded quantities. Both parts of this fixing method could be done from the left or from the right or in any order and were done in various orders. These 1s and xs were described clearly as ten-for-one block trades, so it is apparent that the children understood these marks as multiunit quantities.

Over the 2 days following this linking intervention, the adult supported a transition from block use by encouraging the children to use only the marks but to describe them in block words. The final two magic pad solutions related their own group method to the standard algorithm by writing both. On the final solution, one child also wrote little xs below the 1s in the unfixed sum to show this relation: "to show what I put up there when I carried." The combination of good group interactions, social norms for understanding, linked blocks and marks beginning from the blocks rather than the marks, and support of an adult to reflect on their invented method produced insightful and conceptually sound inventions by this group.

Place-value notation in itself does not explicitly show the multiunit values of numbers. Method N clarified what kind of multiunits were being added by writing each one out as standard marks, for example, writing two thousand as 2000, six hundred as 600, seventy as 70. Method N shows the adding of ones to ones, tens to tens, hundreds to hundreds, and thousands to thousands. Children learning to write multidigit numbers often make the errors of writing number as they sound. For instance, asked to record *fifty-three*, they may write 503 (write 50 to represent *fifty* and then 3 to represent *three*; see discussions in Baroody, 1987; Fuson, 1990; Ginsburg, 1977; Seron, Deloche, & Noel, 1992; Sinclair, Garin, & Tieche-Christinat, 1992). Method N was not such an error, however, but was invented by a girl in one of the high-achieving groups to show their blocks addition more clearly. Writing a problem like this for discussion might provide a memorable perceptual support for children having trouble constructing or using any meaning other than a concatenated single-digit conception of written marks. This form also permits discussion of the multiunit values of the 1s traded to the left. Because this method is also potentially misleading, however, it is important to discuss how it differs from normal place-value numbers. Such discussion emerged in the group that developed this method, when a boy exclaimed about a number written out this way, "It's seven billion!" Alternatively, numbers may be shown in expanded notation, which is just like Method N but includes a "+" between multiunits.

Subtraction

Overview of Block and Mark Multidigit Subtraction. The subtraction methods in most groups were dominated by the traditional U.S. subtraction algorithm, which moves from right to left, alternating "borrowing" (trading of the multiunits in the next-left column for 10 more of a unit) and then subtracting. One member of all but one group initially or eventually remembered borrowing (or *regrouping*, as it was called by some children) and suggested it as a marks method. None of the children already knew a block method of multidigit subtracting with or without borrowing; they had to figure these out. Children invented either a block-*trading* method (e.g., removing a pancake from the hundreds column and adding ten licorices to the tens column) or a block-*putting* method (just putting a pancake with the licorices in the tens column).

Because we were limited to 14 days and had only about 30 functional min a day, four groups did not have enough time to explore subtraction thoroughly. Two groups did not even have time to finish subtraction without zeros in the subtrahend. Thus, it is not clear how many of the difficulties would have been overcome with more time, more linking of numeric methods to the blocks, and more thorough explanations. These factors were effective in producing high levels of understanding of some method of addition. We outline the difficulties experienced by the children, because these need to be addressed in any subtraction learning experience. We observed the following three major types of difficulties: (a) failure to describe the full trading process in clear language (describing clearly both taking away 1 multiunit and adding its equivalent of 10 next-smaller units), (b) directionality of subtraction (the words used to describe it and the physical direction of taking away), and (c) the conceptual integrity of the horizontal multidigit numbers versus the functional separation into vertical column subunits when children wrote the expressions vertically.

During work with the blocks, most children who began with some notion of borrowing connected this conception to multiunit quantities and could describe trades in block or multiunit words. However, most children failed to describe or justify spontaneously what they were doing with the blocks or written marks. Even when adult leaders asked them to describe trading, they usually did not give adequate and complete descriptions. This was problematic because some children only saw half of the trade. For instance, children in several groups objected vehemently to block trading because it "was adding and we aren't adding" (seeing only the adding in of the new ten blocks to the right). These actions may have been more noticeable than the taking away from or moving of one block. In some groups, children acquiesced to trading only when both aspects of the trade were described in block language (e.g., "She took one licorice away, and she put on 10 teeth"). Some children used taking-away language for both aspects of the trade, thus creating confusions.

The use of two-digit tens-and-ones language without block or multiunit names and the double use of "ten" as the number of traded blocks and as a multiunit value led three boys in two of the groups to misunderstand trading as always giving *a ten* (e.g., a licorice block) to each column. This is similar to the incorrect addition Method G in which the 10 from each teen sum was given to the tens column.

Some aspects of the noncommutativity of subtraction and the resulting need to attend to the direction of subtraction caused difficulty. English terminology for subtraction is complex, with different words stating the same subtraction in opposite directions (eight take away two, take two from eight, eight minus two, two from eight, eight subtract two). For single-digit subtraction, one does not have to attend to the language order closely; on hearing two numbers one merely subtracts the smaller from the larger number. But for multidigit subtraction, direction is important. In all groups, the previously described subtraction language was used in all possible combinations of intent and accuracy. For example, a child might say and mean "eight take away two," say "two take away eight" but mean "eight take away two," and so forth. In some groups there were extended discussions about the correct way to say phrases and, in others, little or no such discussion.

Some groups established strong spatial top-to-bottom directionality for subtracting (all groups aligned blocks and written marks problems vertically). In these cases, everyone functioned from a shared perspective that they were subtracting the bottom number from the top number. Most children in these groups were not confused by, or did not notice, reversed subtraction statements. In the group that had the most prolonged discussion about the direction of phrases, everything about subtraction directionality was problematic. These students had questions about the direction of subtraction in the original horizontal problem (left–right subtraction direction) and in the vertical problem written by the children (up–down direction), as well as prolonged controversies about how to say various phrases correctly. Other groups did not have difficulty with any or all of these aspects of subtraction directionality.

The blocks themselves neither inhibited common incorrect methods nor directed correct methods in subtraction as powerfully as they had in addition. Children who carried their vertical perspective from addition into subtraction and who had not worked out strong overall problem directionality based on location (taking bottom blocks from top blocks) often subtracted the smaller number from the larger (e.g., for 342 – 185 got 143). Nothing in the blocks themselves helps to compose them into their two different multidigit numbers or makes the larger of the two multidigit numbers especially salient. This might have been increased by the tendency to show both numbers with the blocks (done by five of the six groups) and use a takeaway, rather than comparison, meaning of subtraction. Children using a comparison meaning in Fuson (1986a) did not make this error.

Within the groups, individuals took away in different ways. Some children subtracted mentally or with fingers using unit-based methods and then just took blocks away until only that answer remained. Some took away the bottom blocks first and then took that number of blocks from the top blocks. Some children took away blocks from those representing the top number first and then took away the bottom blocks. In groups where no blocks were taken away and new blocks were put in to show the answer, subtraction was done mentally or with fingers and the answer was then represented with blocks.

Incorrect Subtraction Methods. Incorrect subtraction methods that arose in the groups are illustrated in Fig. 10.4. They are arranged roughly, from most to

Figure 10.4: Incorrect Invented Multidigit Subtraction Methods

A) Subtract smaller from larger in each column

```
  4 6 5 2
- 1 9 6 8
  3 3 1 6
```

B) Take as many as you can down to zero

```
  5 2 8 6
- 1 6 2 9
  4 0 6 0
```

C) Add instead of subtract

```
  1 1 1
  4 6 5 2
- 1 9 6 8
  6 6 2 0
```

D) Left to right, put 1 digit above each column

```
  4 1 1 16
  5 2 8 6
- 1 6 2 9
```

E) Not cross out borrow; add top numbers then subtract

```
      4 13
  2 5 6 3
-   9 7 8
      3 5
```

F) Do not do a middle borrow; subtract smaller from larger[a]

```
  3 16 4 12
  4 6  5 2
- 1 9  6 8
  2 7  2 4
```

G) Incomplete borrow, left column not decreased

```
  4 12   16
  5 2  8 6
- 1 6  2 9
  4 6  6 7
```

H) Forget to reduce dot column answer by one

```
  . . .
  4 6 5 2
- 1 9 6 8
  3 6 8 4
```

I) All borrows mental; forget a borrowed-from decrease

```
  4 6 5 2
- 1 9 6 8
  2 6 9 4
```

J) Add one column, subtract others

```
  3 12 7 16
  4 2  8 6
- 2 6  2 9
  5 6  5 7
```

K) Borrow tens to each column[b]

```
  1
  3 10 10 10
  4 0  0  0
- 1 7  3  5
```

L) Borrow a ten from thousands column

```
  3     10
  4 0 0 0
- 1 7 3 5
```

M) Borrow 1 thousand for each column[b]

```
  1000
  2000
  3000 1000 1000 1000
  4    0    0    0
- 1    7    3    5
```

N) Borrow 1 thousand

```
  3     1000
  4 0 0 0
- 1 7 3 5
```

O) Forget to reduce dot column answer by one; 0 thought of as a 10

```
  . . .
  4 0 0 0
- 1 7 3 5
  3 3 7 5
```

P) Bottom numbers ones and tens put on top, borrow as usual; several unclear steps (13, 3, 5)

```
       12
  3 13 3 15
  4 0  0 0
- 1 7  3 5
  2 6  3 5
```

[a] Previous problem had only a thousand-to-hundreds and a ten-to-ones renaming.
[b] Blocks showed a "fatty" (a large or thousands cube) put in each top column.

least severe. As with addition, most of these (Errors D to P) were incorrect variations of the standard U.S. subtraction algorithm.

• *Methods A and B.* Children in the United States frequently make the first error (Method A in Figure 10.4; Fuson & Briars, 1990; VanLehn, 1986). Children look at a given column and subtract the smaller from the larger digit without considering the order of subtraction. This method was sometimes done only on a single column, especially on a middle borrow (Method F). Children in the United States make the second error (Method B) less frequently than the first; some Korean children also make this error (Fuson & Kwon, 1992). With Method B, the child preserves the order of subtraction and subtracts as much as possible down to zero. For example, for 6 – 9, six of the nine are taken to use up all of the top number, but the child does not know how to take the other three and so does nothing. The result of taking 6 – 6 = 0, so 0 is written as the answer. In two groups, a child mentioned negative numbers as a solution to combinations such as 6 – 9, but the group ignored this suggestion. These children needed the support of an adult to pursue this notion. Others have reported children inventing such negative number methods in which the negative numbers for given columns are then subtracted from the next larger position (Davis, 1986; Kamii, 1989).

• *Method C.* Error C, multidigit adding instead of subtracting, usually seemed to occur when children looked at the familiar multidigit layout, ignored the sign, and continued to use the method they had used for the past several days (addition). Children sometimes also added one column while subtracting the others (Method J). These errors were very infrequent.

• *Method D.* Error D arose from trying to adapt the traditional algorithm to a left-to-right method without linking it to the block left-to-right method used by the group. Error D seems to be a mixture of writing the traded 1 above the column to the right, writing the decreased number to the right, and making the far-right number look like the traditional algorithm (a teen number). This method was rare.

• *Methods E, G, H, and I.* Four different errors (Methods E, G, H, and I) arose from not crossing out the column from which 10 of that value were taken and then forgetting that the column had been diminished by one. Methods H and I were made by a Chinese-American girl, whose parents had taught her the traditional Chinese method of putting a small dot above a column from which a unit had been taken. The girl usually carried out this method correctly but, occasionally after a trade, forgot to decrease the left-hand column by one (the thousands column in Example H of Fig. 10.4). Method I included a similar lapse in one column, in problems where trading was done mentally.

• *Methods K through P.* Methods K through P arose on items with all zeros in the top number. Only the top two groups worked on these items. Both of these groups tried several incorrect methods before finding one that worked. The children recognized that the Methods K through P were incorrect, but they could not invent correct methods during the first day.

Correct Subtraction Methods. Correct subtraction methods that arose in the groups are shown in Fig. 10.5.

• *Methods Q, Ri, and S.* Method Q is the standard U.S. algorithm. This method arose primarily from children who knew at least the two-digit version.

Figure 10.5: Correct Invented Multidigit Subtraction Methods

Q) Right-to-left borrow, rewrite new value above top digit, and then subtract (standard U.S. algorithm)	R) i) Alternate "one-step" trading and subtraction; ii) All trades first, then subtract	S) Special features to facilitate keeping borrows clear[a]	T) Draw dots to show the borrowed quantities and the top digits
15 14 3 5 4 12 4 6 5 2 − 1 9 6 8 2 6 8 4	3 15 14 12 4 6 5 2 − 1 9 6 8 2 6 8 4	|4|13|6|16| 5 3 7 6 − 1 6 2 9 3 7 4 7	5 3 7 6 − 1 6 2 9 3 7 4 7
U) Fix all top digits first, left to right, then subtract each column	V) Fix all top first, cross out but do top changes mentally before subtracting	W) Decompose borrowed-from columns into the ten for the multiunit to the right and the remaining digit of that multiunit[b]	X) Put a dot above a borrowed-from column, reduce its digit by 1, use teens in top digit when necessary
15 14 3 16 15 12 4 6 5 2 − 1 9 6 8 2 6 8 4	4 6 5 2 − 1 9 6 8 2 6 8 4	· · · 4 6 5 2 − 1 9 6 8 2 6 8 4	6·10 3·10 7 4 2 − 5 6 8 1 7 4
Y) Just record the decreased borrowed-from part; do teens increase mentally	Z) Do borrows mentally: i) decrease top digit by 1; ii) subtract digits, then decrease answer by 1; or iii) increase bottom digits by 1, then subtract		AA) Add a 1 (signifying 10 or 100) to top digit and to bottom digit in next-left column
4 6 5 3 7 6 − 1 6 2 9 3 7 4 7	4 6 5 2 − 1 9 6 8 2 6 8 4		4 $^{1}6$ $^{1}5$ $^{1}2$ − $^{1}1$ $^{1}9$ $^{1}6$ 8 2 6 8 4
BB) Redistribute the 1 thousand as 9 hundreds 9 tens and ten[b]	CC) Successive left to right gives to adjacent right column	DD) A dot method like W, know middle 9s mentally	EE) Give all tens, adjust for borrows, and subtract
3 9 9 10 4 0 0 0 − 1 7 3 5 2 2 6 5	9 9 3 10 10 10 4 0 0 0 − 1 7 3 5 2 2 6 5	· · · 3 9 9 10 4 0 0 0 − 1 7 3 5 2 2 6 5	9 9 15 2 10 10 10 3 0 0 5 − 1 6 4 8

[a] The magic-pad problem stimulating this method was extremely messy, with the columns of some borrowed digits not clear.

[b] Either a quantitative sharing or an anticipation and shortcut of Method CC.

Methods Ri and S are variations of this standard method. In both, after a column had been reduced by a trade, the new 1 ten coming from the left was written to the left of that number (to make a teen number), rather than that number being crossed out and the new teen number written above. Method S arose in response to a magic-pad problem with extremely messy writing so that the columns were not clear; children wrote the vertical lines to separate the columns in order to clarify the location of the traded two-digit numbers.

- *Methods U, Rii, and V.* Methods U, Rii, and V are all methods for fixing everything first in the top number so that borrowing occurs first in all applicable columns before subtraction begins. The fixing is done left to right in Method U and right to left in Rii (Ri and Rii look the same, but Ri alternates borrowing and subtracting). In Method V, the need for borrowing was noted in the top number at the start by crossing out all digits that would be changed by borrowing. These intended trades were then remembered as each column was subtracted, progressing from right to left. Method V arose after children had answered items with all zeros in the top number and thus had experienced fixing everything in the top number before beginning any subtraction. These methods are easier to carry out than alternating methods because the two major components of multidigit subtraction—trading to get more and subtracting—are each done all at once. Method U was taught successfully with base-ten blocks in Fuson (1986a) and Fuson and Briars (1990); it is less subject to occasional top-from-bottom errors.
- *Methods W and X.* Method W is the (correct) Chinese method of placing a dot above any column from which one has been given. Method X is the version that the children wrote the first time they used this method to show the meaning of the dot. This method is similar to the method presented in Korean textbooks, except that in that method the 10 is written above the next-right column (Fuson & Kwon, 1992). These methods that write a 10 support the take-from-ten method of single-digit subtraction taught in China, Japan, and Korea: for 14 – 8, take 8 from the 10 in 14, it is 2 that is then added to the 4 in 14 to get 6 as the difference.
- *Method Y.* Method Y is a clever abbreviation of the standard U.S. algorithm in which only the part that is difficult to remember is recorded. When borrowing, it is easy to remember the total in the increased column, because one is subtracting from it immediately. However, doing this subtraction may cause one to forget to decrease the next-left column by 1. The children who invented Method Y only showed this problematic part (the reduction of the left-hand digit in a trade); the rest was done mentally.
- *Methods Z and AA.* Example Z consists of three different methods done entirely mentally so that nothing extra is written. All three methods involve giving a block (or multiunit) to make 10 in the next right column, but they differ in the location from which that block is taken. Method Zii follows what happens with blocks when one subtracts from left to right; one subtracts within one block size (one column) and then gives one of the remaining blocks to the next-right column if it is needed. This subtraction procedure is analogous to the European left-to-right look-ahead method described previously for addition. Method Ziii users viewed it as an easy subtraction method, equivalent to subtracting one from the top number and then subtracting the bottom number. This method is easy if one is doing the subtraction by a forward method such as counting up from 7 to 15 (in the tens column, for example), because one can see the 5 (15) to

which one is counting up. A version of this method (shown as Method AA) is common in Europe and in Latin America (Ron, 1998), but informal conversations indicate that it is rarely understood.

- *Methods BB through EE.* Methods BB through EE are alternative ways to show trading or putting for expressions with zeros in the top number. Both groups who got to such items needed some support from an adult to devise a method for getting more ones (Fuson & Burghardt, 1997). Each group unsuccessfully spent a whole lesson trying to solve this problem. Their multiunit knowledge was strong enough in each group that they discussed and rejected each incorrect zeros method shown in Fig. 10.4, but they needed a hint to devise a correct method. In one group, the experimenter asked the children, "Where do you usually put the thousands block?" This was sufficient for these children to invent a correct method using successive putting to the right (Method CC). In the other group, two children asked their parents how to solve such problems at home at night, and they brought in written marks methods from home. These methods (BB and CC) were then discussed, carried out with the blocks, and explained. Method BB was taught to one child by his father as looking at the first three digits (from the left) of 4000 as "four hundred," which needed to be reduced by one because one is given as ten to the ones column, leaving 399 (three hundred ninety-nine tens). Although the father's version did not say 400 tens or 399 tens, the children used this language with the blocks. Method BB was also developed as a shortcut to Method CC and as a solution to the erroneous method of giving 1000 to the ones column.

Educational Implications

In this section, we first discuss the result of our study in terms of the issues of adaptive expertise and flexibility and then describe general instructional guidelines.

FOSTERING ADAPTIVE EXPERTISE AND FLEXIBILITY

Different conditions led to the invention and the use of correct and incorrect methods. Understanding these differences can illuminate how to support children's construction of adaptive expertise and flexible problem-solving strategies. In the present study, the problem to be solved was the invention of correct methods of four-digit addition and subtraction with blocks and with written marks. As discussed earlier, participants' use of incorrect methods was repeatedly stimulated by memories of some attributes of (usually two-digit) algorithms or "math rules" learned by rote from teachers, parents, or siblings. Children could not use this unconnected or meaningless knowledge adaptively or flexibly to direct the invention of correct methods or prevent that of incorrect ones. In contrast, the base-ten block environment simultaneously supported children's reasoning about grouping, place value, and multidigit addition (Baroody, 1990). When adults asked participants to link the blocks and written methods, they *self-corrected* their incorrect methods. The blocks-based instruction also enabled they to understand the standard U.S. algorithm, a procedure that—before the study began—they had known only by rote and could not explain. Almost all participants explained this method either during the study or on the posttest.

Working in a group created opportunities for developing adaptive expertise, because different methods, or different views of a given method, arose and needed to be resolved (Cobb & Bauersfeld, 1995). The explainers had to adapt their current knowledge to the issues raised by questioners, and the latter had to adapt to the explanations given. However, children did not spontaneously use clear referential language, which turned out to be crucial in helping everyone in a group understand each method. Such language involved clearly referring to all parts of the block actions, so that a listener could understand the "big picture" (the whole process). Either blocks language (i.e., the names of the different-sized blocks that each group had chosen) or place-value language (ones, tens, hundreds, thousands) was effective as long as the name of the position was used. The word "ten" had special potential for confusion, because it was sometimes a position name ("the tens [place]") and sometimes specified how many of some position ("ten of these"). Complete referential language, such as "ten tens," was helpful in directing attention to crucial features of a method and understanding them.

A striking result was that children's invented methods can be conceptually or procedurally superior to standard algorithms. Creating a learning environment that supports attention to and understanding of the crucial features of the domain can facilitate methods more adapted to the emerging expertise of children. For example, both addition Methods I and M are conceptually superior to the standard U.S. method. Both involve adding like units and then recording the total in a separate "total" space, rather than changing the "problem" by adding to the top addend as the solution proceeds. Changing the problem is difficult for some children to understand. Indeed, a number of participants objected to the standard method when members of their groups modeled it with blocks ("You're changing the problem"). Both Methods I and M can move from left to right as well as from right to left, thus accommodating the many children who prefer to move from left to right. (Note that the latter is consistent with how multidigit numerals and English text, for instance, are read. It is also consistent with the informal left-to-right mental multidigit procedures often used by children and experts; see Baroody & Tiilikainen, chap. 3, this volume.) Moving from left to right is messier than the standard U.S. method, requiring crossing out if you get a new traded multiunit. However, this step can eventually be eliminated with experience by moving to a look-ahead method discussed previously. Alternatively, children might decide to move to a right-to-left method, as did the one group who, as their adult leader suggested, compared such methods with left-to-right methods. Furthermore, for the reasons discussed in the previous section, single-digit addition within each column is also easier in addition Methods I, J, L, and M than in the standard U.S. method.

Some subtraction methods were also superior to the usual U.S. method that alternates borrowing and subtracting. The methods in which all of the fixing (borrowing, regrouping) was done first (in any direction) and all of the subtractions were done second were conceptually and procedurally better. The separation of the steps enabled these two crucial features to be more salient. They also eliminated subtract-top-from-bottom errors that often occur when children move right to left and alternate steps. Seeing a 2 and an 8 in some column automatically produces a 6. Unlike methods that do not alternate borrowing and subtracting, this automatic response must be inhibited when using the standard algorithm if the 2 is on the top.

Differences in the effectiveness of using blocks for addition and subtraction underscore the importance of a conceptual analysis of a domain. A visual or linguistic support is only as powerful as is its match to the underlying crucial features of the domain for which it is a potential support. Multidigit addition has the following two crucial features, both of which the blocks supported: (a) adding of like units (the blocks are very powerful here) and (b) fixing a unit when there is 10 or more of it by trading or putting 10 of the unit with the next unit to the left. The second feature also needs to be coordinated with knowledge of written place value, but the blocks helped greatly in understanding the conceptual shift from "10 ones" to "1 ten" (or "10 tens" to "1 hundred" and so forth) and the recording the new 1 (in various places).

Multidigit subtraction may be more challenging than multidigit addition in two ways. One is the noncommutativity of subtraction. That is, unlike addition, the order of subtracting terms does make a difference. The second is that the coherence of each multidigit number may be more obscured in the subtraction process than in the addition process. Because both the standard U.S. multidigit addition and subtraction algorithms focus attention on the digits of numbers one column at a time, many students do not view the top digits as one multidigit number and the bottom digits as another. By alternating the processes of borrowing and subtracting, the standard U.S. subtraction algorithm may reinforce children's focus on the columns of digits and exacerbate an incoherent view of multidigit addends.

There are several possible changes to our learning environment that might support these features of subtraction better. First, the integrity of each number might be supported by cardboard frames and by a discussion that subtracting can involve breaking apart a number into two smaller numbers. Second, only the number to be broken apart by subtraction need be represented with the blocks. Third, a comparative subtraction context might be used in which the larger number is actually split into a part that matches the other given number and the remaining part that represents the answer. This last method was used effectively in an earlier study (Fuson, 1986a) and has the advantage of encouraging children to count-up to solve the single-digit subtractions, which is easier and more accurate than is counting-down (Baroody, 1984; Fuson, 1984). However, counting-up can also be supported in a taking-away subtraction situation by taking away the entities representing the starting amount (Fuson, 1986b; Fuson & Willis, 1988).

Children had more trouble with subtraction than with addition in resolving how to get more of a unit and how to fix the situation when they had too many of a unit. Several kinds of experiences might help. The suggestions given previously might have helped children resolve the issues by enabling them to see multidigit numbers as wholes. Previous experience with simplifying noncanonical displays of more than 10 of a unit, as was the case for the most productive group, might be helpful. A teacher might raise this issue with the whole class, once a group encountered it. Helpful framing by the teacher could also be useful. Some of the groups needed suggestions from the adult leaders such as, "What else in the top number could you subtract eight little guys from?" or "In addition, when you had too many, you traded or put those over to the next column.

What if you traded or put that back now?" With such minimal help, all groups resolved this issue. Moving 1 unit to the next-right place seemed conceptually clearer than trading 1 unit for 10 of the next-smaller units. Such "putting" made it very clear that one was just moving part of the number from one column to the next. This was also not subject to the confusing "taking-away" language used by some groups to describe trading in subtraction or to noticing only the missing of half of the trade as some children did.

Yet another factor that might help account for the relative difficulty of multidigit subtraction is that less time was spent on this operation, and so participants had less opportunity to construct an understanding of it. That is, the results are consistent with Hatano's observation in the Foreword that hurried instruction frequently does not enable children to effectively apply their learning.

Our discussion of the various correct methods in addition and subtraction and the advantages of some invented methods over the standard U.S. algorithms raises another crucial issue regarding adaptive expertise, namely "accessible algorithms." The superior algorithms were more accessible to children, because such procedures were more adapted to their knowledge. Such procedures facilitated understandings of crucial features of the domain, which, in turn, promoted adaptive expertise. These procedures were also accessible to children using less advanced (but still rapid and effective) single-digit methods and were readily explainable to them. Additionally, these methods were as general and as rapid to carry out as the standard methods. Environments that support children's learning of accessible methods, then, might facilitate adaptive expertise. Domain analyses and empirical work that identify and test accessible methods in various mathematical domains might enable many more children to build flexible adaptive expertise in those domains than at present with standard algorithms that have not been examined for their conceptual and procedural accessibility.

Some educators and researchers strenuously object to teaching any particular algorithms and argue that such teaching inevitably leads to the kind of rote, inflexible knowledge evidenced by some of the children in this study (particularly with subtraction). In our view, a social environment that supports sense making is more important than whether a particular method is or taught. That is, the focus should be on learning with understanding and setting expectations that all methods must be fully explained is central. Less advanced children can learn accessible methods from more advanced children, teachers, or a combination of the two.

INSTRUCTIONAL GUIDELINES

In this subsection, we discuss guidelines concerning the role of the teacher, group work, and manipulatives.

Tasks of the Teacher

An analysis was undertaken of the major teacher tasks assumed by the adults during the study and of those tasks that would have been helpful to assume. Teacher tasks that can support students' invention of calculation methods when working in small groups and using physical objects as pedagogical tools are summarized in Table 10.1.

These tasks may become intertwined when a group disagrees about which method to pursue. Some groups or children within a group may then need support so that the discussion is focused on mathematical issues, not on issues of power or use of rules from past authorities. ("My teacher last year" and "my sister" were both evoked as authorities in this study.) In this study, the pursuit of methods was more often resolved by dominance than by mathematical persuasion (Fuson & Burghardt, 1993). Of course, it is difficult in a brief intervention to create social norms different from those experienced in the regular classroom. Prolonged experience working in groups with the Social Norms 2Aa and 2Ab with an emphasis on using full quantity language to discuss mathematical issues should reduce the need for teacher support. In this study, most groups did adopt these social norms, at least part of the time, and the group inventing the innovative addition Method M conformed to them most of the time. Nevertheless, most groups would have worked better with more adult guidance in implementing these social norms.

As discussed before, most groups neglected to link block and written-mark methods, despite an adult's charge to do so. Children also rarely used block names or English multiunit values or fully described actions with the blocks. Groups needed persistent adult support for linking (Teacher Task 2Aa) and using quantity language (Task 2Ab). The blocks were powerful enough for addition that directing attention to crucial attributes (Task 2Ac) was not really needed. Quantity language, when used, also helped children see crucial attributes of a method and, thus, correct incorrect methods. For subtraction, directing attention (Task 2Ac) was needed for computations involving zero, and it might have been needed for groups where no one had experience with subtracting by taking from a larger multiunit.

Several of the groups were held back by the way they wrote their addition or subtraction solutions. These groups could have benefited from teacher guidance about how to record their work to facilitate reflection (Task 2Ae). The recordings of the socially effective group that invented addition Method M were often messy and all over the page, which led to considerable confusion. In subtraction, some groups completely blacked out numbers they were reducing by one, so it was hard to see what they had done afterward. One high-achieving group also sometimes made fancy curlicue numbers that even they then had trouble reading.

Adult support of a nondominant child (Task 2Ba), namely a suggestion that the group address her objections, was important for the second and final correction of incorrect addition Method G. An adult might also have focused attention on two interesting subtraction methods (getting negative numbers in a column and fixing all top numbers and then subtracting left to right) that were suggested by nondominant children but were not pursued. At the very least, the teacher could have encouraged the children to evaluate the procedure (e.g., explain why they thought it was wrong, checked whether it produced the correct answer).

Occasionally, when children made errors in counting or wrote a wrong number, no one in the group noticed the error, leading later to lengthy searches to recover from or find the error. In these situations, the other students were

Table 10.1: Tasks of the Teacher to Facilitate Mathematically Productive Small Groups

1) Create social norms for children

   A) to understand and explain block and written-marks methods they use, not just get answers, and
   B) to help everyone in your group understand.

2) Monitor each group's interactions and, when necessary, provide support as specified by Points A and B.

   A) Support meaning building by

      a) helping children make the requisite links between the pedagogical objects (the blocks), written marks, and mathematical words;
      b) monitoring and helping children describe, argue, and explain using quantity language (here, block words and multiunit quantities);
      c) directing children's attention to crucial mathematical attributes by priming or eliciting helpful knowledge,
      d) encouraging children to count and write carefully and to check each other as they do these steps; and
      e) monitoring and helping children to record in ways that facilitate reflection.

   B) Support facilitative social interactions by

      a) providing social support to nondominant children to rescue good ideas;
      b) when necessary, directing attention to the task, ascertaining if there are current difficulties that are contributing to off-task behavior;
      c) encouraging discouraged children or groups.

3) Monitor group and individual progress and allow children sufficient time to construct the conceptual structures for the domain and function comfortably with them.

4) Encourage whole-class reflection on, comparisons of, and discussion of different solution methods.

passively observing, rather than actively monitoring, others' problem solving. The teacher's task here would be to encourage children in groups to count and write carefully and, especially, to check each other (Task 2Ad), that is, to share responsibility for producing a correct and reasonable solution.

Tasks 2Bb and 2Bc are related, though also independent. Individual children and whole groups sometimes engaged in sustained off-task behavior. At times, this occurred when children faced a problem they did not know how to resolve (e.g., subtraction with zeros), but it could also occur in response to distractions (e.g., a very hot day, an interesting event in the class just before, or an individual toothache). Some children also seemed to get discouraged at times and withdraw from the group (e.g., when their suggestions were ignored or when they did not understand). These clearly are also related to other teacher tasks (2Ac and 2Ae). We found that even a minimal hint may bring profound change to a discouraged group or child. In subtraction with zeros, a question ("Where did you usually put the ice cube?") was enough to create sustained engagement for the rest of the period by a boy who was easily discouraged and then was mean to other children. Indeed, while solving other problems involving zeros with blocks and marks, he even commented enthusiastically, "This is fun!"

It takes time for children to construct mathematically robust conceptions of a domain and use these conceptions to calculate in that domain. The groups in the present study spent from 5 to 8 days on addition and from 3 to 8 days on subtraction. Some children in four of the groups would have benefited from more time in addition, and all groups needed more time in subtraction. The appearance of multidigit numbers, easily seen as constantly seductive single digits, can be misleading and interfere with fledgling multiunit conceptions and suggest wrong methods. Furthermore, it takes a long time to solve a single four-digit addition or subtraction with blocks; our groups averaged about two such numeric problems per 30-min working session. Of course, they were slower in the beginning and faster at the end. If children are to describe, reflect on, and discuss an extended solution method with objects and written marks, they need quite a few examples before they begin to see and describe the overall pattern across the particularities of given numbers. This was particularly true for some group members, who initially did not understand all or part of the group's method. There were some such children in all groups, and they were supported to some extent by the adults' questions and directives. Such supporting also helped the faster members of the group to see and articulate overall solution methods rather than single steps.

Children with a more typical level of achievement and those with low achievement might have less initial knowledge and slower learning rates than members in the "low" group of our study, all of whom were in a class for students with above-average achievement. Average- and low-achieving children would need even longer to be able to carry out, understand, and explain a group's method. Both in this study, where base-ten blocks were used by children to invent methods, and in whole-class situations, where teachers are supporting a whole class to understanding and using a particular solution method (e.g., Fuson, 1986a; Fuson & Briars, 1990), children need weeks of time, not a single session, to learn. Using blocks for a single session results in little lasting conceptual gain (e.g., Resnick & Omanson, 1987; Thompson, 1992). However, if children have more time, use of Thompson's microworld (described in his 1992 report) can promote considerable learning. For example, in a case study of two children (Fraivillig, Fuson, & Thompson, 1993), one moved from a concatenated single-digit understanding of her correct addition algorithm to understanding and explaining it in quantity terms over days, and the other moved from no correct method to an invented and conceptual method over days. Hiebert and Wearne (1992) also reported that it took weeks for children to construct adequate two-digit addition and subtraction methods using base-ten blocks. All conceptual supports including manipulatives must connect with each child's existing knowledge. (This point has been made by many others; e.g., Baroody, 1989, and Clements & McMillen, 1996.) They must also help each child build new flexible readily used knowledge. These goals take time for such complex methods as multidigit calculation.

Insight, the rapid reorganization of conceptual structures that can result from making connections, might occur in one session, especially as children discuss and compare various methods. But insights assume that the child has already constructed the required multi-unit conceptions. Such construction seems to be for most children a slower process of pattern-making and connecting that leads to the gradual interiorization of mental images of the blocks that can be used to direct and correct thinking. An "instamatic camera" view of visual learning is not an adequate view of learning with visual supports.

We use "interiorization" because this word suggests retention of external characteristics, and children's expressions of internal representations suggested fairly direct perceptual adherences. Participants in our study who were asked by an adult on the final days of their groupwork on addition or on subtraction if they could describe their written marks method in terms of the blocks without using actual blocks. The children could use physical attributes of the blocks in describing or responding to questions about the method, so these attributes seem to have been interiorized and be accessible for use in constraining and directing-marks methods. Fuson (1986a) and Thompson (1992) also both reported such interiorization. Children in Fuson self-corrected errors when told to "think about the blocks." The best microworld student in Thompson reported that he thought about the blocks when he was "stumped or stopped" (p. 142).

This process of interiorization underscores two phases in the linking of physical pedagogical objects to the mathematical written marks and to the words whose meanings the objects are to support (see also Fuson et al., 1992). The first learning phase is the initial construction of meaning. In this phase, the mathematical marks record actions with the pedagogical objects (here, the base-ten blocks) and the words describe such actions. Gradually, these marks and words take on meanings derived from attributes of the pedagogical objects. The marks and words can then be used meaningfully, without the pedagogical objects. However, especially if alternative meanings for the marks or words already exist or can easily be constructed (e.g., treating multidigit numbers as concatenated single digits or the two-digit tens and ones conceptions), a second reverse-linking phase is helpful. In this phase, operations on the marks are occasionally described in terms of the pedagogical objects (here, actions on blocks). Such reverse linkings can serve to keep the correct meanings linked to the marks methods. If pedagogical objects are interiorized sufficiently by most children by the time small groups are ready for whole-class discussion, this discussion could take place quite rapidly in the reverse-linking block-word mode. Blocks themselves could be used when children are not following a particular argument or when a particularly difficult issue is discussed. But some children or groups may continue to benefit from carrying out particular alternative methods with blocks in order to understand them, or such methods could be drawn with block drawings (e.g., Fuson, Smith, et al., 1997).

Teacher Task 4 in Table 10.1 might have been helpful in this study. After students have worked on their group constructions for some time, it seems important for the teacher to encourage full-class reflection on and comparison of the groups' different solutions. This would enable important mathematical issues that arose only in one group to be reflected on by the whole class. Leading such reflective discussion would also provide an opportunity to carry out many of the other teacher tasks listed in Table 10.1 with the class as a whole (e.g., model and support full descriptions of methods). We could not implement Teacher Task 4 because of circumstances beyond our control (e.g., limited access to the participants and physical barriers, such as separation of groups by a stair landing).

Conditions for Learning in a Small Group

Working in a small group with the support of pedagogical objects does not necessarily lead to the construction of correct methods (for a full report on how children's interactions in small groups affect their invention of multidigit methods, see Burghardt, 1992; Fuson & Burghardt, 1993; Fuson et al., 1992). Knowledge, methods, and individual ideas can be socially destroyed (Tudge, 1992) or ignored, as well as socially constructed (Burghardt, 1992; Fuson & Burghardt, 1993; Fuson et al., 1992).

Small groups are stimulating "noisy" environments with multiple simultaneous visual, auditory, and kinesthetic stimuli, only some of which are mathematically relevant. The teacher tasks described in Table 10.1 focus on increasing the proportion of mathematically relevant stimuli and the ability of children to attend to them. In such a busy environment, a great deal is happening rapidly, and redundancy and salience are helpful. Thus, it might be beneficial for all children to verbally describe block or marks actions. Not all children are experienced in providing such descriptions. Helping students learn to ask whenever they do not understand a certain step is also important. When children in our groups did report lack of understanding, they were usually given help by their group (albeit not always with adequate explanations).

Knowledge under construction also can be quite ephemeral. One day a child may carry out a solution with blocks or marks, and the next day it may be gone or, at least, not accessible at that moment. A child in one group, who provided a full adult-like explanation of the group's subtraction method to a boy who had used the method himself the day before, commented on this phenomenon, "He understands one day, and then you have to explain it to him the next." The social conditions of the construction of individual knowledge have to allow for and support repeated opportunities for that construction and reconstruction in a particular domain.

There were at least four different sources of learning in the small group situation; examples of these types of learning are given in case studies of individual learners within the group context in Burghardt (1992). Most children learned by some combination of these sources. First, many children used features of the mathematical entities in the learning situation to direct and constrain their actions with those entities. Features of the blocks directed correct actions, and features of the written marks, sometimes combined with features of a rotely learned and dimly remembered standard algorithm, frequently directed incorrect actions. Second, some children learned some steps in a method, or a whole method, by observation and imitation of that method carried out by others in their group. Third, some children learned by receiving guidance of various kinds from another person, either a child in their group or the adult. This help sometimes was explicitly elicited and sometimes was given because the helper perceived a need for that help. This help ranged from physical actions to verbal directives, questions, and explanations and sometimes was a complex combination of most of these. Fourth, some children learned through cognitive conflict, demonstrated in discussion or argument. They disagreed about the accuracy of a method or some part of a method, and their verbal disagreement led them and sometimes others to change their view of the accuracy or inaccuracy of that part of the method. The first and fourth of these sources of learning are in keeping with Piaget's theories of learning (1965, 1970a, 1970b), whereas the second and third support Vygotsky's (1934/1962, 1934/1986, 1978) theories of learning. They all can contribute to an individual child's construction of knowledge. The balance among them also can be varied by the power of the pedagogical objects present in the situation, the knowledge and dominance relations across individual children, and the power of the social norms concerning helping, discussing, and explaining.

Pedagogical Objects as Sources of Learning

Pedagogical objects can support children's construction of correct meanings for mathematical words and written marks only to the extent such objects have salient features that suggest the correct meaning, do not possess misleading features, and are linked over a sustained period to the target mathematical words and written marks. We have seen that the different sizes of base-ten blocks are powerful in directing the two major components of multidigit addition: (a) adding like values and (b) when necessary, trading or putting 10 blocks of one size with the next larger blocks. In subtraction, the blocks do support subtracting like multiunits and can support understanding of taking from one of the next larger multiunit. However, the problems of directionality of subtraction and the lack of coherence of different kinds of blocks in making a single multidigit number are not addressed by features of the blocks. These require additional supports within the block context, as discussed earlier. The blocks also did not support a subtraction method using negative or minus numbers. One child wanting to try this method asked the adult if she had anything to make a minus four. Some kind of special mark on the blocks would be helpful to show any blocks considered as minus blocks. With such marks, the blocks could support the minus method sometimes invented by children in a supportive environment (e.g., Davis, 1984; Kamii, 1989). These examples indicate that an adequate learning environment might require both pedagogical objects and surrounding supports that together promote correct methods.

The surrounding supports we provided varied in their effectiveness. In the second session, we attempted to facilitate linking of the objects and written marks by asking children to write problems and solutions on a "magic pad" and to beep whenever a blocks action was not recorded on the magic pad. This fantasy-game approach was quite successful: Although unconnected to mathematics, it gave everyone an active role in monitoring the group action, seemed to help the linking considerably in some groups, and was fun and engaging, perhaps in part because it reminded children of video games. In contrast, the digit cards used in the first session often caused groups to splinter into pairs. It was natural for two children to use the cards (one to place and one to find the right card from the card box), leaving the other two children to focus on the blocks. These two pairs then would become separated in their actions and discussions.

Other possible outside supports include teacher directives concerning written marks. Thompson (1992), for example, had fourth graders circle the location of the answer in vertical addition and subtraction problems and calling this, "the place where you write the number of blocks of a place value after you add blocks having that value" (p. 126). Several children in both groups then invented a version of addition Method M, which Thompson termed "add within columns." In this version, each column was added (the 1 was written with the total, but slightly higher and smaller than the other digit) and then the answer was fixed to standard form. This method had been clearly suggested by Thompson's definition of the "answer place" in the introduction.

Any particular pedagogical object will support only certain meanings, though other meanings can sometimes also be supported by further additions to the learning environment. The salience of the separate multiunits in the base-ten blocks supported addition and subtraction methods of operating on separate like multiunits. In contrast, the hundreds board (a chart of numbers in sequence from 1 to 100, arranged in rows of ten) supports sequence methods, in which one begins adding or subtracting with one whole two-digit number and then counts up or down by ones or by ones and tens from that number. The hundreds board could support a sequence interpretation of the common European and Latin American subtraction Method K. For 62 – 34, for example, such a sequence method begins at 34 on the hundreds board, moves right 6 squares to 40, writes the small 1 by the 3 to show that one now is not at 30 but at 40, moves down two rows on the hundreds board to 60 (each vertical move of one square signifying an increase of one ten, from 40 to 50 to 60) and writes 2 in the tens column to show that increase of 20, and then counts up 2 more squares to get to 62, and writes 8 (the original 6 ones up to 40 plus the 2). Alternatively, all of the ones counting could be done at once. A similar sequence interpretation could be given for addition Method J. The 7 ones are added to the 85 to get 92; the 2 is recorded and the 8 is changed to a 9; the 60 is then added to the 92 to get 150; the 50 is recorded and the 600 is changed to 700; the 900 is added to the 700 to get 1600; the 600 is recorded and the 200 is changed to 3000; 3000 and 1000 make 4000. However, such sequence methods also require learning to count by tens, which must be supported somehow within the classroom or home environment. Children in our study did often count the base-ten blocks by tens when they were making block arrays or adding or subtracting blocks. Thus, the blocks can be used either with sequence methods or counting multiunit methods but are more likely to direct the latter without some outside suggestion for trying sequence methods of adding or subtracting.

Teachers do not necessarily understand the use of pedagogical objects to help children construct meanings linked to written marks and mathematical words. Hart (1987) reported that British teachers in the 1980s typically used base-ten blocks alone for an extended time and then moved abruptly to written marks with little (1 day) or no connections made between the blocks and the written marks. If pedagogical objects are well chosen, children do not need an extended time to operate with them. The objects will powerfully and quickly direct correct object methods. However, the meanings represented by the pedagogical objects will not become connected to the written marks or mathematical words, unless children have sufficient experience to overcome any misleading meanings stemming from the written marks (e.g., for multidigit numbers, concatenated single digit meanings) or words (e.g., two thirds, in fractions). Because written mathematical marks for multidigit numbers, rational numbers, and decimals, and the English words for these marks, all have misleading features (e.g., Fuson, 1992), this link may need to be sustained for a considerable time. We found in this study that children may require considerable prodding from their teacher to make this link between representations. In a study of a range of schools and a range of mathematical topics, Hart (1989) also found that the pattern of dissociation between pedagogical objects and written marks was pervasive; children usually operated in two different worlds and perceived these worlds as different. Her informal title for this study was a quote from one child in the study, "Blocks is blocks, and sums is sums" (Hart, 1989).

Conclusions

The results of the present study indicate that a meaningful and inquiry-based (an "investigative") approach to multidigit addition and subtraction instruction can foster children's adaptive expertise and flexibility (see also, e.g., Hiebert & Wearne, 1992; Pengelly, 1988). Specifically, with conceptual supports (e.g., teacher monitoring and feedback, pedagogical objects such as base-ten blocks, and the interaction of cooperative-learning groups), participants quickly devised correct addition methods with blocks. Furthermore, with adult prompting, they translated their concrete procedures into written procedures. Some of these invented written methods were conceptually or procedurally superior to the standard U.S. algorithm.

The results are also consistent with reservations about an "incidental-learning model" (Brownell, 1935) or a laissez faire problem-solving approach (see Baroody, with Coslick, 1998, and Baroody, chap. 1, this volume, for a detailed comparison of this approach, the investigative approach, and the traditional "skills" approach). Specifically, many incorrect written addition methods were invented because children did not link them to their block solutions. In such cases, adults had to insist that they connect their written solutions to their block solutions.

Block-trading methods for subtraction were considerably more difficult to develop than were those for addition. To a large extent, the participants' invention of both written and block subtraction methods followed a few students' memories of written methods. Children's use of imprecise language for blocks and written marks further contributed to difficulties in developing correct subtraction methods. Yet another key difficulty was that the participants did not have sufficient time to explore, analyze, discuss, and otherwise reflect on multidigit subtraction.

Children's invention of multiple methods of adding and subtracting are not just desirable because of the new mathematics learning goals (NCTM, 1989, 2000). The invention of multiple methods is inevitable. Whether with traditional instruction or with the investigative approach, children invent different methods, some correct and some incorrect. Correct inventions are typically directed by the child's conception of multiunit numbers and multiunit addition and subtraction.

Teachers in the United States and other English-speaking countries have a special responsibility to help children construct robust multiunit conceptions for two reasons. One is that English number words for two-digit numbers do not support such conceptions. Another is that few other conceptual supports are experienced outside the school. (See Miura & Okamoto, chap. 8, this volume, for a detailed discussion of both points.) The results of our study suggest that several specific teacher tasks are essential to support students' small-group invention of meaningful calculation methods. These tasks include creating an environment in which children can and do link concrete quantities to written methods; creating social norms in which understanding, explaining, and helping are integral; monitoring groups' interactions to help students to use clear quantitative language and to record their solutions; and supporting less dominant students or dispirited groups cognitively and socially.

This study further indicates that pedagogical objects, such as base-ten blocks, can provide a rich and powerful environment for student invention and learning. However, this is not something that happens automatically (Baroody, 1989; Clements & McMillen, 1996). The power of pedagogical objects is limited by their features and depends on the extent to which they are linked to a conceptual understanding of grouping and place-value concepts. Their power further depends on the extent to which they are linked to mathematical words and written marks and provide a bridge between these symbolic representations and conceptual understanding of grouping and place value.

This study also demonstrates that working in small groups can facilitate children's inventions of computational procedures and help them to deepen their arithmetic knowledge. Again, however, this does not happen automatically. Without the proper social and intellectual climate, working in a group can also hurt children and slow their learning. In a small-group setting with ambitious mathematical learning goals, therefore, a teacher's tasks are many and complex. As discussed earlier, teachers must lay the initial groundwork, constantly monitor the progress of each group, and otherwise support children's conceptual progress. Without such support, students may well invent unlinked and incorrect methods. With a positive climate and adult support, though, children can invent effective methods, some of which are better than the formal methods presently taught in the United States.

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