Fuson, K. C. & Kalchman, M. (2002, April)
Bridging the addition-multiplication learning gap:
Teaching studies in four multiplicative domains.
Symposium presented at the Annual Meeting of the
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The relationship between additive and multiplicative thinking is often considered in the research literature as either developmental or interfering. The developmental relationship rests on the notion that additive thinking precedes multiplicative understanding on the continuum of mathematical knowledge acquisition. The interfering relationship is a well-documented one in which additive thinking predominates in multiplicative mathematical domains such as proportional reasoning and ratio understanding (e.g., Confrey & Harel, 1994; Greer, 1992). Additive thinking and corresponding addition strategies inhibit understanding because many students do not go beyond additive thinking to learn multiplicative approaches. Rather, students persist in using additive thinking and additive operations. The common view is that multiplicative strategies must come to replace these inaccurate additive strategies in multiplicative problem situations, i.e., students must learn to stop adding and start multiplying.

The position we take in the proposed group of papers builds on the developmental approach but considers how additive thinking can be a viable and, in some cases, even a helpful cognitive attribute for understanding multiplicative thinking. Across four mathematical multiplicative domains--multidigit multiplication, multiplication of fractions, ratio and proportion, functions and slope, we discuss how students can use additive thinking to model situations and to solve problems. We describe how students can then build increasingly abstract and efficient versions of the additive strategies to transition to multiplicative perspectives and strategies. Thus, we extend the developmental perspective to examine how students use transitional thinking methods that bridge from addition to multiplication; we call these methods "additive multiplication." Students' thinking in these domains uses these additive multiplicative ways of thinking both to conceptualize the underlying problem situation and to carry out the numerical solution process. Students then can move to more advanced methods for conceptualizing a kind of problem situation, but the level of the method for the numerical solution process depends upon specific knowledge of the numbers involved (e.g., a student may be able to multiply 5 x 3 but not 7 x 8).

In the four papers in this symposium, we present instructional approaches that build on additive thinking so that it can develop into multiplicative thinking. Each of the papers describes research with elementary or middle-school students. Three of the papers involve students from low SES communities as well as from high SES communities. Thus, the identified teaching approaches and student strategies are accessible to a wide range of students. We will give an overview of our teaching approaches and the student strategies we identified. We view it as crucial for the educational research community to identify bridging methods that will help students move from additive perspectives and methods to multiplicative perspectives and methods in ways that permit students to integrate all of their knowledge in these two areas. We believe that research understanding of effective implementations of such approaches will enable multiplicative domains to become widely accessible to all students.

Together these four mathematical domains hold the keys to success in further mathematics. We must find out how to open doors to enable all students to succeed in these areas. The proposed papers demonstrate how to build on student additive knowledge to bridge to multiplicative reasoning strategies and modeling of problem situations via “additive multiplication” approaches. The interactive symposium structure will provide an opportunity for researchers to understand and critique these approaches, share other productive approaches to these problems, and together advance the knowledge of all participants in the symposium.

Paper 1: Izsák, A. (2002, April)
Bridging the Addition-Multiplication Learning Gap in Multidigit Multiplication: From Counting Units to Repeated Addition of Groups to Multiplication of Sub-dimensions

Multidigit multiplication requires complex integrations of additive and multiplicative knowledge and skills. Multidigit numbers need to be understood as totals of tens and ones, and students need to understand and coordinate which tens and ones are multiplied by which tens and ones and be able to find all of these products. Inadequate mathematical experiences in the lower grades result in many fourth and fifth graders, especially in urban schools, having inadequate understanding of place value and of multiplication. Thus, multidigit multiplication is a domain in which prior grade-level goals must be addressed for successful learning to occur.

This paper overviews relationships between additive and multiplicative thinking in classroom studies conducted in six classrooms over a 2-year period (one teacher participated in both years). The students ranged from poor urban students, many speaking English as a second language, to students in a rich suburban community. The teachers implemented teaching approaches that were designed to co-develop student understandings of areas of rectangles and of multidigit multiplication methods. Students began with an accessible numerical algorithm that recorded each conceptual step in a visual way. These steps were linked to drawings of rectangles that varied in detail from every unit square being shown to non-proportional sketches that showed only the outer dimensions as tens or ones and the products of major sub-dimensions of the rectangles (e.g., for 38 x46, the dimensions 30 + 8 and 40 + 6 and the four areas 30 x 40, 30 x 6, 8 x 40, 8 x 6). The less-detailed rectangles were phased in as the teaching progressed, i.e., the students were supported to move from detailed rectangles in which additive perspectives and methods could be used to drawing shells that required multiplicative perspectives and methods.

The theoretical frame for this study coordinates analyses of instruction and student solutions by focusing on strategies for using representational features to accomplish problem solving goals. When analyzing classroom instruction, I analyze whole-class strategies as taken-as-shared means of using representational features for accomplishing problem-solving goals. Such practices emerge as students and their teacher contribute to whole-class solutions. When analyzing student solutions, I analyze strategies as knowledge structures that emerge as students use their existing understandings either to make sense of taken-as-shared class strategies or to construct alternative strategies for accomplishing goals.

Students in each class began the unit at different places on the additive to multiplicative continuum, and many ended up functioning with additive multiplicative or multiplicative strategies. Two different continua were involved: one concerned conceptualizing areas of rectangles and strategies for finding these areas, and the other concerned numerical methods for finding products of given numbers. For the areas of rectangles, conceptualizations moved from 1) initial difficulties in conceptualizing areas of rectangles as consisting of rows of the same-sized groups to 2) understanding 1 but needing to see and count all of the unit squares in a given row and then see that row iterated to make repeated groups of that size (repeated addition) to 3) focusing on the dimensions as describing the repeated groups and coming to a multiplicative perspective in which those 2 dimensional numbers could just be multiplied to find the area. Student difficulties with 3 illustrate the importance of Principle 5 identified by Outhred and Mitchelmore (2000), and their ability to progress along this continuum by making their own drawings underscores the merits of drawing as a teaching and learning tool, as suggested by Outhred and Mitchelmore.

The numerical continuum for single-digit numbers followed the usual steps from 1) counting all to 2) counting repeated groups to 3) knowing parts of a count-by but counting all to get to the needed product to 4) using a related product to 5) just knowing the product rapidly. For multidigit multiplication students began by combining numbers that were not ten and so gave no general patterns. With support students began using the tens and ones in a number to multiply times the tens and ones in the other number, thus leading to general methods that could be recorded in various ways.

Students could come to a multiplicative understanding of using dimensions to multiply while still using additive numerical methods for a given product, and, conversely, they might be at an early level in conceptualizing the area of rectangles but know a specific numerical product (e.g., 2 x 3 = 6). Further complications in dealing with products of ten will be discussed in the symposium. [Outhred, L. N. & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measurement. Journal for Research in Mathematics Education, 31, 144-167.]

Papers related to Paper 1:

Izsák, A. & Fuson, K. C. (2000). Students' understanding and use of multiple representations while learning two-digit multiplication. In M. L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 714-721). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Izsák, A. (2001). Learning multi-digit multiplication by modeling rectangles. In R. Speiser, C. Maher, & C. Walter (Eds.), *Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Vol. 1* (pp. 187–194). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Izsak, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking and Learning, 6(1)*, 37-79. MpDv

Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *Journal of the Learning Sciences, 14(3),* 361-403.

Paper 2: Kalchman and Fuson (2002, April)

Bridging the Addition-Multiplication Learning Gap in Multiplication of Fractions:

Using an Additive Linear Model to Explain Any Multiplication

These classroom teaching studies were designed to enable fourth and fifth graders from a range of backgrounds to understand and be able to explain any multiplication of fractions problem. The studies embodied a Piagetian view of learning as involving students constructing their own knowledge and a Vygotskiian view of teaching as supporting student learning with selected semiotic tools (visual and linguistic representations) and a supportive learning environment. They were carried out in collaboration with classroom teachers in a high SES suburban classroom and in a heterogeneous classroom containing many students who qualified for free lunch.

Introductory work on the meaning and notations of fractions was done first. Students related fraction bars they folded to number lines to see the different kinds of labeling involved and to give a length meaning to the unit fraction lengths on the number line. To multiply, a number-line model of fractions was used in which students drew a long line, partitioned it to show a fraction, and labelled the resulting unit fractions. To multiply by another fraction (e.g., 3/4 x 2/7), they partitioned each of the original unit fractions parts (2 of the 7ths) into the multiplier fraction (made 4ths inside each of the 7ths), found how many fractional parts they now had (in order to name the product fraction: many students labelled each of the new smaller unit fraction parts), and then took the needed number of multiplier parts (3 of the 4ths within each of 2 of the 7ths). They then added these parts to find the total product (e.g., 3/28 + 3/28 = 6/28). Most students were not only able to learn this additive approach to multiplying fractions, they were able to describe each of the steps verbally (see reference below).

Students then went on to a reflective teaching phase in which they looked over examples of such additive linear solutions to see and understand a more general numerical pattern for multiplying fractions: You multiply the bottom numbers because that is what you did when you made one fraction inside another for the whole unit 1 (making 4ths inside each 7th gives you 28ths), and you multiply the top numbers because that is what you did when you took the first top number parts for each of the second top number parts (took 3 for each of the 2 sevenths). Most students were able to function at the multiplicative level at the end of the unit, though some still preferred to use some abbreviation of or the full additive approach. Thus, the additive multiplicative perspective and method was accessible to students and also served as an effective bridge into multiplicative methods for many of them.

A related paper:

Fuson, K. and Kalchman, M. (2002). A length model of fractions puts multiplication of fractions in the learning zone of fifth graders. In D. L. Haury (Ed.). *Proceedings of the twenty-fourth annual meeting of North American chapter of the International Group of the Psychology of Mathematics Education* (pp. 1641 – 1649). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Paper 3: Abrahamson and Fuson (2002, April)

Bridging the Addition-Multiplication Learning Gap in Ratio and Proportion:
Learning to See and to Trust Multiplication in Proportion Problems

The domain of ratio and proportion is very subject to additive perspectives and strategies that do not lead to success. This teaching experiment was designed to bridge students from their additive perspectives to multiplicative perspectives. It began with activities in which students identified ratio matches of pairs of drawings by moving one drawing closer or farther until the pairs looked the same when viewed with only one eye. The heights of ratio matches were recorded in ratio tables in which pairs were separated by horizontal lines. Students found other ratio matches that would fit into the ratio tables. Proportional word problems were presented as involving ratio matches from a ratio table. Students used their knowledge of ratio tables to solve proportional word problems.

This teaching experiment involved students with no previous background in proportion and with a range of understandings of multiplication. Five low-SES minority fourth- and fifth-graders stayed after summer school to participate in the 12 hour-long sessions. All students knew some small numerical products, but none knew rapidly all of the larger products such as 7 x 8. The perspective of the teaching experiment was a cognitive science focus on students' mental representations of the problem situations and solutions as evidenced in their drawings and videotaped discussion and a focus on the representational supports provided by the instructional design and by the teacher during the teaching experiment.

Student strategies initially involved only additive increases/decreases within the ratio tables. With experience and repeated focusing on non-contiguous relationships, students came to understand and express these as multiplications. However, when moving to the new domain of proportional word problems, students fell back to their additive methods and generated complete ratio tables additively until they reached the required ratio pair. With experience students spontaneously generated intermediate transitional representations such as leaving empty rows in a ratio table but only filling in (via multiplication) the needed ratio pair. Accompanying diagrammatic symbols, along with student written and oral explanations, suggest that students were now interpreting proportions multiplicatively, and that this interpretation rested on their earlier notions of repeated addition.

Emotive factors seemed to play an important role in these students' initial refusal to use multiplication strategies. Students professed a sense of insecurity in the context of--and even suspicion of the validity of--multiplication as a means for arriving at solutions pertaining to real-world situations. They preferred to stay in their familiar additive world even if additive strategies took more time to carry out. We attempted to bridge conceptual and calculation problems by encouraging students to interpret a printed multiplication table as a collection of 10 contiguous "count-by-x" vertical strips forming a huge ratio table. Discussions highlighted aspects of proportion problems that could be solved by employing dual "count-by-x" strategies by running one's finger down the two required columns. This bridging context both facilitated solutions and linked their former school multiplication experiences with their proportional work with ratio tables. We conclude that proportional equivalence seemed to demand for some, and perhaps many, students a "leap of faith" and supportive experiences bridging from addition to multiplication in order to function with understanding in multiplicative proportional situations.
Related papers:

Abrahamson, D. (2002a). When "the same" is the same as different differences: Aliya reconciles her perceptual judgment of proportional equivalence with her additive computation skills. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant, and K. Nooney (Eds.), Proceedings of the Twenty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA, October 26–29, 2002: Vol. 4 (pp. 1658–1661). Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Abrahamson, D. (2003). Text talk, body talk, table talk: A design of ratio and proportion as classroom parallel events. Proceedings of the 27th annual meeting of the International Group for the Psychology of Mathematics Education, Honolulu, Hawaii, 2003. Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Abrahamson, D., & Cigan, C. (2003). A design for ratio and proportion. Mathematics Teaching in the Middle School, 8(9), 493–501. Reston, VA: National Council of Teachers of Mathematics.

Fuson, K. C. & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase Problem-Solving Models. In J. Campbell (Ed.), Handbook of Mathematical Cognition (pp. 213-234). New York: Psychology Press.

Paper 4: Kalchman (2002, April)
Bridging the Addition-Multiplication Learning Gap in Functions: Using a Walkathon Model to Relate Graphing and Equations

Students' opportunities for constructing a conceptual framework that integrates numeric and spatial understandings are greatly increased when they experience instruction and curricula that focus on developing and relating both numeric and spatial understandings and doings. Kalchman and Case developed one such curriculum for functions. It has been shown experimentally to help students construct deeper and more flexible understandings of functions than do much older students who learn from textbooks (Kalchman & Case, 1998, 1999). In this integrative curriculum, the context of a walkathon is used to bridge students' spatial and numeric understandings and to help them foster a central conceptual structure for the domain (see Kalchman, 2001, for a description of this curriculum).

To illustrate differences in students' reasoning about functions following a textbook unit and the walkathon approach, we will exemplify differences between an integrated conceptual understanding of select function problems and an understanding that favors the numeric, sequential aspect of the domain. We will use examples of how students in an advanced-level Grade 11 mathematics class (n= 17), who had at least three years of textbook-based instruction in functions, responded to two tasks. These examples will be compared to responses to those same items by students in a high-achieving Grade 6 sample(n = 48), who had experienced three weeks of the walkathon curriculum. The younger students approached the tasks using concepts of the walkathon situation that they related to the graphs. They outperformed the older students on all tasks.

We used Case's theory of intellectual development and related empirical work on the teaching and learning of functions as a guiding framework to show how conceptual and procedural knowledge relate to each other as children construct an integrated conceptual cognitive structure for understanding in the domain. We argue that understandings and doings (procedural and conceptual knowledge) are present in some proportion when students are reasoning about sophisticated mathematical ideas such as those found in functions. We also present the case for how numeric and spatial features of functions must be co-active when creating a central conceptual structure for understanding in a domain such as functions, which includes multiple ways of representing a common concept. Such co-activation may be promoted and facilitated through appropriate curricular and instructional design such as the "walkathon" curriculum presented here, with its meaningful bridge to students 'previous understandings and doings.

A related paper:

Kalchman, M. & Fuson, K. (2001). Conceptual understanding of functions: A tale of two schemas. In R. Speiser, C. S. Maher, & C. Walter (Eds.), *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 195-205). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.