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Children's Conceptual Structures for Multidigit Numbers and Methods of Multidigit Addition and Subtraction

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Researchers from 4 projects with a problem-solving approach to teaching and learning multidigit number concepts and operations describe (a) a common framework of conceptual structures children construct for multidigit numbers and (b) categories of methods children devise for multidigit addition and subtraction. For each of the quantitative conceptual structures for 2-digit numbers, a somewhat different triad of relations is established between the number words, written 2-digit marks, and quantities. The conceptions are unitary, decade and ones, sequence-tens and ones, separate-tens and ones, and integrated sequence-separate conceptions. Conceptual supports used within each of the 4 projects are described and linked to multidigit addition and subtraction methods used by project children. Typical errors that may arise with each method are identified. We identify as crucial across all projects sustained opportunities for children to (a) construct triad conceptual structures that relate ten-structured quantities to number words and written 2-digit numerals and (b) use these triads in solving multidigit addition and subtraction situations.

Traditional mathematics schooling in the United States and many other countries fosters memorization of multidigit calculation procedures, inadequate understanding of the base-ten place-value system of written multidigit numbers, and consequent long-term errors in multidigit calculation procedures (Bednarz & Janvier, 1982; Beishuizen, 1993; Fuson, 1990, 1992a, 1992b; Kouba et al., 1988; Murray & Olivier, 1989; Olivier, Murray, & Human, 1990). A number of research projects around the world have begun to address these deficiencies by trying new approaches that are hypothesized to support children's construction of accurate and robust conceptual structures for multidigit numbers and facilitate the use of these conceptual

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structures in multidigit calculation. For 3 years participants in four of these projects have met on a semiannual basis to discuss approaches, progress, children's thinking, evaluation methods, teacher education, and other pertinent aspects of the projects. These discussions have worked toward mutual understanding of the theoretical and pragmatic contexts of the projects and the coconstruction of overarching views of children's thinking about mathematics.

In this article we report progress in one area of these discussions: children's conceptions of multidigit numbers and their uses of these conceptions in multidigit addition and subtraction situations. Progress in other areas is reported in Hiebert et al. (1996, 1997). We describe here a common framework of conceptual structures children construct for multidigit numbers and categories of methods children devise for multidigit addition and subtraction. The developmental sequences of children's conceptual structures proposed here extend and clarify the theoretical analyses in Fuson (1990), integrate the theoretical perspectives of the four projects (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996; Fuson, Fraivillig, & Burghardt, 1992; Fuson, Smith, & Lo Cicero, in press; Hiebert & Wearne, 1992, 1993, 1996; Murray & Olivier, 1989), and depend substantially on the descriptions of children's thinking as manifested in all four projects. The relationship between our construction of both the framework and the categories was an interactive one over the years. We began with the early separate frameworks, tried to understand children's methods within them, modified and integrated them as our understanding of children's methods grew, and constantly revised our categories of children's methods.

The projects designed to help children learn number concepts and operations with understanding are (a) Cognitively Guided Instruction (CGI), directed by Thomas Carpenter, Elizabeth Fennema, and Megan Franke at the University of Wisconsin; (b) the Conceptually Based Instruction project (CBI), directed by James Hiebert and Diana Wearne at the University of Delaware; (c) the Problem Centered Mathematics Project (PCMP), directed by Piet Human, Hanlie Murray, and Alwyn Olivier at the University of Stellenbosch in South Africa; and (d) the Supporting Ten-Structured Thinking projects (STST), directed by Karen Fuson at Northwestern University.

All four programs take a problem-solving approach to teaching multidigit number concepts and operations. The learning of multidigit concepts and procedures is perceived as a conceptual problem-solving activity rather than as the transmission of established rules and procedures. Teachers do not expect all children to use the same procedure and do not teach only a single expected algorithm. A great deal of lesson time is devoted to allowing children to work out their own procedures and then to share and discuss strategies for solving addition and subtraction problems and tasks involving place-value meanings of numbers (and, for some projects, multiplication and division problems). The intent is to convey to students the importance of working out a strategy for solving the problem and then sharing and reflecting on alternative strategies.

In all four projects, the teacher plays an active role in the classroom by posing the problems, coordinating the discussion of strategies, and joining the students in asking questions about strategies. The intent is to create an environment in which teachers support students' efforts to construct their own solution methods. In traditional instruction, teachers often do more than serve as a resource and guide. They usually teach a standard procedure and may intervene in ways not adapted to children's thinking. In these ways, teachers can easily undercut the goal of children devising and using multidigit methods they truly understand. Some teachers in some projects did occasionally introduce an alternative strategy, but this was considered by the teacher and by the students as just another way one might approach the problem. Teachers in all four projects worked to communicate their expectations that students can figure out methods for dealing with multidigit numbers, that multiple methods exist, and that students do not have to appeal to the authority of the teacher to ascertain the correctness or acceptability of a given procedure.

As more classrooms around the world move toward reform teaching and learning, children will use a wider range of addition and subtraction methods. Our hope is that by sharing our methods, we can accelerate understanding and appreciation of the range of methods children can use. Most methods have advantages and disadvantages. Knowledge of these may enable teachers to decide more intelligently which methods they wish to support in their classrooms.

In this article, we briefly describe aspects of our theoretical perspective and then summarize the conceptual supports used in the project classrooms. Next, the conceptual structures children construct for multidigit numbers are described. Finally, multidigit addition and subtraction methods used by children in the various projects are described and discussed.

THEORETICAL VIEW

Children construct meanings for multidigit numbers through the various encounters they have with these numbers both in and out of school. Elementary school mathematics classrooms encourage or facilitate the development of various conceptions of multidigit numbers through the language that is used by the teachers and students, the type of physical materials that are used, the problems that are to be solved, and the structured class activities. These components act in concert with one another to support children's construction of meanings for multidigit numbers.

We want to clarify our theoretical perspective on the use of such conceptual supports in the classroom, including objects organized or organizable into tens. Dialogue within the research community in the last several years has sometimes been dichotomized as (radical) constructivist vs. non(radical) constructivist, with the latter described as the representational view of mind (e.g., the title of Cobb, Yackel, & Wood, 1992). In this dialogue the use of objects is sometimes identified with the representational view. We believe that this is a false dichotomy that greatly oversimplifies the issues involved.

What is crucial in the use of objects is the theory of learning with which the objects are used (the terms used in the following are taken from a longer discussion in Fuson, Fraivillig, & Burghardt, 1992). Objects sometimes are used with an "instamatic camera" view of learning that assumes that a child needs one or very few "exposures" to a given object and that the child will then almost automatically interiorize and use these objects in mathematical thinking. Children in our projects do indeed sometimes seem to interiorize classroom ten-structured object supports and use them in their problem solving. Children may explicitly refer to "seeing in their minds" base-ten blocks or unifix cubes or a counting frame or Montessori cards and may use these mental images to carry out a multidigit addition or subtraction or at least to direct or constrain such methods. But we also have strong evidence against the rapid and direct interiorization implied by an "instamatic camera" view of learning. Our experience instead supports a "meaning maker" view of learning in which what a child "sees" when looking at objects depends on the conceptual structures used by that child. A given child can be supported toward constructing conceptual structures not yet built by having particular kinds of objects available, by kinds of use and discussion of such use by other children and adults in a classroom, and by activities that help or direct the child in certain ways. But the construction of new conceptual multidigit structures is a *prolonged* process that occurs within classroom social and activity structures that include many elements other than the objects (see Hiebert et al., in press).

Two issues are central to the current debate. One is whether children do or do not have a mental interpreter of what they see and hear in the world. Consensus is gradually growing that all children do indeed have such interpreters; in this paper we call these "conceptual structures." For us, a conceptual structure in use indicates/reflects the aspects of the mathematical situation considered by the user at that moment: it captures what aspects are focused on and how these aspects are interpreted.

A second issue in the debate is the rate at which new conceptual structures are constructed by a child. Our collective experience is that such constructions ordinarily take a rather long time for many children to construct, at least in the domain of multidigit numbers and for children speaking European languages. Thus, construction of conceptual structures is more like concept learning of many types and less like insight, which is characterized by a rapid reorganization of conceptual structures. This is not to say that we have never seen moments of insight. They do occur and are striking and exciting to witness and share. Even insights, however, involve restructurings of concepts children already have.

PROJECT CONCEPTUAL SUPPORTS

In Cognitively Guided Instruction (CGI) classes (Carpenter et al., 1996; Carpenter et al., in press; Fennema et al., 1996), methods for operating on multidigit numbers develop as natural extensions of the methods that children use to solve problems involving single units. Word problems provide the basis for almost all instruction. In the early grades, teachers begin by giving children a variety of word problems that can be solved by modeling and counting using single counters. Teachers do not demonstrate the solution to problems, but a great deal of time is spent discussing alternative strategies for solving each problem. The strategies discussed serve as models for other children, and the discussions provide an opportunity for children to reflect on their own solutions.

Initially, children solve problems involving multidigit numbers by modeling the problems with single-unit counters. These solutions do not require any real conceptions of place value beyond the ability to count. Children learn place-value concepts as they explore the use of base-ten blocks and other base-ten materials to solve word problems and listen to other children explain their solutions with the blocks. Initially the ten-blocks simply serve as convenient collections of unit counters that do not get mixed up. With teacher encouragement, some children come to recognize that they do not have to count all the individual units in the tens block each time they construct a set and begin to construct two-digit quantities by making collections of tens and ones. Place-value concepts emerge over time as children become increasingly flexible and efficient in the use of base-ten materials. As their use of the materials becomes more automatic, they come to depend less on the manipulations of the physical materials themselves. Over time they are able to abstract their solutions with physical materials so that they can add and subtract multidigit numbers without them.

Throughout the year different children in a CGI class operate at many different levels with respect to place-value knowledge. One important consequence is that there is no prevalent strategy that all children use at a particular point in time. Children have the latitude to use a strategy that makes sense to them at the time. A consequence of the variety of strategies in use at any given time is that children have the opportunity to learn more advanced strategies by interacting with other students who are using them. Thus, although children are not asked to relate specific components of different representations to one another, they continuously shift among representations both in their own solutions of different problems and in their discussions with classmates of different strategies for the same problems.

In the Conceptually Based Instruction Project (CBI), the timing and sequencing of place value and multidigit addition and subtraction were matched to that in the textbook to permit a control comparison. As in texts in the United States, activities related to place value were followed several weeks later by activities involving combining multidigit numbers. Students began by grouping objects into sets of ten to facilitate counting and then were encouraged to connect their representation with the recorded written symbol (5 groups of ten and 3 units is written 53). The activities were expanded to larger numbers using base-ten blocks. Students were given word problem situations involving packaging in tens to assist them in constructing meaning for the written symbol. Students developed methods for multidigit addition and subtraction using base-ten blocks and their understanding of the meaning of written numbers. Problems were presented in various contexts, and students shared and discussed their solution strategies.

In the Problem Centered Mathematics Project (PCMP), classroom activities primarily support children's construction of robust counting abilities including counting by tens. Base-ten blocks are not used because many teachers in the past used them to teach a standard algorithm, and a vital component of the PCMP is to stimulate children's own alternative solution methods. First graders spend much time in counting activities: estimating and then counting piles of loose counters, counting large sets of objects into groups (especially groups of ten), counting on a ten-frame of 10 or 12 rows of ten horizontal movable beads (first by moving individual beads and then by moving whole rows of ten while counting by tens), counting on a number chart with rows of ten numbers, skip-counting real objects in twos, fours, fives, tens, and twenties, and measuring activities with body parts and tape measures. Children use Montessori cards to show numbers (putting a 6 card on top of the 0 in the 40 card to show 46 as 40 and 6); some classes later use Montessori cards for multidigit addition and subtraction. Teachers are encouraged to go as far as possible every day in counting activities, aiming to reach the three-digit numbers as soon as possible. They frequently count over tens and over hundreds to facilitate these more difficult counts. A wide range of word problems are given from the beginning (including multiplication and division and a range of addition and subtraction problem types). The size of numbers is adjusted to children's conceptual structures by the teacher working serially with different groups of students and by using small numbers for difficult problems. Simple addition and subtraction word problems for students not yet counting on have numbers such as the following: 18 + 5, 27 + 4, 32 – 3, 28 + ___ = 32 to encourage such counting on. Students counting on but only considering numbers as collections of single objects with no subgroupings have all two-digit numbers in their problems so that they can construct more advanced conceptions while constructing addition and subtraction methods. Children solve word problems on slates in small groups, showing their work in the arrow format we use later in Tables 1 and 2. They then discuss their solutions in their small groups.

There were multiple Supporting Ten-Structured Thinking (STST) projects that moved from supporting children's understanding of a single accessible and generalizable strategy to supporting children's construction of all of the multiunit conceptual structures and their invention of multiple strategies. In addition to standard English or Spanish number words for two-digit numbers, tens and ones words (e.g., "five tens and three ones" or "five groups of ten and three loose ones" for 53) were used in the later projects; these paralleled the written marks and explicitly name the tens as do Asian number words.

In the early projects children moved directly from single-digit addition and subtraction to using base-ten blocks to construct four-digit addition and subtraction methods. Strong connections were made among the quantities presented by the blocks, number words, and written numerals during multidigit addition and subtraction. The blocks were used in most classrooms in a dialogue between teacher and children in which the features of the blocks were used to direct and constrain the written multidigit addition or subtraction methods. These Target Algorithm Studies indicated that our linked conceptually supported instruction resulted in considerably higher levels of correct multidigit addition and subtraction and of explanations of computational procedures using multiunit quantities and one/ten trades conceptions than much of traditional school instruction (Fuson, 1986; Fuson & Briars, 1990).

In the second STST project, the Children's Invented Procedures Study, six groups of four or five high-achieving second graders worked together to use base-ten blocks and written marks to add pairs of four-digit numbers presented horizontally (Burghardt, 1992, 1993; Burghardt & Fuson, 1996; Fuson & Burghardt, 1993a, 1993b; Fuson, Fraivillig, & Burghardt, 1992). However, children often worked in two separate contexts (blocks and written marks) and did not use the objects to help them with their written numeral procedures. In this separate numeral context, children invented many different incorrect numeral procedures. When the experimenters forced the children to link the blocks and the written numerals, the children could and did correct their numeral errors. Therefore, it is clear that teachers who are using multiunit objects to support children's construction of multiunit conceptual structures may have to support the linking of the multiunit objects and any written mathematical methods that are to be facilitated by the multiunit objects. Emphasizing and supporting this linking can provide a powerful context for children's construction of the desired network of conceptual structures and for the invention of accurate and understood multidigit methods.

The most recent STST project, Children's Math Worlds, focuses on Latino children's constructions of arithmetical understandings in urban classrooms (Fuson, 1996; Fuson, Smith, & Lo Cicero, in press; Fuson, Zecker, Lo Cicero, & Ron, 1995). Conceptual supports vary somewhat across classrooms. All classrooms use tens and ones words as well as standard English or Spanish number words. In all classrooms children add and subtract two-digit numbers by making ten-stick and unit drawings. Units are dots, circles, or short horizontal lines. Children build up meanings for the ten-sticks by a succession of activities in which they (a) make columns of ten units to record a large number of objects, (b) join the units by a vertical line to make the ten units into one ten, and c) draw only a vertical stick to show one ten (rather then ten connected units). These drawings are used in real-world problem contexts like a doughnut store, where a unit is a doughnut or a penny, the vertical line is a box of doughnuts or 10 cents (or a dime), and a square is a baking tray of 100 doughnuts (or a tray of ten boxes) or 100 cents (or one dollar). Children talk about boxing or unboxing doughnuts (or other entities of their own designation) or exchanging money if they use a trading process in addition or subtraction. Children also arrange large sets of objects into groups of tens and ones and count by tens as well as count the tens. Conceptual supports used by some but not all classes are two-sided money that show money equivalents (e.g., a strip of ten pennies with one dime on the back), centimeter ten-lengths or square decimeter grids to measure length and area, large classroom thermometers and small versions, vertical line segments ("trees") on which children write addition or unknown addend counting-on solutions, decade and ones cards that show quantities of tens and of ones in which the ones are put on top of the decades as are Montessori cards, and discussion of single-digit addition and subtraction finger methods that use ten. Children work on object and on mental methods.

CHILDREN'S CONCEPTIONS OF MULTIDIGIT NUMBERS

Background

Our work builds on and extends the conceptual structures children construct for four-digit numbers identified by Fuson (1990) in her review of the literature on children's functioning on place-value tasks and multidigit addition and subtraction. These conceptual structures involve two aspects of the written number marks (the visual layout and the increase in value according to relative positional value from the right), two aspects of spoken number words (the number names and the decreasing value as the names are spoken), and six increasingly general and abstract quantity multiunit structures that give meanings to the written marks and spoken words.

A major focus of Fuson's analysis is the learning difficulties caused by the several irregularities in English in the words for two-digit numbers. These contrast with the regularity of the named-value structure of the hundreds, thousands, and larger places in which the number of multiunits is stated and then the multiunit value word is said (e.g., "five thousand eight hundred"). These larger number words relate fairly simply to their written place-value numbers; the value words disappear and are signified by relative position from the right (e.g., in 5800, thousands are in the fourth position and hundreds in the third). In contrast, for two-digit numbers, children must learn a special decade list (ten, twenty, thirty, forty, fifty, etc.) whose cardinal relationship to the first nine number words is masked by pronunciation differences for several decades (two, twen; three, thir; five, fif). The teens pose special problems with their irregular "eleven" and "twelve" that show little sense of their original meanings as "(one) left ten" and "two left ten" (one and two left over ten: Menninger, 1958/69) and the pronunciation irregularities for thirteen and fifteen (not threeteen and fiveteen). Furthermore, the order of the words and of the marks is reversed: We say "fourteen" with the four first, but write the four second (14). Finally, the suffixes "-ty" and "-teen" do not clearly suggest ten, so these quantity meanings are not clear from English number words. Most European languages have such irregularities that mask ten-structured meanings.

Some children come to conceptualize multidigit addition and subtraction as adding or subtracting quantities grouped into multiunits of ones, tens, hundreds, thousands, and so on. Two different such multiunit conceptualizations were identified by Fuson in the earlier literature. Children may add or subtract multiunits within the counting word sequence (i.e., use *sequence methods* such as counting on by tens and ones), or they may add or subtract the multiunits directly (i.e., use *collected multiunit methods*, e. g., count or add the hundreds, then count or add the tens, then count or add the ones). Children also may conceptualize such problems as involving *concatenated single digits* and operate as if they were adding and subtracting separate columns of single digits. This conception of multidigit numbers is error prone because it does not direct or constrain the methods children use sufficiently, and it leads to many different errors (see Burghardt & Fuson, 1996, for a recent report of many such errors using this conception and VanLehn, 1986, for an earlier summary). A given child may have all three of these conceptions and use them in different situations; the concatenated single-digit conception is used especially with vertical numeral problems.

When children move rapidly in their learning from single-digit numbers to four-digit numbers, the irregularities for two-digit numbers have less impact because the regularities of the hundreds and thousands enable children to construct general multiunit concepts that can regularize the tens somewhat. For example, some children in the PCMP experiencing difficulties with two-digit numbers perform with more understanding after exposure to three-digit numbers. PCMP teachers are therefore encouraged to move to three-digit numbers as rapidly as possible. In the more usual practice of spending quite a long time on two-digit numbers and two-digit addition and subtraction, the decade structure of two-digit numbers in most European languages is very salient, and children construct various decade conceptions of two-digit numbers. This paper focuses on such two-digit conceptions, though three-digit addition and subtraction methods are also discussed. Therefore this paper does not use the more generalized aspects of Fuson's (1990) treatment (the pattern of increases/decreases in value of positions and number words and the last four quantity multiunit structures).

Our Developmental Sequence of Conceptual Structures

Through our work with children in the various projects, we have identified five different correct conceptions of two-digit numbers that children use (see Figure 1). We call this the UDSSI Model after the names of these five conceptual structures (unitary, decade, sequence, separate, integrated). Several different conceptions may be available to a given child and be used in different situations. Thus, new conceptions are added to rather than replacing old conceptions. A sixth conception, the incorrect concatenated single-digit conception discussed above, is also included in Figure 1. This conception may exist alongside any of the correct conceptions and be used in certain situations (especially those in which numbers are written vertically) until another conception becomes more likely to be used in such situations.

Each of our conceptions involves a triad of two-way relationships between number words, written number marks, and quantities. Each of these is connected to the other two. These triads are shown in Figure 1 as triangles with a solid two-way arrow representing each pair of unidirectional relations that must be established. This triad structure is shown in the bottom right-hand corner for single-digit numbers, where the structure begins. Multidigit numbers build on and use the *unitary single-digit triads* of knowledge for single-digit numbers. Thus, before children can learn about two-digit numbers, they must have learned for one to nine how to read and say the number word corresponding to each number mark, write the numeral corresponding to each number word, and count or count out quantities for each mark and number word one to nine. Because the number words for single-digit numbers in most languages and the corresponding written marks are arbitrary, most children learn most of the unitary single-digit triads as rote associations. The six relationships in the triad come in at different times for a given number, and a given relation will be learned later for some numbers than for others (e.g., children will be able to make a quantity of nine later than a quantity of three). Children usually learn many number words before they learn many number marks. Therefore the quantity–number mark relationships will often involve counting the quantities, and they will be generated as a two-step process using the bottom mark to number-word relationship and then the number-word to quantity relationship. The direct link between quantity and number mark can only occur by associating patterns of entities directly with a numeral; counting necessarily involves the number words. As the numbers get larger, counting obviously is used more frequently because it is more difficult to see or use patterns of quantities.

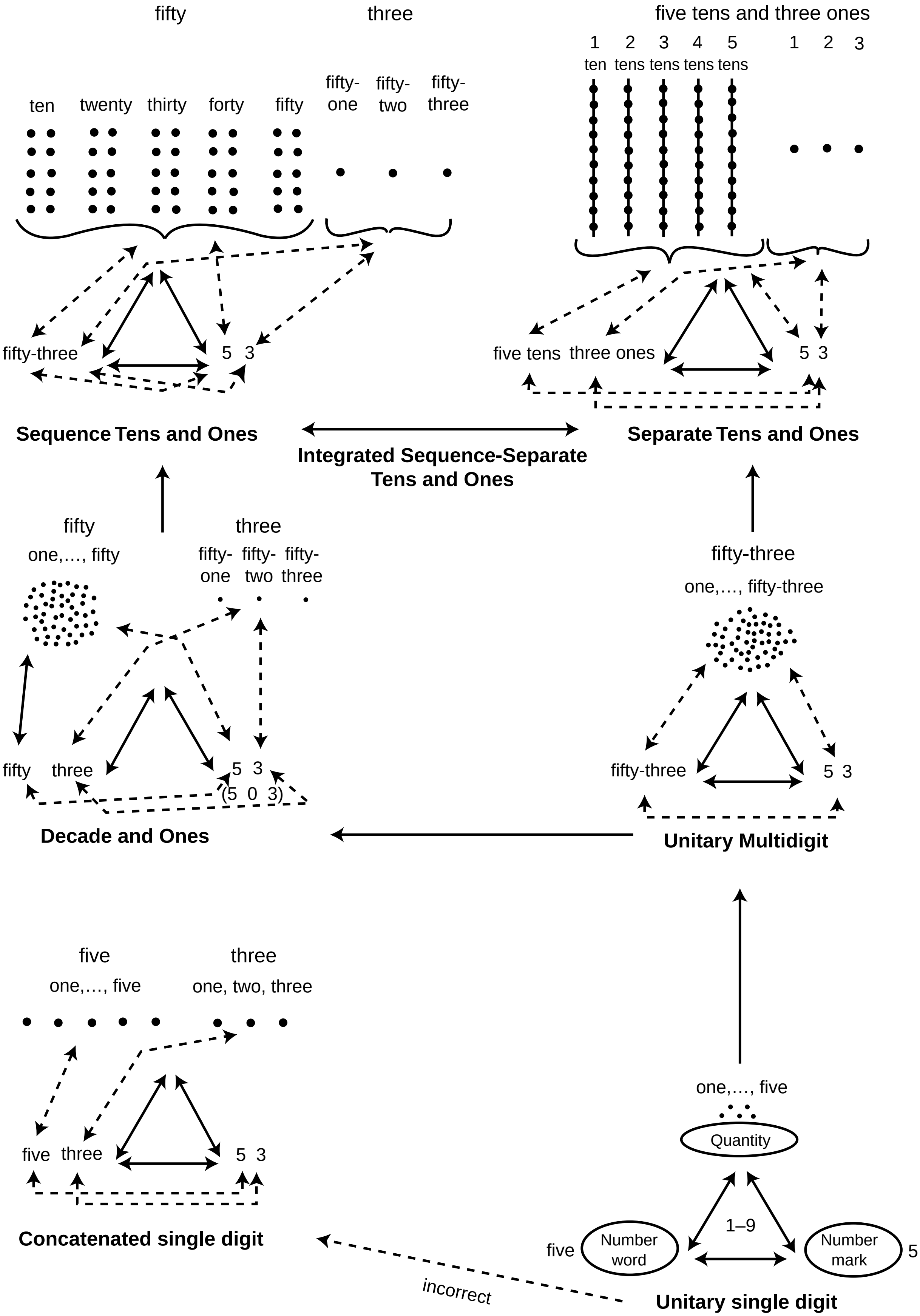

Figure 1. A developmental sequence of children's two-digit conceptual structures: The UDSSI Triad Model

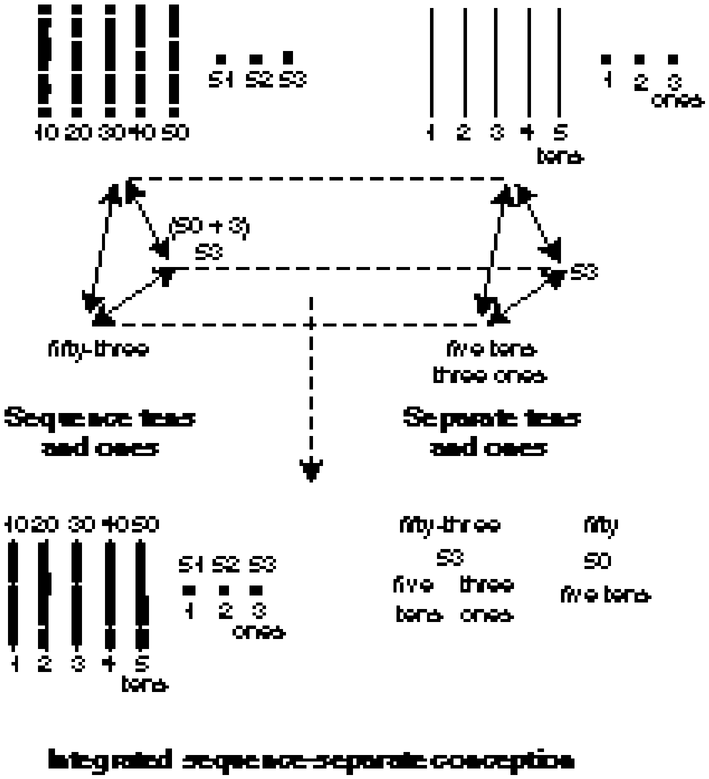

Figure 1—continued. A developmental sequence of children's two-digit conceptual structures

Unitary multidigit conception. The *unitary multidigit conception* is a simple extension from the *unitary single-digit conception.* In the unitary multidigit conception the triad relationships relate a whole word to a whole quantity to a whole mark (e.g., Bergeron & Herscovics, 1990; C. Kamii, 1985; M. Kamii, 1981, 1982; Sinclair, Garin, & Tieche-Christinat, 1992). Quantities are not differentiated into groupings, and the number word and number marks are not differentiated into parts. So for 15 doughnuts, for example, the 1 is not related to "teen" in "fifteen," and the quantities are not meaningfully separable into 10 doughnuts and 5 doughnuts.

Decade and ones conception. The *decade and ones conception* is built by children using English number words or languages with a similar decade structure. Many English-speaking children begin to recognize the decade structure in number words by age $4\frac{1}{2}$ (Fuson, Richards, & Briars, 1982). They count using this structure, but they may not for as much as $1\frac{1}{2}$ years learn the order of the particular decade words. Instead, they cycle after twenty-nine through repeated and seemingly random decade counts (e.g., twenty-nine, fifty, fifty-one,…, fifty-nine, thirty, thirty-one,…, thirty-nine, seventy, seventy-one,…, seventy-nine, sixty,…, etc.). As children begin having experiences counting large numbers of objects and perhaps especially seeing written number marks for the decades (20, 30, 40, etc.), they may begin to separate the decade and the ones parts of a number word and start to relate each part separately to the quantity to which it refers: "fifty" to fifty objects and "three" to three objects. They may make the same separation and try to link the decade and the ones parts of a number word to written marks. All of these separated links are shown by the dashed arrows connecting these parts. So for 53 doughnuts, for example, fifty is understood to be the 5 written first and three to be the 3 written second.

This decade conception often leads in the beginning to a particular error of writing number marks in which the features of the number words are extended to the marks (Bell & Burns, 1981; Fuson & Drueck, 1994; Power & Dal Martello, 1990; Seron, 1994; Seron, Deloche, & Noel, 1992). Because the number words are concatenated (the ones word follows the decade word), the child also concatenates the numerals: writes fifty (50) followed by three (3), yielding 503. Children eventually do learn that the 0 is not written and the 3 is written immediately following the 5. This may be learned rotely or as a response to realizing that 503 is wrong (503 is five hundred three, not fifty-three). Or it may be learned through conceptual supports in the classroom, such as Montessori cards in which a 3 card is placed on top of a 50 card to make a 53 (the South African PCMP project), or the 0 in 50 is made with faint dots and the 3 is written on the faint 0 to show the fifty "hiding" in the 53 (STST Latino project).

Sequence-tens and ones conception. Children who have experiences learning to count by ten within the number-word sequence and learning to group objects into tens and count these groups by tens (e.g., "ten, twenty, thirty, forty, fifty") may construct a ten-structured version of the decade-and-ones conception, a *sequence-tens and ones conception,* in which each decade is structured into groups of ten (see Fuson, 1990, for early literature concerning this conception). This conception requires the skill of being able to count by tens, but it also requires "seeing" the groups of ten within a quantity and choosing to count these by tens. Analogous to the count-to-cardinal shift children make when counting by single-digit numbers, children must shift from the sequence-tens referent of the last counted group of ten ("fifty" said while pointing to the last, fifth box of ten doughnuts) to its decade meaning as all the doughnuts counted so far ("fifty" doughnuts so far).

Separate-tens and ones conception. In a quantity situation with grouped tens, a child focusing on and counting the groups rather than the objects in the groups (e.g., counting the boxes of ten doughnuts) is using a *separate-tens and ones conception.* This meaning is not supported by European number words, because their irregularities do not explicitly and clearly name the ten (see Fuson & Kwon, 1992; Menninger, 1958/69). In contrast, Asian number words based on ancient Chinese use the same pattern English uses for hundreds and thousands: 53 is said "five ten three." When the first digit is 1, a slight irregularity occurs: the one is not said, so 12 is said "ten two." These number words facilitate easy links between each part of a number word and each part of the written mark: a child need only drop the ten when writing the mark and insert the ten when reading a mark. These number words clearly suggest the quantitative meaning of the number words and marks as referring to separate-tens and single units. Children making a multidigit quantity using a separate-tens and ones view count the groups of ten using single-digit numbers (one ten, two tens, three tens, four tens, five tens) or omit the word *tens* while counting, leaving it as understood (one, two, three, four, five tens). Some Chinese mothers help children as young as 5 years of age form this triad of conceptions by grouping quantities to show the multiunits of ten and the units and then relating the words to these multiunits of ten and individual units (Ana Lo Cicero, personal communication, February, 1994; Yang & Cobb, 1993). For children speaking number words based on Chinese, building this separate-tens and ones triad is relatively simple. Children speaking European languages may need more support to see and focus on the groups of ten as tens, rather than as just collections of units to be sequence counted. However, children do not need to learn any special counting list to use a separate-tens conception.

Integrated sequence-separate tens conception. Children who have the opportunity to construct both a sequence-tens and a separate-tens conception of two-digit numbers may go on to relate these two conceptions so that they are able to shift back and forth between them rapidly. In this *integrated-tens conception*, bidirectional relations are established between the tens and the ones component of each of the three parts (numbers words, marks, quantities) of the sequence-tens and the separate-tens conceptions (see Figure 1). With the integrated-tens conception, a child is able to answer immediately that fifty has five tens because these two multiunit tens components are related. Before such a conception is constructed, a child with the sequence-tens conception has to count by tens to fifty and keep track of how many ten counts there are to find five tens in fifty. A child with the separate-tens conception has to count five tens to find out that they make fifty. This integrated-tens conception allows children considerable flexibility in approaching and solving problems using two-digit numbers because they can rapidly shift attributes of the ten-structured situations to the background or foreground, for example, *fifty* doughnuts, the five open boxes of *ten* doughnuts (five groups of *ten ones*), and the *five* closed boxes (five *tens*).

Concatenated single-digit conception. Even when children have one of the adequate multidigit conceptions and use this conception to add or subtract numbers meaningfully and accurately when these are presented in a word problem or horizontally, they may use a *concatenated single-digit conception* for the same computation presented vertically and make an error (Cobb & Wheatley, 1988; Davis, 1984; Murray, unpublished data). The vertical presentation elicits an orientation of vertical slots on the multidigit numbers that partitions these numbers into single digits. The physical appearance of the written multidigit marks as single digits and the nonintuitive use of relative left-right position as a signifier may combine to seduce children to use a concatenated single-digit conceptual structure even if they have a more meaningful conception available. For this reason the CSD conceptual structure was also labeled as the *constantly seductive digits* conception (Fuson & Burghardt, 1993a).

Relative Difficulty of the Learning Task for European Children

The learning task for a child speaking a European language is quite daunting. The European number words require some decade conception, and the written marks require some conception of separate tens and ones. For full understanding of the words and marks, children need to construct all five of the UDSSI multidigit conceptions. Children speaking Chinese-based number words that are regular and name the tens have a much easier task. They need only construct the unitary and the separate-tens conception, and the relationships in the separate-tens conception are easy to construct because their number words regularly name the tens. They also have an easier time learning the counting sequence to 100 than do children speaking English or Italian because of the regularity of the Chinese-based number words (Miller, 1990; Miller, Agnoli, & Zhu, 1989; Miller & Stigler, 1987). Therefore, children speaking European languages need considerable quantitative support and a long time to construct all of these conceptions.

In the four projects reported here, children were supported in various ways to construct a sequence-tens or separate-tens or both conceptions before or while they were inventing conceptual methods to add and subtract multidigit numbers. Addition and subtraction frequently provided learning activities in which more advanced multidigit conceptions could be constructed.

Complexity of children's construction of triads. Figure 1 is deceptively neat in several ways. First, as discussed earlier, children usually learn the six relationships for a given number at different times (e.g., a child may be able to read 12 before being able to write it). Second, children may start to learn triads for different numbers at quite different times. Third, children may not construct the last triad relationships for all numbers to 99 for one kind of conception before the first triad relationships for another conception are constructed. Thus, for a given child, one should think of stacks of possible triads for each conception for numbers from 10 to 99, and within each stack (and even within a triad) some of the relationships are automatic, some are newly built, and some may not yet exist.

Children's use of the conceptions may also be more complex than indicated in Figure 1. Children who have more than one multidigit conception may use different conceptions in different situations or combine parts of different triads in a single situation (e.g., count 53 objects unitarily but use a decade conception to write how many as 503). Furthermore, not all children construct all conceptions; these constructions depend on the conceptual supports experienced by individual children in their classroom and outside of school. Therefore, children's multiunit conceptions definitely do not conform to a stage model, except that the earliest relationships in each conception do follow the paths in Figure 1.

Triads for Three-Digit and Four-Digit Numbers

European languages vary somewhat in the regularity of the words for the third and fourth digits, the next two multiunits. English is regular for the multiunits of hundred and thousand: the number of multiunits of hundred or thousand is ordinarily followed by the word *hundred* or *thousand* (e.g., 7500 is said as "seven thousand five hundred"). Therefore children do not need to construct special "hunade" or "thousade" conceptual structures based on special series of such words; they need only construct a separate-hundreds and separate-thousands conceptual structure. The relationships in these triads are explicit in English, just as the two-digit triad is explicit in Asian languages. Some European languages such as Spanish do have some irregularities, but they do not have full special lists for hundreds and thousands. These irregularities may make it more difficult for children to see the regular patterns in these triads. For example, Spanish uses "quinientos" instead of "cincocientos," "setecientos" instead of "sietecientos," and "novecientos" instead of "nuevecientos."

English also has some irregularities for four-digit numbers. These are sometimes read as two 2-digit numbers (e.g., 2648 might be read as "twenty-six, forty-eight"). Dates are always read this way, and street addresses are frequently said this way. For example, 1918 is said as "nineteen eighteen" as a date and as an address. Such a partitioning may be useful for memorizing addresses or other noncardinal number uses, but it carries the irregularities for two-digit numbers into four-digit numbers. Dates also may contribute to confusion concerning the order of subtraction because time intervals are written using the subtraction symbol as 1850–1965 (a possibility pointed out to us by a reviewer of this article). But to find the difference in these dates, one must count/add up or reverse the numbers to subtract. Four-digit amounts are also partitioned this way for money in the United States: $19.18 is read "nineteen dollars and eighteen cents." Finally, we also sometimes do not say "thousand" but instead say a two-digit number and then the word *hundred*, for example, 7500 as "seventy-five hundred."

In a later section we discuss some addition and subtraction methods for three-digit numbers. The conceptual structures for three-digit numbers are extensions of the conceptual structures in Figure 1 except that the hundreds have more separate than sequence characteristics because of their regular named structure (e.g., three hundred).

CHILDREN'S METHODS OF MULTIDIGIT ADDITION AND SUBTRACTION

Children's multidigit addition and subtraction methods are generalizations of, and for more advanced methods, depend upon and use methods for adding and subtracting single-digit numbers. For this reason we first provide a brief overview of these methods.

Children's Single-Digit Addition and Subtraction Methods

With experience, children's single-digit addition and subtraction methods become more complex, abstract, interiorized, embedded, and abbreviated. Several developmental levels in these methods have been identified, and this progression of methods can be described at varying levels of specificity. We use only the three levels described by Fuson (1992b) in a recent review of the literature (see Fuson, 1988, 1992a, for more details). These three developmental levels vary in the conceptual units children use to make quantities, in the conceptual operation children use to gather separate entities into the cardinal sets that present specific quantities in a given situation, in the cardinal conceptual structures children use to relate the three quantities in an addition or subtraction situation, and in the solution actions children carry out to find the unknown quantity.

At Level I, children must construct addition or subtraction situations using physical objects of some kind. These objects are used to model directly the addition or subtraction operation given in the situation. At a given moment each object can only be part of an addend or part of the total, though the role can change over time (an object can first be part of an addend and can later be considered as part of the total or vice versa). Children count all the objects to add, and they take away and count the remaining objects to subtract.

At Level 2, a child can simultaneously consider all three quantities in an addition or subtraction situation by embedding the addends within the total and considering objects as being simultaneously part of the addend and part of the total. Children can now count words in the number-word sequence instead of only counting objects, and they can abbreviate the count of the first addend. Thus, to add, they can count on from one addend word while keeping track of the other addend words counted on, or they can add on by adding objects for one addend onto those for the other addend while counting on. To subtract, they may count back from the total, keeping track of the addend counted back; count back from the total to an addend; or count up from the known addend to the total, keeping track of how many are counted up.

At Level 3 the addends no longer have to be embedded within the total but exist outside in a numerical triplet structure in which the two addends are seen as equivalent to the total. Quantities are composed of ideal chunkable unit items that can be combined and separated in flexible ways. A given numerical triplet can be recomposed into a related triplet. In this way children can transform a given triplet with one unknown member into a triplet of known facts. These "derived fact" solutions commonly use doubles ($a + a$) in the United States. For example, 7 + 6 = 6 + 6 + 1 = 12 + 1 = 13. In Asian countries children learn to recompose numbers into ten-structured triplets (Fuson, Stigler, & Bartsch, 1986; Fuson & Kwon, 1992a). For example, 7 + 6 = 7 + 3 (to make ten) + 3 = ten three (13). For subtraction one can "take from ten" (13 – 7 is ten three – 7: take 7 from the ten is 3 plus the 3 in ten three is 6) or go down over ten (ten three – 7 is 3 down to ten and 4 more from the 7 goes down to 6). Such ten-structured methods are particularly useful in multidigit addition and subtraction, where each next larger multiunit is related by ten. Thus, ten-structured methods enable children to (a) recompose ten or more of one multiunit into one next larger multiunit and the leftover of that multiunit or (b) recompose a larger multiunit into ten of the next smaller multiunit in order to subtract. These methods are used much less frequently in the United States, though some children do use them, especially for an addend of 9 (e.g., Steinberg, 1984, 1985).

Children eventually memorize many single-digit addition combinations. Because new facts are memorized during each of the three levels (e.g., 2 + 2 is learned very early), using a known fact is not really a special conceptual level. Rather, use of known facts occurs at all three levels. Children gradually learn more and more number combinations as known facts.

Children's Two-Digit Addition and Subtraction Methods

Unitary methods. Children at Level 1 who can count above 10 can use a unitary multidigit conception to add two 2-digit numbers by making objects for each number and counting all of the objects. They can subtract by making objects, taking away from those objects, and counting the remaining objects. Children at Level 2 can count on by ones, add on objects by ones, or verbally count all by ones to add. To subtract, they can count back or count up to by ones. However, keeping track of the number counted on, up, or back may be difficult because it will be so large. These methods are constrained only by how high a child can count and keep track accurately.

Kinds of methods using tens. Children in our project classrooms with sequence-tens or separate-tens conceptions used many different methods for adding and subtracting two-digit numbers. Some of these methods were carried out with objects (e.g., base-ten blocks or unifix cubes showing individual units and multiunits of tens, counting frames with rows of ten beads, drawings showing units and multiunits of ten), some were done verbally (out loud, subvocally, or with inner speech), and some were done with written numerals on paper to record object or mental verbal methods (fingers might also be used with these written numeral methods).

We have classified the methods children used into four kinds: methods that begin with one number and move up or down the sequence by tens and by ones, decompose-tens-and-ones methods in which the tens and the ones are added or subtracted separately from each other, mixed methods in which the tens are added or subtracted and then a sequence number is made with the original ones and a sequence method is used to add or subtract the other ones, and methods in which both numbers are changed to make easier numbers. The begin-with-one-number and mixed methods were typically done with a sequence-ten or integrated-tens conceptions, and the separate-tens and change-both-number methods used a sequence-tens, separate-tens, or integrated-tens conception.

These two-digit methods have versions that correspond to the single-digit methods, though of course they all involve using multiunits of tens as well as individual units. All but the change-both-numbers methods may be done by Level 1 methods counting all or taking away. We have not shown these methods to save space and because they are readily replaced by counting on/up/down methods, which we do show. We also show methods involving addition and subtraction. These are more complex methods involving chunking both numbers in various ways.

In Table 1 we exemplify methods used by children across the various projects. Methods are shown by counting words or with an arrow recording method used in the South African classrooms. This notation is used to record sequential problem-solving actions, which cannot be shown easily by use of equations without violating usual conventions. The result of an operation is written immediately after an arrow, and then the number that is added or subtracted next is written. Thus, each number after an arrow shifts from being a result of the previous operation to being the start of the next operation. We have written independent operations horizontally; children may write these lines under each other.

Table 1

Two-Digit Addition, Subtraction, and Unknown Addend Methods Using Sequence-Tens and/or Separate-Tens

38 + 26 = __	**64 – 26 = __**	**38 + __ = 64**
Begin-With-One-Number Methods: Begin With One Number and Move Up or Down by Tens and Ones		
Count on/add on tens, then ones 38, 48, 58, 59, 60, 61, 62, 63, 64 38 + 20 → 58 + 6 → 64	Count down/subtract tens then ones 64, 54, 44, 43, 42, 41, 40, 39, 38 60 – 20 → 44 – 6 → 38 or $\begin{array}{r} 64 \\ -\underline{26} \\ 44 \\ 38 \end{array}$	Count up/add up tens, then ones like count on; keep track: count up 26 like add on; keep track: add up 26
Overshoot and come back[a] 38 + 30 → 68 – 4 → 64	Overshoot and come back[a] 64 – 30 → 34 + 4 → 38	Overshoot and come back[a] like addition, added up 26
Count on/add on to make a ten count on/ add on tens, then rest of ones 38, 39, 40, 50, 60, 61, 62, 63, 64 38 + 2 → 40 + 20 → 60 + 4 → 64	Count down/subtract to make a ten, count down/subtract tens then rest of ones 64, 63, 62, 61, 60, 50, 40, 39, 38 64 – 4 → 60 – 20 → 40 – 2 → 38	Count up/add up to make a ten, count up/ add up tens, then rest of ones like count on; keep track: counted up 26 like add on; keep track: added up 26
Mixed Methods: Add or Subtract Tens, Make Sequence Number With Original Ones, Add/Subtract Other Ones		
Count on/ add on tens, add ones, count on/ add on other ones 30, 40, 50, 58, 59, 60, 61, 62, 63, 64 30 + 20 → 50 + 8 → 58 + 6 → 64	Count down/subtract tens, add original ones, count down/subtract other ones 60, 50, 40, 44[b], 43, 42, 41, 40, 39, 38 60 – 20 → 40 + 4 [b] → 44 – 6 → 38	Count up/ add up tens, add original ones, count up/ add other ones 30, 40, 50 + 8[b] → 58, 59, 60, 61, 62, 63, 64; 26 30 + 20 → 50 + 8[b] → 58 + 6 → 64; 26
Change-Both-Numbers Methods		
Move some from one number to the other to make a tens number[c] (maintaining the total) 38 + 2, 26 – 2 → 40 + 24→ 64	Make subtracted number a tens number, change other to maintain difference 26 + 4, 64 + 4[d] → 68 – 30 → 38 Latin America, Europe: add a ten to both numbers $\begin{array}{r} 6^{1}4 \\ -{}^{1}\underline{2\ 6} \\ 3\ 8 \end{array}$	Make initial number a tens number, change other to maintain difference 38 + 2, 64 + 2[d] → 40 up to 66 → 26

(table continued)

Table 1—continued

38 + 26		**64 – 26**	
Decompose-Tens-and-Ones Methods: Add or Subtract Everywhere, Then Regroup			
Add tens, add ones, make 1 ten from 10 ones	Add ones, add tens make 1 ten from 10 ones	Subtract tens, subtract ones, combine totals This requires some notion of negative numbers of owing, or ones “in the hole”	Subtract ones, subtract tens, combine totals same as preceding method
$\begin{array}{r}38\\+\underline{26}\\50\\\underline{14}\\64\end{array}$ $\begin{array}{r}38\\\underline{+26}\\514\\6\ \ 4\end{array}$	$\begin{array}{r}38\\\underline{+26}\\\cancel{5}4\\6\\\enclose{circle}{64}\end{array}$ $\nearrow$ or erase 5 $\begin{array}{r}38\\\underline{+26}\\14\\\underline{50}\\64\end{array}$ $\begin{array}{r}38\\\underline{+26}\\514\\\enclose{circle}{6\ \ 4}\end{array}$	$\begin{array}{l}\ \ 6\ 4\\-\underline{2\ 6}\\\ \ 4\ -2\rightarrow 38\ (40-2=38)\end{array}$	4 – 6 = 2 is difficult so 4 – 6 = 2 → 42 is a typical error
Decompose-Tens-and-Ones Methods: Regroup, Then Add or Subtract Everywhere			
Look to see if total ones ≥ 10, record or remember, then make 1 ten from 10 ones, add tens, add ones or make 1 ten from 10 ones, add ones, add tens		Make 10 ones from 1 ten [open a ten], then subtract tens, subtract ones or subtract ones, subtract tens	
$\begin{array}{r}38\\\underline{+\underset{1}{2}6}\\64\end{array}$ $\begin{array}{r}4\ \\\cancel{3}8\\\underline{+26}\\64\end{array}$ $\begin{array}{r}38\\\underline{+26}\\64\end{array}$ $\begin{array}{r}38\\\underline{+26}\\64\end{array}$		$\begin{array}{r}5\,{}^{1}4\\\cancel{6}\ \ \cancel{4}\\\underline{-2\ \ 6}\\3\ \ 8\end{array}$ $\begin{array}{r}5\,{}^{1}0\\\cancel{6}\ \ 4\\\underline{-2\ \ 6}\\3\ \ 8\end{array}$ $\begin{array}{r}5\ \ \\\cancel{6}\,{}^{1}4\\\underline{-2\ \ 6}\\3\ \ 8\end{array}$	$\begin{array}{r}64\rightarrow 50\\\underline{-20}\\30\end{array}$ $\begin{array}{r}14\\\underline{-6}\\8\end{array}$ $\begin{array}{r}64\\\underline{-26}\\\rightarrow\end{array}$ 38
Decompose-Tens-and-Ones Methods: Alternate Adding/Subtracting and Regrouping			
Alternate adding and making another ten:		Alternate subtracting and opening a ten:	
Add tens, look to see if there is another ten, add ones	Add ones, make 1 ten from 10 ones, add tens	Subtract tens, open a ten, subtract ones	Subtract ones, open a ten, subtract tens
$\begin{array}{r}38\\\underline{+26}\\\cancel{5}\\64\end{array}$ $\leftarrow$ or erase $\begin{array}{r}38\\\underline{+26}\\64\end{array}$ look before writing tens	these look just like regroup, add ones, add tens methods above	$\begin{array}{r}14\\6\cancel{4}\\\underline{-26}\\\cancel{4}\\38\end{array}$ $\leftarrow$ or erase $\begin{array}{r}64\\\underline{-26}\\38\end{array}$ open before writing tens	$\begin{array}{r}64\\\underline{-26}\\-2\end{array}$ $\begin{array}{r}58\\\underline{-20}\\38\end{array}$ [we have not seen this]

Table 1—continued

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[a] The reverse of this method is also used occasionally: increase the first number to make an easy addition/subtraction, add/subtract, and decrease the answer to compensate (38 becomes 40 + 26 = 66 –2 = 64 or 64 becomes 66 – 26 = 40 – 2 = 38).

[b] Forgetting to add back in the original ones (the 4 from 64 or the 8 in 38) or subtracting them are (in a subtraction problem) frequent errors. The ones from the 26 sometimes are subtracted first and then the ones from the 64 are added back in; forgetting to add the 4 or subtracting it are also frequent errors.

[c] This is a different way to think of the method just above (sequence add on to make a ten) in which the rest of the number is added at once instead of in two steps as tens and ones.

[d] This step is difficult; the number is often subtracted rather than added, confusing what must be kept constant in addition (the total) and in subtraction (the difference).

Note. All separate-tens-and-ones methods may also be written horizontally. All methods in the table are for problems requiring regrouping (making another ten from ten ones or opening of ten to make 10 ones). This is done explicitly or by counting or adding/subtracting over a ten. Problems without regrouping are much simpler. Unknown addend methods for separate-tens and ones can be done by writing the addition problem with the second number empty and then adding up to the total to find that number. Single-digit subtraction for separate-tens-and-ones methods may be done as an unknown addend method (forward count up/add up methods).

We have found that children's invented methods almost always begin at the left with the largest multiunits. Adding the larger multiunits first has also been reported by others (e.g., Kamii, 1989; Labinowicz, 1985), though most invented methods reported before our projects began are only for two-digit calculation.

The strategies for addition and subtraction require at least implicit knowledge of properties of operations (commutativity, associativity). These are not discussed explicitly because we have no direct data concerning our children's understanding of these properties. It seems likely that much of our children's use of such properties is best characterized as theorems-in-action (Vergnaud, 1988). It is possible for such use to be focused on in discussions of such strategies. This may be especially helpful for subtraction situations, where incorrect generalizations from addition strategies may lead to errors.

Issues concerning subtraction. For subtraction problems, both backward take-away and forward unknown-addend methods are given. We have included the latter for several reasons. First, these are rarely taught, so their use by children almost always is a meaningful one generated by the conceptual structures of that child. Therefore, problems that elicit them (such as word problems with an unknown change number) can be useful diagnostic tools for understanding children's thinking. Second, all four projects used various real-world situations for computational problems, so both unknown-addend situations and take-away situations, and methods modeling these, arose in the classrooms. Third, having some experience with both kinds of methods may enable chidren to relate multidigit addition and multidigit subtraction as inverse operations that undo each other. Establishing this inverse relationship can permit more flexibility in problem solving. Finally, forward unknown-addend strategies are considerably easier than take-away counting-down strategies because counting down is so much more difficult than counting up. Teachers in the PCMP Project who used unknown-addend word problems reported that many children construct unknown-addend solution methods and then adopt such a method as a general subtraction method for numeral problems and that such methods are especially safe methods for weaker pupils. This is similar to the result for single-digit numbers reported by Fuson (1986) and Fuson and Willis (1988): Children encouraged to interpret written subtraction items such as 14 – 8 with how-many-more meanings as well as take-away meanings learned to count up for such problems, and their subtraction became as accurate and rapid as their addition. Most children who learn only a take-away interpretation of subtraction have a long lag between single-digit addition and single-digit subtraction competence with such numeral problems. This difference may be exacerbated for multidigit numbers under traditional and even some alternative classrooms approaches. For example, C. Kamii (1989) reported that multidigit subtraction was much more difficult than multidigit addition for her children, and she recommended delaying it by a year until third grade. Her children were using take-away interpretations of subtraction.

The unknown-addend method always is related to its counterpart addition method. For sequence methods, counting on or adding on for addition sounds just like counting up or adding up for an unknown addend. The difference is that, for addition, known multiunits are being added on to get an unknown total and the keeping-track process controls when the addition stops (it stops when the second addend has been added). In unknown-addend subtraction, multiunits are being added until a given known total is reached; the keeping-track method for the second addend then provides the answer: how many need to be added to reach the desired total. For the separate tens and ones methods, unknown addend methods can be written as addition with the second addend missing. When finished, they would look like the addition methods. To save space, we did not write them all again in Table 1.

Multidigit subtraction seems to be more difficult for children than multidigit addition. Some difficulties at this point seem to be inherent, and some may result from particular aspects of classroom activities, such as an emphasis on a take-away meaning. Children also may incorrectly generalize attributes of addition methods to subtraction; this may be exacerbated if addition is experienced for a long time before subtraction. How many of these difficulties could be reduced by changes in classroom activities is an important issue for future research.

One inherent source of difficulty in subtraction is the lack of commutativity of subtraction and the appearance of multidigit numerals as constantly seductive single digits, especially in vertical form. This combination results in many children (and even adults, occasionally) subtracting the smaller from the larger number in a given column either consistently or occasionally. In multidigit addition, children also permute the order of adding digits in various positions, but addition is commutative so the answer is not wrong. Thus, use of a correct addition strategy does not ensure that the child is even considering the order of the addends; subtraction does test this.

Subtraction posed some difficulties for most classes of strategies our project children invented. The subtraction begin-with-one-number methods initially require counting down, which is difficult and also has certain typical errors (e.g., 43, 42, 41, 40, 30, 39, 38, Fuson, Richards, & Briars, 1982). The mixed method requires a combination of addition and subtraction, which poses difficulties for some children. Methods in which a ten is opened up to make ten ones may need initial support, either in originally thinking to get more ones or in ways to record such a method in numerals.

The particular difficulties of each method are discussed more specifically below. What seems to be common across all of the subtraction methods is the difficulty of organizing subgoals into a coherent process without getting lost within particular subgoals. All of the methods (except unitary take away) require addition notions, if only for the composition of the initial or final quantities as decades and ones or tens and ones. Some methods require actual addition, as when the new ten ones are added to the old ones. Negotiating these shifts between addition and subtraction without losing the sense of the whole task, especially in take-away situations, may be initially difficult and require support.

Begin-with-one-number methods. The first method (count on/down/up by tens then by ones) in Table 1 is simple conceptually, but it requires that a child be able to count on/down/up from an arbitrary number (e.g., 38, 48, 58) and not just count by tens (30, 40, 50). The former is quite a bit more difficult for children (see the review of this literature in Fuson, 1990). The count on/down/up of ones also frequently goes over a decade word, which can be especially difficult for subtraction. The third sequence method requires only the simpler kind of counts by ten beginning with a decade, but one must shift between counting on by ones, to counting on by tens, and back to counting on the rest of the ones. This is easy to do when objects or drawings or numbers are used to keep track of the counting or adding/subtracting.

The overshoot-and-come-back methods can be done mentally because the problem is changed to require only easy decade addition or subtraction. The reverse of this method is also used occasionally: increase the first number to make an easy addition/subtraction, add/subtract, and decrease the answer to compensate (38 becomes 40 in this sequence: 38 + 26 is 40 + 26 – 2 or 66 – 2 or 64. Likewise, 64 becomes 66 in 64 – 26, so the problem becomes 66 – 26 = 40; then 2 is subtracted to get 64 – 26 = 40 – 2 =38).

Mixed methods. The mixed methods begin by separating both numbers, adding or subtracting the tens, and then moving into the sequence by adding the original ones, and then adding or subtracting the other ones. The mixed subtraction method is prone to errors arising from (a) overgeneralizing the addition method in which you add both ones or (b) not clearly differentiating it from the methods that begin with one number (in which you subtract the tens and subtract the ones). The step of making the sequence number by adding in the original ones comes in the middle of the method, so it is especially confusing. Some children forget to add in these original ones, or they subtract the ones rather than add them (because "in adding you added everything, so in subtraction you subtract everything"). In the PCMP project, the latter error was found to be increased considerably if teachers gave only addition problems before they gave any subtraction problems, even for as short a period as 2 weeks. For the add-on-up-to mixed methods, children also may forget to add in the orginal ones.

Change-both-numbers methods. The change-both-numbers methods can be easily confused if children do not understand what must remain the same in each method. In addition the total must stay the same; this method can be thought of as just moving some entities from one number to the other to make one number easy to add. In subtraction, the difference must be maintained, so the same number must be added to (or subtracted from) each number. However, children sometimes subtract the second number, as in addition. We have also shown a subtraction method that is frequently brought from home by children whose parents were educated in Latin America or Europe. This method appeared frequently in the Latino classrooms (Ron, in press). But most parents had not been taught it in a meaningful manner, so they did not realize why it worked (that it was adding a ten to the ones in the top number and a ten to the tens in the bottom number). Some adults learn it without writing any extra numbers. Knowing about this method can be helpful to a teacher who otherwise will not understand what a child is doing when she or he increases the bottom tens by one.

Decompose-tens-and-ones-methods. The methods in which the tens and ones are decomposed and then operated on separately must deal explicitly with regrouping: in adding, a unit of ten must be made from ten ones, and in subtracting, a unit of ten must be opened to make ten ones. In the begin-with-one-number methods, this could be dealt with implicitly by counting up or down (or adding or subtracting) over a ten. With the decompose methods, the regrouping step can be done after both the tens and the ones additions (or subtractions) are done, before both, or alternating with the additions (or subtractions). The tens can be added (or subtracted) first and then the ones, or vice versa. The total (or difference) for the ones or for the tens can be written down as it is found, or it can be remembered and written all at once. Most of the methods in Table 1 can be done with the numbers written horizontally or by using the arrow notation, but vertical writing makes it easy to see the like units that are being added or subtracted. The methods can be done using sequence-ten conceptions in which the tens are thought of as groups of ten and counted on by tens or added as decades (e.g., 30 + 20), using separate-tens conceptions (e.g., 3 tens + 2 tens is 5 tens), or using integrated-tens conceptions. Children in the projects often wrote these methods horizontally or did some steps in some methods mentally, without recording them.

Unknown addend methods of subtraction can be done in addition format with the second addend to be filled in by adding up steps or in a subtraction format in which the single-digit subtraction done for the ones or the tens is thought of as counting or adding up to. The solved problems will look like those given in Table 1 for addition and subtraction, so unknown addend methods are not given separately in Table 1.

Children may need initial support to think of regrouping. In addition, this can usually be stimulated by pointing out how failure to so do leads to an answer that cannot be correct (e.g., 38 + 26 = 514) or by asking, “You have too many ones. What could you do with some of them without changing the number?” In subtraction, children may need to be asked, “You need some more ones. Where can you get some without changing that number?” With such support, these methods can become accurate and well understood.

All of the subtraction regrouping methods involve recomposing the total (a ten is opened to be available for the subtraction of some or all of the ones). The addition methods in which the regrouping is done below the total line also involve recomposing the total; these show the inverse relationship between addition and subtraction quite nicely (e.g., the last addition method in the top row in Table 1 is just the inverse of the first subtraction method in the second row). The traditional addition algorithm for the United States (third method in the third row in Table 1) instead adds the ten back into the problem by writing it above the top tens digit. This is understandably confusing to some children (Burghardt & Fuson, 1996) because it seems to be changing the problem. Methods in which the regrouped ten is written down in the total to be added later (e.g., the first method in the second row) are conceptually clearer.

The addition methods that add everything first and then regroup are quite straightforward and clear conceptually. The subtraction methods require some sense of negative numbers because one is subtracting rather than “fixing” the ones to get more, and they require a firm sense of the directionality of the subtraction. Both of these are a problem for many children, though some children do invent or readily understand this method (e.g., Davis, 1984; Kamii, 1989). The common error is to switch the order of the ones subtraction because it "doesn't make sense" the other way or because the order of subtraction is considered unimportant and find 4 – 8 or 8 – 4 to equal 4, so 30 + 4 = 34 is given as the answer instead of 30 + –4 = 26.

The methods in which regrouping is done first make a lot of sense in subtracting because you need more ones from which to subtract. In adding, the parallel methods require one to look ahead and decide whether the total of the ones will be over ten and if so, to add another ten. The fourth method shown in the second row is a common addition method in Europe in which you add from left to right, looking ahead (to the right) after each addition to see if you need to add another of that unit to your total. This method, and the methods in which you record any regrouping in the total, also have easier additions than the traditional U.S. algorithm because you add the two numbers shown and then increase the total by one mentally. When the one is written above a column, it is often added to the top number, and then one must add two numbers, one of which is not visible but must be remembered.

Two-digit problems with sums over 100. The numerical example in Table 1 has a sum of the tens that is less than 100. Problems with the sum of the tens over 100 are more difficult. English-speaking children in the United States experience considerable difficulty counting by tens over one hundred, with many children in traditional third-grade classes still not doing so accurately (Labinowicz, 1985). The begin-with-one-number tens methods require counting over 100, so children need opportunities to learn such counting associated with quantities. Counting down over 100 may be even more difficult. The decompose-tens-and-ones methods require adding or subtracting tens over 100, which also must be learned.

Addition and Subtraction of Three-Digit Numbers

The strategies for two-digit addition and subtraction are extended to those for three-digit addition and subtraction in Table 2. Unknown addend methods are not included because of space constraints, but they can be identified by reading the addition problems as if they were 478 + ? = 834. The numbers in Table 2 do not require adding over 1000. Such problems are more difficult because one must be able to count up or down by hundreds over 1000 or add or subtract hundreds over 1000, both of which require additional learning.

The begin-with-one-number sequence methods are straightforward but very cumbersome to do by counting on or down or up if one carries along the whole sequence value as one counts. These also involve difficult counts up or down over hundreds. Sequence adding or subtracting and writing partial sums or differences is fairly easy as long as one can add or subtract over a ten or a hundred. The method of subtracting by adding up to make a hundred is particularly rapid and easy: 478 + *22* is 500 and *334* more to make 834; *334* and 22 is 356.

The methods involving adding or subtracting hundreds, tens, and ones separately and regrouping when necessary are simple extensions of the two-digit methods. These methods are easier if written in vertical form because this aligns like multiunits (but, of course, this also can facilitate thinking of the numbers as concatenated single digits). The methods that do not alternate between addition or subtraction and regrouping are simpler than

Table 2

Three-Digit Addition and Subtraction Methods

478 + 356 = __	**834 – 356 = __**
Begin-With-One-Number Methods: Begin With One Number and Move Up or Down by Hundreds, Tens, and Ones	
Count on/add on hundreds, then tens, then ones 478, 578, 678, 778, 788, 798, 808[a], 818[a], 828, 829, 830, 831, 832, 833, 834 478 + 300 → 778 + 50 → 828[a] + 6 → 834	Count down/subtract hundreds, then tens, then ones 834, 734, 634, 534, 524, 514, 504[a], 494[a], 484, 483, 482, 481, 480, 479, 478 834 – 300 → 534 – 50 → 484[a] – 6 → 478 or 834 –356 534 484 478
Overshoot and come back 478 + 400 → 878 – 44[a] → 834	Overshoot and come back 834 – 400 → 434 + 44[a] → 478
Count on/add on to make a hundred, add the rest[b] 478, 479, 480, 490, 500, 600, 700, 800, 810, 820, 830, 831, 832, 833, 834 478 + 22 → 500 + 334 → 834	Count down/subtract to make a hundred, subtract the rest[b] 834, 833, 832, 831, 830, 820, 810, 800, 700, 600, 500, 490, 480, 479, 478 834 – 34 → 800 – 322 → 478[a]
Mixed Methods: Add or Subtract Hundreds, Make Sequence Number With Rest of First Number, Add or Subtract Other Tens and Ones	
Count on/add on hundreds, make sequence number with first tens and ones, count on/add on other tens and ones 400, 500, 600, 700, 778, continue as in first method 400 + 300 → 700 + 78 → 778 + 50 → 828[a] + 6 → 834 or 700 + 70 → 770 + 50 → 820[a] + 8 → 828 + 6 → 834	Count down/subtract hundreds, make sequence number with first tens and ones, count down/subtract other tens and ones 800, 700, 600, 500, 534, continue as in first method 800 – 300 → 500 + 34 → 534 – 50 → 484[a] – 6 → 478 or 500 + 30 → 530 – 50 → 480[a] + 4 → 484 – 6 → 478
Change-Both-Numbers Methods	
Move some from one number to the other to make a hundreds number[c] (maintaining the total) 478 + 22, 356 – 22 → 500 + 334 → 834	Make subtracted number a hundreds number, change other to main difference 356 + 44, 834 + 44[d] → 878 – 400 → 478

(table continued)

Table 2—continued

Decompose Hundreds, Tens, and Ones Methods: Add or Subtract Everywhere, Then Regroup

Add hundreds, add tens, add ones
make 1 hundred from 10 tens
make 1 ten from 10 ones

$$\begin{array}{r} 478 \\ \underline{+356} \\ 700 \\ 120 \\ \underline{14} \\ 834 \end{array} \qquad \begin{array}{rrr} 4 & 7 & 8 \\ \underline{+3} & \underline{5} & \underline{6} \\ 7 & 12 & 14 \\ 8 & 2 & \\ & 3 & 4 \\ 8 & 3 & 4 \end{array} \quad \begin{array}{c} \\ \\ \\ \text{or} \\ \\ \\ \end{array} \quad \begin{array}{rrr} \\ \\ 7 & 12 & 14 \\ & 3 & 4 \\ 8 & 3 & 4 \\ \\ \end{array}$$

add ones, add tens,
add hundreds
← like these

$$\begin{array}{rrr} 4 & 7 & 8 \\ \underline{+3} & \underline{5} & \underline{6} \\ & 1 & 4 \\ 1 & 2 & 0 \\ \underline{7} & \underline{0} & \underline{0} \\ 8 & 3 & 4 \end{array} \qquad \begin{array}{rrr} 4 & 7 & 8 \\ \underline{+3} & \underline{5} & \underline{6} \\ 7 & 12 & 14 \\ 8 & 3 & \\ 8 & 3 & 4 \end{array}$$

Subtract hundreds, subtract tens, subtract ones,
combine totals (some notion ofof negative numbers)

$$\begin{array}{rrr} 8 & 3 & 4 \\ \underline{-3} & \underline{5} & \underline{6} \\ 5 & -2 & -2 \\ 4 & 8 & \\ 4 & 7 & 8 \end{array}$$

May also be done by beginning subtracting with the ones

← 5 hundreds – 2 tens = 4H2T or 500 – 20 = 480
← 480 – 2 → 478

Decompose Hundreds-Tens-and-Ones Methods: Regroup, Then Add or Subtract Everywhere

Make 1 hundred from 10 tens ⟷ make 1 ten from 10 ones
[Look to see if ≥ 10 tens, ≥ 10 ones; record or remember]
add ones, add tens, add hundreds
add hundreds, add tens, add ones

$$\begin{array}{r} 478 \\ \underline{+356} \\ {\scriptstyle 1\,1} \\ 834 \end{array} \qquad \begin{array}{rrr} 5 & 8 & \\ \cancel{4} & \cancel{7} & 8 \\ \underline{+3} & \underline{5} & \underline{6} \\ 8 & 3 & 4 \end{array} \qquad \begin{array}{r} 11 \\ 478 \\ \underline{+356} \\ 834 \end{array} \qquad \begin{array}{r} 478 \\ \underline{+356} \\ 834 \end{array}$$

Make 10 tens from 1 hundred ⟷ make 10 ones from 1 ten
subtract ones, subtract tens, subtract hundreds
subtract hundreds, subtract tens, subtract ones

$$\begin{array}{rrr} & 12 & \\ 7 & \cancel{13} & 14 \\ \cancel{8} & \cancel{3} & \cancel{4} \\ \underline{-3} & \underline{5} & \underline{6} \\ 4 & 7 & 8 \end{array} \qquad \begin{array}{rrr} 7 & 2 & \\ \cancel{8} & \cancel{3} & 4 \\ \underline{-3} & \underline{5} & \underline{6} \\ 4 & 7 & 8 \end{array} \qquad \begin{array}{rrr} 7 & 2 & \\ \cancel{8} & \cancel{3} & 4 \\ \underline{-3} & \underline{5} & \underline{6} \\ 4 & 7 & 8 \end{array} \qquad \begin{array}{r} 834 \\ \underline{-356} \\ 478 \end{array}$$

Decompose-Hundreds-Tens-and-Ones Methods: Alternate Adding/Subtracting and Regrouping

These methods look just like those above, but they are carried out for addition by adding a column and regrouping, adding the next column and regrouping, and adding the last column and, for subtraction, by regrouping if necessary in subtraction, subtracting that column, regrouping and then subtracting the next column, and subtracting the last column. These methods can be done from right to left or from left to right if you look ahead to see if regrouping will involve that column. The first two subtraction methods are written as left-to-right methods.

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[a] Finding this answer is difficult and error-prone because it involves going over a ten or hundred.
[b] This can be done in two steps: to make a ten and then to make a hundred.
[c] This is another way of thinking about the method just above.
[d] This step is difficult; the number is often subtracted rather than added, confusing what must be kept constant in addition (the total) and in subtraction (the difference).
Note. Subtraction unknown addend methods (478 + __ =834) look just like the addition (478 + 356 = __) methods except that one is adding up ro 834 and keeping track of how much was added up. Adding up to make an even hundred and then adding on the rest is particularly easy: 478 + $\underline{22}$ → 500 + $\underline{334}$, so 334 + 22 = 356.

those that do alternate, because children do not have to switch back and forth between different kinds of steps. The adding methods with regrouping last and the subtracting methods with regrouping first are clear inverses of each other. The subtracting methods that involve doing all subtracting first require negative numbers and a good understanding of the units in each place.

Addition or Subtraction of Four-Digit and Larger Numbers

Because our number system has consistent ten-for-one relationships between adjacent multiunit quantities or multiunit names or positions, the decompose methods that add or subtract like multiunits and recompose (regroup) to make a larger multiunit (or open a multiunit to make ten of the next smaller multiunit) are particularly easy to generalize to numbers of four digits and more. Recomposing any multiunit sum over nine as one of the next larger multiunit and the rest of those multiunits (or opening a multiunit to make ten of the next smaller multiunit) can be accomplished within a tens-and-ones conception of each place as single-digit numbers or by thinking of the multiunit values. The former is the rapid automatized way to calculate with large numbers. As long as a child can think of the multiunit values if necessary, using the simplified adjacent tens-and-ones conception is a sensible approach for adding and subtracting very large numbers.

Children who have educational experiences supporting such a general “recompose” strategy may go through two levels of thinking. First, in a study of U.S. children using base-ten blocks (Fuson & Briars, 1990), and in a study of Korean children’s multidigit competence (Fuson & Kwon, 1992b), some first and second graders were found to use a separate-tens meaning for the sums of all positions. For example, they described a trade from a hundreds sum of 12 hundreds to the thousands as trading the 10 from the 12. Older children integrated this view with the multiunit values involved and described this trade as 10 hundreds from the 12 hundreds giving one thousand that needed to go with the thousands in the next left column.

Currently there is relatively little research on the extension of most of the methods in Table 2 to four-digit and larger numbers because the projects have followed only second or third graders in any depth. Some children in all projects did pose and solve larger problems, but systematic data on these methods are not yet available. Children in the CBI project who worked with numbers of four digits and more generally abandoned left-to-right methods and moved from the right to the left. The sequence counting methods that begin with one number would seem to get quite burdensome with large numbers if done verbally without recording, because one would have to carry along a whole multidigit number. The separate multiunits methods generalize easily to larger numbers.

Experiences in some of our projects suggest that it is important for students eventually to record their verbal methods. Some fourth- and fifth-grade PCMP students who were top achievers in the first few grades suddenly seemed to come unstuck and experience to difficulty in these higher grades. They had never been encouraged to record their methods and could cope very well mentally. However, in fourth or fifth grade, the problems moved beyond their mental arithmetic ability, but they had not acquired the necessary recording skills. Recording also seemed to help some CGI students to use accurate methods for larger numbers.

Methods Used by Children in Each Project

Within each project, the conceptual structures constructed by children varied, and many children varied in their solution methods across different problems and problem settings. But there were also some widely used conceptual structures that were related to the conceptual supports used in the classroom. To summarize and oversimplify, children in the Problem Centered Mathematics Project (PCMP) project frequently used for two-digit addition or subtraction sequence methods that begin with one number and mixed methods. For three-digit problems, adding up to make tens and hundreds was popular, and "mixed taking" from larger multiunits was used frequently. Children in the Conceptually Based Instruction (CBI) project most frequently used methods in which multiunits were added or subtracted separately. Children in the Cognitively Guided Instruction (CGI) project varied considerably from one classroom to another and within classrooms, with many children using methods in which multiunits were added or subtracted separately. Sequence methods that begin with one number were also often used for two-digit problems. In the early Supporting Ten-Structured Thinking (STST) projects, children predominantly used methods in which multiunits were added or subtracted separately. In the Latino project, children tended to look more like Cognitively Guided Instruction children, with considerable variability between and within classrooms, though adding or subtracting multiunits separately predominated.

In all projects, most children understood the multidigit addition and subtraction methods they used (e.g., Carpenter et al., in press; Fuson et al., in press; Hiebert & Wearne, 1996). Thus, clearly, the nature of the conceptual supports available in a given classroom directs and constrains the conceptual structures built by individual children, though this building is a long, slow process in which considerable variability arises from the learning history and individual learning experiences of each child. Children attend to, hear, and see the "same" classroom experience (e.g., a class discussion of alternative solution procedures) according to each child's conceptual structures at the time. Of course, children also have different mathematical experiences even in the same classroom (e.g., working with different partners or small groups).

What is common to the conceptual supports in all classrooms, in addition to the use of word-problem situations and an emphasis on discourse about problem solutions, is the provision of sustained opportunities (a) to construct triads of connected conceptual structures that relate ten-structured quantities to number words and to written two-digit numerals and (b) to use these triads in solving multidigit addition and subtraction situations. The multidigit addition and subtraction methods given in Tables 1 and 2 can be carried out as counting or adding or subtracting actions on multiunit quantities (objects or drawings), as verbal abbreviations of these counting or adding or subtraction actions in which quantities are not perceivable by an onlooker (but may be generated mentally by the child using the method), or as written numerals recording quantities and/or counting or adding or subtracting actions on quantities.

Written recordings have two advantages over verbal methods. First, they serve as memory supports and allow a child to do the several steps involved in most multidigit solutions without needing to remember the results of each step. This may be important for weaker children initially because they may have difficulty keeping track of multiple steps. For some very able children, as discussed earlier, this will be less necessary initially because they can construct mental methods, but it is important to record these methods, at least sometimes, to support their later extension to larger numbers. Second, written recordings remain after the problem solution, and thus support reflection and discourse about that solution method. However, the use of numerals does run the concatenated single-digit risk, that the constantly seductive digits will elicit a concatenated single-digit conception that will insufficiently direct and constrain a solution. This risk may be reduced if problems are written in a horizontal fashion or if vertical problems are linked tightly enough to adequate ten-structured conceptions.

Experiences in two projects suggest that it may be better to intermix multidigit addition and subtraction fairly early. There is direct evidence from the PCMP that sustained experience solving problems requiring multidigit addition before solving problems requiring multidigit subtraction will for some children support an incorrect generalization of an addition solution method to subtraction (subtract the decade and subtract both ones). In the CBI project, such a separation of addition and subtraction problems (created by the constraints of traditional textbook sequencing, which the CBI project followed to permit their control comparison) might also have contributed at least somewhat to the CBI children's greater difficulty in devising a written method for subtraction than for addition. Their adding-like-multiunit frame suggests "adding and then fixing the total if necessary," but this frame does not generalize to subtraction. In subtraction, one has to get more multiunits (fix the large number) before one can subtract. With objects, one can just subtract from the next larger multiunit (take ones from a ten), but translating this into a written method seems to have been difficult for some children. Children with support did devise written methods through discussions of their actions on blocks.

CONCLUSION

We have found that the deep, and sometimes heated, discussions over the several years of the working group have enabled us to clarify the solution methods used in each project, articulate the conceptual supports available in the classrooms, and move toward an understanding of the conceptual structures children are using in the wide variety of place-value and multidigit addition and subtraction situations encountered in the various projects. There is still much to learn and to articulate concerning how conceptual supports in the classroom enable children to construct and use robust multidigit conceptual structures. This learning and articulation must be accompanied by increased understanding about supportive (and interfering) roles of classroom discourse and of the teacher, understandings we are presently trying to articulate for all our projects (Hiebert et al., 1997). We hope that this articulation of children's multidigit conceptual structures and multidigit addition and subtraction methods will stimulate others to engage in the complex task of understanding how classroom mathematical activities enable children to engage in accurate and robust mathematical thinking and doing.

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