More Lessons Learned from Research

Helping All Students Understand Important Mathematics

Volume 2

more4U
More resources available online
www.nctm.org/more4u
Look inside for your access code

NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

More Lessons Learned from Research

Helping All Students Understand Important Mathematics

Volume 2

Edited by

Edward A. Silver
University of Michigan, Ann Arbor

Patricia Ann Kenney
University of Michigan, Ann Arbor

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

ISBN 978-0-87353-725-4

Library of Congress Control Number: 2015413341

The National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Printed in the United States of America

In E. A. Silver & P. A. Kenney (Eds.), 2016. More lessons learned from research Volume 2: Helping all students understand important mathematics. (pp. 101-108). Reston, VA: NCTM.

Chapter 9

Multiplication Methods in the Context of the Common Core State Standards

Bruce Sherin and Karen Fuson
Northwestern University

This chapter is about the strategies that children use to multiply single-digit numbers. Given problems that require them to figure out the products of 3×5 or 6×9, how do they go about figuring out the solution? What must they learn in order to be fully proficient at this type of task?

A big question here has to do with the role of "rote memorization." In the United States, learning all single-digit multiplications has often been called "learning the multiplication tables" or "memorizing the multiplication tables." As mathematics educators, we are committed to teaching mathematics with understanding. Given that commitment, how should we understand the crucial task of learning single-digit multiplications and their related divisions? Looking at a multiplication table can be helpful in thinking about this task (see fig. 9.1). We immediately see that there are many number-specific patterns in the multiplication table. Each column shows a vertical list of the products in numerical order for a given factor. These lists are often called "count-by" lists, because children can learn to say these patterns aloud. Some of these count-by lists have an easy pattern: 5, 10, 15, 20, 25, ... Others have a more difficult pattern: 7, 14, 21, 28, 35, ... The 9s list is particularly rich in patterns. The nature of the pattern depends on the relationship of the count-by number to ten. Identifying and discussing such patterns is a worthwhile and important mathematical endeavor. The lists can also be seen in the rows of the table, but the numerical patterns are a bit easier to see vertically.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Fig. 9.1. A multiplication table

This chapter is adapted from B. Sherin & K. Fuson (2005), Multiplication strategies and the appropriation of computational resources, *Journal for Research in Mathematics Education*, *36*, 347–395.

Students can use a count-by list to find specific products, such as six fives (6 × 5) is 5, 10, 15, 20, 25, 30. But generating products in this slow way is not sufficient for most multiplication needs, so students must progress over time to produce specific products rapidly. This is a complex process we discuss next. We can see that there is a great deal of number-specific learning required and that considerable practice will be involved. But it is also clear that this activity need not be simply "rote memorization"; there are patterns here, and children can learn to attend to and use these patterns.

We now turn to how children approach single-digit multiplication. This chapter summarizes our *Journal for Research in Mathematics Education* (*JRME*) article published in 2005, which was based on data collected in earlier years. In that article, we devoted much of our effort to exploring children's invented methods, particularly those that make use of drawings and finger counting. However, much has changed in the intervening years. In the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NCA Center & CCSSO], 2010), teaching and learning single-digit multiplication and division begins and comes to fluency in grade 3. This means that we must help students move rapidly from slow and possibly inaccurate methods to meaningful but more rapid methods that can culminate in fluency for multiplication and division of single-digit numbers.

We draw here on the extensive research discussed in the original article (Sherin & Fuson, 2005), which was based on a corpus of 230 interviews with third-grade students that were conducted before, during, and after instruction in multiplication. These interviewed students were in classrooms of the Children's Math Worlds Project (CMW), which combined the design of curricular materials and professional development for teachers with a range of more traditional research activities such as interviews and intensive observations of classrooms. The relevant portions of our interview data were digitized, transcribed, and coded for the methods used. Insights from experience with *Math Expressions* (Fuson, 2009/2013), the published form of the Children's Math Worlds program, are also included here.

Computational Strategies in Addition

We believe it is helpful to start by looking briefly at computational strategies that children use for addition, in part because the story for multiplication differs in some important ways. For addition, children progress through three conceptual levels. These levels capture changes in children's ability to conceptualize the relationships among quantities that are at the heart of the addition task. The list below gives these three levels (we use the language in Fuson, 1992, p. 250), and examples of each follow the list:

1. **Perceptual unit items.** Children must present addition or subtraction situations to themselves using things they can see, such as drawn quantities.
2. **Embedded integration.** All three quantities involved—the two addends and the total—can be simultaneously represented by embedding entities for the addends within the total.
3. **Ideal unit items.** The addends are not embedded within the total but can be conceptualized as outside and can be compared to the total. Numbers become units that comprise numerical triads—two known addends and a known total. This permits recomposition of the addends so that a problem can be transformed into an easier total of different addends.

As children move through the levels, they develop computational strategies that are characteristic of the level. At each level there are general strategies that work for all numbers that a child might be given to add or subtract. Children at the first level use a *count-all* strategy: They count out items for each of the addends, then they count all of the items, starting at

1 and proceeding to the total. In contrast, students at level 2 are capable of using a *count-on* procedure. They begin with the first addend and count on from there, stopping when they have counted on the number of the second addend. For example, 8 + 6 would be solved as 8, 9, 10, 11, 12, 13, 14. Finally, at level 3, students can use a recomposing procedure: One addend is broken apart to make a related addition problem whose total is known. For example, 8 + 6 would be solved as 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14.

The important point here is that changes in the addition strategies are produced by fundamental changes in conceptual understanding, and not only practice with specific number problems. Children come to understand numbers and addition differently, and this drives the changes in strategies.

Computational Strategies in Multiplication

As we outlined above, the learning of addition strategies is driven primarily by fundamental conceptual developments in how children conceptualize relationships among quantities. In contrast, in our research we found that the learning of multiplication strategies is driven primarily by the learning of specific knowledge about specific numbers. In our 2005 paper, we called this new knowledge *number-specific computational resources*. Such knowledge was important for all but the most time-consuming and earliest of our six main types of strategies. We also found that as children approached fluency for specific numbers, the main strategies merged so that they were not easily identifiable. Children seemed to have developed a web of related knowledge on which they drew to give answers. The six strategies we identified are summarized below.

Count-all

When children first start to learn multiplication, they can make use of knowledge that they have acquired during their time learning addition. In the first type of strategy, count-all, a student can be seen counting from 1 to the product as he performs the computation. Example 9.1 describes an incident in which a student, Danny, was presented with the task of finding the total number of children, given that 4 children are seated at each of 3 tables. He solved this problem by first drawing a picture, and then counting all of the children he had drawn.

Example 9.1. Danny, pre-interview

Task:	There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?
Description:	Initially Danny was unsure how to proceed. Following the suggestion of the interviewer, he drew the situation. When the interviewer asked, “So, how many children are there altogether?” he counted quietly without pointing, but his head moved and he nodded a bit, as if in the direction of each drawn child.

An important feature of count-all strategies is that they are time-consuming and difficult to enact correctly, especially when the factors are large. To multiply using count-all, a child must keep track of three simultaneous counts. Consider the task of multiplying 3×4. One way to do this is to count to 4 three times, and then count the total. This approach requires that we enact and coordinate the three counting sequences shown in figure 9.2: (1) We need to count from 1 to 3 to keep track of the number of groups; (2) we need to count from 1 to 4 three times, to keep track of where we are within each group; and (3) we need to count from 1 to 12, thus keeping track of the running total.

1				2				3				Count of the number of groups
1	2	3	4	1	2	3	4	1	2	3	4	Count of entities in a group
1	2	3	4	5	6	7	8	9	10	11	12	Count of total

Fig. 9.2. The three coordinated counting sequences for multiplying 3×4

Children employ different techniques to keep track of the three separate counts. For that reason, children's use of count-all strategies can look different in different circumstances. In example 9.1, Danny used a drawing to help him keep track. Children also use their fingers to track counts. Or they make use of *rhythmic counting* and emphasize each value that is associated with the completion of a group. So a student multiplying 3×4 might say: "One two three four, five six seven eight, nine ten eleven twelve."

Additive calculation

Students also have prior learning experiences using addition. This knowledge can provide the basis of strategies that are less time-consuming and easier to enact than count-all strategies. We call these strategies that are based on addition-related techniques *additive calculations.* Example 9.2 shows this strategy. In this example, Ellen multiplies 3×4 by first adding 4 + 4 to get 8, and then 8 + 4 to get 12. This is clearly different than a count-all calculation. Ellen had to keep track of the three 4s she added, but she did not have to count from 1 to the total.

Example 9.2. Ellen, pre-interview

Task: There are 3 tables in the classroom and 4 children are seated at each table. How many children are there altogether?

Description: Ellen added two 4s to get 8, and then added an additional 4 to get 12.

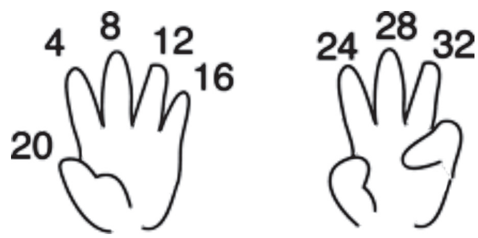
Count-by

The first two strategies make use of knowledge about numbers that students have before they receive instruction in multiplication. Such instruction emphasizes the meaning of multiplication as repeated groups. Then students begin the extended task of learning the various number-specific computational resources that can support more efficient and accurate strategies. One such resource is the collection of *count-by sequences*; students learn to say sequences such as "5, 10, 15, 20, . . ." and later "6, 12, 18, 24, . . ." and "9, 18, 27, 36 . . . ." Knowing these sequences makes it possible for students to use count-by strategies to multiply single-digit numbers. In example 9.3, we describe an episode in which a student used a count-by strategy to multiply 8×4: She counts by 4s to 32, putting up a finger on each hand to keep track of the number of groups.

Example 9.3. Linda, post-interview

Task: 8×4

Description: Linda counted by 4s to 32. She said: "4, 8, 12, etc." putting up a finger as she says each number. She uses only her left hand, so she must reuse some fingers.



As in count-all, count-by strategies require children to keep track of multiple counting sequences. However, in the case of count-by, there are only two sequences, a reduction that greatly lessens the difficulty of accurately enacting count-by strategies. The tradeoff is that a count-by sequence must be learned for each number. Figure 9.3 depicts a count-by sequence for the case of 8×4.

4	8	12	16	20	24	28	32	Count of total
1	2	3	4	5	6	7	8	Number-of-groups count

Fig. 9.3. Two sequences to be coordinated for multiplying 8×4

Pattern-based

Many single-digit multiplication problems become easy once children learn to recognize certain simple patterns. These *pattern-based* strategies are another type of strategy children use. There are clear patterns associated with multiplication by 0, 1, and 10. These three patterns allow students to produce certain results rapidly and without visible work. For example, when students are asked to multiply 7×1, they may very quickly respond by saying "seven."

Beyond the 0s, 1s, and 10s patterns, students may learn other patterns that may support them in multiplication computations. These patterns are visible in the multiplication table and can be the focus of continued discussion by students. The 9s products are particularly rich with useful patterns, and students' recognition of these patterns can reduce the difficulty of multiplication tasks involving 9. In CMW, students first considered 9s patterns based on 9 as $10 – 1$ (e.g., $6 \times 9 = 6 (10 – 1) = 60 – 6 = 54$). After working through all of the related patterns and discussing them using tens and ones, students summarize these using a finger shortcut that captures the patterns. The pattern shortcut works in this way: If a student wants to multiply $9 \times N$, the student holds up both hands and puts down the Nth finger, counting from the left. The tens digit of the result is then given by the number of fingers to the left of the finger that was put down (because it is always 1 less than the number of tens in $10 \times N$), and the ones digit is given by the number of fingers to the right (because those fingers show how many ones remain after N ones are taken from $N \times 10$).

Learned product

The learned product strategy requires number-specific resources that are multiplication triads, such as $9 \times 6 = 54$ and $4 \times 7 = 28$. These triads can be remembered as results from any other strategies. The count-by sequences especially are rich sources for learning multiplication triads as students link the product and the multiplier used in the keeping-track process to the multiplied factor (e.g., 7, 14, 21, 28, four 7s are 28). The multiplication-triad resources are acquired bit by bit, with some triads being learned earlier than others, especially those for smaller or easier factors. Learning all such multiplication triads takes time and practice because the triads are number specific.

Hybrids

Because the two main strategies, count-by and learned product, are learned gradually, we sometimes saw hybrid strategies that combine the use of a count-by or learned product strategy with count-all or additive calculation. For example, for 7×6 one student said, "6, 12, 18, 24, 32, 36, 37, 38, 39, 40, 41, 42," and another student said, "6 times 6 is 36, and 4 more is 40 and 2 more is 42."

Relationships among the strategies

Our discussion above presented the individual, number-specific computational resources—such as count-by sequences and multiplication triads—as if they are clear and distinct elements of knowledge. For example, we implied that each count-by sequence is learned separately, and each number triad is a separate multiplication "fact" to be learned. However, relationships exist among all of the strategies. When students solve problems with additive calculations or with rhythmic counting, these strategies can support the learning of specific count-bys. Analysis of patterns can underlie any of the strategies. Learned products can emerge from the use of any of the other strategies. So students are always building a web of integrated and related knowledge rather than separate, discrete bits of knowledge. This web of knowledge rests on the meanings for multiplication and division that students are developing from the Common Core Operations and Algebraic Thinking (OA) standards 1 through 7; these meanings (e.g., equal groups, arrays, area) help students relate strategies.

Is It All about Rote Memorization?

We now return to the question with which we began this chapter: *Is the learning of single-digit multiplications and divisions just about rote memorization?* No. Instead, we have seen that it is about acquiring knowledge that is linked to specific numbers—number-specific computational resources, such as count-by sequences and multiplication triads—and that these resources are related to form an integrated generative conceptual web. A student may respond that 7×5 equals 35 by drawing on many visual, oral, and reasoning experiences from different parts of this web: 35 is one more 5 past 6 fives are 30; the product ends in 5 because 7 is odd and 35 seems about far enough along; five 5s is 25 and 30, 35. These thoughts may occur very quickly and may not functionally differ much from a learned-product response. Similar points have been made elsewhere in the research literature. For example, Heege (1985) states that students can become so skilled "that the border between 'figure out' and 'know by heart' seems to blur" (p. 386).

The OA learning progression, written by members of the Common Core Standards Writing Team (2011), came to a similar conclusion:

> All of the understandings of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single-digit numbers and 10. Such fluency may be reached by becoming fluent for each number (e.g., the 2s,

> the 5s, etc.) and then extending the fluency to several, then all numbers mixed together. Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. As should be clear from the foregoing, this isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning. All of the work on how different numbers fit with the base-ten numbers culminates in these "just know" products and is necessary for learning products. Fluent dividing for all single-digit numbers, which will combine just knows, knowing from a multiplication, patterns, and best strategy, is also part of this vital standard 3.OA.7. (p. 27)

Instruction

What does all this mean for instruction? It is important for us to emphasize, once again, a point made in the introduction. In this chapter, we have presented a variety of strategies, including students' invented strategies. But given the need to move from understanding to fluency in grade 3, it does not seem to us to be possible or wise to spend instructional time on a leisurely exploration of student strategies. Third grade instruction must support students quickly to meaningful, accurate, and efficient strategies. We hope we have made clear, however, that though number-specific knowledge must be practiced in order to be learned, this knowledge does not need to be acquired through rote memorization. The goal can be met by helping students see and relate pattern and structure in numbers.

Before concluding, we want to introduce a few additional thoughts about instruction that arise from the second author's extensive experience with the *Math Expressions* program and with the Common Core standards. Additional discussion of multiplication/division instruction and the Common Core standards can be found in the OA progression (Common Core Standards Writing Team, 2011) and in a twenty-minute webcast developed by the second author: "Math Expressions and Operations and Algebraic Thinking (OA) in the Common Core State Standards Part 3: The Grade 3 Learning Path for OA x/÷" (available at http://www.brainshark.com/hmhsupp/vu?pi=134411335).

First, multiplication and division learning needs to begin intensively and early in grade 3. Group and individual practice needs to continue throughout the year. Initially the class can move through each number, from the easier numbers (e.g., 2, 5, 10, 9) to medium (3, 4) to difficult (6, 7, 8), with the general patterns for the 0s and 1s folded in somewhere along this path. Practice needs to occur for each number separately for count-by sequences, pattern analysis and discussion, and known products. Then, students need to practice known products mixed across numbers—for example, mixed across multiples 2, 5, 10, and 9. Practice on new larger numbers separately and then mixed numbers continues throughout the year. This learning path requires a complex and sustained social organization of support and motivation for students to maintain their focus throughout the year. Students learn at different rates. Students who learn more slowly must be given support in learning the easier numbers so that they do not fall completely behind the class. This support needs to be given outside of and in addition to class so that students can participate in the discussions of problem solving and reasoning about new numbers that occurs in class.

Second, division strategies are closely related to multiplication strategies. Counting-by to divide is the same process as counting-by to multiply, but the student monitors the count-by sequence and stops when she or he hears the known product. For example, 32 ÷ 4 looks and sounds like Linda's method in example 9.3, but one would listen for and stop at 32 and then look at the 8 fingers raised to find the unknown factor 8. This is actually easier than multiplying because the student can just look at the product while saying the count-bys to help keep track of when to stop. The strong relationship between these multiplication and division count-by methods means that students can practice multiplications and divisions involving the same count-by in a related fashion, and these can strengthen each other.

Finally, it is vital that student practice be focused on the individual learning zone of the student: on the count-by sequences or individual known products that the student does not yet know firmly. Practice time is often wasted by using resources such as a page of 100 multiplication problems or a general computer game. Both of these resources often have many products known to a given student and only a few problems that the given student needs to learn next. Student time is better spent on their next most difficult problems, whether those involve the 9s or the 3s or the 7s. Individualized piles of not too many flash cards can provide individualized practice on what a given student needs to practice. Supports such as count-by sequences written on the back of a flash card can provide the grounding a student needs to advance more quickly.

References

Common Core Standards Writing Team. (2011). *The operations and algebraic thinking (OA) progression for the Common Core State Standards in Mathematics*. Retrieved from http://ime.math.arizona.edu/progressions/.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York, NY: Macmillan.

Fuson, K. C. (2009/2013). *Math Expressions K to grade 6*. Boston, MA: Houghton Mifflin Harcourt.

Heege, H. T. (1985). The acquisition of basic multiplication skills. *Educational Studies in Mathematics, 16,* 375–388.

National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA Center & CCSSO). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/.

Sherin, B., & Fuson, K. C. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for Research in Mathematics Education*, *36*, 347–395.