In M. L. Fernandez (Ed., 2000), Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 2 (pp. 714-721). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

STUDENTS' UNDERSTANDING AND USE OF MULTIPLE REPRESENTATIONS WHILE LEARNING TWO-DIGIT MULTIPLICATION

Andrew Izsák
Northwestern University
izsak@northwestern.edu

Karen Fuson
Northwestern University
fuson@northwestern.edu

We report the results of implementing a two-digit multiplication unit that relied on modeling areas of rectangles. We worked in one urban and one suburban fourth-grade classroom to determine whether such an approach could support diverse students as they learned a general computation method invented by urban students. Both classrooms outperformed U.S. fifth-graders in traditional curricula, and the suburban classroom was comparable to Japanese and Chinese classrooms. Results also suggested ways to make the unit more accessible to low-achieving students.

Introduction

We report the results of implementing a two-digit multiplication unit in urban and suburban fourth-grade classrooms. The work is part of Children's Math Worlds (CMW), a project that develops instructional materials for elementary school mathematics and that conducts research on teaching and learning as teachers use those materials in their classrooms. A main objective of CMW is to make the goals of the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) accessible to all students. The standards and principles about number and operations, representation, problem solving, communication, and equity are most relevant to the study reported here.

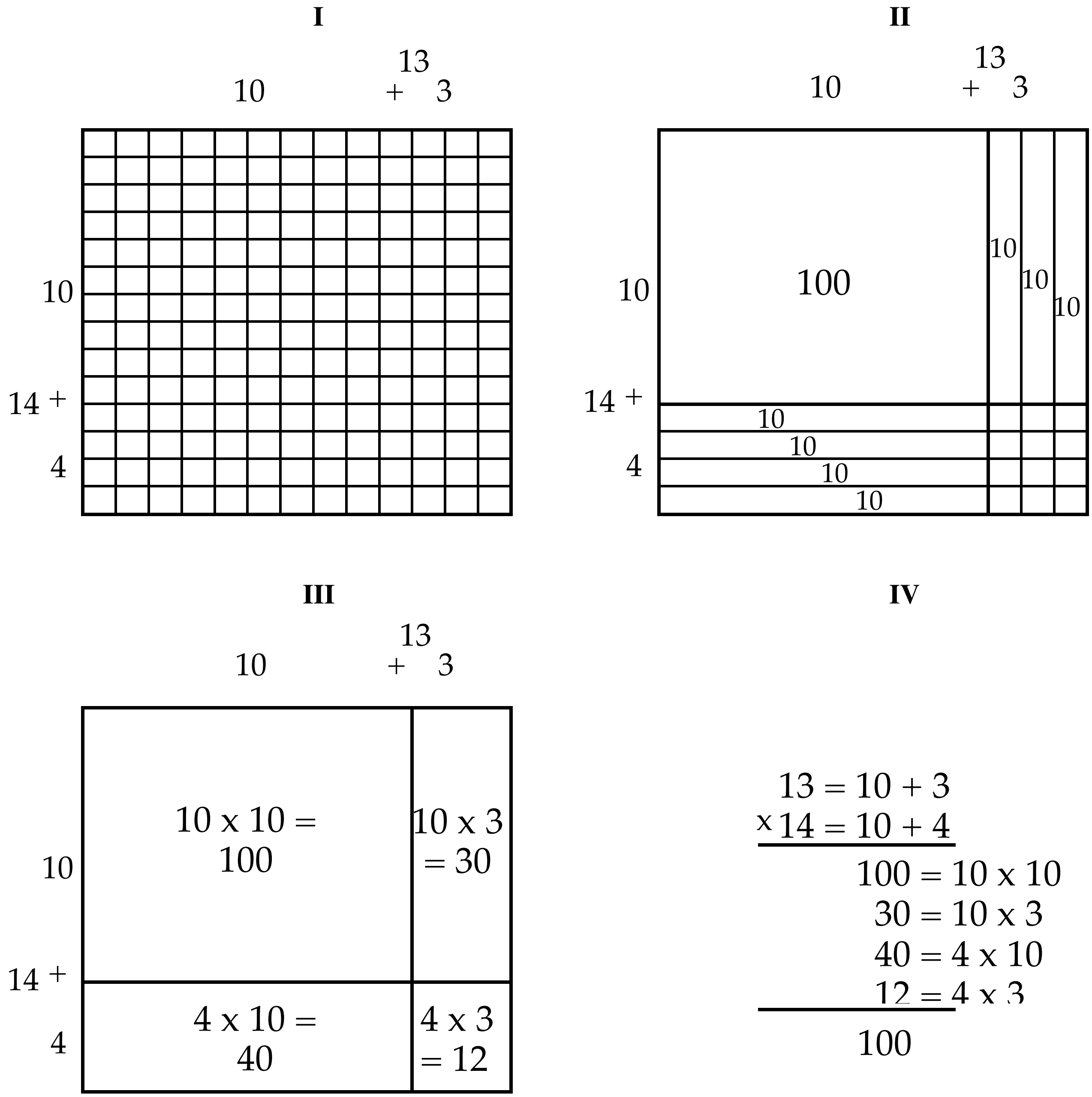
CMW combines a Vygotskiian (1978, 1986) view of teaching with a constructivist view of learning. In particular, the project investigates means by which teachers can help students take what they already know and construct culturally adapted conceptions of mathematics. Central to all CMW units (including the two-digit multiplication unit) are activities in which students use drawn representations of situations to solve problems and explain solutions to others in the class. In the course of such activities, teachers help students connect their experiences and understandings to traditional mathematical symbols, words, and procedures. Equity Pedagogy (Fuson et al., 2000) describes in more detail the principles that guide our design efforts.

The work reported here integrates and extends three areas of research: that on multiplication and division, that on place value and its role in multi-digit addition and subtraction, and that on students' understanding of representations. Extensive research has already been done in each of these literatures individually. Although existing research has investigated ways in which students might conceptualize multiplication as a model of equal groups, multiplicative comparison, Cartesian product, and rectangular area situations (see Greer, 1992 for a review), to the best of our knowledge, existing research has not examined the following question: Can whole classrooms of diverse students, including inner-city students, master two-digit multiplication using a modeling approach?

Methods and Data

We based our two-digit multiplication unit on modeling areas of rectangles both because area is a core meaning for multiplication and because this approach allowed us to help students build a general computation method based on their prior experiences modeling areas of smaller rectangles with single-digit numbers. We used a progression of three area representations that built on students' strategies for tallying unit squares and directed their sense-making toward our target computation method, a method invented by urban fourth- and fifth-grade students. We chose the target method because it shows all four quantities and all four sub-products involved in two digit-multiplication (i.e., $42 \times 36 = 40 \times 30 + 40 \times 6 + 2 \times 30 + 2 \times 6$). The rectangles afforded drawn representations of the quantities involved in the target method and so could potentially support students' understanding that the product of 2 two-digit numbers is the sum of four sub-products.

We use the example 13 x 14 to outline the progression of activities linking area representations to the target method. The first area representation (see I below) showed all of the unit squares in a 13 by 14 rectangle. We wanted teachers and students to propose and discuss strategies for grouping and counting the total number of unit squares, and then build on those contributions that led to the second area representation (II). This transition was important because the second area representation supported connections among area, base-10 place value, and the target method. (Note that for problems with larger factors, such as 23 x 34, students constructed II



by breaking apart the factors into 10 + 10 + 3 and 10 + 10 + 10 + 4 and by drawing a representation that grouped unit squares into six "100 squares," tens, and ones.) To prepare students for work with larger numbers, we had them abbreviate their work in II to create a third area representation (III). Finally, we had students connect the third area representation to the target method for multiplying two-digit numbers (IV). As students moved to problems with larger factors and products over 1,000, they found II cumbersome to work with and increasingly relied on just III and IV, and finally just on IV. We used different colors to help students connect the four sub-products of the area representations to each other and to the computation method.

We piloted the unit in one urban and one suburban classroom, each with 25 to 30 students. The teachers in these classrooms faced challenges common in the United States. Many of the urban students, and their parents, were recent immigrants and struggled with English as a second language. Nearly a quarter of the suburban students were main-streamed students with learning disabilities, and about the same proportion were bilingual.

To gather data on implementation and learning, we worked intensively with both teachers in and out of class. We observed lessons in both classrooms at least twice a week. Videotapes and field notes from classroom observations provided data on how the teachers used the materials, and hence how students actually experienced the unit. We met with teachers after school to discuss aspects of the unit that students understood, aspects that were hard for students, and ways in which the materials could be adapted to better support students' learning. We also gathered students' written work, primarily tests, and conducted in-depth, videotaped interviews with students at the end of the unit. For these forty- to fifty-minute interviews, we selected a cross-section of students from low- to high-achieving and asked them to work problems similar to those that they had done in class and for homework. These data provided access to students' strategies, understandings of the three area representations, and connections among the area representations and the expanded algorithm (IV) (or abbreviations of the algorithm).

Analysis and Results

To assess how well diverse students mastered two-digit multiplication using our modeling approach, we first analyzed the accuracy with which students multiplied two-digit numbers at the end of the unit. To put our results in context, we say more about where each class of students began at the start of the school year.

Many students at the urban school began the year still having difficulties with place value and multi-digit addition and subtraction. For example, many students lined up left-most digits when adding and could not borrow across zero correctly when subtracting. Students could perform some single-digit multiplication either by recall when the factors were small (i.e., 2 x 3) or by counting repeated groups (often on their fingers).

When analyzing the accuracy of the urban students' two-digit multiplication at the end of the unit, we found the following percent correct by item: 17 x 12 (94%), 45 x 26 (61%), 37 x 24 (56%), and 92 x 78 (56%). We traced many of the errors to near, but faulty, products of single-digit numbers (i.e., 6 x 4 = 20) and to faulty place value when multiplying multiples of 10 (i.e., 30 x 20 = 60). We gave students additional practice with single-digit multiplication and place value, re-tested, and got the following results by item: 26 x 7 (83%), 65 x 43 (61%), 40 x 9 (87%), 50 x 6 (91%), 80 x 70 (65%), and 423 x 3 (87%). By way of comparison, Stigler, Lee, and Stevenson (1990) reported international performance by fifth-grade students on multiplication problems. Percentages for Japanese, Chinese, and U.S traditional students on 30 x 60 were 73%, 74%, and 35%, respectively. Fuson and Carroll (1999) reported percentages for U.S traditional and U.S. reform (Everyday Mathematics) fifth-grade students on 45 x 26 as 54% and 78%, respectively.

Students at the suburban school began the year much better prepared than the urban students. About half had been in third-grade classes that used CMW materials the year before.

These students already had a good start on single-digit multiplication, could add and subtract multi-digit numbers accurately, and were used to working with drawn representations of situations. When analyzing the accuracy of students' two-digit multiplication at the end of the unit, we found the following percent correct by item: 17 x 12 (88%), 45 x 26 (80%), 37 x 24 (80%), and 92 x 78 (64%). Many of the errors were similar to those made by the urban students: near, but faulty, products of single-digit numbers and faulty place value when multiplying multiples of 10. We note that these results compare favorably with the international comparison of fifth-grade students cited above.

In analyzing students' understandings of the area representations and connections to the computation method, we found that some low-achieving students (mostly urban) began with a shaky understanding of basic geometry concepts. A number of urban students did not understand that opposite sides of rectangles have the same length, and some urban and suburban students were confused by the distinction between area and perimeter. By the end of the unit, both teachers reported that such students had developed a much better understanding of these properties and concepts.

When analyzing the extent to which students used the area representations as supports for multiplication methods, we found that many could use the rectangles to find products, but that this became harder as the numbers got larger. By the end of the unit, many students were using at least two methods in class, because they learned the traditional algorithm at home. In such cases, we found that students could only explain why the expanded algorithm worked. We also found cases in which students' strategies were closely tied to particular features of the area representations. For example, some students could count unit squares along edges in I to determine correct sub-products, but could not use III to determine correct sub-products. High-achieving students at both schools could articulate connections among all four representations.

Analyses of our data suggest several refinements that should make the unit more accessible to low-achieving students. One set of refinements revolve around the sequence of activities. Introducing the unit with activities to insure that students understand basic properties of rectangles and placing greater emphasis early in the unit on problems of the form 30 x 20 = 60 should reduce many of the errors that we saw during our first implementation. A second set of refinements revolve around the design of the representations. Students seemed to have a better grasp of the connections between I and IV than between III and IV. Eliminating II and redesigning III so that it contains unit squares along the top and left hand edges of each region should help students connect initial to subsequent methods for computing products of two-digit numbers.

Conclusions

We are taking steps toward curricula that provide all students the opportunity to achieve at those levels articulated by the National Council of Teachers of Mathematics. The results of our pilot study suggest that diverse students, including inner-city students, can master two-digit multiplication using a modeling approach if activities and representations are carefully designed and students are expected to understand and explain their computational methods.

The research reported in this paper was supported by the National Science Foundation under Grant No. REC 9806020. The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

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