10 *Snapshots Across Two Years in the Life of an Urban Latino Classroom*

Fuson, K. C., Lo Cicero, A., Hudson, K., & Smith, S. T. (1997). In Hiebert, J., Carpenter, T., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A., Human, P., Making sense: Teaching and learning mathematics with understanding (pp. 129-159). Portsmouth, NH: Hienemann.

In this chapter we will follow a class of first graders through first and second grade, observing them in typical lessons.[1] The lesson topics have been selected to demonstrate key aspects of classrooms that support understanding. The lessons and approaches come from the last set of classroom teaching experiments in the Supporting Ten-Structured Thinking (STST) project directed by Karen Fuson at Northwestern University. These studies were funded under a National Science Foundation project called Latino Children's Constructions of Arithmetical Understandings in Urban Classrooms That Support Thinking. This project focused on word problems, single-digit addition and subtraction methods that use ten, and multidigit addition and subtraction in grades 1 and 2. We also did some work in kindergarten and third grade. Curriculum materials (*Children's Math Worlds*) were developed over a three-year period. We worked with children in both English-speaking and Spanish-speaking classes. Most children are from Spanish-speaking backgrounds, but some children are African American and some are European American.

The Project Approach

We begin the year with two major foci: One is decomposing numbers below ten into many possible pairs of numbers (often set within sharing situations). The other is generation of mathematical stories by children that are then mathematized by the teacher into various simple kinds of word problems. Children show numbers to themselves and to others with objects, fingers, and mathematical drawings (simplifications of the situation into dots, circles, etc.). Decomposition allows children to begin seeing flexible combinations of numbers that compose other numbers. They also can see relations between pairs and overall patterns. Decomposition also helps children learn the basic addition/subtraction facts below ten in a related way. Finally, decomposition allows children to begin building one of the prerequisites for ten-structured methods of addition and subtraction: being able to break a number into a given number and the rest. Eliciting mathematical stories from children allows the teacher to get to know the children (because most of the children's stories are about themselves and their families). These stories relate early mathematical processing to children's lives, and they facilitate children's mathematical understanding because the situations are familiar and engaging.

We help children learn how to use their fingers to show numbers in the mathematical story situations. Fingers are an inexpensive, powerful, and natural tool for problem solving. Children in almost all cultures learn to use their fingers to solve numerical situations. There is a progression of more advanced finger methods that children use if they have frequent opportunities to use their fingers in mathematical class. Our experience with many different classes of children is that fingers do not become a crutch that slows children down in later grades. When we help children move on to more advanced methods, many children do not need fingers by third grade. The few who still need them have rapid and reliable finger addition and subtraction methods. Many children from Latin American countries, and from other parts of the world, put numbers on their fingers in ways different from the typical method in the United States (raising the index finger first and the thumb last). All of these methods can be accepted, and all children can be helped to move on to more advanced methods (see Fuson [1992] for a description of these methods).

Our word problem work begins with the simplest kinds of addition, subtraction, multiplication, and division (equal sharing) problems. Children show these situations with drawn dots, circles, and fingers. Children tell word problems, and they solve problems posed by other children and by the teacher. Later children solve, tell, and write more difficult problems in which the unknown is not the final quantity. Traditional mathematical tools such as number sentences are introduced during this time, but children use them to show *the situation*—not the solution method. We also use problems with irrelevant information and two-step problems. Over time, numbers in the problems also increase in size, but we keep numbers in more difficult problems smaller than those in simple problems.

Another method is to give a situation involving several different numbers and ask children to pose many different questions about that situation. Children also pose and solve problems about information graphs, and they make graphs. Thus, children do a lot of problem posing, question asking, and describing of their problem-solving methods. They also solve a much richer range of problem types than is typical in most textbooks (see Fuson [1992] for a description). This whole approach emphasizes thinking about the underlying situation and then mathematizing that situation (stripping it down to its mathematical elements).

We introduce tens and ones and how to write two-digit numbers by having children use penny strips that show ten pennies. A dime is shown on the back to clarify the meaning of a dime as ten pennies. Children use the penny strips to learn to count to 100 by tens and by ones. They also record the penny strips by drawing columns of ten circles. Later they draw a line through the column to show the strip, and still later just draw the stick as a ten. Children use penny strips and drawn ten-sticks and circles (ones) for multidigit adding and subtracting.

Children initially use penny strips or draw columns of tens or ten-sticks when adding and subtracting multidigit numbers. Penny strips allow children to work with the meanings they have. Some children count by ones, and others count by tens (10, 20, 30, 40, 50) or count the tens (1, 2, 3, 4, 5 tens). Both of these ways of thinking of tens have been built up by activities in which children count in each of these ways and read a number such as 53 as both fifty-three and five tens and three ones. By the end of first grade, almost all children can use addition methods involving tens, not just ones. Many children also invent mental methods that do not use penny strips or drawn quantities, and by the end of second grade most children do two-digit and three-digit multidigit addition and subtraction using numbers (and perhaps fingers). The purpose of all of this work is that children understand the number words and written multidigit numbers as collections of thousands, hundreds, tens, and ones. Unless these meanings are very strong, children may begin to use methods based on how the numbers look instead of what they mean. Children make many errors when they see multidigit numbers as separate ones digits.

Children develop their own methods of multidigit adding and subtracting using tens and ones. They explain their methods to other children, try to use other children's methods, and compare different methods. These all help children to reflect on their own and on others' methods. The latter two kinds of activities also ensure that children listen to and try to understand other children's methods. Solving problems in different ways is emphasized, but children in different classes sometimes have certain kinds of methods that dominate. This frequently is because the activities in that class have enabled the children to think more rapidly in one way than in another (e.g., counting by tens might be emphasized). We have found that different children do have preferences for different methods even within the same class, so we view it as important that the teacher support different ways of thinking about multidigit addition and subtraction problems. Such different methods also enable children to build more complex and generalizable understandings of multidigit situations. Children who are making errors are helped by peers or the teacher with whatever part of their process they are doing wrong.

The examples of classroom activities in this chapter have more formal structured activities in the beginning, when children have to construct initial mathematical knowledge. Later activities demonstrate the varied ways in which children use the knowledge they constructed in such activities. In all activities, the teacher tries to set expectations that children will be active and autonomous learners and that they have responsibilities to share their own thinking and to help their classmates develop their own thinking.

The Examples Chosen

To compose this chapter, we selected examples from our English-speaking and Spanish-speaking classes to give readers some sense of major aspects of our approach. These examples are typical of many of our past classes and children. We changed the names to make one hypothetical class to follow through two years. Some vignettes are taken verbatim from videotapes, and others are constructed from classroom observation notes. We have not included all of the bumps and glitches of ordinary urban classes of inner-city children (e.g., interruptions), and we portray a teacher who has created a safe and nourishing learning environment. Furthermore, informal activities that arise from children, and discussions of children's thinking, can sometimes go extraordinarily well. Substantial insights may be made by several or many participants, often including the teacher. Sometimes such discussions do not go well. Things may get muddled or tangential. We have avoided examples of either of these extremes.

In this chapter we initially see this class near the beginning of first grade. The class of twenty-four children has a wide range of mathematical knowledge, but most can count to ten and write the numerals one through nine. A few can count to 100, but most cannot. Some of the children have not had any preschool or kindergarten experience. The teacher is working to identify someone in each home to help with mathematical homework every day. She has spoken to some of the parents of the children who cannot count or write numbers to ten about helping their child at home and plans to continue making such parent contacts. She also is trying to organize some time for a teacher aide to work with these children during school. The teacher has been using word problems as a way to get to know her children and their mathematical knowledge. She tries to adapt word problems that come from the children to her mathematical goals.

Classroom Vignettes

End of First Week of School

MS. LO CICERO: Roberto, come and share with the class a story. A story about you. Be sure to speak loud and clear so we may all hear. Students, please listen carefully to Roberto's story because I will ask questions to see how well you can remember his story.

ROBERTO: In Mexico I have a dog. My dog's name is Paco. My uncle takes care of the dog. He feeds him bones and water. When I visit Mexico, I play with him.

MS. LO CICERO: Great story! Roberto, how many bones does your uncle feed him?

ROBERTO: He feeds him five bones every day.

MS. LO CICERO: If yesterday your uncle fed him five bones and today he feeds him another five bones, how many bones does he eat in two days?

ROBERTO: Five. *(Some students say ten.)*

MS. LO CICERO: Can you go and draw this on the board?

Roberto goes to the board and draws five bones and, after a space, another five bones.

MS. LO CICERO: Ah! So five one day and five another day. Could you please draw the dishes your uncle uses to feed Paco? Draw the plates around the bones.

ROBERTO: You want me to put the bones on plates?

MS. LO CICERO: Yes, Roberto.

Roberto encircles each group with one plate.

MS. LO CICERO: Roberto, ask the classroom questions about your story. *(Silence)*

MS. LO CICERO: Who would like to ask a question about the story? Yes, Karina.

KARINA: What was the dog's name?

MS. LO CICERO: Very good. Karina, choose a student to answer your question.

Karina chooses Sandra.

SANDRA: Paco.

MS. LO CICERO: Now *you* can ask a question.

The teacher helps students ask questions that other students answer: Who feeds the dog? Where does Paco live? How many bones does he get?

MS. LO CICERO: Class, answer the questions I am going to ask by showing me with your fingers. Now pay attention to the questions we are going to ask about the story. How many groups of bones do we have here?

Most say ten. Only three show two fingers.

MS. LO CICERO: Listen to the question, "How many *groups* of bones do we have here? How many dishes do we have with bones in them?" *(relating the mathematical question to the real-world story).*

Most students show two fingers.

MS. LO CICERO: How many bones did Paco eat in two days?

Most students show ten fingers.

These two questions are then asked several times to several individual students to be sure they were listening and understanding. The teacher then asks another child to tell a story and another discussion ensues.

The Next Day

While students are playing in the playground, the teacher draws 17 bones on the board. She plans to continue Roberto's story but to use it as a transition to the topic of the day. The focus today is to discuss various kinds of groupings that occur in the world, and to examine various decompositions of the number eight. When the students come in, the class talks about yesterday's lesson, reviewing some parts of the story.

MS. LO CICERO: Without counting, just by looking at all these bones, can anyone try to guess how many bones I drew here?

MOST STUDENTS: 20, 22, 13. *(One student counted quickly and said 17.)*

MS. LO CICERO: You all made very good guesses. Chantelli, you counted them really quickly. Carlos, can you put these bones in dishes? Let's find out how many dishes of five bones can we draw to see how many days we can feed Paco.

Carlos encircles five bones.

Ms. Lo Cicero: Carlos, choose a friend to draw the next dish.

Finally three dishes are drawn.

Ms. Lo Cicero: So how many groups of five bones do we have here? Show me with your fingers.

Most students show three fingers.

Ms. Lo Cicero: So we have three dishes and how many bones are not in a plate?

Most students show two fingers.

Ms. Lo Cicero: So we have three *groups* of five and two *extra* bones. So how many do we have here?

Many students say 17.

Ms. Lo Cicero: Now we are going to think about things that have many parts. One thing with parts that look alike. In *one* hand, how many fingers do we have?

Many students say five, but some students don't know.

Ms. Lo Cicero: Remember to show me the answer with your fingers. Don't look at other students. Take your time. *(The teacher does this to practice numbers on fingers and to keep children from hearing the answer from the fastest child.)*

Ms. Lo Cicero *(showing a picture of a bicycle)*: In this *one* bicycle, how many wheels do we have?

Most students show two fingers.

Ms. Lo Cicero *(showing a picture of a tricycle)*: And in this *one* tricycle, how many wheels?

Most students show three fingers.

MS. LO CICERO: Who can give me examples of how one thing has many parts? These parts have to look the same. Look around the classroom.

VIVIANA: One chair has four legs.

MS. LO CICERO: Good thinking. Who can find in the classroom something else with four legs?

CARLOS: The tables.

MS. LO CICERO: That is right. One table has four legs. Who can think of another thing that has four parts that are the same?

LUIS: A wagon has four wheels.

MS. LO CICERO: What kind of wagon?

LUIS: A train wagon.

MS. LO CICERO: Can you come and draw the train?

Luis draws a rectangle with four wheels below it.

MS. LO CICERO: Now let's talk about the number eight. Yesterday we talked about finding seven of something. Does anyone know what has eight parts that belong to one thing?

EDGAR: Some trucks have eight wheels.

MS. LO CICERO: Very good!

DANIELA: I know one! A spider has eight legs.

MS. LO CICERO: Any more groups of eight or eight parts?

NANCI: Two train wagons would have eight wheels. I'll show you. *(She draws another rectangle with four wheels beside Luis's wagon.)*

Already children are building on the contributions of others. The teacher has worked hard in the beginning to structure participation so that it requires hearing and making sense of what other children say and do, not just listening to or talking to the teacher.

MS. LO CICERO: Do you know what comes in front of the train? *(Silence)*

By asking questions the teacher discovers that not one student knows the Spanish or English word for engine *(locomotora)*, so she discusses that word in both languages. Building vocabulary is a natural and important aspect of this approach.

MS. LO CICERO: Let's pretend that we are going to check the wheels to see that everything is right. The mechanic has to check all eight wheels. Let's see, Linda, can you come and check the wheels?

She extends this fantasy situation to use one-to-one correspondence to show a how-many-more situation. Linda comes up front.

MS. LO CICERO: Check a wheel. *(Linda does nothing.)* Just pretend.

Several boys offer to help.

MS. LO CICERO: Linda can do it. Just see if the wheel is working. Once you check it, put a line under the wheel. *(Linda puts a line under the first wheel, gestures toward each of the next two wheels and draws lines under them.)* Linda, wait a minute before you continue checking. Class, Linda has to check eight wheels. She has checked three so far. How many more does she need to check? Show me with your fingers.

Many students show five.

MS. LO CICERO: Linda, count how many more you need to check to show everyone.

LINDA: One, two, three, four, five. Five more.

MS. LO CICERO: Now check one more.

LINDA: Now I have checked four.

MS. LO CICERO: She has to check eight in all. How many more does she need to check?

Most students show four fingers.

This checking and asking process continues through eight and zero. This scenario grew from the children's stories and was spontaneously co-invented by the teacher and the class. For example, Nanci used Luis's train wagon idea to respond to the teacher's request about eight. This class provided children experience with decomposing the number eight and with *how many more* language in a repeated context so that children can begin to make sense of the words *how many more* (which some children do not yet differentiate from the words *how many*).

The Next Day

MS. LO CICERO: Now let's pretend that you have a garden with many flowers and that your mother asks you to cut eight flowers and put them in two vases, one for the dining table and one for the TV. So let's draw the eight flowers. *(She draws them on the board.)* Let's see. Cinthia, how would you put some flowers in one vase and some in another vase?

Cinthia circles five flowers and the remaining three flowers.

The teacher goes on to have children do various decompositions of eight and begin to write numbers beside each decomposition. Each child then draws a pattern of eight circles and decomposes this pattern into various numbers "hiding inside eight."

Two Days Later

The teacher has children work with a partner to show pairs of numbers hiding inside ten. Children will do many activities so that the pairs to ten become very fast. They need to know these combinations to do adding and subtracting using ten. The first step in solving eight plus five by using ten is knowing that eight plus two is ten. Then they need to be able to break five into two plus three; decompositions of numbers help with this. Finally, they need to know that ten plus three is thirteen.

Fourth Week of School

Classes for the past week have been split between posing and solving simple word problems and decomposition activities. Since school began, children have done many activities concerning decomposition of ten and of smaller numbers on their fingers. For many days they have drawn all the decompositions they could think of for a number as homework, using their own patterns for numbers and horizontal rows of circles. Children have engaged in these activities at their own levels, but all are doing more advanced or more thorough approaches than at the beginning. By now many are beginning to memorize some of the combinations. Children have made various observations about some patterns in decompositions throughout this work. In this discussion the teacher is trying to enable them to express and reflect on many of these observations. Most children are still drawing groups of circles and circling them and then recording their numbers, but some children are working just with numbers.

MS. LO CICERO: Class, today we're going to discuss your homework. Let's talk about the numbers you found that made ten. You've been working with ten for a long time. Let's see how many different things we can say about the numbers in ten. Look at the board. Let's write some of your work. Who found a pair with one? Carlos? Tell me what you know.

CARLOS: One and nine are inside ten.

Ms. Lo Cicero draws a row of ten circles.

MS. LO CICERO: Carlos told me he had one *(shades the first circle)*. Then there are nine left *(counts the rest)*: one, two, three, four, five, six, seven, eight, nine *(writes 1 and 9 to the right of the row)*. Who else?

Mrs. Lo Cicero elicits until she has all pairs. Children give these in various orders but she draws and shades them on the board in order from zero plus ten to one plus nine to two plus eight and so on, to eight plus two to nine plus one to ten plus zero. She leaves spaces for the missing ones to be reported and drawn.

MS. LO CICERO: Can someone tell me if they see something interesting here? [In English we ask, "Do you see a pattern?" In Spanish there is not a word for "pattern," so you have to talk around this concept.]

The teacher points to children to share what they see.

ANDRÉS: If the first number is small, the other is big.

MARTA: Eight and two is ten. Two and eight is also ten.

VIVIANA: That's true of other numbers. There's one and nine, and nine and one, and three and seven, and seven and three.

ROBERTO: And four and six, and six and four.

JORGE: But there's only one five and five.

RUFINA: Ten plus zero is ten. And zero and ten are ten. Zero plus a number is the number. Zero doesn't add anything.

KARINA: The numbers on the left go zero, one, two, three, four, five, six, seven, eight, nine, and on the right it is the opposite.

JOSUÉ: Numbers on the left get bigger by one and on the right they get smaller by one.

JORGE: I can make one row from another. I know six and four make ten, and one more is seven. So the other number is one less so it is three.

MS. LO CICERO: You are really getting good at noticing things and saying them so other children can understand you. In math it is really important to notice things. There are lots of interesting things in math. Now let's tell a math story and see what problems we can make from it.

Mid October

The class moves from the general grouping activities to focus more heavily on groupings of ten. This begins a series of activities supporting children's construction of a connected web of different meanings for two-digit numbers. All children initially count objects by ones, but they gradually build understandings of two-digit numbers as groups of ten and loose ones (53 is five groups of ten and three loose ones) and as sequence counting words (53 is made by counting by tens five times: 10, 20, 30, 40, 50, and then by counting on three more: 51, 52, 53).

MS. LO CICERO: Remember how we have been talking about groups: a group of two, a group of six, a group of five. Today we will make groups of ten. Groups of ten are especially important because our numbers use groups of ten. Also our money uses groups of ten. Today we are going to make groups of ten with pennies. We are going to find out how many groups of ten pennies we have. I will show you with my pennies, then you will do the same with your pennies. We are going to use a special worksheet and these penny strips to help us collect ten pennies. On each strip you can put ten pennies *(each strip has pictures of ten pennies with a space between the two groups of five so children can see eight as five plus three or six as five plus one).* We are learning to count pennies in a special way. We can also count other things in this way. We can count seeds, beans, or little stones. We will be working for many days learning to count money and to write it down. After we learn how to count money, we will be able to go to the market. Remember I told you we will learn about money so we can buy bones for Paco's dog and we can buy other things.

Ms. Lo Cicero grabs a bunch of pennies and goes around showing them.

MS. LO CICERO: Try to guess how many *groups* of ten pennies I have in my hands. In your worksheet you have the word *estimate*; that is like guessing but you think before you guess. Write down your *estimate* of how many *groups* of ten I have. *(Pause)* Now we'll find out. Teacher puts a penny strip on the board. *(Figure 10–1 shows ways in which children in a later lesson showed 53 with penny strips. One side of each strip has ten pennies with a space to show the five and five and the other side shows a dime.)* We will use this penny strip to help us see when we have ten. I will put my pennies on top of the pennies on this strip. How many pennies are on this strip? Let's count together. One, two, three, four, five, six, seven, eight, nine, ten. So there are ten pennies here. When I cover all of these pennies on the strip I will have ten pennies, *one group* of ten and *no* pennies extra, *no* loose pennies. When pennies are not in a group of ten, we say they are the loose ones, or that they are the extra ones. So let's see how many groups of ten I have and how many loose ones I have. How many *strips* can I fill? We will find out how many *groups* of ten I have and how many *loose* ones. Count with me.

The teacher begins filling the strip and counting as she sticks each penny on top of the penny on the strip.

STUDENTS: One, two, three, four, five, six, seven, eight, nine, ten.

MS. LO CICERO: So I have one ten, one *group* of ten, so far. Let's see if I can make another ten. *(She puts up another strip.)* Let's count together.

STUDENTS: One, two, three, four, five, six, seven, eight, nine, ten.

MS. LO CICERO: Yes! How many tens do I have?

STUDENTS: Two tens. 20.

MS. LO CICERO: How do you know we have 20?

VIVIANA: Because ten and ten is 20.

MS. LO CICERO: So how many *groups* of ten do I have?

STUDENTS: Two.

MS. LO CICERO: So I have two *groups* of ten and 10, 20 *(she points to each strip)* pennies. How many loose ones? Let's see. *(She puts up pennies.)* One, two, three, four. Do I have enough to make a ten?

STUDENTS: No.

MS. LO CICERO: So how many groups of ten?

STUDENTS: Two.

MS. LO CICERO: How many loose ones?

STUDENTS: Four.

MS. LO CICERO: That's right.

JUAN: You have 24 pennies.

MS. LO CICERO: How do you know that?

JUAN: Because 10, 20, 21, 22, 23, 24.

MS. LO CICERO: That is right. I have 10 *(gesturing across whole first strip)*, 20 *(gesturing across whole second strip)*, 21, 22, 23, 24 *(pointing to each penny)*. Twenty-four pennies. That makes one, two *(pointing to each strip)*—two *groups of ten* and *four loose ones*, four ones extra *(pointing to four pennies)*. Let's count them by ones just to make sure *(the class counts all of the pennies by ones as Ms. Lo Cicero points to each penny)*. Juan was right. We have 24 pennies, *two* groups of ten *(pointing to the two strips and writing the numeral 2 under the two strips)* and *four* loose ones, *four* ones extra *(writing the numeral 4 under the four pennies)*. We write *two* groups of ten, one, two *(pointing to each strip and then to the numeral 2)* in the tens place here on the left, and we write *four* loose ones, one, two, three, four *(pointing to each penny and then to the numeral 4)* in the ones place here on the right. Now I will give you some pennies, and I will help you do the same. Then we are going to learn to draw the pennies and write the number. There's a special way to write numbers bigger than nine. It uses these groups of ten.

Late November

The children have done penny strips and other grouping activities for three of the last six weeks. They were working on word problems and graphing (reading graphs that compare two quantities and making word problems from them) most of the rest of the time. Different children have constructed different parts of the two-digit web of meanings. Most can now count to 100 by ones and by tens because many activities focused on doing this in a meaningful way (e.g., discussing the structure of the English or Spanish number words and how the *ty* in *forty* or the *enta* in *cuarenta* means tens, so *forty* means *four tens;* putting up a finger as you say each ten and using the fingers to help you generate the next ten if you get stuck). Some children tend to emphasize the number words; they do many activities by counting by tens and ones. Others focus more on the written numbers and see them as so many groups of tens and loose ones. The activities are designed to help everyone build both kinds of meaning and eventually connect all of these meanings. Some children are still struggling with left-right issues, and so sometimes reverse the tens' and ones' position. Some children still prefer to count by ones.

MS. LO CICERO: Today, first make 53 using your penny strips.

She writes 53 on the board. Children work individually. (All of these methods are shown in Figure 10–1.)

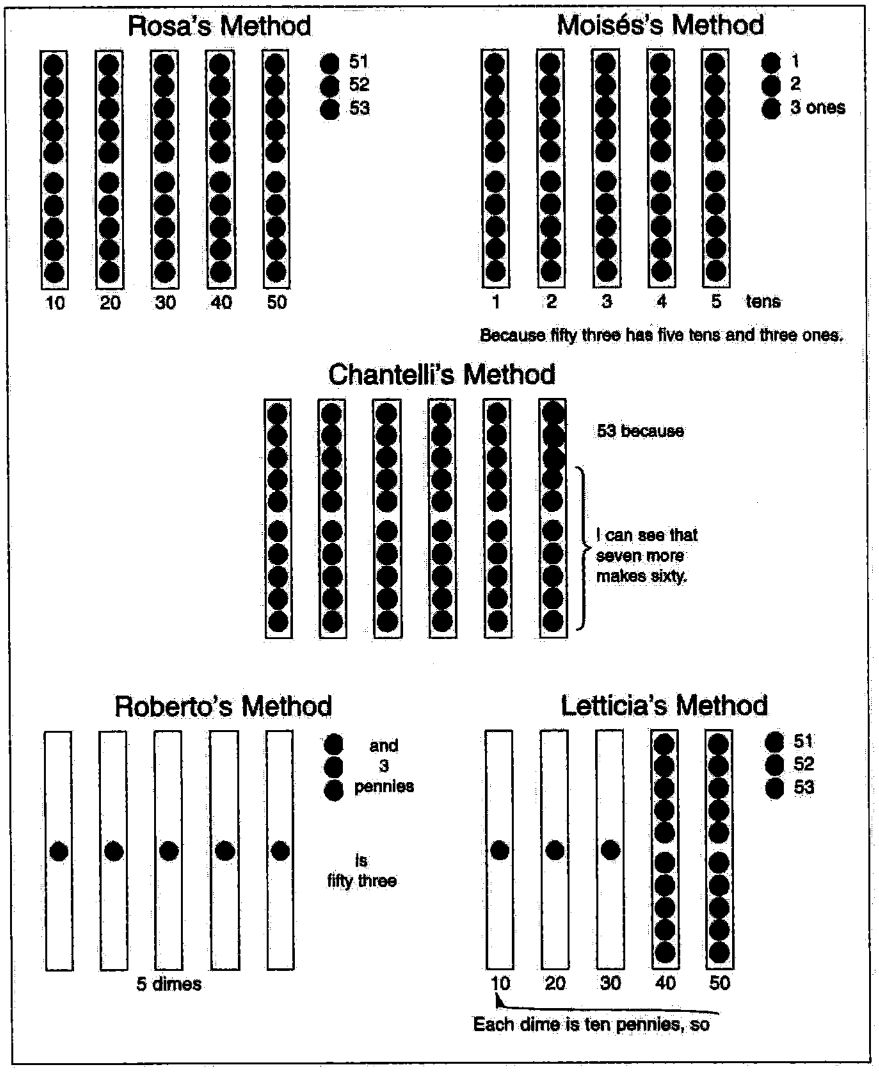
MS. LO CICERO: How did you do it, Rosa? Come to the front and show us.

Rosa uses the large strips and pennies that stick on the chalkboard.

ROSA: I made five strips and three ones. See, 10, 20, 30, 40, 50 *(she points to each strip)*, 51, 52, 53 *(pointing to her three loose pennies)*.

MS. LO CICERO: Did anyone do it a different way? Moisés?

MOISÉS: One, two, three, four, five tens *(he counts as he puts up each strip)*, and one, two, three *(counting each penny as he puts it up)*. I knew 53 has five tens and three ones.



10–1 Methods of making and counting 53 with penny strips

MS. LO CICERO: Did anyone do something different? Chantelli?

CHANTELLI: I put my three pennies on another ten strip *(putting up six penny strips and puts three loose pennies on top of the three pennies at the top of the strip)*. So I can see how much more to 60 *(pointing to the penny pictures without pennies covering them)*. See, seven. Two more and five is seven.

MS. LO CICERO: So you can see that 53 *(gesturing to all coins on strips)* needs seven more to make 60 *(gesturing to printed pennies on last strip not covered by actual pennies)*. Anyone else?

ROBERTO: With 53 pennies we have five dimes and three pennies. *(His five strips show the dime side, and he puts up three pennies.)*

MS. LO CICERO: How did you know that?

ROBERTO: Each group of ten pennies is a dime *(he holds up a penny strip and shows the ten pennies and then the dime on the back)*, so 53 is five dimes and three pennies. We have 53 cents.

MS. LO CICERO: Class, did you all listen to how Roberto made his 53? Who can tell me what Roberto said just now? Carlos.

CARLOS: He said he made 53 with five dimes and three pennies. That's like five groups of ten and three loose ones.

MS. LO CICERO: Yes, you're saying that Roberto's way is kind of like how Moisés did it by counting the tens because one dime is one group of ten pennies. Anyone else? Letticia.

LETTICIA: I used dimes *and* groups of ten pennies. I had three dimes *(she puts up three strips showing the dime side)* and two groups of ten pennies *(putting up two strips showing the penny side)* to make five tens and then three loose pennies *(putting up three pennies)*.

MS. LO CICERO: Can you show us how you know it makes 53 pennies?

LETTICIA: Each dime is ten pennies, so 10, 20, 30 *(pointing to each dime)*, 40, 50 *(pointing to each ten-penny strip)*, 51, 52, 53 *(pointing to each penny)*.

MS. LO CICERO: OK. That's kind of a hard way. Does everyone understand how Letticia's way works? Gabriela, can you come show us again? We'll help you if you need help.

GABRIELA: 10, 20, 30, 31 *(first tens strip)*, 32 *(second tens strip)*, 33, 34, 35 *(she shifts to counting the pennies on the penny strip by ones but only counts each strip of ten once as 31, 32 instead of counting all of the pennies by ones)*.

LETTICIA: There are *ten* pennies here *(pointing to the first strip of pennies)*. You have to count all ten or say 40. You can't just count it 31.

GABRIELA: 10, 20, 30 *(counting the 3 dimes by tens and then shifts to counting all the pennies on the penny strips by ones)*, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53. *(She prefers to think of the pennies as ones instead of counting two different kinds of tens.)*

MS. LO CICERO: Good counting, Gabriela. Can someone count it the other way Letticia said? Count this as *one* group of ten *(gesturing to first strip of pennies)*? Pedro, come write the numbers for us while you count.

Pedro comes and counts and writes the numbers as he counts.

PEDRO: 10, 20, 30 *(three dime strips)*, 40, 50 *(two ten-penny strips)*, 60, 70, 80 *(the three loose pennies). (He succeeds in counting each penny strip by ten but then keeps counting by tens when he counts the loose pennies, a typical early error. Some children initially find it hard to shift from counting by tens to counting by ones.)*

MS. LO CICERO: Are these loose pennies tens?

PEDRO: Oh, no.

MS. LO CICERO: So how do we count?

PEDRO: 10, 20, 30, 40, 50, 51, 52, 53.

MS. LO CICERO: Can you count again and count the groups of tens?

PEDRO: One group of ten, two groups of ten, three groups of ten, four groups of ten, five groups of ten, and one, two, three loose ones.

MS. LO CICERO: Very nice. So this 53 *(gesturing to the 53)* is *five* groups of ten *(gesturing to the 5 in 53 and then to the five strips)* and *three* loose ones *(gesturing to the 3 in 53 and the three loose pennies), five tens (gesturing to the 5)* and *three ones (gesturing to the 3)* is 53. As Roberto said, we have 53 cents. (Teacher connects the quantities, words, and written numerals.)

May, First Grade

The children have been solving situations requiring the addition of two-digit numbers with regrouping on and off since late February. These problems were the first problems introduced so that from the beginning children would think about whether they get another ten in such situations. Children used the penny strips or drawn recordings of the strips in which a vertical line is a ten-penny strip and dots or circles are the loose pennies.

Most children can successfully solve and explain such problems if they use drawn quantities. Through such activities, many children are still constructing or automatizing their sequence meanings of number words (10, 20, 30, . . .) or their meanings of two-digit written numbers as groups of tens and loose ones (48 as four groups of ten and eight loose ones). Some children are still having trouble remembering whether the tens are on the left or on the right. New children who have entered the class since January missed a lot of the early quantity work and some are still behind, though the teacher has been trying to organize extra help for them to catch up. Some children do so quickly, and others more slowly.

Today the teacher is beginning the reflective process of shifting children from methods using objects or drawn quantities (penny strips and pennies or drawn ten-sticks and dots) to methods relying on mental versions of the operations on physical or drawn quantities. In inventing these mental methods, children use their meanings for two-digit numbers. Some children use sequence meanings and methods (count by tens), some use collected-tens methods (count the tens), and some use a combination, shifting flexibly between these kinds of meanings. Between a third and a half of the children are not yet ready; they still need to use drawn quantities. In the second grade many children will need to fall back to using drawn quantities when they encounter new and difficult problems.

MS. LO CICERO: Can you discover a way to add all the tens and all the ones without having to draw them all and count them all? Try it for this problem.

She writes 49 + 25 horizontally. The children work independently and sometimes in consultation with each other or the teacher.

MS. LO CICERO: OK, let's see some of the ways you have discovered. Nanci.

The methods are shown in Figure 10–2. Nanci comes to the blackboard. She looks at the tens (the four and the two) and explains.

NANCI: A four and a two in my mind.

MS. LO CICERO: Two what?

NANCI: Four tens and two tens.

Nanci writes a 6, looks at the ones, erases the 6.

NANCI: I can make another ten, and then you count the ones and it's . . . *(Pauses, counts five fingers onto nine, writes 74.)*

MS. LO CICERO: Can you tell us how?

NANCI: Nine, and another ten *(first finger of the five more)* is seven tens, and one, two, three, four *(putting up four new fingers for the rest of the 5)* is 74.

Nanci has invented a precursor to the look-ahead method widely taught in Europe. In this method you start on the left, add a column, look ahead to see if you're getting another ten and increase your total if you are, write the tens, then add the next column (to the right) in the

Nanci's Method

49 + 25 = 6

Four tens and two tens (writes 6).
(Looks at the ones; erases the 6.)
I can make another ten, and then you count the ones (fingers count 5 on to 9), writes 74.

Later she invents a way to record the new ten:

3
49 + 25 = 74

Cinthia's Method

25 + 47

I took three from the five and put it with the seven. Then I counted two plus four is six. Then there is another ten, so seven tens, and there are two left, seventy-two.

Later she invents

5
48 + 27 = 75

Viviana's Method

48 + 23

Forty and two tens makes sixty.
Eight in my mind. 68, 9, 10, 11, 71.

Martha and Rufina's Methods

1
37 + 26 = 63

48 + 16 = 64

1
25
+ 49

Jorge's Method

56 + 27 =

I know these are tens.
50, 60, 70. Then I counted 7 (7 fingers up): 71, 72, 73, 74, 75, 76, 77. Then I counted 6 more (6 fingers up): 78, 79, 80, 81, 82, 83.

Karina's Method

37 + 56

Eight. Eighty seven
(counts on fingers, 6 fingers)
8, 9, 10, 90, 93

Methods of Marking Tens and Ones

TO TO
34 + 19 = __ __

4|8 + 1|6 = __ __

47 + 28 = __ __

25 + 47 = __ __

10–2 Mental and written numeric two-digit addition methods

same way. This method generalizes to problems with large numbers and can be done meaningfully. Most children's methods work from left to right.

The teacher then asks the class to try to figure out other ways to add all the tens and ones together without having to draw them all and count them all. The teacher gives some more problems to experiment on and tours the class while they do. This first phase gives the teacher direct feedback on where the children are, what they're ready for, and what sort of help they'll need. Most important, there usually are some children, from the very beginning, with productive invented methods. This further reinforces the general orientation of the class toward inventive problem solving because the invention is focused upon and credited to the inventor.

MS. LO CICERO: Now let's have some more children explain their discovery of how to add without drawing and counting all of the ten-sticks and ones. Jorge has a good method. Jorge, come explain what you discovered.

The problem is 56 + 27 written horizontally. Jorge has invented a mental counting-on method for the tens; he then puts up fingers for the ones and counts them on from the tens' total. He uses the written form of the problem on the board to point out what he was grouping orally and with his fingers.

JORGE: I know these are tens *(pointing to the 56 and 27)*, 50, 60, 70. Then I counted seven *(putting up one hand and two fingers and then counts them)* 71, 72, 73, 74, 75, 76, 77. Then I counted six more *(putting up one hand and one finger and then counts them)* 78, 79, 80, 81, 82, 83.

Two other children's hands shoot up, and they volunteer that they do it like that as well. Juan describes his method and does it pretty much like Jorge. But Viviana shows a way to collapse the counting on of the initial ones from the tens (she chooses to explain 48 + 23).

VIVIANA: Forty and two tens makes 60. Eight in my mind. 68 *(she just adds the 8 to 60)*, 9, 10, 11 *(she counts these three without using her fingers)*, 71.

MS. LO CICERO: Can you explain how you got the 71?

VIVIANA: Forty and two tens makes 60. Eight in my mind. 68, 69, 70, 71.

The teacher asks the class to try it Jorge's, Juan's, and Viviana's way. About half the class does, and the rest continue doing it the way they had been. This is fine. At this stage many still need to operate by drawing and counting all the quantities. The suggestion to "Try it ___'s way" is best thought of as an invitation for those who feel ready to try that method.

In this class, about one-third of the class immediately invent a wide range of correct and efficient methods, another one-third struggle in various ways over several days to figure out some way not to lose track of the extra ten (in contexts where they no longer have it present before them in the form of drawn ones counted and regrouped into a ten), and about one-third at first prefer to continue drawing and counting all the tens and ones, as they had been doing.

Over the next several days, the focus is on various inventors showing the class their ways of solving regrouping problems mentally and numerically. The class tries out some of these ways. Small groups of the children who are struggling toward workable methods work together and compare their approaches, usually with some help from the teacher sometime during a class period.

The Next Day

While the teacher is touring the class observing problem solving, she notices that Cinthia and Karina have each invented a ten-structured method, splitting the ones into the part needed to make a ten and the ones remaining. Cinthia solves the problem and explains 25 + 47 like this: "I took three from the five and put it with the seven. Then I counted two plus four is six. Then there is another ten *(this shows she is thinking of the two plus four equals six as tens)*, so seven tens, and there is two left. 72." Karina starts from the tens first, on the problem 37 + 56. She points at the tens numbers 3 and 5 while saying, "Eight, *(then points at the 7 in 37)* 87, *(points at the 6 in 56 and counts on her fingers)* 8, 9, 10, 90 *(80 plus that 10 was really 90)*, 93 *(chunking the three from the six over 90)*."

Next Several Days

The children in the bottom third of the class have been struggling unsuccessfully with remembering the extra ten when solving without drawn quantities. Some add the tens first, and some add the ones first. Sometimes these children remember the extra ten, sometimes not. Many of them also have trouble with what turns out to be one of the main underlying problems when numbers are written horizontally: sorting out which are the tens and the ones. Giving problems vertically right away may allow children to work without really understanding tens and ones. The teacher gathers the strugglers together into a group to see what they can figure out among themselves (and with teacher help) that would be effective. Pedro, having begun to learn the traditional carry method from home, lobbies for it as a reminder of the extra ten. Everyone in the group agrees that this is a good way to do things, but when they begin to work on their own, most immediately start to do quite different things. Martha and Rufina break into a working pair of their own and devise three different ways (see Figure 10–2): underlining tens when using the horizontal equation form, making loops that connect tens' places in one number to tens in the other number, or (in the vertical case) vertical lines from tens to tens. Martha prefers to store the extra ten mentally, Rufina by writing it, but by the end of this session both make errors less frequently. Linda sometimes stores mentally, sometimes with a carry mark, but also from this point on, rarely makes mistakes. Pedro, noticing that few were following the carry procedure, attempts to lobby for it more, but no one pays attention, so he goes back to working on his own. Gabriela and Luis get the main attention of the teacher because they are having trouble consistently. Across several sessions Gabriela steadily becomes more accurate in remembering an extra ten. Luis consistently needs to rewrite a horizontal equation in vertical form, and he uses a carry procedure only partially understood at that time.

During these days, Nanci and Cinthia go on to invent symbolic numeric supports for their methods (see Figure 10–2). These might be especially helpful as they move to larger numbers where the mental load of remembering everything is greater. Later in first grade and in second grade, many of the children still drawing and counting tens and ones can only shift to numeric or mental methods when they visually mark which are the tens and which are the ones as Martha and Rufina do, or in the other ways shown in Figure 10–2. They can distinguish tens from ones and use these when they are visually clear as with drawn ten-sticks and ones, but need some kind of written support when only using numbers because the numbers just look like single digits.

December, Second Grade

MS. LO CICERO: We have been talking about money all year. We studied the value of our coins. We talked about some of your experiences in buying and selling. We learned how to give change from a dime and from a quarter. We used the penny strips, and now we are using money. Remember we acted out Mercado [a small store] and worked in pairs. Today we will learn to give change from a dollar. We need to help everyone learn to give change before we can have our bake sale. I will pass you each a bag [sandwich bag] containing coins. Fold a sheet of paper in four equal parts. Draw different ways of making up one dollar.

Students work for ten minutes. Some converse and help each other. The teacher observes methods and asks questions of students to find out their thinking, or to help them understand, describe, or correct their method, or to see if a child is ready to move on to a more advanced method.

MS. LO CICERO: Now imagine that you are buying a pencil that costs 37 cents. How much change will you get back? You may draw, write down numbers, or use ten-sticks and dots. I want to see how you work this out. Try different ways. Remember how we drew the coins before.

Students work for twenty minutes. The teacher goes around checking and helping the students. She is also looking for good examples that later may be used by the whole class and for typical errors that may need to be discussed.

MS. LO CICERO: Let's have Juan come up and show his way. Draw your first dollar. Juan, explain what you did in your first drawing. [See Figure 10–3.]

JUAN: I drew ten dimes. Ten dimes make one dollar. 63 is the answer.

MS. LO CICERO: Can you explain what you did with this dime?

JUAN: One dime has ten pennies, or a nickel and five pennies. I took 37, and now I have 63.

MS. LO CICERO: You took 37 from what? From how much?

JUAN: From one dollar.

MS. LO CICERO: How do you know you have a dollar?

JUAN: Ten dimes is the same as one dollar.

MS. LO CICERO: How many cents do you have in one dollar?

JUAN: 100.

MS. LO CICERO: Who else would like to show their way? Viviana, yes, you may come. Let's see, show us the second solution *(looking at her paper)*.

VIVIANA: Shall I draw it? [See Figure 10–3.]

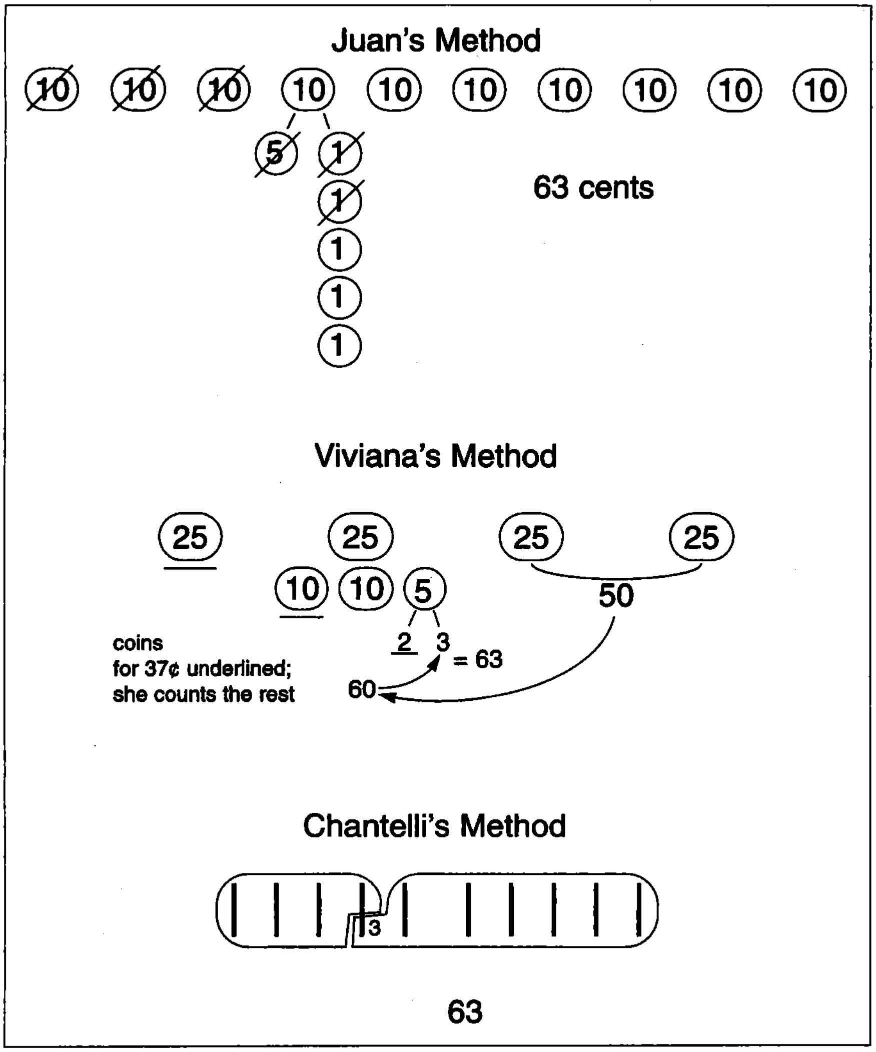
MS. LO CICERO: Yes, please. Tell us how you thought about all this.

Viviana: 25 plus 10 is 35 plus 2 *(pointing to her underlined coins)* is 37. Then I counted what I had left: 25 plus 25 is 50 plus 10 is 60 plus 3 is 63.

MS. LO CICERO: Good thinking! Who would like to show another way?

Chantelli: These tens *(the ten-stick drawings of the penny strips)* are my dimes [see Figure 10–3]. There's three left in this dime *(pointing to the split ten-stick)* and six more dimes.

MS. LO CICERO: Can you count up to give me change with this money here?



10–3 Methods for finding the change for $.37 from $1.00

CHANTELLI: Here is three pennies to make 40 cents, and 60 more is one dollar. Here is 10, 20, 30, 40, 50, 60.

MS. LO CICERO: Very nice. You found lots of ways to show how you give change from a dollar. Tomorrow we will do Mercado [children work in pairs buying and selling] and practice giving change from a dollar for our bake sale. [Such practice is important because counting up with money is more difficult for children than counting up with penny strips or ten-sticks, because the money does not look like tens—the dimes are even smaller than the pennies.]

January, Second Grade

The class is working on problems comparing heights. They are using inches so that the heights will be two-digit numbers. Later they will use centimeters to get three-digit numbers. They have been finding the difference in height between Jorge and Cinthia's little brother Paulo: 62" and 37". They also are working on different ways to ask the difference questions. Here several students explain how they solved the problem. (See Figure 10–4.)

GABRIELA: *(She has drawn three dots, then two ten-sticks, then two dots, and written 25; the other numbers are what she says in her explanation.)* I said, "How much does Paulo have to grow?" so 37 plus 3 more *(pointing to three dots)* is 38, 39, 40, and 50 *(pointing to a ten-stick)*, 60 *(pointing to another ten-stick)*, 61, 62 *(pointing to the two dots)*. So this is 23 *(gesturing to the three dots and two ten-sticks)*, 24, 25 more he has to grow to catch Jorge.

ROBERTO: I shrunk the big guy down by taking away the little guy from him *(gesturing to his drawing of the little guy beside the big guy and the line he drew across from the top of the little guy to the big guy)*. So 62 minus 37 is 25 [see Figure 10–4]. I took three tens from the six tens and seven from this ten. That leaves three and these two are five and two tens left is 25.

MS. LO CICERO: I am going to ask someone to tell me how Josué explains it. You all need to listen to him. I know it is really hot in here today *(the heating system in the building malfunctions, and the room is stifling; some children had put their heads down and were not watching)*. Let's all take five big deep breaths before Josué explains so we can all concentrate. Ready, ooooone, twoooooo, threeeeeeee, fouuuuuurrrrr, fiiiiivvvvve. Good. Josué.

JOSUÉ: I did it like Gabriela but I wrote three and then my ten-sticks and two and then added them to get 25 more the little guy needs. [See Figure 10–4.]

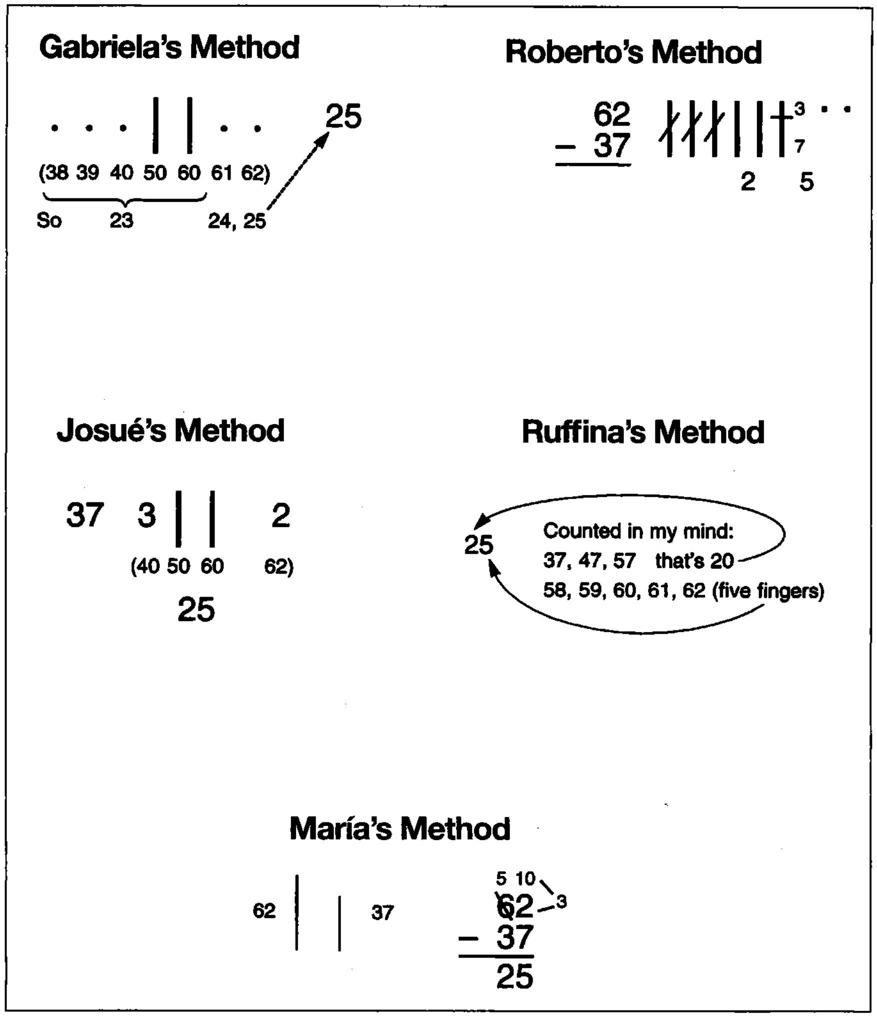
MS. LO CICERO: Karina, how did Josué do it?

KARINA: Like Gabriela but he used numbers and sticks.

MS. LO CICERO: What did he start with?

KARINA: 37 and made 25 more.

MS. LO CICERO: Can someone else say in their words how Josué did it?



10–4 Methods for finding the difference in height between 62 inches and 37 inches

NANCI: He used numbers and sticks to go 37 plus 3 is 40 plus 2 tens is 60 plus 2 to get to Jorge. So 2 tens and 5 is 25.

MS. LO CICERO: OK. Ruffina.

RUFFINA *(She has just written 25 above her little guy drawing, [see Figure 10–4])*: I just counted in my mind 37, 47, 57, that's 20 *(she points to the 2 in 25)*, then 58, 59, 60, 61, 62 *(raises five fingers as she counts)*, so that's 5. 25.

MARÍA: I subtracted Paulo from Jorge like Roberto did, but I used numbers. I took one of the tens to get enough to take away the seven, so that was three and two more was five ones, and there were two tens left so 25.

Ms. Lo Cicero: Can someone else tell how Roberto's and María's methods are alike?

Carlos: They both took away the little guy to get the difference. They took away 37 from 62.

Ms. Lo Cicero: Anything else?

Jazmin: They both had to open a ten because there weren't seven ones to take away. So Roberto took his seven from that ten-stick. *(Teacher points to show the ten-stick Roberto separated into seven and three, and looks questioningly at Jazmin.)* Yes, there he took seven and left three. And María took a ten from the six tens and wrote it with the ones and then took the seven to leave three. *(Teacher points to the top right part of María's problem.)*

Ms. Lo Cicero: So they were both thinking about taking ones from a ten but they wrote it in different ways?

Several students: Yes.

Letticia: And we know other ways to write subtraction, too.

Ms. Lo Cicero: Yes, you have lots of ways you show taking away and comparing, too. Whose heights should we compare next?

Discussion

Tasks

This chapter contains many examples of tasks that children problematized: decomposing numbers, searching for patterns in the decomposed pairs, different ways of showing 53, adding multidigit numbers, finding 49 + 25 without drawing objects, making change, and finding differences in heights. Initially, children contributed to generating mathematical situations and contexts that were familiar, and the teacher supported them in learning to mathematize these situations. As the children's mathematical thinking developed, their thinking, and the various methods children devised, increasingly became the focus of classroom discussions. Over time, children took increasing responsibility for the ideas discussed and for helping their classmates in their mathematical thinking.

In order to make mathematics problematic, teachers need to select tasks that are accessible to children but are not already automatic. However, sometimes the teacher needs to do activities to help children make certain things automatic. For example, after children understand and have drawn decompositions to ten for some period of time, Ms. Lo Cicero had children practice these so that they would be available for mental ten-structured methods of adding and subtracting. Children also practiced counting by tens: This must be automatic if children are to use addition or subtraction methods involving counting by tens. Children regularly do distributed practice on homework: They have a range of problems they have done earlier as well as the current problems. But most of the tasks done in class are ones that are problematic for many students. Helping other children solve a problem in their own way can also be problematic and can extend the helper's mathematical thinking because of the new perspective.

Role of the Teacher

A vital new role is to understand children's thinking about mathematics. Teachers have told us that one of the most exciting things about changing from a traditional mathematics classroom to teaching mathematics with understanding is that the teacher constantly has a chance to learn more about children's thinking and about mathematics itself. The teacher no longer has to be the sole authority, but can participate with children in figuring out some mathematical situation. Our teachers say that they would never go back to teaching in the old ways. Earlier, neither they nor most of their children understood some or even much of the mathematics that was being taught and learned. Understanding mathematics, and looking forward to understanding even more about mathematics and about children's mathematical thinking, is now a vital and energizing part of our teachers' professional self-image. This attitude and the feeling of involvement and self-confidence that it generates enable classrooms to be exciting and interesting places for children and for teachers.

Once a teacher experiences a year focusing on children's thinking, she or he is in a position to have a vision of progressions in children's thinking and various solution methods children use. This enables the teacher to set learning goals throughout the year for children who are progressing at different speeds and to select tasks and tools that support children through these learning goals. In the examples here, Ms. Lo Cicero used (with the introduction of the penny strips) the different kinds of language she knew that the children would need to carry out different addition and subtraction methods. She talked about groups of ten so that children could understand the meaning of the tens' place. She counted these tens as tens (one, two groups of ten) and also as sequence counting words (10, 20), thus relating these two meanings. She counted by ones, the counting that all children could do, to relate these new ways of thinking to children's present ways of thinking and present skills. She repeatedly used all forms of language and linked these to the quantities (the strips and loose pennies) and to the written numerals so that children could begin to build quantity meaning for the numerals. Later, some children used each of the kinds of language she had introduced: Some used methods in which they thought of units of ten and counted tens (Moisés, Nanci, and Roberto), and other children counted by sequence tens (Rosa, Viviana, and Gabriela).

The penny strips are also an example of another vital role of the teacher: to share essential mathematical information and to design activities to help children understand this information. The penny strips were introduced to help children understand the value of money and to help children build quantity meanings for written two-digit numerals. Both money and written place value are arbitrary mathematical or social-cultural conventions that the teacher must help children come to understand.

Classroom Culture

Ms. Lo Cicero consciously worked to establish a classroom culture in which children felt safe and confident enough to share their thinking. Many children, especially those who do not come from middle-class backgrounds, need models of the desired kinds of classroom discourse; they are not used to talking about their thinking. Some of our children had never been in a school classroom of any kind before first grade. The early examples show how a teacher might get started involving children in classroom discussions. Ms. Lo Cicero did a great deal of talking in these early episodes. She was doing many different things: building a class feeling, modeling how children might talk, helping children describe their thinking, ensuring that children speak loudly enough for their classmates to hear, keeping children involved, focusing attention, and helping children learn to listen to their classmates. She was very patient with children and waited for the long time it sometimes took them to find words, while gently offering encouragement if needed. She constantly began with some situation that was familiar to many of the children, mathematized that situation with respect to her mathematical goals at the time (which might necessitate changing some aspect of the situation), and then wove that mathematical thinking back to the situation. She tried to create coherence and continuity over many days by returning to earlier situations and extending them to new mathematical topics.

As children build the desired mathematical concepts and methods, they come to be able to take over more and more of the discussion. The later examples demonstrate that the teacher can now elicit methods from many children without helping them to clarify nearly as much as was necessary earlier. Some children have become quite sophisticated about seeing methods that are similar, even if there are differences in the methods (e.g., Jorge, Juan, and Viviana for their invented two-digit addition methods; Karina in describing Gabriela's and Josué's methods of finding the difference in heights). Children even take over some of the teacher's initial roles, such as helping others understand (e.g., Letticia explaining to Gabriela when Gabriela counts a tens strip by ones instead of tens). This cycle of teacher modeling and greater support will need to be carried out again whenever a class enters a mathematical domain in which considerable new knowledge must be constructed. As one of our first-grade teachers summarized this process at a recent meeting for other teachers, "It is a great deal of work at the beginning to build the classroom environment you need. But then the children just take over and do it mostly themselves. And it is so wonderful to see what they can do then!"

Tools as Learning Supports

The figures all show tools children learned to use to support their mathematical thinking and their communication about their thinking. These tools became quantitatively meaningful for individual children through activities in which they counted, added, and subtracted the quantities in the tools. The penny strips and the earliest forms of the drawn ten-sticks (ten dots or short horizontal dashes with or without a line drawn through them) allowed children to carry out the most primitive methods, counting by ones. Classroom activities during this time helped children build the more advanced views of these tools as supporting sequence counting methods or methods counting groups of tens by ones. In all of these activities, the quantities were linked to written numerals and to spoken number words (both the sequence counting words and the tens and ones words). The drawn ten-sticks and dots enabled children to write numbers beside them to clarify the meanings and the method used. Such linking supported reflection by children on their own methods, and it facilitated communication to others about that solution method. Children followed discussions of other children's thinking better if that child or the teacher pointed to objects or numbers during the description of the method. Having children work at the board or on individual blackboards was one way to have the child's work visible as they explained. The examples given in the figures show how the tools, and the numbers themselves, could and did take on different meanings for different children. These meanings were supported by activities using the tools in the classroom.

Equity and Accessibility

Our project uses activities that can be solved in multiple ways ranging from the more primitive to the more advanced so that all children can participate. Acceptance of such a range of methods is important, as is monitoring by a teacher of students' readiness to move on to a more advanced method. Teachers often directly support such moving on by activities in which everyone tries a given method (e.g., the class tried Jorge's, Juan's, and Viviana's method of two-digit addition) and with tutoring with adults or peers. Helping children use the most advanced method of which they are capable at a given time, while not undermining their autonomy or confidence, is an important aspect of equity for us. It is also important to try to make such help available for children who cannot initially do even the simplest method in a given domain.

Supporting nontraditional gender roles in mathematical thinking is also valuable. The teacher needs to monitor whether she or he is enabling girls to participate as capably and as frequently as do boys. A classroom climate of all learners as teachers, and of all children as responsible for their classmates, can also increase equity and access to mathematical thinking with understanding. We have found that all children can progress substantially in their mathematical thinking, in their confidence and clarity in describing their thinking, and in their helping behavior.