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GROUP CASE STUDIES OF SECOND GRADERS INVENTING MULTIDIGIT SUBTRACTION METHODS

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Four groups of second graders explored the subtraction of horizontally presented 4-digit subtraction problems using base-ten blocks. The blocks afforded children's inventions of several variants of separate-multiunit methods involving trading or putting one multiunit in the next-right adjacent position to make ten of that multiunit. When dominant group members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. Language describing subtraction is complex because different phrases reverse the direction of subtraction. Trading is also complex because some children focus only on one part of the trade. Descriptions of block and numeric operations were often too general to follow, but children could explain more clearly when asked. Important functions of a teacher are to increase such specific full descriptions using quantitative language, to facilitate linking of numerical and block (or other referent) operations, to help overcome impasses, and to create a simpler problem or use another method to focus children on available mathematical meanings.

This research was undertaken to increase our understanding of how individual conceptual and social competences affect individual learning within social learning settings. This was addressed by studying children's invention of multidigit subtraction methods within a small-group setting. Brief case studies of four groups of second graders overview interactions of individual personalities and varying mathematical understandings of group members that created different patterns of group interaction and different group and individual learning paths.

Perspectives and Theoretical Framework

The authors of this paper take a constructivist view of learning as individual meaning making by each participant and a Vygotskiian view of teaching as assisting the performance of learners by adapting to the perspective of the learner while helping the learner move toward more culturally adapted conceptions. This study was designed to allow teaching to arise mainly from the group interactions of the children, though the adult group supervisors did some scaffolding. The analysis of Fuson (1990), Fuson and Kwon (1992), and of Fuson, Smith, and Lo Cicero (in press) concerning conceptual structures children use in multidigit situations is used to analyze children's multidigit thinking. The base-ten blocks used by the groups afforded primarily separate-multiunit conceptions of subtraction as requiring trading from the adjacent-left position to get more of a given multiunit. The two conceptions identified by Fuson and Kwon (1992) in Korean children's subtraction thinking were the conceptions used in most verbalizations of children's thinking in this study. The multiunit quantities conception used block words or multiunit quantity words (e.g., "I put this Big Mac [or thousand] here to make ten plates [or hundreds]"). The regular one/ten trades conception was used for all places; it described the action above as "I took this one and made ten here." The resulting teen number was viewed as consisting of one ten and some ones and not just as concatenated single digits (e.g., 13 as a one and a three). As in the Fuson and Kwon (1992) study, some children in this study also integrated these two views for a full conception that could use the multiunit values along with the tens/ones numeral view of each column.

Methods and Data Sources

Four groups of four or five children were asked to subtract horizontally presented 3- and 4-digit numbers using base-ten blocks and to make numeral recordings of their block methods. The children were from the highest-achieving of three second-grade classes. These subtraction sessions followed a 2- to 3-day introductory period with the blocks, and a 5- to 8-day period solving 4-digit addition problems (Fuson & Burghardt, 1993). Groups spent between three and eight days on subtraction. The subtraction time was not sufficient, but was limited by the total time the school was willing for the groups to participate in the study.

Each group session was videotaped. Each group had an adult who oversaw the videotaping and took live notes. Each videotape was transcribed and then checked by a second transcriber. Mathematical and group interaction verbalizations were transcribed verbatim. Mathematical and social/emotional actions were described in the transcript. A third person made separate drawings of all operations with blocks and all writing of problems in numerals either on the group writing pad or on individual papers; these records were related to transcript line numbers.

Results

Group A

The group did not discuss or have difficulties with the direction of subtraction or of words describing subtraction. U immediately saw and verbalized the difficulty: "We have a problem, though. We can't take 2 away from 6. Take 6 away from two." (this was the required hundreds subtraction). He correctly inferred directionality from the horizontal problem. T repeatedly suggested "borrowing" and even described a ten/ones borrow with the blocks more than once. But her English was not very good, and she never used the blocks to explain or justify her suggestion. She was not an assertive group member, so this suggestion was ignored repeatedly. The children did not resolve their posed problem (how to take 6 from 2) on that day. On the second day T again suggested borrowing and finally put in 10 teeth [ones] and took out 1 licorice [1 ten]. Da objected that she was adding (focusing just on the *adding in* of ten teeth). U used a 2-digit tens and ones conception to describe a thousand/hundreds trade, "Take one ten away from this (5 in the thousands column)." This trading attempt was cut short by Da's insistence that you have to go from right to left in math. T showed the licorice/teeth [tens/ones] trade again, but never explained it. Da again objected that "it is adding." Finally T said that she was not adding but taking away "because I have to take ten of these here (pointing to tens)" (focusing Da on the taking-away part of the trade but not discussing the whole trade). Da said, "That's a good idea. Let's do the same with the pancakes (immediately generalizing the trading method to the hundreds)." N finally articulated the whole trade in block words, "She took one licorice away, and she put on ten teeth." This seemed to reduce objections, and the three girls (Da, T, and N) made trades with the blocks linked to numeric recordings.

The group then showed their acceptance of the norm to explain actions in order for everyone to understand by a long period of repeated explanation by the girls to the boys. However, these lacked clarity because of the lack of full multiunit or block words (lots of "here" and "there") and no explicit justification of the fairness of the trade. Also, U throughout wanted to subtract from left to right, and all of the others insisted on subtracting from right to left. This group for addition had worked in either direction, and even started in the middle, in their method of adding everything and then fixing the answer (Fuson & Burghardt, 1993). They might have invented a "fix everything first and then subtract" inverse of this procedure if some of them had not heard the "math rule" to work from the right in subtraction.

On the next day U again tried to begin from the left and do a thousand/hundreds trade; he was stopped by T. He never got a chance to try out a whole problem from left to right. This difference in approach, combined with lack of clear language, continued to plague the group's discussions and eventually led to some withdrawal by U. Most of the time the trades were not fully articulated. But in the final two days Dh did explain both aspects of a trading action, "She took one of these (pointing to the three remaining licorices [tens]) away and put ten teeth [ones]." and Da explained about T that "She is putting the same thing out but in different places."

Group B

This group had with addition developed block and digit-card methods in which the blocks and digit cards for each column were removed and replaced by answer blocks or digit cards (index cards each with a number used to show the numeric problem). The adding was usually done mentally or with fingers. Children did show trades with blocks (e.g., removing ten tinies [ones] and adding in another rectangle [ten]), but they never evolved a method for showing this trade with numerals: Three of the four children added in the trade mentally and did not write it. This group developed similar methods for subtraction. They placed blocks and digit cards vertically and then moved from right to left subtracting mentally or by counting up. Problem blocks and digit cards were removed, and answer blocks and digit cards were put into that column. Until the third and last day when the adult suggested that they relate their block and digit-card trades, children predominantly used a method of making mental teens in order to subtract and did not really focus much on the blocks.

They began by adding the first problem, and then M did smaller from larger with the blocks. N said that the smaller from larger (e.g., 2 - 8) columns should all be 0. Three of the children then did smaller from larger on their paper, but N subtracted to give zeroes. The adult then focused them to look just at the rightmost two columns, which showed 62 - 38. D said, "Maybe you can take a ten from the 6 column and put it with two. You get twelve minus eight." The adult asked D if she could do that with the blocks; D put a long from the 6 longs with the 2 tinies. No one was paying any attention, so the adult asked D to do everything again. Then each child solved the problem again numerically on individual sheets. D and X showed the thousand/hundreds trade and the tens/ones trade, i.e., they immediately generalized the trading. M wrote no trades but did them mentally; she forgot to reduce the thousands by 1. N probably just copied the answer from M, but he might have done part of the problem himself.

D was absent the next day. M worked hard all day and did try to explain issues to both boys. A couple of times M took out one of the next-left top blocks when removing all blocks of one kind in order to put in the answer blocks (e.g., took out one rectangle and all of the tinies before putting back the answer in tinies). But she never said what she was doing (taking out the top 16 and bottom 9) or explained, so this subtle version of putting a larger multiunit with the next smaller column was not noticed or understood by the boys. All three children did help correct each other on various columns about both parts of the group's strategy: They each initiated the statement of the correct subtraction in a column requiring more (e.g., said 3 - 6 as thirteen minus six), and they each initiated or corrected the subtraction in a column from which putting had occurred. Therefore, the boys each showed understanding at some points. Children evolved two different methods for subtracting a traded-from column. M usually first subtracted the numbers and then took away one if a teen had been made to the right. N added one to the bottom number and then subtracted. The children never used multiunit words and hardly ever used block words; their brief and rare descriptions or discussions used 2-digit tens and ones or concatenated single-digit language. They were all focused on the written numeral problem and worked orally and mentally from this. Each frequently understood their mental "make a teens" method, but each also sometimes made an error (usually not decreasing the traded from column by the one traded).

Group C

The group set up blocks and numbers vertically. N said, "You have to borrow because you can't take 2 away from 8, I mean take 8 away from 2." When asked what to do with the blocks to show borrowing, she said, "Ask Mr. Ten if you could borrow a pretzel [a ten] and then you would take it - one of them - and you would take ten sugar cubes [ones] and put them down." This was one of the fullest descriptions of the next four days spent on subtraction; block words were rarely used after this. Other children did pick up the theme of "Ask Mr. X" and the language "take one." Block trades were done for all columns. The language describing the trades usually used the word "take" or "take away" for both the one taken and the new 10 blocks put in ("Now take ten of these."), but occasionally the word "put" was used for the 10 blocks put in. The equivalence of these block actions was never discussed. Some children clearly indicated by various comments that they understood this equivalence, while others did not indicate so clearly. Children occasionally said the subtraction words backwards, but everyone understood the direction of subtraction (the bottom number was being subtracted from the top number). J once said, "2 take away 6 is minus 4, negative 4," but this was never pursued.

T dominated the block trading over the first three days, either doing it or telling other people what to do before they could think it out for themselves. N continued to show her understanding throughout. B initiated block trades and numeric recordings for various columns, but she was slow and T often told her what to do. K did initiate some block moves and numeric recordings, but he had difficulty explaining what he had done. He finally did so at the adult's insistence. On the next-to-final day, they all wrote on individual papers as the problems were done with blocks. Everyone wrote both problems correctly, often before the block trading. Everyone by the end could do accurately and could record numerical borrowing, but K and J were not so consistently clear about the block actions for borrowing. J at one point argued that he should add <u>one</u> pretzel [one ten] to the tens rather than <u>ten</u> pretzels from the 1 bread [hundred] taken away; this seemed to reflect thinking (at least at that moment) that they were always trading <u>a ten</u> to a given column. The group's tens and ones language suggested this, and their failure to use block words or multiunit names facilitated this view. T, at that time in response to J, said the only clear multiunit description of trading, "You need to borrow one hundred; and hundred's are ten tens."

N kept wanting to subtract from the left, and the others from the right. This was the one group that did addition problems from both directions and had discussed relative advantages of each. They had decided that going from the right was faster because they did not to cross out their first answer in a column. T used this in arguing that they should subtract from the right, "Remember, it is faster." On the final problem on the final day, at the suggestion of the adult, they did all of the trades first (from the right as usual) and then subtracted from each direction. They saw that they got the same answer.

Group D

This group especially at the beginning had great difficulty with, and long controversies about, correct language to express the direction of subtraction. The work of this group was heavily led by one girl C, who knew the standard U.S. subtraction algorithm and a standard Chinese algorithm. Her blocks subtraction methods looked like these algorithms, with a block put into the next column to the right when trading was necessary. C did little explaining, though her occasional comments indicated that she was using both multiunit values and a 2-digit tens and ones conception. The two boys E and N had only partial understanding of her method, initially doing a smaller-from-larger method on some columns when they wrote their own problem. L several times throughout the subtraction proposed that the difference of a column be a "minus 6" or "negative 6," but this idea was never pursued by the group. C also introduced her Chinese subtraction method, which was to put a dot above a column from which a multiunit was borrowed. Usually, nothing else was written. But in the first use of this method, she recorded each aspect: A dot was put above a column, a 10 was written to the right of the dot (i.e., it was then 10 of the next-right multiunit), and the top number was reduced by one.

On the last three days problems with zeroes on the top were introduced. C and N were absent for two days. E and C struggled with how to get more ones, coming up with several unsatisfactory ideas (e.g., putting a thousand block in the ones, putting a thousand block in each column), and E was his usual unpleasant and aggressive self. On the second day the experimenter asked them where the thousand block usually was traded. This was all the two needed, and E worked productively and collaboratively for the rest of the day, even saying, "This is fun," when they had solved all of the top trades. Their block method was to make all of the usual trades (put a thousand above the hundreds, a hundred block above the tens, and a tens block above the ones) and then to compensate for the trades from the hundreds and the tens (the thousands block in the hundreds was changed to 9 hundred blocks and the hundreds block in the tens was changed to 9 tens blocks). Then all of the subtraction was done right to left. This method was done with the numerals and the blocks.

On the final day a regular problem without zeroes was done because the other two children returned. N did with the digit cards a fix-everything-first version in which he crossed out the top digit cards with carrots, but he did not put new digit cards to show the changes. The changes were said in multiunit values (e.g., "We have thirteen hundreds."), and the subtraction was then done right to left. All children then individually wrote that problem on their papers. The boys did the traditional algorithm correctly. C did the Chinese method writing only dots. L invented a new dots method in which she wrote ten dots above each of the traded to columns and also made dots for the top number. This method conceptually shows a 2-digit tens and ones conception underlying her method.

This group would have been much more productive if E had been both encouraged and constrained, and if appropriate support had been given to elicit full explanations. These children could probably have articulated a fully integrated multiunit and 2-digit tens and ones conceptually-based subtraction method, even for zeroes in the top number. Furthermore, adult support of L's negative-number idea might have led to a negative-number method (get positive and negative multiunit differences, then fix the answer). The group lacked a strong leader. Neither girl was strong enough to stand up to E's negative behavior, precipitating frequent boy-against-girl battles. N would have been a good collaborator in a group without E's negative influence.

Conclusions

Personality factors combined with the mathematical strength of individual children to create different group learning paths and different subtraction methods with the blocks and the written numerals. Official leader and checker roles rotated daily among children in a group. Most children were adequate leaders, but "natural" leaders also emerged in all groups. When dominant members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. These groups exemplify groups at the beginning of learning to work together or groups whose teacher has not worked to establish powerful social norms or group interaction competence. They indicate that good mathematical ideas (e.g., negative number approaches) can get lost in group processes if the ideas do not come from dominant children.

Directionality in the language of subtraction was a difficulty for some groups but not for others. Different English ways to describe subtraction are opposite to each other: for example, 2 take away 8, take 8 from 2, 2 minus 8, 8 from 2, 2 subtract 8. Children frequently said such subtraction phrases backwards (e.g., 8 take away 14). Sometimes everyone seemed to know what was actually meant, and sometimes such reversals confused discussion or operations.

Several groups did not adopt very well the social norm to assist all group members to understand or were not very good at such assistance. There were in some groups sustained efforts to help everyone understand, and there were isolated incidents of effective explanation in other groups. However, overall the spontaneous explanations were quite limited. Most explanations did not use multiunit quantity language ("thirteen hundreds minus six hundreds is seven hundreds") or block language. Many explanations did not even use numbers ("We took away these from those."). This lack of verbal clarity made it difficult to follow what a child was saying.

Children's objections and misunderstandings of trading illustrate its complexity. Children need to be able to see both parts of the trade: the taking from one place—and the subsequent reduction of those multiunits by one—and the adding to the adjacent-right place—and the subsequent increase of those multiunits by ten. Explanations and demonstrations that focus on both these parts—and the numerical consequences of each—are necessary. Given the paucity of full verbal explanations in most groups, and the too brief learning period, it perhaps is surprising that most children came to understand trading and to use some numerical method of recording such a trade.

Several vital functions of a teacher are clear from the above case studies: Supporting children to describe their mathematical actions using quantity language, to link numeric and block (or other referent) operations, to explore deeper aspects of an operation, to focus on meaning (e.g., in looking at 62-38), and to overcome an impasse. It is also important for teachers to help the voices of non-dominant children be heard because they may contain productive mathematical ideas.

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