In R. Speiser, C. Maher, & C. Walter (Eds., 2001), Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Vol. 1 (pp. 187–194). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

LEARNING MULTI-DIGIT MULTIPLICATION BY MODELING RECTANGLES

Andrew Izsák
Northwestern University
izsak@northwestern.edu

Abstract: I report the results of interviews with 3 pairs of 5th-grade students during the course of a 2-digit multiplication unit implemented in a heterogeneous suburban classroom. The goal of the unit was for students to connect multiplication, arrays of unit squares, base-10 place-value[1], and the distributive property by modeling areas of rectangles. In the interviews, the students made such connections by (a) distinguishing between representational features for determining perimeters and areas and (b) refining contexts in which they applied criteria for using rectangular representations to solve multiplication problems.

Introduction

U.S. students often perform multi-digit multiplication poorly. In one international comparison, Stigler, Lee, and Stevenson (1990) reported that only 54% of U.S. 5th grade students in "traditional" courses could solve 45 x 26 correctly. The study reported here is a fine-grained analysis of three pairs of 5th grade students learning to solve similar two-digit multiplication problems with understanding through modeling activities. Modeling activities as contexts for learning are promoted by the National Council of Teachers of Mathematics (2000), and by others.

Existing research has established that learning to multiply involves recognizing classes of situations that can be modeled by multiplication (Greer, 1992), developing numerical strategies (Anghileri, 1989; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997), coordinating psychological operations (Confrey, 1994; Confrey & Smith, 1995; Steffe, 1994), and restructuring whole number understandings when extending multiplication to rational numbers (Greer, 1992, 1994). This research has examined students' performance when using a range of external representations–including blocks, number lines, pictorial diagrams, tables, and rectangles–but has not examined in detail how students learn to use such representations for modeling and solving problems about multiplicative situations. Moreover, this research has not examined how students extend multiplication from single- to multi-digit numbers.

At least two mathematical issues arise when multiplication is extended from single- to multi-digit numbers and suggest that multi-digit multiplication is an important, though underrepresented, area of research. The first issue has to do with magnitudes of factors and products. Some single-digit products (e.g., 2 x 3 = 6) are of the same magnitude as both factors, but related products (e.g., 2 x 30 = 60 and 20 x 30 = 600) differ in magnitude from at least one factor. Thus students must coordinate understandings of place-value and single-digit multiplication to determine products. The second issue has to do with place-value and the distributive property. Efficient multiplication methods that generalize to arbitrary numbers of digits rely on multiplying each term in the expanded form for one factor by all terms in the expanded form for the second whether or not the expanded forms are made explicit. Thus students have also to coordinate place-value with the distributive property. Coordinating magnitudes of partial products, place-value, and the distributive property distinguishes multi- from single-digit multiplication and demonstrates that multiplication extends to multi-digit numbers in ways that differ from addition.

This study is part of the Children's Math Worlds project (CMW), an on-going project that develops instructional materials for elementary school mathematics and then conducts research on teaching and learning as teachers use those materials in their classrooms. Izsák and Fuson (2000) reported on an earlier design of the two-digit multiplication unit, implementation in one urban and one suburban 4th grade classroom, and strong whole-class results in both classrooms. This study extends Izsák and Fuson by examining in more detail how students can learn with the CMW materials. In particular, the study examines how 5th grade students refined their understandings of rectangular area representations and extended their understandings of single-digit multiplication. The students were in a second suburban classroom that was using a subsequent version of the unit. All classrooms were heterogeneous in terms of ability.

The version of the CMW two-digit multiplication unit used in this study relies on arrays of dots printed on whiteboards. Students use the whiteboards to draw rectangles and the unit squares inside. The goal is for students to coordinate magnitudes of partial products, place-value, and the distributive property by modeling areas of rectangles. Phase 1 lessons focus on single-digit multiplication problems, such as 4 x 7 (see Figure 1), and review connections among single-digit multiplication, repeated

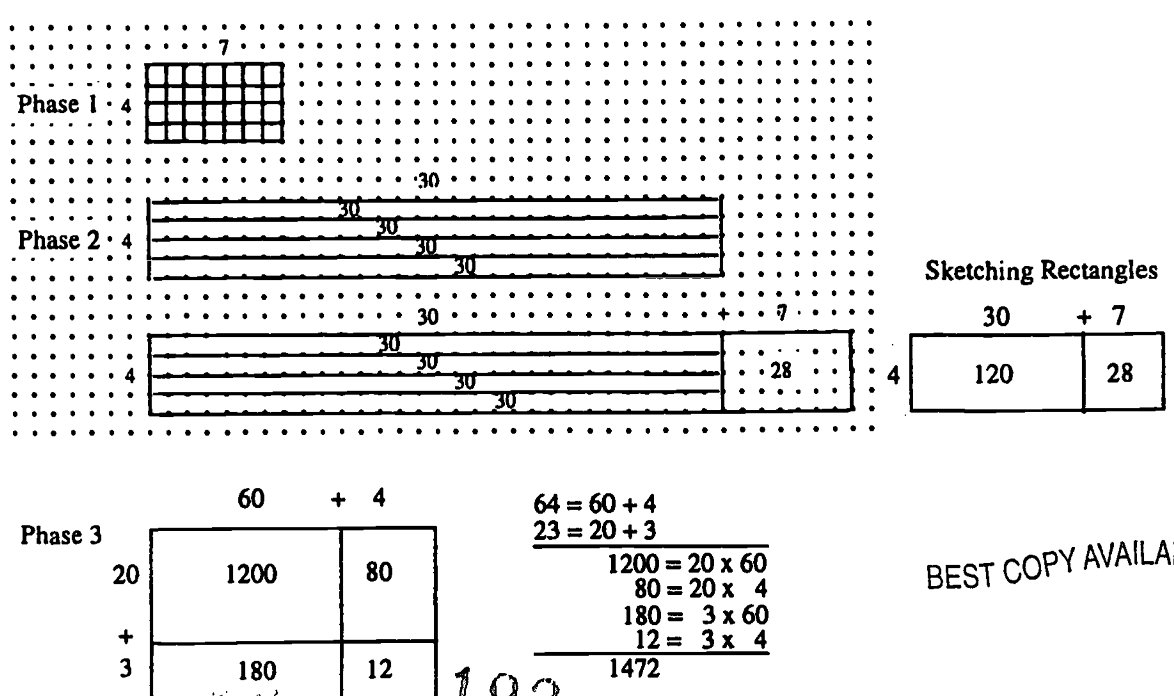

Figure 1. The three phases in the CMW two-digit multiplication unit.

addition, arrays of unit squares, and areas of rectangles. Students draw rectangles on dot boards, focus on lengths connecting dots when thinking about perimeters, and focus on unit squares when thinking about areas. Phase 2 lessons focus on single-digit times two-digit multiplication problems. Students first examine the special case of a single-digit times a decade number (e.g., 4 x 30), and then the general case (e.g., 4 x 37). To prepare for work with larger numbers, students also sketch rectangles at the end of Phase 2. These rectangles are not to scale, and students are to understand that sketched rectangles can be used to multiply numbers of any size. Phase 3 lessons focus on two-digit by two-digit problems. For the example 64 times 23 shown in Figure 1, students sketch a rectangle; divide it into four parts that corresponded to 60 x 20, 60 x 3, 4 x 20, and 4 x 3; and add the partial products. Thus the CMW two-digit multiplication unit starts with connections among single-digit multiplication, repeated addition, arrays of unit squares, and areas of rectangles and develops methods that coordinate magnitudes of partial products, place-value, and the distributive property.

Theoretical Framework

This study is grounded in perspectives on learning found in (a) Smith, diSessa, and Roschelle (1993/1994), who described learning as the refinement and reorganization of prior knowledge that is useful in some contexts, (b) Izsák (2000) and Schoenfeld, Smith, and Arcavi (1993), who have emphasized the features of representations to which students attend as they construct knowledge structures through problem solving, and (c) diSessa, Hammer, Sherin, and Kolpakowski (1991), who coined the phrase meta-representational competence to describe students' understandings about the design and use of representations for solving problems. I focus on how students learn to model areas of rectangles with multiplication by refining and reorganizing the representational features to which they attend and the contexts in which they apply understandings about the design and use of representations.

Methods and Data

I interviewed three pairs of 5th grade students. In their teacher's experience, the students were articulate and worked well together. Naomi and Candis struggled with mathematics lessons, Jill and Eli were strong, and Sam and John were very strong. [2] I interviewed Naomi and Candis and Jill and Eli once a week for five weeks beginning in Phase 1 and continuing to the end of the unit. Due to scheduling constraints, I interviewed Sam and John at the middle and end of the unit. During the interviews I asked students to solve multiplication problems using dot paper that resembled the dot boards they were using in class. I used videotapes of whole-class lessons as a source of questions to pursue during the interviews and as a resource for understanding what students said.

I recorded the interviews using two video cameras, one to capture the students and one to capture what they wrote. I transcribed the interviews in their entirety and added notes indicating what students wrote and what hand gestures they used. I also kept all of the students' written work in case the videotapes did not capture important aspects clearly. I performed fine-grained analyses (Izsák, 1999; Schoenfeld et al., 1993) of line-by-line utterances, hand gestures, and evolving written work to determine how students used area representations to solve multiplication problems.

Analysis and Results

All of the students that I interviewed had used Everyday Mathematics (The University of Chicago School Mathematics Project, 1995) in the fourth and fifth grades. They could multiply two-digit numbers using the lattice method (see Figure 2), the method used in Everyday Mathematics, but could not explain the representation of place-value and the distributive property in the method. Thus their understanding was limited to the sequence of steps in the method. They had studied arrays using Everyday Mathematics activities that focused on counting dots and had limited experience with area, primarily in the third grade. Finally, they demonstrated connections between single-digit multiplication and repeated addition during the interviews when working on word problems such as the following: How many tomato plants are in Jim's garden if there are six plants in each of three rows? Analysis of the interviews led to three main results about how students can develop conceptual understanding of multi-digit multiplication by modeling areas of rectangles. I focus on Jill and Eli because data on these students led to all three results. Data on Naomi and Candis provided additional data for the first result, and data on Sam and John provided additional data for the third result.

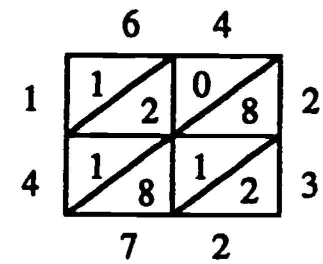

Figure 2. Lattice multiplication for 64 times 23.

1. Students distinguished representational features for determining perimeters from those for determining and areas.

When drawing and examining rectangles on dot paper, the students I interviewed focused at different times on dots, vertical and horizontal line spaces between adjacent dots, and square spaces between dots. The first interview with Jill and Eli occurred after initial CMW lessons. The students began with two methods for connecting multiplication to rectangles drawn on dot paper. Both methods used a single representational feature to answer questions about both perimeters and areas. The first method used dots, and the second used line segments that connected adjacent dots. Using the first method, the pair thought that the dimensions of a 5 by 7 rectangle drawn on dot paper were 6 and 8. They discussed the 6 rows of 8 dots and stated that the area was 48. Jill and Eli were explicit about their past experiences in which they had been told to always count dots when working with arrays.

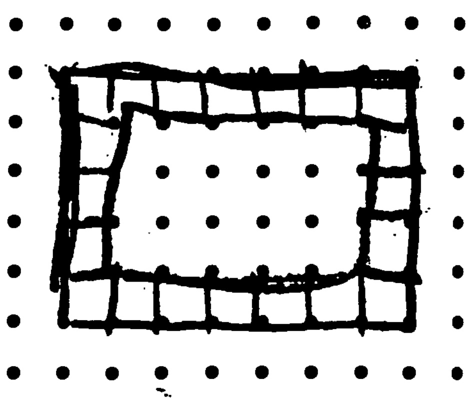

Figure 3. Jill and Eli used line segments to determine the area of a 5 by 7 rectangle.

Using the second method based on CMW lessons, Jill and Eli determined the correct dimensions of the same 5 by 7 rectangle and expected to count 35 line segments inside the rectangle as well. They drew horizontal line segments starting on the top edge and experienced conflict when they reached 35 before they were done (see Figure 3). The students stated that they had still to count the vertical segments. When I asked if they could count something else, Eli proposed unit squares and the students got their expected answer of 35. Learning to use line segments to determine dimensions, but unit squares to determine areas, was a significant representational accomplishment that allowed Jill and Eli to connect multiplication to areas of rectangles.

Naomi and Candis also had trouble distinguishing between representational features for determining perimeters and areas. Like Jill and Eli, they discussed two methods for determining perimeters and areas of rectangles drawn on dot paper, one based on dots and one based on line segments. Using the dot method, these students thought that the dimensions of a 5 by 7 rectangle were 6 and 8 and that the area was 48. Figure 4 shows how, when determining area using the second method, these students drew 20 line segments from the perimeter. The gestures on videotape made clear that the students counted line segments, not squares. They added the 24 line segments on the perimeter and got 44 as the area. In subsequent interviews, Candis continued having trouble selecting appropriate representational features when determining perimeters and areas. Sam and John had no trouble focusing on line segments when determining perimeter and unit squares when determining areas.

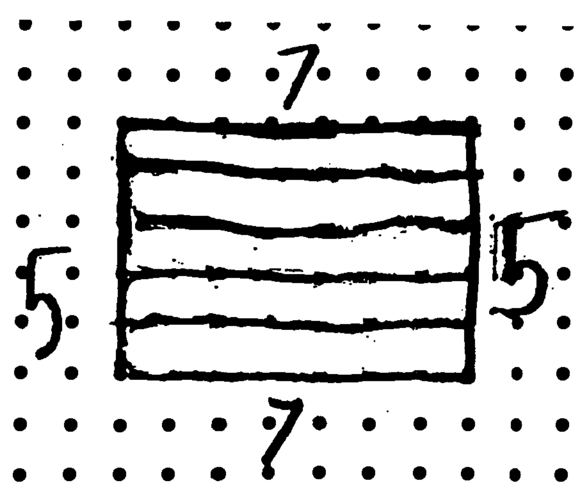

Figure 4. Naomi and Candis' method for counting line segments to determine area.

2. *Students refined the contexts in which they applied criteria for representations.*

Jill and Eli began the second interview, which occurred at the end of Phase 2, stating that rectangles, even when sketched on blank paper, had to be drawn to scale. The videotape showed the students drawing rectangles by stopping the movement of their pencils momentarily at the end of each imagined unit line segment. Eli eventually distinguished between multiplying "for fun," for example in an out-of-school game context, and multiplying to determine area, a school activity. She explained that rectangles had to be drawn to scale only when determining area.

In the third interview, Jill and Eli measured the dimensions of the rectangular interview room and drew representations on blank paper to determine the area. At first they drew rectangles to scale once more, but eventually focused on the array of unit rectangles formed when rectangles are not drawn to scale. Jill and Eli dropped their "scale" criterion for rectangular representations, a second significant representational accomplishment. I did not pursue issues of scale with Naomi and Candis because I continued to focus their interviews on representational features for determining perimeters and areas when rectangles were drawn on dot paper. Sam stated immediately in his second and last interview that rectangles did not have to be drawn to scale to determine the area of the interview room, and his comments evidenced understanding that the underlying array structure would give the same answer.

3. *Students demonstrated conceptual understanding of multi-digit multiplication.*

In the final interview, Jill and Eli developed their own correct method for multiplying two- by three-digit numbers that coordinated magnitudes of partial products, place-value, and the distributive property. They sketched rectangles, explained their method in terms of imagined arrays of unit rectangles, connected the rectangle and lattice methods, and explained how place-value is represented in the lattice method for the first time. Sam was also able to extend the rectangle method to two-digit times three-digit numbers and to connect the rectangle method to the lattice method. His explanations evidenced connections among magnitudes of partial products, place-value, and the distributive property.

Conclusions

This study extends research on multiplication by examining multi-digit multiplication and by applying fine-grained analytic techniques that have recently led to insights into students learning to use representations in other domains for solving modeling problems. The study examines how 5th grade students developed conceptual understanding of multi-digit multiplication through modeling areas of rectangles, and how they learned to model areas of rectangles by refining the representational features to which they attended and the contexts in which they applied criteria for rectangular representations. Both refinements were significant representational accomplishments. Such insights can inform future curricula that make conceptual understanding of multi-digit multiplication accessible to all students.

Note. The research reported in this paper was supported by the National Science Foundation under Grant No. REC 9806020. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*, 367-385.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State University of New York Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26* (1), 66-86.

diSessa, A., Hammer, D., Sherin, B., & Kolpakowski, T. (1991). Inventing graphing: Meta-representational expertise in children. *Journal of Mathematical Behavior, 10* (2), 117-160.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 61-85). Albany: State University of New York Press.

Izsák, A. (1999). *How Amy and Kate used criteria for algebraic expressions to develop knowledge structures for modeling the physical world*. Manuscript submitted for publication.

Izsák, A. (2000). Inscribing the winch: Mechanisms by which students develop knowledge structures for representing the physical world with algebra. *The Journal of the Learning Sciences, 9*, 31-74.

Izsák, A., & Fuson, K. (2000). Students' understanding and use of multiple representations while learning two-digit multiplication. In M. Fernández (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 715-721). Tucson, AZ.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20* (2), 147-158.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28* (3), 309-330.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA.

Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993/1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences, 3* (2), 115-163.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Stigler, J. W., Lee, S,, & Stevenson, H. W. (1990). *Mathematical knowledge of Japanese, Chinese, and American elementary school children*. Reston, VA: National Council of Teachers of Mathematics.

The University of Chicago School Mathematics Project. (1995). *Everyday mathematics*. Evanston, IL: Everyday Learning Corporation.

Notes

[1]Heretofore I will use the term "place-value" to mean base-10 place-value.

[2]All names are pseudonyms.