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# Pedagogical, Mathematical, and Real-World Conceptual-Support Nets: A Model for Building Children's Multidigit Domain Knowledge

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A model of a conceptual-support net is exemplified in the domain of multidigit addition. This model can provide guidance in making curricular and classroom teaching choices about what kinds of conceptual supports to provide in a particular setting. An outline of a theory of children's multidigit conceptual structures and an analysis of classes of multidigit conceptual supports used in classrooms provide the background for the Conceptual-Support Net model. Issues of matches and mismatches among conceptual supports, solution methods, language of the learner, mathematics achievement level of the learner, and conceptual and procedural accessibility of various multidigit algorithms are then discussed. The presentation of the Conceptual-Support Net model specifies how this model differs from traditional instruction and from usual ways of using manipulatives in the classroom. Summaries of several different multidigit addition solution methods indicate how different quantity conceptual supports might facilitate different solutions. How to facilitate children's reflection and movement along developmental trajectories to more advanced and coherent numerical understandings and methods is then discussed. An overview of the importance of social processes in using conceptual-support nets closes the paper.

In this paper I present a model of conceptual-support nets that can be used to help children build mathematical domain knowledge. This model is exemplified here in the domain of multidigit addition. The model arose from a 15-year programme of research in U.S. classrooms using different kinds of pedagogical (e.g. base-ten

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blocks) and real-world (e.g. money) referents for multidigit numbers and multidigit addition and subtraction calculation. The goals of this research were to build a model of children's understanding in the domain of place value and multidigit addition and subtraction and to devise classroom meaning supports and participant structures that would enable children to construct this domain knowledge and learn personally meaningful multidigit calculation methods. Multidigit addition and subtraction were viewed as contexts in which children could build powerful and general understandings of our cultural ways to say, write, and use whole numbers greater than or equal to 10.

The first phase of the research focused on teachers using base-ten blocks to help children learn to understand potentially meaningful standard U.S. addition and subtraction algorithms (Fuson, 1986; Fuson & Briars, 1990). The second phase examined second graders working in small groups to invent base-ten block and written numerical methods of adding and subtracting horizontally presented four-digit numbers (Burghardt & Fuson, 1998; Fuson & Burghardt, 1993, 1997; Fuson, Fraivillig, & Burghardt, 1992). The third phase was an intensive six-year effort in urban Latino classrooms (the Children's Math Worlds Project) to develop conceptual supports that fit into the ecology of urban schools and that were adapted to the lives of urban Latino children (Fuson, Hudson, & Ron, 1997; Fuson, Lo Cicero, Hudson, & Smith, in press; Fuson, Lo Cicero, Ron, et al., 1998; Fuson, Perry, & Kwon, 1994; Fuson, Perry, & Ron, 1996; Fuson & Smith, 1995; Fuson, Smith, & Lo Cicero, 1997). During all three phases, I carried out extensive discussions with researchers around the world on the teaching of place value and multidigit addition and subtraction in their languages and their countries and read many research studies from other countries. Also, together with Youngshim Kwon, I carried out research with Korean children (Fuson & Kwon, 1992a, 1992b) and analysed effects of different language structures on multidigit thinking (Fuson & Kwon, 1992/1991, 1991). Together these three phases, and the extensive discussions with others, provide a depth of experience that allows me to generalise across many different kinds of conceptual supports, leading to the model proposed here.

My work on conceptual-support nets is situated within other work on Vygotskiian theory. In the Children's Math Worlds Project, we examined in our classroom research various ways in which Vygotsky's (1934/1962, 1934/1986) hypothesised interaction between spontaneous everyday concepts and formal scientific concepts can occur. We described a referential classroom in which concrete and drawn references for mathematical words and symbols helped children link their spontaneous concepts to their developing mathematical concepts (Fuson, Lo Cicero, Ron, et al., 1998). We developed a mathematics equity pedagogy that helped teachers bridge from where children were to facilitating understanding of ambitious above-grade-level mathematical topics

(Fuson, De La Cruz, Smith, et al., in press). We carried out analyses focused on domain analyses of real-world situations, conceptual analyses of children's thinking in a domain, and analyses of the cultural semiotic tools (the mathematical words and written notation) in a domain. These kinds of analyses were then compared to ascertain characteristics of everyday situations and of the formal mathematical words and notations that can direct and constrain the mathematical understandings that children construct. We view these as examples of the more general tension between support and constraint offered by various signifying systems (Sinha, 1988, 1989). When everyday examples did not offer sufficient direction for mathematical words or symbols that are new, unclear, or misleading, we designed pedagogical supports to help children build correct meanings. Our model of conceptual-support nets describes the relationships among three concentric related triads: real-world words, notation, and referents; mathematical words, notation, and referents; and meaningful words, notation, and referents. Such a net can be developed for any mathematical quantity domain (e.g. multiplication of single-digit numbers, fractions).

The choice of conceptual supports in a curriculum or in the classroom depends on an analysis of the mathematical structure of the real-world quantities in the domain and an analysis of the structure of the cultural words and of the written notation for those quantities. It also depends on some theory of the conceptual structures used by children in that domain. We first briefly outline the theory of children's multidigit conceptual structures that arose from the above research. Our analysis of multidigit real-world quantities and our analysis of the structures of the multidigit cultural tools (words and written notation) are then interwoven with a presentation of our analysis of the kinds of multidigit conceptual supports used in classrooms. Issues of matches and mismatches among conceptual supports, solution methods, language of the learner, mathematics achievement level of the learner, and conceptual and procedural accessibility of various multidigit algorithms are then treated. This will provide an indication of the complexities of using conceptual supports in this domain.

I then present and discuss the Conceptual-Support Net model and discuss how it differs from traditional instruction and from usual ways of using manipulatives in the classroom. This model, along with an understanding of children's conceptual structures in a domain, can provide guidance in making curricular and classroom teaching choices about what kinds of conceptual supports to provide in a particular setting. I then summarise several different multidigit addition solution methods to indicate how different quantity conceptual supports might facilitate different solutions. These methods then allow me to address several issues concerning how to facilitate children's reflection and movement along developmental trajectories to more advanced and coherent numerical understandings and methods. I close by overviewing the importance of social processes in using conceptual-support nets.

# ANALYSIS OF THE MATHEMATICAL DOMAIN

## A Model of Conceptual Structures Used in the Domain

I think of conceptual structures as hypothesised categories of quantitative activity that seem useful in understanding teaching and learning in a domain. For me, a conceptual structure for multidigit numbers is (a) a structuring of—a particular viewing of—the quantities, number words, and written numerals so that these can be understood, counted, added, or subtracted in particular ways and (b) the knowledge required to understand, count, add, or subtract in those ways. My model of conceptual structures has evolved from progressive reflections on extensive experiences interacting with children in this domain and on the experiences of others as communicated in papers and in conversations. The model presented here is a revision of earlier models; it uses simpler language to describe children's conceptual structures.

Earlier literature identified three correct conceptions used by children in the United States: a unitary conception in which children count a two-digit quantity by ones, a sequence-tens conception in which they count by tens and then by ones, and a separate tens and ones conception in which the units of ten and the units of one are counted separately (see Fuson, 1990, for an extensive review of this literature and Fuson, 1992a, 1992b, for later briefer reviews). For example, when counting 3 ten-bars (each made from 10 unifix cubes) and 2 extra cubes, children using a unitary conception would count all 32 of the unifix cubes (1, 2, 3, …, 32), children using a sequence-tens conception would count "10, 20, 30, 31, 32", and children using a separate-tens conception would count "1, 2, 3 tens and 1, 2 ones. 32".

Children also use a concatenated single-digit (CSD) conception (see Fig. 1) in which the two-digit number is thought of as two separate single-digit numbers. Because any single-digit number can be added to or subtracted from any other, this meaning cannot direct or constrain addition or subtraction methods. It leads to many well-documented errors (e.g. see VanLehn, 1986, for a discussion and examples). This concatenated single-digit meaning arises when insufficient opportunities are given to children to link accurate multidigit quantity meanings to the written numerals used in adding and subtracting. Many teachers and children have this conception. It is exemplified by the remarks of three very good second-grade teachers after they had taught a multidigit unit from the Children's Math Worlds (CMW) Project: "We loved the multidigit unit. Now we understand what those little 1s we borrow and carry really are! They are 1 ten or 1 hundred or 1 thousand. We never understood that before." These teachers, as many others, learned multidigit addition and subtraction as a "dance of the digits" in which each digit was thought of as that many ones (a CSD conception). Birch Burghardt and I also call CSD the "constantly seductive digits" conception, because even children who have quantity meanings for the place-value digits may fail to use them in new

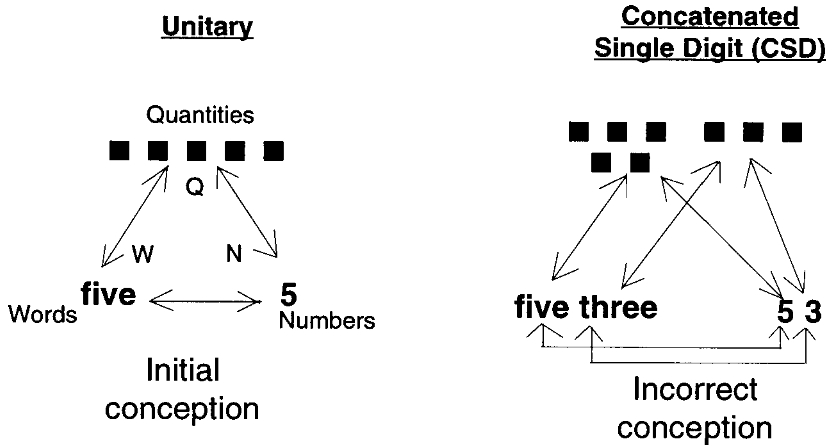



FIG 1. Unitary triad (quantity, number word, written notation) and concatenated single-digit triad (common incorrect multidigit conception derived from the appearance of the written numbers).

situations (and sometimes in old situations) because the digits look like single digits: they constantly seduce children into the CSD conception, which is prone to many errors (Fuson & Burghardt, in press). For example, when adding 327 + 84, many children write the 8 under the 3 because they both look like single digits. Thinking of the meaning of 3 hundreds and 8 tens (or eighty) could direct a correct alignment.

In Fuson, Wearne, et al. (1997) and in Fuson, Smith, and Lo Cicero (1997), we extended this earlier work to a model named for the five correct conceptions described in that model: unitary, decade, sequence-tens, separate-tens, and integrated conceptions (the UDSSI Triad Model). A revised model, renamed the Ct-PIV triad model, is presented in this paper. All of these conceptual models assume a triad of relationships between quantities, number words, and written number notation. With single-digit numbers, there are three 2-way links in the triangle formed by these quantities, words, and marks (see the left side of Fig. 1). Each 1-way link relates an aspect initially seen or heard to the elicited aspect: For example, I hear "five" and think/see/can write 5 (bottom left-to-right arrow) or I see five cookies and think/can say "five" (left arrow from top to bottom). The user of the CSD conception (right side of Fig. 1) constructs these six relations for each of the pairs of single digits in a two-digit number.

The Ct-PIV triad model is shown in Fig. 2. The outer three conceptual structures arise from children's experiences with counting; they are primarily different quantity meanings for the words of the counting sequence that also become linked to written numbers. The innermost triad, the Place-Value-Tens and Ones, primarily arises from meanings for the written two-digit numbers (e.g. 53 as meaning 5 tens and 3 ones). A final mature conception, the Integrated Tens and

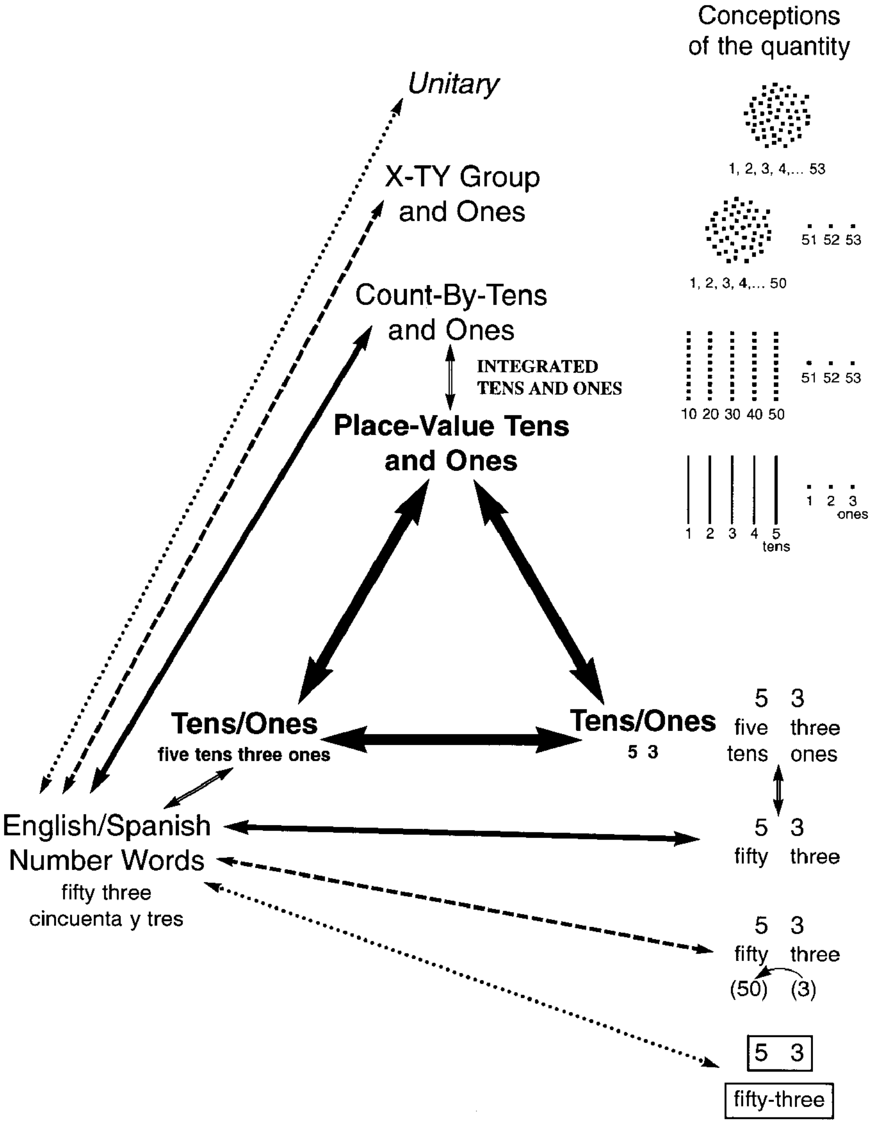



FIG. 2. A developmental sequence of conceptual structures for two-digit numbers: The Ct-PIV triad model.

Ones conception, relates the Place Value conception to the most advanced counting conception, the Count-By-Tens and Ones conception.

Children's earliest conceptual structure, the unitary conception, is on the outside of Fig. 2. All children begin with a unitary conception that is a simple extension from the unitary triad for single-digit numbers: The separate number words (e.g. sixteen) and the two digits (e.g. 16) do not have separate quantity referents. The whole number word (e.g. sixteen) or whole numeral (e.g. 16) refers to the whole quantityas a group of that-many things.

As children learn the English counting words to 100, they begin to notice the X-ty, X-ty–one, X-ty–two, …, X-ty–nine pattern in these words. These number words have a meaning as a group of X-ty things plus some ones (e.g. "twen-ty five" is a group of twenty things plus five things, "eight-ty two" is a group of eighty things plus two things). Murray and Olivier (1989) called this a decade and ones conception, a term I used in the UDSSI model. Now I propose to call this conception the "X-ty group and ones" because "decade" ordinarily means "ten years" rather than all of the words that name the tens words in English: twen-ty, thir-ty, four-ty, etc.: the X-ty words.

The written numbers for this X-ty group and ones conception have a meaning of the ones as written on top of the X-ty group (the arrow in Fig. 2 shows the 3 going on top of the 0 in 50). This conception leads some children to write numbers as they sound: as 50 and then a 3, so 503. This error is frequent for children speaking English, French, Afrikaans, Dutch, and Spanish (literature reviewed in Fuson, 1990; in Fuson, Wearne, et al., 1997; and in Power & Dal Martello, 1997). In the CMW project, we talked about an invisible zero hiding under the ones for children using this meaning.

The X-ty group and ones conception, and the count-by-tens and ones conception that develops out of it (discussed next), are constructed only by children whose languages have special words for the x-tens groups. Many European languages do have such structures, though of course the words for the x-tens groups vary, and the suffix is not -ty except in English (e.g. it is -*enta* in Spanish). East Asian children whose number words are modelled on the Chinese words have number words that say the place values: 53 is said "five ten three" and 12 is said "ten two." English, and other European languages, are more regular for hundreds and thousands than for tens and do name the place values for those places: 5832 is said "five *thousand* eight *hundred* thir*ty* two" (some, like Spanish, may still have some irregularities for these places). Thus, the educational teaching and learning task for East Asian teachers and children is much simpler than for European teachers and children: There are only the unitary and the place-value meanings of the number words, and the place-value meanings direct meaningful addition and subtraction. European children must learn a more complex counting sequence, must come to understand multiple meanings of the words and of the written numbers, and then integrate these various meanings. For simplicity, we use here the term X-ty groups to mean such a conception for any language that has special words for the x-tens groups.

The count-by-tens and ones conception develops out of the X-ty group and ones conception as children become able to count by tens and to form conceptual units that are groups of ten single units. The count-by-tens conception was called the sequence-tens conception in the UDSSI model to emphasise its source within the counting sequence. When children first use a count-by-tens conception, they may not know that there are five tens in fifty, but they can find out by counting "10, 20, 30, 40, 50" while keeping track of the five counts.

Some children have experiences in which they come to think of a two-digit quantity as composed of two kinds of units: units of ten and units of one. When adding or subtracting two-digit numbers in this way of thinking, children decompose numbers and count, add, or subtract the units of ten and then count, add, or subtract the units of one (or vice versa), leading to our initial designation of this way of thinking as the separate-tens and ones conception. In Fig. 2, we show these units of ten as a single line to stress their (ten)-unitness, but the user of these units understands that each ten is composed of ten ones, and can switch to thinking of ten ones if that becomes useful. I now propose to call this conception the place-value-tens and ones conception to emphasise its origin in the meaning of our place-value notation: 53 as meaning 5 groups of ten and 3 single ones.

Whether children construct the count-by-tens or the place-value-tens conception seems to depend heavily on their learning environment, though individuals in the same classroom may construct either first. Which is first may partly depend on whether a child focuses on the words, which facilitate the count-by-tens conception, or on the written numbers, which facilitate the place-value-tens conception.

Children may eventually construct both the count-by-tens and place-value-tens conceptions and relate them to each other in an integrated conception (these connections are shown in Fig. 2 as the double arrows). Such children connect fifty and five tens, and the written marks 53 can then take on either quantity meaning (fifty-three or five tens three ones).

Each of the two-digit conceptual structures was originally conceptualised as a triad of six relations. However, only the place-value-tens and ones conception has direct links between quantities and marks, and these only occur where the quantities of tens and ones are small enough to be subitized (immediately seen as a certain number of units) or are in a pattern. The other conceptions must relate quantities to written marks via the number words by counting. Therefore the link between quantities and marks is not drawn in Fig. 2 for these conceptions.

There are many individual pathways that children take to construct all of the conceptions and all of the links in the Ct-PIV triad model. For those children in classrooms that do use conceptual supports, pathways are influenced by the supports. However, there also is much variation in what children construct from the same conceptual supports: Children see and hear things differently, and they enter experiences with different knowledge. Sadly, many children in the U.S., and in other countries speaking European languages, do not build many of these conceptions. Instead, they use the concatenated single-digit conception while carrying out multidigit addition and subtraction, and they make many kinds of well-documented errors, particularly in subtraction. The Ct-PIV triad model is not a model of development (except that all children begin with the outer unitary triad), because many children do not build all of the conceptions and many different learning paths exist for building up the many links relating the different conceptions. The model is, rather, an outline of the conceptual structures necessary for full understanding of two-digit numbers if one speaks a European language having an X-ty structure for the two-digit number words.

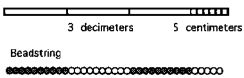
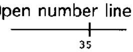
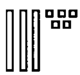
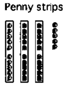
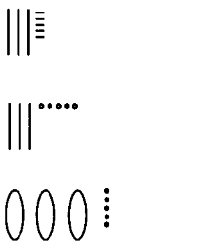
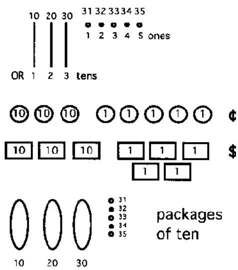
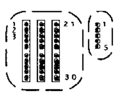
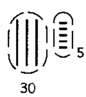
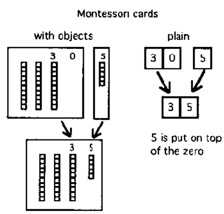
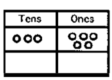
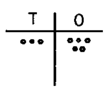
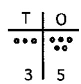
## CLASSES OF QUANTITY CONCEPTUAL SUPPORTS

Quantity conceptual supports are designed to present the mathematical attributes of the quantity so that these attributes can be apprehended and understood by learners. Some such referents are necessary if children are to understand the quantity meanings of the two-digit words and notations. The kinds of multidigit quantity conceptual supports used in schools typically present attributes of the number words or of the written numerals. Each kind of conceptual support potentially facilitates understanding of some aspects of the Ct-PIV triad model but not of others. These facilitations are discussed in the next section. Issues concerned with moving from use of conceptual supports to understood and explainable numerical methods are discussed after the model of the conceptual-support net is presented.

Classes of common kinds of two-digit quantity conceptual supports are shown in Table 1. The first class, the size tens and ones, shows the quantitative meaning of groups of ten units and single units through their differing size. The X-ty groups and ones embodiments stem from the special structure of the European number words as X-ty ones plus some ones (thirty-eight is thir-ty plus eight). The Montessori cards are designed to help children relate this meaning of the words to the appearance of the written numerals as not showing the 0 by having the ones cover the 0 in the decade number.

The concatenated-single-digit embodiments look like written numbers in that the tens and the ones quantities look identical and are differentiated only by location. Some school place-value conceptual supports use colour as well as location to differentiate tens and ones (e.g. an abacus or coloured chips in which ones are red, tens are green, hundreds are blue, etc.). However, this introduces an extraneous meaningless feature that can be used by children instead of left–right position. Colour is not present in place-value numerals, and colour has no quantity meanings. Therefore, either size embodiments that clearly present the quantity meanings of ones, tens, hundreds, and thousands or a CSD embodiment that uses only one colour of chips (so that the poker-chip computer looks maximally like

TABLE 1
Classes of Object and Drawn Conceptual Supports for Two-digit Numbers

| *Type of Embodiment* | *Embodiment Medium*: *Objects* | *Drawn Objects* | *Drawn Numbers & Objects* |
|---|---|---|---|
| SIZE TENS & ONES | | | |
| Length |  |  | Meter stick<br>Number line<br>Open number line <br>Thermometer |
| Area (folded length) | Gold bars <br>Bead Frames: ten rows of ten movable beads | | 100 number grid: 10 rows or columns of ten numbers each |
| Groups | Base-ten blocks or unifix cubes <br>Penny strips  |  |  |
| X-TY GROUP & ONES GROUP | Any of the objects above can be thought of as decade objects: 30 + 5 where both are counted by ones.  |  | Montessori cards  |
| CONCATENATED SINGLE-DIGIT TENS & ONES | Fingers as ones and as tens<br>Abacus<br>Colored chips<br>Poker chip computer  |  |  |

written two-digit numbers) seem superior to embodiments that use a meaningless attribute like colour.

Within each major subclass in Table 1 are shown three possible conceptual-support media: objects, drawn objects, and drawn numbers and objects. We began to use drawn objects rather than objects in the urban Latino project for several reasons. They were much easier for teachers to manage, cost only the paper on which children drew, revealed variations and problems in children's thinking because children produced the drawings, were available as records of thinking after class so that the teacher could see everyone's thinking, facilitated communication about particular solution methods, and allowed children to link mathematical notation to their drawings (numbers with drawn objects). Children initially had some experience with objects grouped into tens, and then they began activities to build up meanings for their size drawings (e.g. Fuson & Smith, 1998; Fuson, Smith, & Lo Cicero, 1997).

Drawings allow children to link multidigit numbers to size quantity drawings. Thus, they address one of the major problems in using object embodiments to support children's mathematical understanding. Many teachers do not necessarily understand that the major function of embodiments is to help children construct meanings for mathematical words and written notation (e.g. multidigit numbers). Such construction of meanings requires that the objects become the quantity referents for the words and the numbers, so they must be tightly linked to these. Teachers rarely do such linking. Instead, they use the embodiments for several or many days alone without numbers and then shift very rapidly (often in one day) to using the numbers alone. Hart (1987) reported this for British teachers using base-ten blocks; she also found that this pattern was typical for all embodiments. She characterised (Hart, 1989) this separation of use by a child's summary, "Blocks is blocks and sums is sums" (the British use of "sums" here means all kinds of written calculations).

The use of drawings of, or at least drawings on, a conceptual support can clarify aspects that are not so clear just by counting activity. For example, the rather considerable difficulty many children have in conceptually understanding the tenness of vertical jumps on a hundreds grid (e.g. Cobb, 1995; Fuson, 1996) might be repaired by drawing on a hundreds grid. For example, to see 10 more than 36, a child could mark the next four squares to 40 and the next six squares to 46 rather than just count the ten squares. Seeing the four and the six squares together making a ten might visually support a conception of tenness. Doing the same drawings repeatedly (perhaps in alternating colours to differentiate the tens) would show the repeating 4 + 6 patterns that unfold the 36, 46, 56, 66 sequence of ten more each time. This pattern is less obvious in the actions of counting.

Many more conceptual supports have been used in research and in classrooms than we are able to show in Table 1. The table shows conceptual structures from seven intensive classroom projects focused on helping children build multidigit meanings; four projects were in the United States, two in the Netherlands, and one was in South Africa. All of these projects resulted in higher than usual levels of understanding of place-value and of multidigit calculation as measured on a range of place-value and calculation tasks including explaining methods (see the individual reports for more details). In the United States, the Cognitively Guided Instruction (CGI) Project (Carpenter, 1997; Carpenter, Ansell, Levi, Franke, &

Fennama, 1995) and the Conceptually Based Instruction Project (Hiebert & Wearne, 1992, 1996) used size-tens-and-ones groups (base-ten blocks or unifix cubes in groups of ten). The Problem Centered Project used size-tens-and-ones area (100 number grid: Cobb, 1995) and groups (unifix cubes in groups of ten and drawings of packages of ten candies and single candies: Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). In our Children's Math Worlds Project, we have at various times used length supports (base-ten blocks as lengths; thermometers), groups supports (base-ten blocks, drawn blocks, and drawn blocks with numbers; penny/dime strips, drawn penny strips and pennies, drawn dimes and pennies with numbers; collections drawn as dots and ten-sticks of dots), X-ty group and ones (Montessori cards with objects, numeral Montessori cards), and CSD conceptual supports (finger ones and finger tens: using fingers to count by ones or count by tens). Recent Dutch approaches (Beishuizen, 1993, 1997; Gravemeijer, 1997) have used length and area supports (the bead string: ten beads of red alternating with ten beads of yellow, the open number line: a line segment drawn on paper and then labelled with numerical jumps backwards or forwards, and the 100 number grid). The South African Problem Centered Mathematics Project has used groups (bead frames), and some classes have used Montessori cards without object drawings (Fuson, Wearne, et al., 1997).

Clearly all of these conceptual supports vary greatly in their complexity and in the aspects of words or place-value numerals they seek to support. The rest of the paper seeks to clarify some of these complexities and to indicate how individual children may interpret and use a given support differently. This underscores the fact that conceptual supports are interpreted by their user according to the aspects of the support noticed by the user and the conceptions that the user brings to that use. Thus, social processes are vital in acculturating users to correct mathematical meanings and in ascertaining meanings that different users are bringing to their use. Such social processes are discussed at the end of the paper. For an overview of other kinds of conceptual supports used in various mathematical domains, see English and Halford (1995).

## INTERACTIONS OF KINDS OF CONCEPTUAL SUPPORT WITH SOLUTION METHODS, LANGUAGE, AND ACHIEVEMENT LEVELS

Considerable research has focused on differences in children's understanding according to the three subclasses within the size embodiments (length, area, groups). An overview of this research will indicate the complexities of choosing conceptual supports by outlining some interactions of conceptual supports, solution methods, language, and children's achievement level. I first outline five reasons for such variation. I then analyse some relationships between solution methods, conceptual supports, and children's native language. I then describe research reporting relationships among children's mathematics achievement level, conceptual supports, and solution methods. These examples provide the backdrop for the introduction of the conceptual-support net model.

Length conceptual supports (e.g. the Dutch empty number line) especially facilitate *sequence counting solution methods* in which children begin with one number and move up or down the number-word sequence. This is because children can just move to the right (up the sequence) or to the left (down the sequence) on the length number line. The groups conceptual supports especially facilitate *decomposition solution methods* in which groups of tens from both addends are combined separately from the combining of the ones from both addends because the groups supports show all the tens and all the ones at the beginning. Across various projects that use either length or groups supports, many children do use the methods facilitated by their instructional supports. However, some children in all projects use methods that are not so obvious with their conceptual supports. For example, some children in our CMW classrooms and in CGI classrooms use the groups conceptual supports for sequence counting solutions that build up a running total within the sequence. Decompose tens and ones methods also appear in Dutch classrooms on the empty number line. Important sources for the variations are discussed below.

## Sources of Individual Variation of Solution Methods

First, a focus either on number words or on the written number marks may lead a child to a particular class of methods. Some children seem to be pulled by the number words and thus think predominantly (or at least initially) with count-by-tens conceptions and methods. Other children seem to be pulled by the written numbers and think in terms of the place-value-tens and single ones and use decompose methods. Whether these preferences reflect more general dispositions toward oral versus visual thinking is not clear, because so few data exist concerning these individual differences in uses of methods. These differences clearly also could stem primarily from the learning history of a child (e.g. being focused on the count-by-tens conception or methods by family members or by a peer at school).

Second, how well children can count to 100 by tens and ones can constrain the initial method used by a child. For children who cannot count to 100, using place-value-tens and ones words and conception is easier. A child cannot begin to construct a count-by-tens conception until she or he knows the X-ty words in sequence. Furthermore, we know that the counting sequence to 18 must be quite automatised for children to begin to count on/up within it. Therefore, the count-by-tens list, and the count-by-tens list embedded within the unitary count by ones to 100, may need to be well automatised for children to use it in solution methods. This may be one reason why less-advanced Dutch pupils use the place-value decompose-tens-and-ones methods instead of the sequence methods (see below).

Third, the two-digit conceptual web is very complex, and children have to construct it piece by piece. Early successes on the count-by-tens path or the place-value-tens path may start a child down that path. Brief comments by a peer, sibling, or parent may be enough to facilitate an initial path. Even in a classroom where activities are designed to help children construct both count-by-tens and place-value-tens, children still cannot think both ways simultaneously. A child begins with one meaning first.

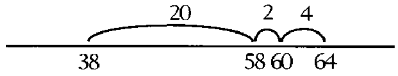
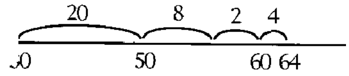
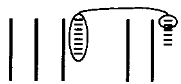
Fourth, the sequence of problems given may affect the solution paths taken by individual children. This is discussed below for Dutch children in the Beishuizen (1993) study. The South African Problem Centered Mathematics Project found that teachers who gave many two-digit addition problems before giving two-digit subtraction problems had many more children who made subtraction errors. They incorrectly generalised from the addition method "Add the X-ty groups and add to that both ones" to the subtraction method "subtract the X-ty groups and subtract both ones" instead of adding on the ones from the initial total and subtracting the ones of the subtrahend.

Fifth, the number words used by children sometimes facilitate one kind of method more than another. European X-ty and ones words suggest methods that build up or take away totals within the counting sequence. Place-value-tens and ones words suggest decomposition methods that add the tens and ones separately. Most classrooms do not emphasise the place-value-tens and ones words as we do in Children's Math Worlds, so many children speaking European X-ty-groups-and-ones number words may invent sequence methods by default or have difficulty understanding a decomposition method taught by the teacher. European number words with the ones said first also may suggest decomposition methods, at least to some less-advanced children (see below).

## Relationships among Solution Methods, Quantity Conceptual Supports, and Language

Table 2 shows one of the most frequent two-digit addition solution methods that build up within the counting sequence and one of the most frequent methods that begins by decomposing the tens and the ones. Each method is carried out with a length and a groups conceptual support. Each method is also shown in three different kinds of words (all translated into English): (1) X-ty and ones words (e.g. English or Spanish), in which the X-ty word precedes the ones word (the form is "twen-ty six"), (2) ones and X-ty words (e.g. Dutch or German), in which the ones word precedes the X-ty word (the form is "six and twen-ty"), and (3) place-value-tens and ones words that state the number of tens directly (e.g. Chinese, Japanese, Korean: the form is "two ten six"). In our Children's Math Worlds Project, we used English (or Spanish) tens and ones words that expanded the East Asian form to "two tens and six ones" as well as using the ordinary English (or Spanish) X-ty and ones words. Some children used each form in carrying out their multidigit addition or subtraction.

TABLE 2
Two-digit Addition Solution Methods for 38 + 26 by Quantity Conceptual Support and Language

| *Type of Conceptual Support*<br>*Language* | *Count/Add-up Method: Begin with One Number, Add Tens, Then Add Ones* | *Decompose Method: Decompose Tens and Ones, Add Tens, Then Add on Ones* |
|---|---|---|
| Length: Empty number line |  |  |
| English/Spanish:<br>X-ty + ones words | 38 + 20 → 58 + 2 → 60 + 4 → 64 | 30 + 20 → 50 + 8 + 2 → 60 + 4 → 64 |
| Dutch/German:<br>ones + X-ty words | 8and30 + 20 → 8and50 + 2 → 60 + 4 → 4and60 | 30 + 20 → 50 + 8 → 8and50 + 2 → 60 + 4 → 4and60 |
| East Asian:<br>number of tens + ones words | 3T80 + 2T → 5T80 + 20 → 6T + 4 –> 6T40 | 3T + 2T → 5T + 80 → 5T80 + 2 → 6T + 4 → 6T40 |
| Groups: Drawn ten-sticks + dots |  |  |
| English/Spanish:<br>X-ty + ones words | 38 + 10 → 48 + 10 → 58 +2 → 60 + 4 → 64 | 30 + 20 → 50 +10 → 60 +4 → 64 |
| Dutch/German:<br>ones + X-ty words | 8and30 + 10 → 8and40 + 10 → 8and50 + 2 → 60 + 4 → 4and60 | 30 + 20→ 50 +10 → 60 + 4 → 4and60 |
| East Asian:<br>Number of tens + ones words | 3T80 + 10 → 4T80 + 10 → 5T80 + 2 → 6T + 4 → 6T40 | 3T + 2T → 5T + 1T → 6T + 4 → 6T40 |

With each conceptual support, the counting or adding for a given solution method can be carried out at different levels of abstraction/abbreviation. Initially, counts will be only of individual tens and ones. In this case, the top-left solution method in Table 2 would show two hops of 10 followed by six hops of 1. Gradually children become able to count on or add on larger numbers of tens or of ones, leading to the advanced methods shown in Table 2. The two methods on the left show the advanced step of using an X-ty word as a pivot in adding the ones: When adding on the 6 to 58, a child adds on 2 to make 60 and then adds the rest of the 6, 4 more, to make 64. The same method is used on the top right when adding on the ones after first combining the two tens numbers (30 + 20 = 50). In the method on the bottom right, a child can physically make another group of ten to make the 60 rather than counting up to the 60. This method thus shows the general strategy of making another ten (when possible) which is used in written numeral methods.

In order to advance to more general or abbreviated methods such as those in Table 2, many children need learning experiences in the classroom to help them learn prerequisites. For example, learning the first step of the left methods (38 + 10) is not an easy one initially. Most children need to count on ten ones from 38 to find 48. The pattern is obvious to adults, but not to many children. Seeing, and talking about, the numerals as tens and ones can be helpful: 38 is 3 tens and 8 ones, so 1 more ten would be 4 tens plus the same 8 ones. Similarly, children need opportunities to learn and know rapidly how many more to the next ten (e.g. 58 plus 2 more will get to the next X-ty word) in order to carry out the method of using the decade as a pivot. Thus, conceptual supports can be useful in enabling children to carry out and communicate solution methods, but teachers must also use them in class activities that are structured to help children learn knowledge that is required for more advanced methods. Without such activities, children may continue to use unitary methods of counting all of the ones, even in third or fourth grade and even in projects focused on meaning-making (Cobb, 1995; Drueck, 1997; Lo, Wheatley, & Smith, 1994; Steinberg, Carpenter, & Fennema, 1994).

The groups embodiments do not lend themselves as well to counts other than by ones or tens, because all of the quantities are there. With the empty number line, children generate only the numbers of the counting sequence they need, so they can abbreviate and collapse steps more easily. However, the groups methods do generalise easily to written numeral methods of adding instead of counting (some of these are shown and discussed later in Table 3). This is because all parts of both numbers can be made or drawn first and thus are present to be combined or counted in various ways.

How the different order of the X-ty words and ones words in English/Spanish and Dutch/German affect children's thinking is not yet clear. This issue was discussed at a conference in the Netherlands by researchers from the Netherlands, Germany, the United States, and the United Kingdom (Beishuizen, Gravemeijer, & van Lieshout, 1997). To many native speakers of English, the reversed ones-and-then-X-ty words of Dutch and German (hearing "eight and thirty plus six and twenty") seem to emphasise the separateness of the X-ty and ones portion of the number: The problem sounds as if you need to add four separate numbers. In contrast, hearing the same problem as "thirty eight plus twenty six" sounds more like adding just two numbers. Therefore, Dutch and German words may predispose children to use decomposition methods. This may be especially true for less-advanced children whose count-by-tens conceptions may be weaker and therefore less able to overcome the separating suggestion of the words themselves. In contrast, English words (and others like them in which the X-ty and ones portions elide together to sound more like a single number) may suggest more those methods that begin with one number—that is, the sequence counting on/adding on methods. This effect of number words may be one reason that many Dutch researchers have used a length conceptual support (the empty number line) and many U.S. researchers have used groups conceptual supports: Each is (unconsciously) trying to support children to construct the multidigit conceptual structure that is less clear in the child's number words. However, at the conference there was strong disagreement within the Dutch and German native speakers concerning this analysis. Some agreed with it, and some disagreed with it (i.e. did not think that Dutch or German words predisposed children to the separating decomposition methods). These different intuitions may indicate that individual differences exist—and therefore would exist in children—in what the number words do and do not suggest and facilitate.

This contrast between methods suggested by the words and those suggested by the conceptual support may, then, be a reason that children in most countries invent and use both kinds of methods, even in a classroom situation in which many other children are doing a different method. It may, of course, be even more complex than this. Each kind of word may facilitate particular kinds of multidigit knowledge. For example, the reversals in German and Dutch may make it easier for children to see or to say the pattern in jumping by tens: "Eight and thirty, eight and forty, eight and fifty, eight and sixty" seems easier than "thirty eight, forty eight, fifty eight, sixty eight" both conceptually and procedurally (you can elongate the "eight" while thinking of the next X-ty word). Or the order of English and Spanish may make it easier to learn the initial cardinal knowledge that "50 + 8 is 58 (fifty plus eight is fifty eight)". Many urban children in our Children's Math Worlds Project initially do not know this and count on from fifty to find out that the sum is fifty eight. So even such a pattern involves complex integrations of sequence counting meanings (fifty eight as the word I say after fifty seven and before fifty nine) and cardinal meanings (fifty as telling how many in a group, eight as telling how many in a group, and fifty eight as telling how many in the total group made from those two groups).

Most adult native speakers of European languages reading this paper have constructed the whole web of multidigit conceptual structures in their language. Therefore, reading (saying to yourself) the East Asian solution methods in Table 2 sounds awkward. But for a learner who has not yet constructed multidigit conceptual structures beyond the unitary structures, the East Asian words (or, better yet, the completely regular tens and ones words used in the Children's Math Worlds Project) are helpful for certain steps. They make adding the tens much easier: "three tens plus two tens is five tens" is much clearer than "thirty plus twenty is fifty". As mentioned above, they also make the 38, 48, 58, etc. pattern clearer because these are stated as "three tens eight ones, four tens eight ones, five tens eight ones". These number words also clarify written numerical methods (see later section) because they tell the quantity meanings of those written numerals.

## Interactions of Children's Math Achievement with Preferred Solution Method

The results of Beishuizen (1993) elucidate interactions among conceptual supports, preferred solution method, and level of mathematics achievement. This study compared the use of base-ten blocks, the hundreds board, and no pedagogical objects by Dutch second-grade classes. At issue was whether children used the cumulative sequence method in the left column of Table 2 or a variation of the decompose method shown in the right column of Table 2. The decompose method was called 1010. In it, the tens were added (e.g. 30 + 20 = 50), the ones were added (8 + 6 = 14), and then these were added to each other (50 + 14 = 64).

The hundreds-board instructional sequence used the board as sequence counting numbers, not as cardinal numbers: Only one square was darkened to show the meaning of a given number (e.g. to do 47 + 8, the square 47 was darkened rather than the first 47 squares darkened). Many exercises were given to support children's learning to count by tens, learning that jumping vertically one row involved counts by ten, and learning that jumping horizontally one square involved counts by one. Arrow notation was used to show horizontal and vertical jumps; children could use such notation on paper to record their sequence counting solutions. Children did many exercises linking pictures with darkened squares and arrows showing solution sequences on the hundreds board to verbal sequence solutions and sometimes to recorded numeral sequence solutions. The classes using base-ten blocks and those using no pedagogical objects also taught the sequence counting method shown in the top left of Table 2. Classroom observations and interviews with children indicated that many children used the blocks as answer-getting devices in which they made the problem (e.g. 42 + 5) and then just read off the answer by counting all of the blocks. Furthermore, many children in this condition made errors on subtraction problems involving borrowing (they mainly subtracted the smaller from the larger numbers for the ones). The base-ten block experiences were supposed to have facilitated sequence counting methods that could be interiorised into mental sequence counting methods (e.g. for 42 + 5, counting on 5 from 42: 42, 43, 44, 45, 46, 47).

Conceptual support did have an overall effect on solution method, even though all classroom teaching focused on the sequence counting method. About 60% of the hundreds-board groups used the sequence method, but about 60% of the block groups used the decompose 1010 method. In the no-objects groups, 50% decomposed, 30% used a sequence method, and 20% used both. The decompose method seems to have been used in this condition because it is so easy to do with the non-trade problems (e.g. 34 + 5) that were given for a long time before any carry problems were given (e.g. 38 + 6). Thus, another aspect of surrounding supports of learning that can influence what is learned is the sequence of problems that are given. The problems were chosen to facilitate the sequence method, but instead they suggested the decomposition method to many children even on the hundreds square because they did not go over a ten and few were two-digit plus two-digit problems.

More of the children who were low-achieving in mathematics used the decompose than the sequence method. High-achieving children in the no-objects condition were as effective with the sequence method as were high-achieving children in the hundreds-board condition. Together these results seem to indicate that activities with the hundreds board were not necessary for high-achieving children to learn sequence methods and that such activities were not sufficient for low-achieving children, many of whom instead learned decompose methods. Decomposition methods are accessible to children who have not learned count-by-tens conceptual structures, and such methods readily generalise to several digits. Thus, their use in this case by the less-advanced children does not mean that they are less desirable in general.

Cobb (1995) reported similar difficulties in learning with a hundreds board. In a case study of eight second graders in mid-year, not one could solve two-digit addition or subtraction problems by tens and ones, and none of the children learned to do so using the hundreds board. Two children did construct count-by-tens methods using other pedagogical objects (e.g. groups of ten tallies) and then became able to use such methods on the hundreds board. The combination of Cobb's and Beishuizen's results suggests that the hundreds board may be somewhat problematic as a learning support even for sequence methods, for which it seems quite natural. This may be occurring because the teaching activities, or the use of the hundreds board itself (e.g. darkening one square), do not help children integrate their sequence meanings with their cardinal meanings of numbers. Furthermore, as discussed above, the multi-units of ten are not very clear on the hundreds board, especially when one is jumping down in the middle of a row: The ten-group involved in moving from 43 to 53 is split across the 7-more in the 40 row and the 3-more in the 50 row. Thus, children, especially lower-achieving children, are at risk of learning sequence jump methods on the hundreds board as rote methods in which the vertical jump does not really signify ten units to the child. This is complicated by a related difficulty of the hundreds board that children (and teachers) also experience with the number line: where to start and whether to count the beginning number or not. In observations of first-grade classes using an Everyday Mathematics (a U.S. reform curriculum) lesson on the hundreds chart, this mistake was made by at least one child in every class (Fuson, 1996). This error arises from not connecting the sequence jumping methods to cardinal meanings of the number line or of the hundreds chart: the 42 is not thought of as the first 42 squares on the hundreds board or as the length of the number line from 0 to 42, but only as the square or number line mark that says 42.

## THE MODEL OF A CONCEPTUAL-SUPPORT NET

Our model of a conceptual-support net has three triads, each consisting of quantity referents, words, and notation. The outer real-world triad surrounds the mathematical triad of a drawn maths model, maths words, and maths notation. The central meaningful triad consists of potentially meaningful object referents, words, and notation designed to help learners build correct meanings for the maths and real-world referents, words, and notation. Such a net for two-digit quantities is shown in Fig. 3. All entries shown there are used in our Children's Math Worlds Project. The model shares many elements with other approaches using conceptual supports for mathematics learning. It differs in the systematicity of its inclusion of carefully chosen and empirically tested quantity, word, and notation elements at all three levels.

The central core pedagogical triad relates a meaningful quantity referent to meaningful words and to meaningful notation. All elements of the triad were designed by us to present the central features of the given math quantities as clearly as possible visually, verbally, and notationally. The inner triad of meaningful referents, words, and notation is needed for many mathematical domains because the standard cultural math words and math notation are either not clear or are actively misleading. We have already discussed how, unlike the East Asian number words that clearly state the tens and ones (52 is said "five ten two"), European teen and decade words do not reveal their composition as tens and ones. "Twenty" does not say "two" or "ten" clearly, although one can see the remnants of meaning in the "twen" related to "twin" (two children born at the same time) and "ty" as the beginning sound of "ten". The teens are even more difficult, reversing the direction of the tens and ones (say "fourteen" but write 1 then 4) and obfuscating the tens completely in "eleven" and "twelve". Our mathematical place-value notation also is not clear. It looks like single digits beside each other (32) and does not show in any way the meaning of the tens place as 3 tens. Other mathematical domains have similar problems. For example, a fraction such as 3/8 is said in English as "three eighths" using the ordinal word "eighth" (thus creating confusion) in contrast to the more meaningful Chinese fraction words "out of eight parts, (take) three".

Although we use the word "meaningful" for all elements of the central triad, these of course are only potentially meaningful for a given individual. Each person

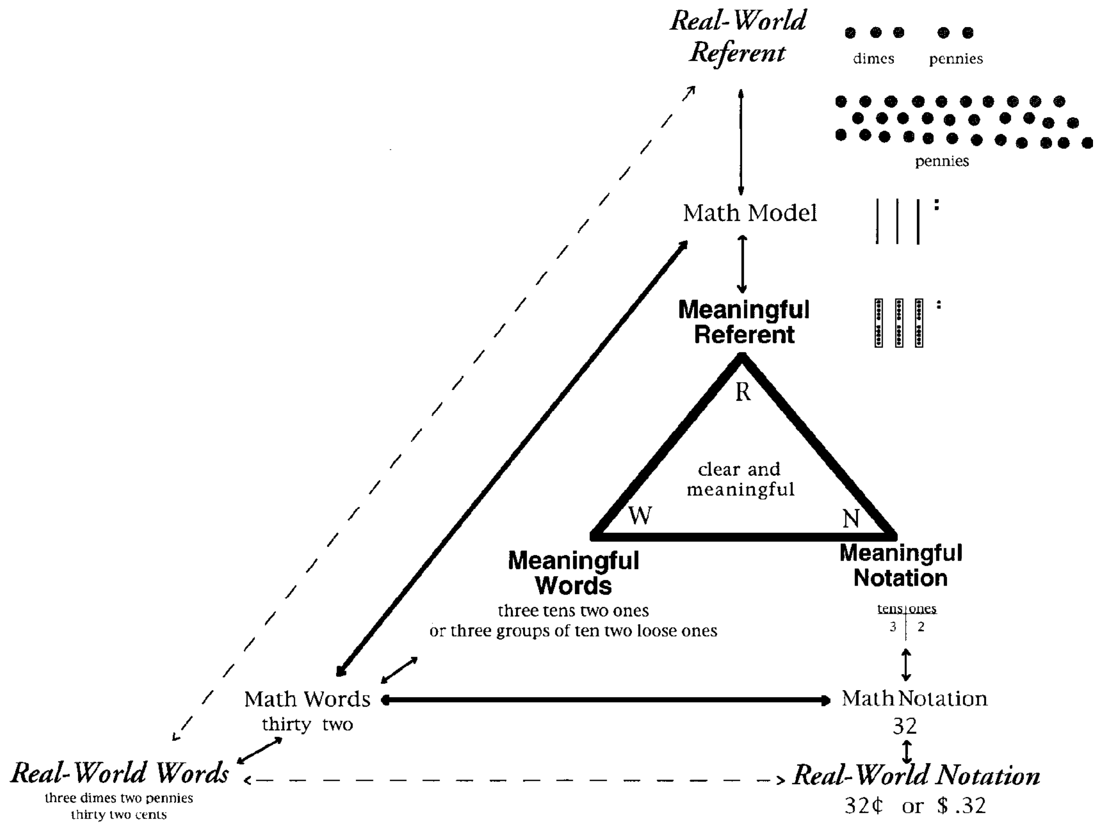



FIG. 3. Conceptual-support net: Real-world, mathematical, and meaningful referent-word-notation triads

must construct personal meanings for the referent, words, and notation. The examples given in Fig. 3 are powerful in our classrooms in helping children construct accurate meanings for the other elements of the net. The meaningful referent is cardboard penny strips that show pictures of ten pennies on one side and a dime on the other. These penny strips can be thought of as 32 pennies, thus linking to the maths words “thirty two”, or as 3 groups of ten pennies and 2 loose extra pennies, thus linking to the written maths notation 32. Our meaningful words were modelled after the East Asian number words that clearly name the tens and the ones (we said 52 as “five tens two ones”), and the meaningful notation labelled the tens and the ones (children did this themselves in various ways—we show one frequent method that we and they used).

The middle triad in a conceptual-support net is the standard maths triad. It relates the standard maths words to the standard maths notation. We have added a drawn maths model as the quantity referent for that triad, because mathematical modelling is a core mathematical behaviour that is too often delayed and used only in more advanced mathematics. As discussed above, meaningful drawn models of quantities are a particularly powerful class of conceptual supports. The drawn model shown in Fig. 3 is what our children eventually drew for the penny strips. They initially drew columns of ten dots, later connected them with a ten-stick, and still later just drew the ten-stick and the loose ones. Fig. 3 shows 3 ten-sticks and 2 loose ones. The connections of the mathematical triad to the inner (potentially) meaningful triad enables children to construct those meanings for the standard maths words and maths notation and to connect multidigit quantity referents to these words and notation.

The outer real-world triad relates real-world words, referents, and notations (if these differ from the maths notation). We show money as the real-world referent because we used it frequently in the Children’s Math Worlds Project (Fuson, Lo Cicero, Ron, et al., 1998). Quantity referents and words from the real world also are often unclear or misleading. For example, dimes are smaller than pennies even though their quantity is ten times that of pennies. Furthermore, the words “dime” and “penny” give no clue to their meanings (at least not to children), and the word “dime” is shorter than the word “penny” even though one dime has ten times the value of one penny. Real-world notations such as $.32 may involve complex mathematical meanings (32 hundredths of a dollar) that cannot really be understood by primary-school children, though such notations can be introduced as another way to write 32 cents and be explored more fully at a higher grade. Such problems with the real-world and mathematical triads require the meaningful triad to help children build correct meanings.

Figure 4 provides a visual comparison of a conceptual-support net, the typical use of manipulatives or other meaningful referents, and traditional teaching with no conceptual supports. The typical use of manipulatives does provide a meaningful quantity referent—the manipulative material—but it omits the meaningful words and meaningful notation and usually omits the drawn maths models. The manipulatives are linked directly to the maths words and maths notation. This can of course be helpful, but we have found that the use of meaningful words and meaningful notation is very important to many children in building understanding, and that these also facilitate meaningful conversations in the classroom. Traditional teaching omits most meaning supports and attempts to link real-world maths referents directly to the maths words and maths notation. A major difficulty with this approach is that the real-world referents, the maths words, and the maths notation all may possess attributes that under-specify and even obfuscate the mathematical meanings involved (as exemplified above for two-digit numbers).

Children require extensive meaning supports if they are to build meanings in the face of so much interference from the standard cultural maths words and notation and from real-world referents, words, and notation. Conceptual-support nets allow them to do so. Our Children's Math Worlds children who learned with the conceptual supports shown in Fig. 3 had a robust knowledge of place value on a range of tasks given by various researchers; they looked more like East Asian than U.S. children in their predisposition to see and use groupings of ten in various tasks (Fuson, 1996; Fuson, Smith, & Lo Cicero, 1997). They carried out accurate multidigit addition and subtraction and explained their methods using tens and ones. Their understanding of money was considerably above grade level. On all of these tasks, they compared quite favourably to samples of U.S. children containing many more middle-class children. For further discussion of issues concerning the importance of meaning supports, see English and Halford (1995, chapters 4 and 5).

## MOVING FROM OBJECT OR DRAWN-OBJECT CONCEPTUAL SUPPORTS TO NUMERICAL METHODS

The eventual goal of using object or drawn-object conceptual supports is for children to carry out mathematical processes and mathematical thinking using the cultural mathematical tools—the maths words, maths notations, and maths methods—of the domain. Helping all children reach this goal requires a systematic instructional sequence of activities using the conceptual supports. These activities help children link the quantity referents to the maths words and notation and then use these maths words and notation in ways that become personally meaningful. Before discussing issues concerning this transition to numerical methods, we look at various methods of multidigit addition invented by or taught to children. These will provide background for the subsequent discussion of issues concerning how to help children move from experiences within the conceptual-support net with object quantities to using personally meaningful numerical methods.

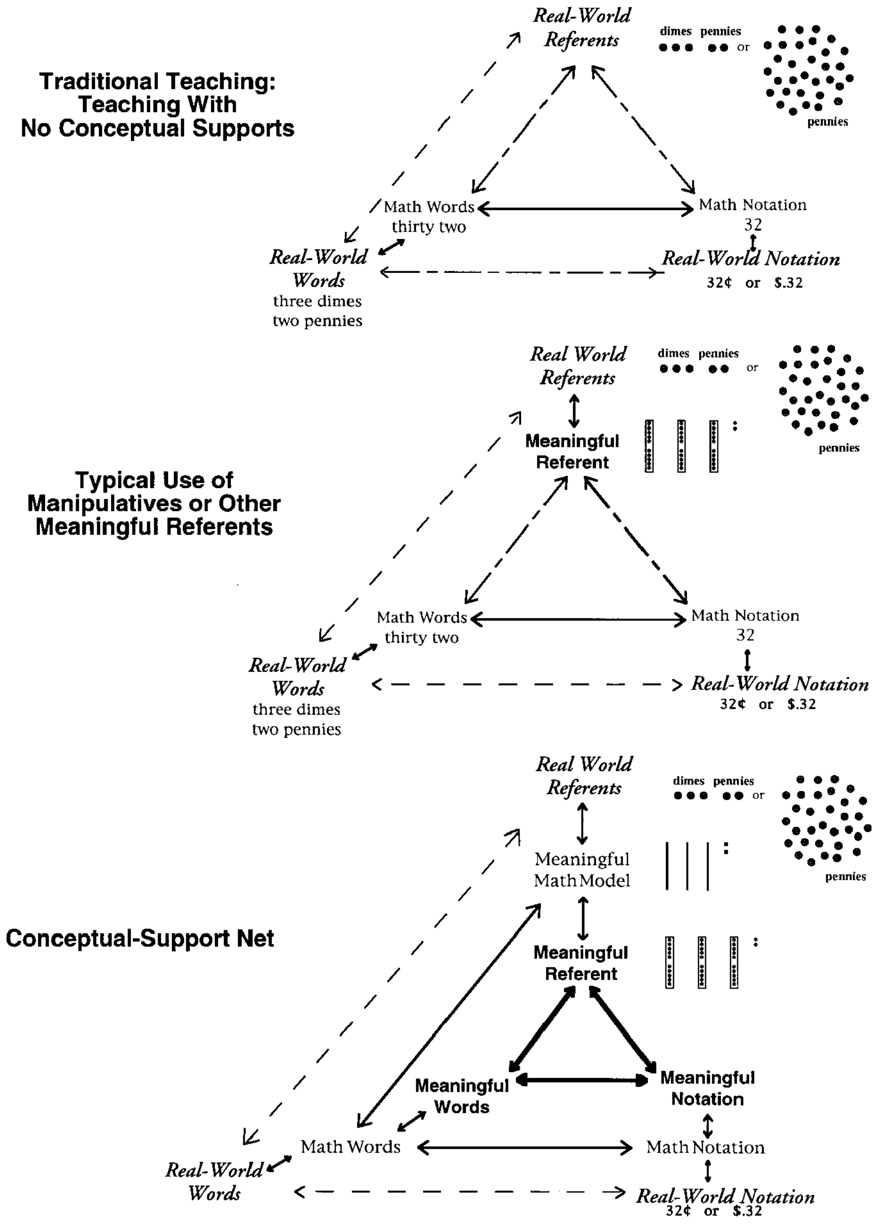



FIG. 4. Traditional teaching, manipulative use, and conceptual-support net

## Issues Concerning Different Numerical Solution Methods and Quantity Conceptual Supports

We have discussed a range of conceptual supports for multidigit addition and subtraction, but only the two solution methods in Table 2. Children invent and use many different addition and subtraction methods (many of these are discussed in Fuson & Burghardt, 1997, in press; Fuson, Wearne, et al., 1997). Different cultures use different standard algorithms (e.g. Ron, 1998). To capture at least a little of this richness, several different written numeral addition methods, all of which could arise from the same quantity support, are presented in Table 3. We show drawn ten-sticks and dots as children use them in the Children's Math Worlds Project by encircling ten ones to make another ten, but some of the written methods can be done with or without that step. Encircling the ten does show the meaning of the written 14 as 1 ten and 4 ones, with that 1 ten ready to be added to the other tens.

All of the methods shown are methods that decompose the tens and the ones in each addend and add them separately. These decomposition methods all readily generalise to many places. We have written one method of each kind with a four-digit problem to show that generalisation. This issue of generalisation of a solution method to many places is one that is, in our opinion, frequently not considered enough when deciding on conceptual supports or on methods that children might learn. Many sequence methods do not generalise easily, and some quantity conceptual supports (like the hundreds grid) also do not generalise easily, even to three-digit numbers. Having children explore different methods in order to build understanding of place value or to construct important elements of the web of multidigit conceptual structures seems sensible. But children do need an instructional sequence that will help all of them understand at least one multidigit addition and one multidigit subtraction method that will generalise to many places.

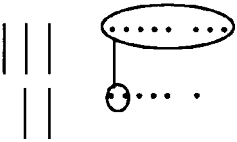
The first two methods in Table 3 vary in whether the tens or the ones are added first (i.e. whether one moves from the left or from the right in adding). We showed the four-digit method as moving from the left because most methods invented by children do move from the left. This method is quite accessible to children because each step is separate and its result is written out in full. The third method shows on the first line a typical early addition error of children. However, children using quantity supports can be helped to see that both the 5 and the 1 tell how many tens there are, so these numbers need to be added for the final answer. A group of second graders using base-ten blocks invented this method for four-digit numbers. They knew that the first sum was temporary and had to be "fixed" (Burghardt & Fuson, 1998; Fuson & Burghardt, 1993, in press).

The second row of Table 3 shows methods that separate the two major steps in multidigit addition: adding each kind of multi-unit and carrying/trading/making another of the next larger multi-unit if you have ten of some unit. All of these methods move from the left. First, one looks through the whole problem to see

## TABLE 3
## Generalisable Symbolic Two-digit Decompose Addition Methods

| *Groups Quantity Conceptual Support: Drawn Ten-sticks & Ones* | *Symbolic Methods* |
|---|---|

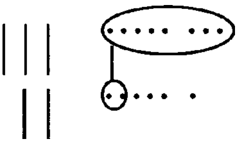
### *Add Everywhere, Then Regroup (or Add Subtotals)*



| left to right | right to left | either direction | left- to right |
|---|---|---|---|
| 38 | 38 | 3 8 | 7838 |
| + 26 | + 26 | +2 6 | + 5626 |
| 50 | 14 | 514 | 12000 |
| 14 | 50 | 64 | 1400 |
| 64 | 64 | | 50 |
| | | | 14 |
| | | | 13464 |

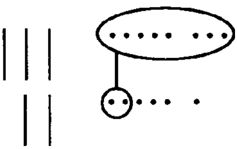
### *Regroup (Get Another Ten), Then Add Everywhere*

Look ahead to see if there is another ten (hundred, thousand, etc.)
and record or remember; then add each column (go either direction)



| | | | | |
|---|---|---|---|---|
| | 4 | 1 | | |
| 38 | 38 | 38 | 38 | 7838 |
| + 26 | + 26 | + 26 | + 26 | + 5626 |
| 64 | 64 | 64 | 64 | 13464 |

### *Alternate Adding and Regrouping*



These methods look like those just above but either

R → L:

Ones are added, a ten is traded/moved to the tens, tens are added, a hundred is traded/moved to the thousands, thousands are added, etc. This is the standard U.S. method.

or L → R:

Thousands are added, look ahead to see if the next-right column total ≥ 10 and therefore you get another thousand so increase the thousands total and write it; add the hundreds, look ahead to see if you will get another hundred from the tens (if tens column total ≥ 10), if so increase hundreds total by 1 and write it, etc. This is the standard method in several European countries.

where one will get another multi-unit and records each such multi-unit. In the first such recording, the new 1 ten is written below the problem on the line waiting for the addition above it. In the second example, this new 1 ten is added into the top number so that there will only be two numbers to add. In the third method, the new 1 ten is written above the tens column (as in the standard U.S. method). In the fourth method, the new 1 ten is remembered and not written (obviously a method that is difficult to do with problems of many digits). In the second problem phase, all of the multi-unit adding is done from either direction. We showed the general method of writing the small ones on the addition line because that method is conceptually and procedurally easier than the others. The second and third methods actually change the problem by adding parts of the answer into the original problem. The third method is also difficult because you must mentally increase the top number by one (here, think 4), remember that number in the face of the number one less than it (still see 3), and add that remembered number to the bottom number (do 4 + 2 while seeing 3 + 2). In the first method, you add the two numbers you see and then just increase that total by one. This distinction is not so important for easy numbers like 3 + 2, but for many students it makes a substantial difference for larger numbers like 8 + 7.

The methods on the bottom row of Table 3 alternate these two kinds of steps. At the end they all look like the methods in the second row. The standard U.S. algorithm looks like the third method in the second row, but children must begin from the right, add the ones, record the ones over ten and write any new 1 ten above the next-left column (which many children think is the far-left column, leading to errors), add in the complex way described above (remembering the number 1 larger than the top number while seeing the top number and then add this mental total to the bottom number), and so on. Some children find this standard U.S. method conceptually confusing, arguing that you are changing the problem by putting the 1 on the top (Burghardt & Fuson, 1998; Fuson & Burghardt, 1993, in press). Others find it difficult to add the numbers with this method because you have to remember a number that is not there and add to it while still looking at that number (as described above). Many others in the United States can carry out this method, but they cannot explain the values of the 1s they write on top because the teaching they received did not adequately help them understand that.

Quantity supports can clarify that you are adding like multi-units (tens to tens, ones to ones) and that you can make 10 ones into 1 new ten to add to the other tens. Drawn objects with numbers (the right-hand column of object supports in Table 1) can help children understand and link quantity meanings to a written method because the groupings of ten units into one of the next larger multi-unit are there for reflection after adding: A child can see both the original 10 ones and the 1 new ten made from these ones. With objects such as base-ten blocks, the new traded-for bigger multi-unit (e.g. the ten) is indistinguishable from the original tens, so it does not leave a record of the trading activity. The record in drawn quantities facilitates discussion of this crucial trading step and allows the teacher and children to discuss the conceptual shift from 10 ones to 1 ten. In all of our work with children, we have found that this conceptual shift is a huge step that poses major problems for many children. This conceptual shift keeps re-occurring in different situations, and it poses difficulties to many children in each situation in which it appears. It seems trivial to adults, but flexible shifting from 10 ones to 1 ten or back is the *sine qua non* of powerful place-value understanding. The later shifts from ten hundreds to and back from one thousand, from ten thousand to and from one ten-thousand, etc. are also crucial for understanding place value and multidigit methods.

## Issues Concerning Moving to Personally Meaningful Numerical Methods

Many of the world-wide movements towards reform of mathematics teaching place an emphasis on problematising calculation so that children find/invent their own calculation methods from a range of methods supported in a given classroom (e.g. Beishuizen, Gravemeijer, & van Lieshout, 1997; Cobb & Bauersfeld, 1996; Hiebert et al., 1996; Fuson, Wearne, et al., 1997). However, a prerequisite for such invention is that all children experience systematic conceptual learning activities that support their construction of advanced domain knowledge for that calculation domain. For multidigit addition and subtraction, children need sustained and systematic activities within a conceptual-support net that will enable them to construct the related web of Ct-PIV triad conceptual structures shown in Fig. 2. Such activities enable children to construct what we call “knowl-skills”: skills based on knowledge. Knowl-skills are rapid and flexible, and they rest on a system of knowledge that can be entered and used when new situations are confronted. Knowl-skills can be explained verbally.

Children need to build knowl-skills for both the count-by-tens conception and the place-value-tens conception and then integrate these two conceptions. Research literature indicates that some reform curricula and research programmes do not successfully enable some/many children to build such ten-structured conceptions in a timely fashion. Sometimes this seems to result from limitations in the pedagogical objects chosen or in the way these objects are used. For example, in a five-year longitudinal study of the Everyday Mathematics curriculum, most teachers we visited had a hundreds grid in the classroom. But the ways in which the grid was used did not enable some/many children in a given classroom to use it as a meaningful model of counting, adding, or subtracting tens and ones. Rather, many children used it only as a model of counting ones. Similarly, second graders in Cobb (1995) could not use the hundreds grid until they had constructed tens-units with some other model. The way the Dutch textbooks used the hundreds grid (reported in Beishuizen, 1993, and discussed above) by darkening only one square was particularly problematic.

The failure of substantial numbers of children to construct ten-structured numerical conceptions that they can use in calculation also sometimes stems from a lack of systematic classroom activities that move *all* children from unitary conceptions to count-by-tens and place-value-tens conceptions. Although, as Carpenter (1997) points out, a class may collectively have a great deal of tens knowledge, the task of the teacher is to help every child have and use such knowledge. The Everyday Mathematics curriculum does contain counting activities to practise counting by ones, tens, and fives and to practise writing large numbers. But it does not contain a sustained sequence of activities that help children count or combine or separate tens and ones quantities. Consequently, the more-advanced children in a class do invent mental methods, but the less-advanced children have no methods except sometimes the standard algorithm that they have learned somewhere (often from teachers just before standardised tests). These children almost always use the standard algorithm with no understanding of tens and ones quantities, and they make many of the typical errors, especially in subtraction (Murphy, 1997). Most CGI teachers and the teachers in the earlier Cobb projects (e.g. Cobb, Wood, & Yackel, 1993; Cobb & Bauersfeld, 1996) did use models of tens and ones (often unifix cubes stored in columns of ten), but few teachers seem to have used systematic activities with such models to facilitate all children's construction of generative tens conceptual structures. Some children in third grade (Lo, Wheatley, & Smith, 1994) and in the fourth grade (Steinberg, Carpenter, & Fennema, 1994) were still using unitary methods. In our experience, all second graders, even those coming from backgrounds of urban poverty, can come to carry out and explain generalisable addition and subtraction methods involving tens by several months into the school year if they have experiences to help them construct the conceptual prerequisites for such methods (Fuson, 1996; Fuson & Smith, 1995; Fuson, Smith, & Lo Cicero, 1997). In our Children's Math Worlds Project, we also help children learn one of the general methods on the right of the top two rows of Table 3 if they have not invented a method for themselves.

Our Children's Math Worlds conceptual-support nets begin with objects (penny strips and pennies) as the quantity referents. These allow children to make, move, and count tens quantities and ones quantities while they are initially constructing the Ct-PIV web of two-digit domain knowledge. However, we rapidly move to using drawn-object models, the ten-sticks and dots (some teachers or children prefer to write ones as small horizontal dashes arranged vertically as in a thermometer). Drawn objects facilitate the links within the meaningful triad, and all of the links among the triads in Fig. 3, because the maths words or maths notation can be written on or beside the drawn objects. The consequent record of a child's problem-solving process permits the teacher to examine children's methods and look for children's errors after a class is over. Such monitoring can provide daily feedback loops that permit teachers to select students to demonstrate particular methods or choose errors that would provide useful discussion for many children in the class. If some students work at the board while others work at their seats (or all work on individual chalkboards/white boards), their solution is available for reflection by their classmates. No waiting is necessary while children put their method on the board, a management consideration of importance in many schools. Working at the board also seems to facilitate children helping each other more than working in a smaller scale on a piece of paper. The teacher can observe such helping more easily than at seats. For the less-advanced children, having a record of their method physically present after their solution also seems to facilitate their explanation of their method. They can rely on gesture as well as on words, and the drawing helps them remember and sequence the steps of their explanation or learn to do so if they require help from the teacher or a peer.

A crucial aspect of using a conceptual-support net, or any meaning referents for mathematical words and notation, is that many children need repeated experiencing of the links in the conceptual-support net for the mathematical words or notations to take on the quantity meanings. The results of Resnick and Omanson (1987) are frequently used to argue that linking is unsuccessful or that manipulatives do not work. But that study used a single session of linked supports to try to help upper-grade children correct errors they were making in written computation. That session did help most of them to overcome the errors, so linking can be conceptually powerful. But the errors reappeared for many in a delayed post-test. Only someone with an "instamatic camera" theory of learning would expect one session of meaning-building to overcome years of rote computational learning (for further discussion, see Fuson, Fraivillig, & Burghardt, 1992). Building conceptual structures that organise attention to several aspects takes many children days or weeks (and a few children, months) of sustained or distributed repetitive experiencing. We take this term from Cooper (1991), who used it to describe a necessary requirement for preschool children to build mathematical concepts. Repetitive experiencing does not mean that the experience is the same each day for a given child. It assumes, rather, that learning in a complex environment requires familiarity and sustained interactions with the features of that environment so that the meanings and functionalities in that environment can be noticed, understood, and used independently.

Learning in any mathematical domain involves a learning trajectory (Simon, 1995) of increasingly abbreviated, abstract, and advanced conceptual solution methods or ways of thinking in a problem situation. Many of these have been well documented in the research literature (e.g. see the reviews for many mathematical domains in Grouws, 1992). Repetitive experiencing is required for building the necessary links for conceptual structures at any given level and for moving along through the learning trajectory to more-advanced methods. Although there is considerable research about such learning trajectories of children's methods, there is considerably less research about teaching/learning approaches or curricula that have tried to move children through such learning trajectories. That is a central focus of our own research programme. The realistic approach in the Netherlands has also had a similar research agenda. Partly in response to the Dutch research concerning children's difficulties in learning with the hundreds grid, a revised approach was developed that used an empty number line. In this instructional sequence, there were explicit activities with a beadstring and the empty number line to support construction of count-by-tens conceptions, and children were moved through a learning trajectory of more-advanced methods (Beishuizen, 1997; Gravemeijer, 1997; Treffers, 1991, 1996). Gravemeijer (1997) summarised this work in the Netherlands concerning vertical mathematisation and reflective cycles in which models of situations become models for mathematical reasoning about methods and numbers.

All such learning trajectories that move from using objects to eventual internal mental operations using words are examples of Vygotsky's hypothesised transition from the inter-psychological to the intra-psychological plane (Vygotsky, 1934/1962, 1934/1986). Children learn first by interacting in a socially mediated fashion with external mathematical objects. The directive words accompanying such learning potentially become internalised as directive inner speech. The actions carried out by counting or adding external objects eventually become internal mental operations carried out verbally in inner-speech. Throughout this process, the internaliser transforms the objects, actions on objects, and words involved in these actions via the concepts that she or he is building about these objects, actions, and words. The internalisation process and the resulting concepts depend on the learning history of the individual.

Gal'perin (1957) amplified this construct by describing the following five levels in the internalisation process, from the inter-psychological to the intra-psychological plane:

1. creating a preliminary conception of the task;
2. mastering the action using objects;
3. mastering the action on the plane of audible speech;
4. transferring the action to the mental plane;
5. consolidating the mental action.

Our experience in tutoring many low-achieving children (e.g. Fuson & Smith, 1995, 1998) is consistent with Gal'perin's ordering of Levels 2 and 3. For most of our tutored children, the ability to describe in words followed, rather than preceded, their ability to manifest knowledge in actions with quantities or in their drawings of quantities. Therefore, many children can "tell their mathematical thinking" in actions with objects or in drawings before they can do so in words.

However, our work with many children in classrooms and in tutoring situations also underscores the importance of practice and consolidation at all of the first three Gal'perin levels. Children vary considerably in the rapidity with which they grasp and remember an overall conception of a complex multi-step solution method, which all multidigit methods are. They may need support to remember steps in the overall method (Level 1), to carry out a particular step (Level 2), or to make a full explanation (Level 3). First graders are able to help their peers with all of these steps (Fuson & Smith, 1995). However, if a classroom is not organised socially so that such help can be available to those who need it, from the teacher or peers or other adults, then less-advanced children may not be able to learn and remember a method involving tens and instead may use a primitive unitary or concatenated single-digit method, both of which are vulnerable to errors.

As the mathematical operations become mental and internalised, we have found that it is crucial to initiate in the classroom new kinds of backward linking to keep the quantity meanings connected to the mathematical notation. Because so much mathematical notation is ambiguous or even actively misdirecting, incorrect meanings can creep in when quantity referents are no longer around to suggest correct meanings. This happened in one of our early classroom studies with base-ten blocks (Fuson, 1986). Some children after some months began making errors in multidigit subtraction problems with zeros in the top number (e.g. finding 6000 – 2874 to be 1236 by giving a 1 from the 6 to each top 0 to make each be a 10). The constantly seductive digits of multidigit numbers had led them to begin to use the concatenated-single-digit conception. Asking these children to "think about the blocks" (they had used base-ten blocks initially to build their understanding of their subtraction method) was sufficient for them to self-correct their errors in their written numeral problems. ("Oh, that is a thousand. A 1 from there is not ten. I have to trade one thousand to make 10 hundreds, 1 hundred to make ten tens, and 1 ten to make a ten in the ones place. I get 3126.") It is helpful for teachers to facilitate such backward linking for a long time after children are only using numeral methods. This can be done by asking children to describe their method as if they were moving the quantity supports (e.g. base-ten blocks, penny strips, etc.). Such descriptions can also help children who are not yet as advanced in the learning trajectory follow a numerical method being demonstrated by a more-advanced child.

As a final example demonstrating that children may see a given conceptual support or a numeral problem in different ways, consider the most difficult subtraction problems with multiple zeros in the top, such as 6000 – 2874. Some children using base-ten blocks did the adjacent trading to each column described above. Others (Fuson & Burghardt, 1997) conceptualised 1000 as the big cube but thought about trading it in simultaneously to make some in each other column: 9 flats (hundreds), 9 longs (tens), and 10 tiny cubes (ones). Still others thought about 1000 numerically and knew that one thousand equalled nine hundred ninety and ten. Korean children also showed a range of different ways of thinking about and explaining this trade numerically (Fuson & Kwon, 1992b). Children also may use their place-value knowledge of 6000 as 600 tens, which can then be thought of as 599 tens and 10 ones (L.D. English, personal communication, April, 1998).

## The Importance of Classroom Communication Processes in Using Conceptual-Support Nets

The implementation of a conceptual-support net in a classroom requires a range of social communication supports organised and enacted by the teacher. To enable children to make all of the multiple links in most mathematical domains, the teacher must lead attention across various components to help children make connections across them. Using conceptual supports alone in the classroom without social support orchestrated by the teacher is not sufficient for many children to build advanced conceptions or to move along a learning trajectory to more advanced thinking. We do not have space here to discuss such classroom communication processes very extensively. Such issues are, however, addressed in Fuson and Burghardt (in press), Fuson and Smith (1995), Fuson, Smith, Lo Cicero, et al. (1998), and Fuson, De La Cruz, Smith, et al. (in press); see also Cobb and Bauersfeld (1996) and earlier papers referenced therein.

Although some analyses of such classroom communication processes have been published, the understanding of the field is at present far from sufficient. This is a major frontier of needed research, especially given the stress on classroom discourse in most reform movements around the world. What is clear is that most teachers need more support to carry out such classroom processes than is presently available to them in most reform curricula. For example, learning to describe one's mathematical thinking in words is a major new component in many reform mathematics programmes around the world. Such descriptions do serve to allow children to lift a method from the plane of action to what Gal'perin (1957) called the "plane of audible speech". These descriptions can facilitate children's more general understanding of their mathematical thinking and provide another conceptual support for linking the quantity referent to the maths words and maths notation.

However, the longitudinal study of one of the U.S. reform curricula, the Everyday Mathematics curriculum from the University of Chicago, has indicated that many teachers carry out such classroom discussions of children's methods in ways that limit the access of many children to understanding this discussion (Mills, 1996; Mills, Wolfe, & Brown, 1997). First, many student descriptions were very brief and not explicit (e.g. "I used the hundreds grid."). Teachers did sometimes extend these descriptions, but most extensions were also very brief. Second, teachers only rarely used any referent for a method described, so the discussion was oral only. For example, they almost never enacted or had a student enact a method with the hundreds grid or any other referent. A few teachers did record on the board in numerals whatever method a child described. This had the advantage of leaving a record of all methods described by children, so such methods could potentially have been compared or contrasted. This was rarely done in the observed lessons, however. Methods were elicited but not discussed. Because most discussions were oral only, without even numerals written on the board, some/many children were not able to follow such descriptions (Murphy, 1997).

Third, two-digit numbers within addition and subtraction problems were frequently said as concatenated single digits rather than as meaningful X-ty words or as tens and ones words. For example, when a child was solving 34 + 26, she or he would say, “I added three and two to get five”, instead of saying “I added thirty and twenty to get fifty” (a count-by-tens method) or “I added three tens and two tens and got five tens” (a place-value-tens method). Even some teachers used such single-digit language, thereby emphasising a rote procedure of adding digits rather than facilitating children’s understandings of the quantities actually being added. We have found in our classroom research that using X-ty words and also using place-value-tens words is necessary to enable all children in a class to follow the discussion conceptually because initially some children only use one or the other of these words. Using both of these kinds of words can also help children begin to construct the other tens meaning. Coming to think with and use both kinds of words also gives children flexibility in understanding various kinds of addition or subtraction methods, and it helps children begin to build the advanced integrated conceptual structure.

Classroom communication processes are involved in the initial activities that help children construct the related web of Ct-PIV triad conceptual structures, in the explorations and discussions after children begin to use addition and subtraction methods, and in helping children who do not yet have an accurate method learn an accessible fall-back method. At the later two points, a typical classroom has children spread out along the whole learning trajectory. Our current Children’s Math Worlds classrooms in urban schools almost always contain children at the beginning levels in the learning trajectory who need help to catch up because they have just transferred into the school and missed all of the earlier activities. Therefore, teachers face a complex role in organising helping structures within the classroom that can facilitate the movement of children along their own learning trajectory and in weaving discussions and classroom activities across the various locations in the learning trajectory occupied by various children.

A vital role at this stage is to relate a given child’s described solution method to more advanced and to more primitive methods so that children at different levels can understand the described method. It is not necessary that a teacher do this for every method given by a student, but doing it frequently can help. Carpenter (1997) describes skilled CGI teachers doing such relating along the learning trajectory of abbreviated and abstract solution methods. In one part of the discussion, a teacher was trying to help a given child advance by asking her to describe her blocks method without using the blocks. Such experiences can help children to move from an objects method to an oral verbal method. An important and natural step after such a verbal description without blocks present would have been to do the description again showing the oral actions accompanying actions on the blocks. This would have made the oral description accessible to most students and potentially have facilitated children's movement from Gal'perin's Level 2 to Level 3. In the third-grade example in Carpenter's paper, the reverse happened. The student said that she did not need blocks to describe her method, but the teacher insisted that the first description of the method be with blocks. This allowed the less-advanced children to follow that method. If the teacher had followed that blocks description with an oral description not linked to the blocks, some of the listeners might have been helped to move away a bit from the blocks to using words (i.e. move to Gal'perin's Level 3). The third related method in the Carpenter example then moved to using the words as the objects, and fingers were used to keep track of the tens counted on and then of the ones counted on. Juxtaposing these methods in this fashion can help children see the relations between them and facilitate their moving ahead to more-advanced methods.

As more examples of classrooms focused on helping children build meanings appear around the world, it will be possible to study ways to help children move through learning trajectories to more-advanced methods. Understanding the classroom communication processes that must accompany the use of a conceptual-support net in any domain will then enable us to help many more teachers teach mathematics in ways that are personally meaningful for children.

## CONCLUSION

The information revolution has irrevocably changed the kinds of mathematics teaching and learning that need to occur in classrooms. Such teaching now must be focused on meaning-building by each individual so that mathematical words, notations, and operations become personally meaningful. This does not mean that children invent their own mathematics free of the cultural constraints of standard mathematical tools (i.e. of the standard cultural mathematical words, notations, and methods). Rather, children must learn the meanings of these mathematical tools so that they can use them in flexible and new ways. Given the present rate of technological change, we cannot even anticipate some of the mathematical needs of the children presently in schools. Therefore, flexibility and meaning are necessary bases for future changes in mathematical thinking by these children.

Our 15-year programme of classroom-based research has indicated that a conceptual-support net can be very helpful in designing and implementing, in the classroom, teaching/learning activities that rapidly bring all children to more-advanced mathematical functioning. Much of this research has focused on the less-advanced children and on ways to help them build an integrated web of ten-structured multidigit conceptions. Coherent and sustained repetitive experiencing with social supports orchestrated by the teacher can help almost all children learn ambitious grade-level goals. A centering triad of a meaningful quantity referent, meaningful words, and meaningful notation and drawn maths models arising from the meaningful referent can be specified in any mathematical domain. These elements provide the bases for meaning-building by children as they connect them to the standard cultural maths vocabulary and notation and to the frequently confusing real-world triad of referents, words, and notation. I hope that this model stimulates other researchers to specify such meaningful triads in various mathematical domains as a major descriptive and design step in researching and implementing meaningful mathematics teaching and learning.

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