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# Supporting Latino First Graders' Ten-Structured Thinking in Urban Classrooms

Karen C. Fuson, *Northwestern University*
Steven T. Smith, *Northwestern University*
Ana Maria Lo Cicero, *Northwestern University*

Year-long classroom teaching experiments in two predominantly Latino low-socioeconomic-status (SES) urban classrooms (one English-speaking and one Spanish-speaking) sought to support first graders' thinking of 2-digit quantities as tens and ones. A model of a developmental sequence of conceptual structures for 2-digit numbers (the UDSSI triad model) is presented to describe children's thinking. By the end of the year, most of the children could accurately add and subtract 2-digit numbers that require trading (regrouping) by using drawings or objects and gave answers by using tens and ones on various tasks. Their performance was substantially above that reported in other studies for U.S. first graders of higher SES and for older U.S. children. Their responses looked more like those of East Asian children than of U.S. children in other studies.

The larger contexts for the problems addressed by this study are the considerably lower level of primary school mathematics learning by children in the United States compared with that of children in China, Japan, and Korea (Fuson & Kwon, 1992a, 1992b; Geary, Bow-Thomas, Liu & Siegler, 1996; Miller, 1990; Miller & Stigler, 1987; Song & Ginsburg, 1987; Stephenson & Stigler, 1992; Stigler, Lee, & Stephenson, 1990) and the reform efforts of the National Council of Teachers of Mathematics (NCTM) to bring about mathematics classrooms appropriate for the future needs of society (NCTM, 1989, 1991). The special challenges are the particularly low levels of mathematics achievement by poor urban children, especially those who do not speak English or who have parents with low levels of education (e.g., Secada, 1992).

These issues were addressed by undertaking a year-long teaching experiment (or developmental research, Gravemeijer, 1994a, 1994b) in two first-grade classrooms in an urban school with a predominantly low-socioeconomic-status (SES) Latino population. The backgrounds of children in English-speaking and Spanish-speaking classrooms

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differ (e.g., children in the latter are more recent immigrants). For this reason, and to explore issues of language differences, we selected one classroom in each language (Spanish and English). We did not envision an experimental contrast between these two classes, but only a test of our approaches in both languages and with a range of Latino-background children. The mathematical foci were addition and subtraction of quantities expressible by two-digit numbers and place-value concepts (the conceptual relationships among number words, two-digit numerals, and quantities). This study was an extension of earlier theoretical (Fuson, 1990) and instructional development work (Burghardt & Fuson, 1997; Fuson & Briars, 1990; Fuson, Fraivillig, & Burghardt, 1992; Smith, 1994). We sought to develop learning activities that fit the ecology of urban Latino classrooms.

The first purpose of this article is to describe two aspects of the research that may be particularly helpful to others: (a) a developmental sequence of conceptual structures for two-digit numbers that guided the instructional design work and (b) conceptual supports used to assist children's construction of these conceptual structures, especially a method for children's drawing of quantities organized by ten that solved major pragmatic and instructional-assessment issues and afforded multiple solution methods. The developmental sequence of conceptual structures was modified as it was clarified for the several drafts of this article. The second purpose of this paper is to describe the learning of the children in the two classes as it compares with that of East Asian and U.S. samples. We first briefly summarize our theoretical perspectives on learning and on teaching.

## THEORETICAL PERSPECTIVES ON LEARNING AND ON TEACHING

The theoretical perspectives on learning and on teaching used by the research team were a constructivist view of learning (e.g., Steffe & Gale, 1995) and a Vygotskiian (1934/1962, 1978, 1934/1986) view of teaching. We view children as meaning makers who use conceptual structures to interpret what they see, hear, and feel (see Fuson et al., 1992, for a fuller discussion). We view teaching as (a) assisting children (Tharp & Gallimore, 1988) to build conceptual structures and conceptual methods that are useful within a cultural domain (such as two-digit addition and subtraction) and (b) assisting children to build more advanced conceptual structures and methods as they become able to do so. Assistance includes using activities that can enable children to construct knowledge that is required for more advanced methods.

This combination of views leads to a conception of teaching and learning in action in the classroom at a given moment as assisting the performance of children with their own individual meanings-in-the-making. Such assistance needs to be adapted to a given child's conceptual structures; this adaptation requires that the teacher ascertain what conceptual structures each child is using. Therefore, teaching requires continual learning about the knowing states of children in assessing-assisting cycles that may be as short as a few seconds (e.g., when the teacher is assisting an individual child) or much longer (e.g., looking at written work to try to understand children's thinking).

Assessment is often used in schools to rank order children for various purposes, and many assessments do not give much insight into children's thinking. We use the term *assessment* to mean any teacher-child interaction (directly or by means of a child's work) that enables the teacher to learn about the child's thinking in order to adapt teacher assistance to the state of the child's conceptual structures. This assistance may take the form of helping a child to develop a more robust structure or to move on to more advanced conceptual possibilities. In new or complex domains of knowledge that have multiple arbitrary cultural aspects (e.g., place-value writing of quantities from 10 to 99), the teacher may need to model or lead knowledge-building activities initially, gradually withdrawing as children become more capable but still monitoring to offer assistance as needed. In domains in which most children already have knowledge (e.g., single-digit addition), the teacher can begin near the end point of this assistance continuum by allowing children to invent their own solution methods, with assistance from the teacher or peers for those who need it.

Our Vygotskiian view of teaching also leads us to consider very carefully the conceptual tools that might help children understand the cultural words and written symbols of a domain (the mathematical referrers) as well as understand the quantities and operations of a domain (the mathematical referents). The cultural referrers in mathematics sometimes have complexities that render them difficult for some children to understand, and quantities likewise may not be so nicely organized as to be clear in the real world. Furthermore, children can construct meanings for the mathematical referrers only by relating them to their referents. Therefore, some kind of quantity referent needs to be present in the classroom. Our developmental research focused heavily on the invention of conceptual tools—conceptual supports—that would help children build meanings in the domain. We considered each of these conceptual tools to be only a potentially meaningful tool whose actual meaning would depend upon its interpretation by each individual child.

## ANALYSIS OF THE MATHEMATICAL DOMAIN OF MULTIDIGIT NUMBERS

### *A Model of Conceptual Structures Used in the Domain of Two-Digit Numbers*

We think of conceptual structures as hypothesized categories of quantitative activity that seem useful in understanding teaching and learning in a domain. These categories have been identified by reflecting on our own extensive experiences with children's activity in this domain, from others' published reports of their own experience, and by conversations with others about their experiences. To us, a conceptual structure for multidigit numbers is (a) a structuring of—a particular viewing of—the quantities, number words, and written numerals so that these can be understood, counted, added, or subtracted in particular ways and (b) the knowledge required to understand, count, add, or subtract in those ways. As researchers, we make judgments that children are demonstrating a particular conceptual structure at a particular time, but of course we can only make inferences about the point of view of, and the knowledge used by, another. We have found the following model of conceptual structures for two-digit numbers to be useful in organizing our understandings of the research literature, designing the teaching-learning activities, and analyzing our data about children's mathematical activity.

Earlier literature identified three correct conceptions demonstrated by children in the United States: a *unitary conception* in which children count a two-digit quantity by ones, a *sequence-tens conception* in which they count by tens and then by ones, and a *separate-tens-and-ones conception* in which the units of tens and the units of ones are counted separately (see Fuson, 1990, for a review of this literature). An example is counting three bars (each made from 10 Unifix cubes) and 2 extra cubes: Children demonstrating a unitary conception would count all 32 of the unifix cubes (1, 2, 3,…, 32); children demonstrating a sequence-tens conception would count "10, 20, 30, 31, 32"; children demonstrating a separate-tens-and-ones conception would count "1, 2, 3 tens and 1, 2 ones. 32."

These conceptions are also demonstrated in solving two-digit addition and subtraction problems. Each conception gives rise to more than one method (see Fuson, 1990, and Fuson, Wearne, et al., 1997, for discussions and examples). However, many children in the United States instead view two-digit numbers as two separate single-digit numbers. This concatenated single-digit meaning does not suggest or constrain correct addition or subtraction methods. Instead, some children just combine arbitrary single digits (e.g., even adding all four digits in two 2-digit numbers), leading to various well-documented errors (e.g., see VanLehn, 1986, for a discussion). This concatenated single-digit meaning is constructed when too few opportunities are given to children to link accurate multidigit quantity meanings to the written numerals used in adding and subtracting.

Our view of these conceptions is that each involves a triangle of relationships among quantities, number words, and written numerals. For single-digit numbers, 3 two-way links form the triangle (see Figure 1). Each one-way link relates the aspect initially seen or heard to another aspect. For example, I hear five and think or see or write the numeral 5 (bottom left-to-right arrow), or I see five birds and think or say five (left arrow from top to bottom). In the concatenated single-digit conception of two-digit numbers (see Figure 1), these six relations are constructed for each single digit in a two-digit number.

Our triad model of two-digit conceptions is shown in Figure 2. We call this model the UDSSI triad model for the names of the five conceptions (unitary, decade, sequence, separate, integrated). This model extends the earlier classification of unitary, sequence, and separate conceptions by adding the decade and integrated conceptions. The conceptual structures in Figure 2 are arranged developmentally from the outside in, except for the innermost two (sequence and separate). These two depend heavily on learning opportunities and may be learned independently of each other. The UDSSI model was presented in a developmental form in Fuson, Wearne, et al. (1997); that form showed more clearly how the conceptual structures grew one from another developmentally and experientially. The form shown in Figure 2 emphasizes the relationships among all the conceptual structures. Stimulating children to construct that related web was a primary goal of our teaching experiment.

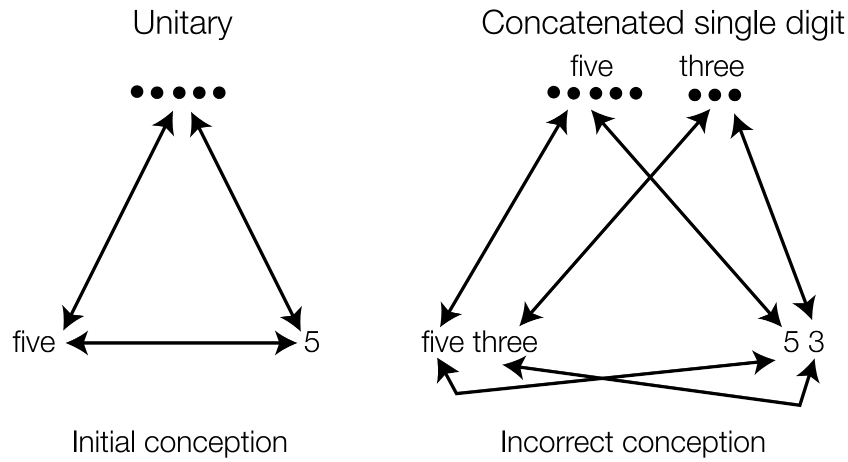


*Figure 1.* Unitary triad (quantity, number word, written numeral) and common incorrect multidigit conception derived from the appearance of the multidigit numbers

All children begin with a unitary conception that is a simple extension from the unitary triad for single-digit numbers. With this conception, the separate number words and the two digits have no quantity referents by themselves. The entire number word (e.g., sixteen) or numeral (16) refers to the whole quantity.

With time and experience, each number word and each digit does take on a meaning as a decade or as the extra ones in the decade-and-ones conception. For example, in 53, the 5 means fifty and the 3 means three. This conception was identified by Murray and Olivier (1989) and discussed in Fuson, Wearne, et al. (1997). Some children write numbers as they hear the decade word: "I hear fifty and then a three, so I write 50 and then 3, so 503." The numerals for this conception can be understood more clearly if we think of the ones as written on the decade numerals: The arrow in Figure 2 shows the 3 going on the 0 in 50.

In the sequence-tens-and-ones conception, an extension of the decade conception, units of ten single units are formed within the decade part of the quantity. These sequence-tens units are counted by tens (e.g., 10, 20, 30, 40, 50). Initially with the sequence-tens conception, the person counting has no immediate way to know that there are 5 tens in fifty. A user of this conception can find out how many tens by counting "10, 20, 30, 40, 50" while keeping track of the five counts.

The separate-tens-and-ones conception is built through experiences in which a child comes to think of a two-digit quantity as comprising two separate kinds of units—units of ten and units of one. Both kinds of units are counted by ones (e.g., "1, 2, 3, 4, 5 tens and 1, 2, 3 ones"). In Figure 2, we show these units of ten as a single line to stress their (ten)-unitness, but the user of these units understands that each ten is composed of 10 ones and can switch to thinking of 10 ones if that approach becomes useful.

Children's construction of the sequence-tens and separate-tens conceptions depends heavily on their learning environments. However, individuals in the same classroom may construct one or the other of these first. Which is first may depend partly on whether a child focuses on the words, which facilitate the sequence-tens conception, or on the written numerals, which facilitate the separate-tens conception.

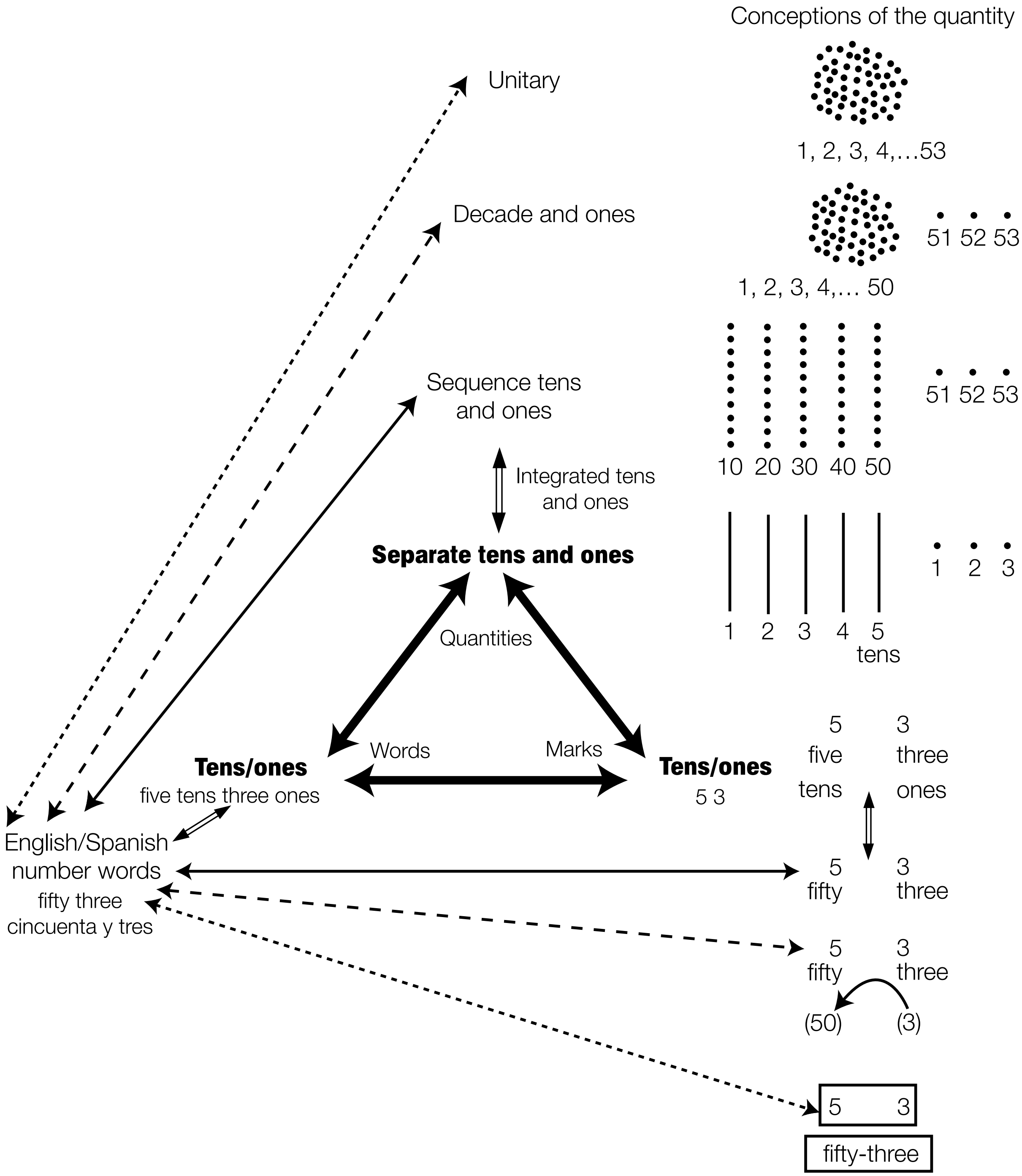


*Figure 2*. A developmental sequence of conceptual structures for two-digit numbers: the UDSSI triad model

Children may eventually relate the sequence-tens and separate-tens conceptions to each other in an integrated sequence-separate conception (these connections are shown in Figure 2 as the short double arrows). In the integrated conception, children connect fifty to 5 tens, and the written numeral 53 can take on either quantity meaning (fifty-three or 5 tens, 3 ones).

We originally thought about each two-digit conceptual structure as a triangle (a triad) of six relations. However, it later became clear that only the separate-tens-and-ones conception has direct links between quantities and numerals. Such a direct link can occur only where the quantities of tens and ones are small enough to be subitized (immediately seen as a certain number of units) or are in a pattern. The other three conceptions must relate quantities to written numerals through the number words by counting. Therefore the link between quantities and numerals is not drawn in Figure 2 for these conceptions.

### *Use of the Triad Model in Our Study*

Our analysis of the structure of Spanish words for two-digit numbers suggests that Figure 2 describes the main conceptual structures that Spanish-speaking children construct, albeit with some small advantages and disadvantages compared with English speakers. Spanish uses a list of decade words (diez, veinte, treinta, cuarenta, cincuenta, sesenta, setenta, ochenta, noventa). The relationship of these to the words for one to nine (uno, dos, tres, cuatro, cinco, seis, siete, ocho, nueve) is not very clear. This lack of a clear relationship is similar to the rather opaque relations of the English words *twenty* and *two*, *thirty* and *three*, and *fifty* and *five*. A special issue for Spanish is that *sesenta* and *setenta* sound very much alike. This similarity makes learning these words difficult for some children (Fuson & Smith, 1997). Spanish words from 21 on use the construction *cincuenta y tres* (fifty and three), which would seem to support children's construction of the decade-and-ones conception. Spanish words in the teens are irregular at first and then begin to name the ten at 16: once, doce, trece, catorce, quince, dieciséis, diecisiete, dieciocho, diecinueve. A less common alternative spelling is conceptually clearer: *Diez y seis* means "ten and six," whereas *dieciséis* means "tens six." Thus, the conceptual contribution of the words for 16 through 19 may depend heavily on the emphasis given by the teacher to the "ten and six" meaning. Most European systems of number words for two-digit numbers have a decade structure, although with various irregularities (see Fuson & Kwon, 1991/1992, for an analysis and Menninger, 1958/1969, for details). The UDSSI triad model, therefore, would seem helpful for these other European languages also.

Our goal in this study was to help children in both the English-speaking and Spanish-speaking classrooms construct all of the triad conceptual structures. But in what order were we to help them and how? Two other perspectives contributed major elements to the design of the learning activities.

The first perspective is the literature on East Asian children's multidigit learning. Two conceptual structures in Figure 2 (the decade and the sequence-tens structures) come from the decade structure of the English number words. Some countries have number words with no separate list of decade words. For example, in East

Asian countries in which the number words are based on ancient Chinese number words, children say 12 as "ten two" and 53 as "five ten three." These children need to construct only the unitary and the separate-tens-and-ones conceptual structures. The construction of the latter triad is supported by the presence in children's lives of ten-structured cultural artifacts and experiences (e.g., the abacus, the metric system), by the teaching in schools of ten-structured methods for adding and subtracting single-digit and multidigit numbers (Fuson & Kwon, 1992a, 1992b; Fuson, Stigler, & Bartsch, 1988), and by the support of parents and teachers in demonstrating ten-structured quantities (Yang & Cobb, 1995). Cross-cultural work on East Asian children's numerical thinking indicates that they build highly effective ten-structured conceptions of numbers; use these very well in their addition and subtraction; and far outperform U.S. children on single-digit and multidigit addition, subtraction, and place-value tasks (Geary et al., 1996; Fuson & Kwon, 1992a, 1992b; Miller, 1990; Miller & Stigler, 1987; Miller & Zhu, 1991; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Song & Ginsburg, 1987; Stigler et al., 1990).

The simplicity of this inner triad suggested to us that it might be easier for English-speaking children to construct than the decade and the sequence-tens conceptions. Also, with the separate-tens-and-ones conception, children could participate in classroom activities involving ten-structured quantities before they learn the English (or Spanish) sequence to 100, which can take months or years (Fuson, Richards, & Briars, 1982). To facilitate the separate-tens-and-ones conception, we decided to use tens-and-ones words to describe quantities. It also seemed possible that the construction of the decade and of the sequence-tens conceptions might be facilitated by activities that also simultaneously supported the construction of the separate-tens-and-ones conception, such as arranging quantities in groups of tens, counting them by tens and ones (a sequence-tens conception), and counting the units of ten and the units of one separately (a separate-tens-and-ones conception). Using both tens-and-ones words and ordinary English or Spanish words would help focus children on each conception (i.e., on the separate-tens and on the sequence-tens conceptions, respectively).

Several design decisions stemmed from our experience with earlier studies of children using base-ten blocks to construct multidigit conceptions and then either learning traditional algorithms with understanding (Fuson, 1986; Fuson & Briars, 1990) or inventing calculation methods (Burghardt & Fuson, 1997; Fuson et al., 1992). First, we found that some children needed a long time (weeks or months) to construct the conceptual structures shown in Figure 2 (though the earlier studies concerned four-digit calculation and so were more complex). Second, doing problems requiring regrouping enabled children to use the ten-for-one relationship between adjacent numeral positions. For these two reasons, rather than try to ensure place-value understanding before moving to addition, we viewed addition and subtraction work as important settings for children's continued construction of place-value conceptions. Third, doing problems without regrouping does not use the ten-for-one quantity values and, in subtraction, sets up the common top-from-bottom error. We began, therefore, with two-digit problems requiring regrouping. Fourth, our experience indicated that children needed to explore two-digit addition and subtraction using ten-structured quantity referents, not just numerals, to construct robust ten-structured conceptions. Therefore we had to select and design such referents.

### *Instructional Design Issues and the Teaching-Learning Activities*

Our instructional-design work focused on the goals previously stated: build activities that would help children construct all the conceptual triads simultaneously and then move to two-digit addition and subtraction by using ten-structured quantity referents. This approach also had the pragmatic advantage of permitting a whole class of children to participate in activities even though they were using different conceptual structures.

We faced various pragmatic constraints concerning our choice of quantity referents to use in the classroom because we wanted them to be usable in any inner-city classroom. Therefore they had to be easy to manage, inexpensive, and require minimal teacher-preparation time. Although we did not plan to use the referents to teach standard algorithms, we felt that they had to be easily usable for these methods because some teachers might be under pressure to use them in this way, at least until reform efforts reached their schools or their communities.

Our Vygotskiian view of teaching also influenced the design of the teaching-learning activities. We were trying to help children build complex related webs of new knowledge. Therefore, for the initial triad activities, the teacher at first led simultaneous performance by all children together; each child did the activity at his or her seat, following or with the teacher. We began with activities designed to help children (a) see objects grouped into tens and (b) relate these ten-groupings and the leftover ones to number words and written numerals. For eight class sessions in the fall, children used a penny frame into which they fitted pennies in rows of ten; after they put a penny into the frame, they wrote below that penny the new total number of pennies they had placed in the frame. Other triad referents were tried out several times each for part of a class period during the fall: a hundreds matrix of numbers in rows of tens, number cards (invented by the second author) with tens as bars of ten connected squares and ones as a connected partial bar, pennies counted into groups to make nickels or dimes, and $1 bills counted into groups to make $10 bills.

In all these activities, the teacher used each of the triad conceptual structures successively in various orders. For example, pennies were placed individually into the penny frame while the teacher counted each one unitarily. As the activity progressed over days, an increasing number of children became able to count with the teacher. As each ten row was filled, the rows of ten would be counted by tens (e.g., 10, 20, 30) and then by ones (1, 2, 3 groups of ten). All the pennies might then be counted again by ones to verify that there were 30 pennies in the three groups of ten. Written numerals would be read as English words (in the Spanish-speaking class, as Spanish words) and as tens-and-ones words: "So 36 is thirty-six pennies, three groups of ten pennies and six loose pennies left over. We write *three* tens here on the left and *six* ones here on the right." Assessing was done by sometimes letting children respond first and by looking at the numbers that children wrote. Gradually in each of these activities, the whole class or individual children would take over more of the task. Finally, individuals would do the task alone with informal help from peers and the teacher.

Base-ten blocks and other object quantities leave no records after class to help teachers assess their students' understanding. We therefore introduced a system of recording quantities as ten-sticks and dots, which children could count by using any of the conceptions. Initially, children made dots in columns of 10 to make a record of objects the class was collecting. They counted by ones as they made these columns of 10 dots. When they had fewer than 10 left, they made a horizontal row of dots (often with a space between the first five dots and the last four dots to facilitate seeing how many dots there were). To check a quantity, children could then count all the dots by ones (unitary conception), count the columns by tens (sequence-tens conception), or count the columns as tens (separate-tens conception). These different ways to count were all modeled by the teacher and by individual children at the chalkboard. These ten-structured arrangements were potentially more accurate than the unitary drawings ordinarily made by children because the ten-columns gave feedback about the making and counting at each ten. This feedback at the end of each ten also helped a child begin to construct conceptions of counting groups of ten. When many children could make such drawings confidently, the columns of 10 dots were connected; children drew a line through them as the counting by tens or of tens was done. Some children had already spontaneously begun to do this. Eventually only the vertical stick was drawn to show a ten. These activities occupied part of the class period for about 2 weeks.

Most of the new teaching activities were initiated by two full-time members of the research team for several reasons. One was that we designed and adapted the teaching-learning activities throughout the year in many cycles of design-try-reflect-redesign; this required more meeting time than was available to the classroom teachers. Another is that we wanted the activities to be led initially by teachers who shared our theoretical views of teaching and of learning and with our understanding of the conceptual structures we were attempting to support. One research teacher worked in the English-speaking class, and the other research teacher worked in the Spanish-speaking class. As the regular classroom teacher became familiar with an activity, she would take over some or all of the teaching. The research-team teachers were in the classroom 5 days a week from late September through November and then on Mondays, Tuesdays, and Wednesdays for the remainder of the year. On Thursdays and Fridays the classroom teachers continued with their usual approaches to nonproject topics. During the fall, activities on single-digit addition and subtraction to ten and word problems occupied much of the time.

Activities varied somewhat between the English-speaking and Spanish-speaking classes for two reasons. First, the standardized tests differed for the two classes. Chicago Public Schools gave the Iowa Test of Basic Skills (ITBS) to English-speaking classes and La Prueba (Riverside Press) to Spanish-speaking classes. The Iowa test had two-digit subtraction with trading and no three-digit problems, and La Prueba had three-digit addition but no two-digit subtraction with trading. We felt pressure to help children prepare adequately for the tests. Therefore, we varied the nature of the activities after two-digit addition to match the tests. The second reason was that the Spanish-speaking research teacher used three new conceptual supports to assist children over conceptual hurdles she saw arising in class. Two of these facilitated the decade conception, and the other labeled the left-right tens-ones writing of two-digit numbers. First, she several times discussed an invisible zero hiding under the ones digit; a dotted 0 was drawn in the ones place to show a ten as 10 (or four tens as 40), and the number of ones was written on the dotted 0. Second, she briefly used number cards with bars of ten and the decade at the top (e.g., 60) and unit cards that fit over this 0. Third, she drew a particular support used in some South American mathematics textbooks: A long vertical line segment was crossed near the top by a horizontal segment, creating a space to label the columns as *dieces* and *unos* (or later, just D and U). Children frequently wrote this on their own papers.

After considerable discussion with the classroom teachers, the research team decided on "five tens and three ones" and "cinco dieces y tres unos" for the tens-and-ones words. In Spanish-speaking classrooms, the words *decenas* and *unidades* are typically used for the tens and ones positions. But these words are rarely used outside school, and so they have no meaning for children. We therefore used the meaningful words for tens and ones (*dieces* and *unos*). We did begin to use *decenas* and *unidades* before the standardized test.

We also explored the extent to which we could support children in learning for single-digit addition and subtraction the mental ten-structured methods used by children in East Asia. Various activities were tried beginning in January, including vertical-number-line or number-bar activities. None seemed particularly powerful or interesting to the children. In mid-February, rows of dots were made for each addend; 10 dots were enclosed, if possible, and the answer was then recorded as "one ten and $x$ ones." During the winter and spring for a few minutes on many days, children solved addition and subtraction problems with sums in the teens; they then demonstrated various finger methods they were using. Explicit practice activities focused on the prerequisites for the ten-structured methods (How many more to make ten with a given number and ten plus $x$ = ?). By midyear most children were counting on or using fingers in other ways that could lead to ten-structured methods (Fuson, Perry, & Ron, 1996), but most were not yet doing mental ten-structured methods, as most Korean children do by midyear (Fuson & Kwon, 1992a). The results of the finger work are reported in Fuson, Perry, and Ron (1997).

We decided to use the ten-sticks and dots for two-digit addition and subtraction for their advantages as written records and for cost and management reasons. Teachers used this activity periodically from late February through May. In the English-speaking class, children did triad review activities for 2 days and then had four classes on two-digit addition. Six classes on two-digit subtraction (in April and May) were followed by six sessions of mixed addition and subtraction problems with and without trading (in May). The time spent on these two-digit activities averaged 30 minutes for each class. The Spanish-speaking children spent about this same amount of time on triad activities, two-digit addition with trading, three-digit triad activities (drawing squares for hundreds), three-digit addition with no trading, and two-digit subtraction with no trading.

Addition and subtraction lessons typically began with instructional conversations in which the teacher elicited children's ideas about methods. Each method was then carried out by the whole class together. Children then worked alone, solving problems by any method. Conversations to facilitate children's reflections on, and comparisons of, methods were intermixed with periods during which children worked alone (or spontaneously together). As errors arose, the teacher also identified and discussed them. Children explained why these examples were errors and how to correct them. Showing errors was sometimes done in a playful "Can you catch me?" mode that the children greatly enjoyed. Sometimes several solve-then-discuss cycles occurred in one class period. Assessment was done by individually monitoring children during the work periods, by eliciting children's responses during discussion, and by looking at children's ten-sticks and dot records of problem solving. In the English-speaking class, we also tried one-item or two-item quizzes at the beginning of class to monitor the children's progress.

To facilitate meaning making by children, we used problem contexts involving packaging for addition and subtraction problems. For example, doughnuts packed in boxes of 10 were bought and sold, so a new box was packed (addition) or opened (subtraction) as necessary. Children could use either sequence-tens or separate-tens conceptions in adding or subtracting; they counted the "stick and dot" ("las barras y los puntos") quantities they drew by tens and ones (10, 20, 30, 40, 41, 42, 43) or as tens and as ones (1, 2, 3, 4 tens and 1, 2, 3 ones). Figure 3 shows examples of children's addition and subtraction drawings. Usually in addition dots were combined to make another ten when possible; in subtraction, a ten-stick was opened (its 10 dots were drawn within an ellipse or rectangle) when necessary. Many individual variations appeared in the drawings and in the links that children made to written numerals.

This work with multidigit addition and subtraction was an opportunity for children to continue to construct and to use in more complex activities their fledgling two-digit conceptual structures and to fill in pieces that might be missing or weakly understood. The two-digit activities were viewed as complex, multistep, goal-directed sequences requiring conceptual quantity understanding linked to the domain words and written numerals, that is, to the social-cultural semiotic domain tools.

At the end of the fourth addition class, seven children in the English-speaking class still had substantial misunderstandings of some aspect of tens and ones or of multiunit addition. We decided to explore the efficacy of individual tutoring sessions to see how much additional assistance would be necessary to bring all these children to some minimum level of understanding. Multiple individual tutoring sessions were carried out with these seven children; most participated in three to five sessions. This tutoring also covered multidigit subtraction for some children. Five other children received one or two sessions of tutoring for specific errors that arose later in their addition or subtraction. The main focus of tutoring sessions was to ascertain a given child's errors within his or her chosen method and then to support the child's correction of that error by an instructional conversation about the error. The tutoring yielded rapid progress in every case in which it could be used. However, some children needed reviews on tutored concepts, and some children were not in school regularly enough to receive sufficient tutoring. Details of errors children made and of tutoring methods are given in Fuson and Smith (1997).

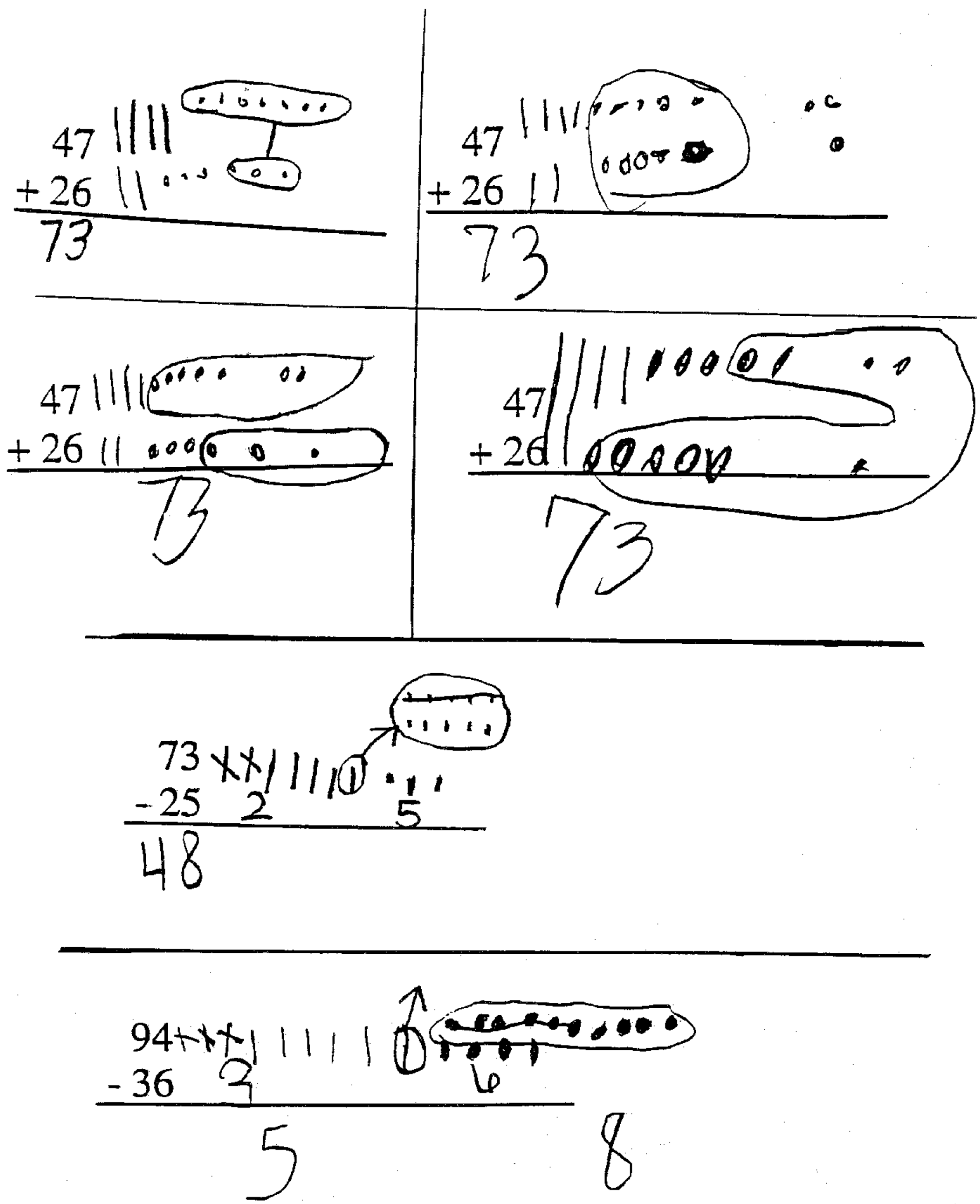


*Figure 3.* Addition and subtraction methods using ten-sticks and dots.

## METHOD

### *Participants*

The K–8 school in which the study took place is located in a predominantly Latino neighborhood. Most families have at least one wage earner, but wages are low. Consequently, 87% of the students qualify for free or reduced lunch. Each grade level from 1 through 5 has a Spanish-speaking and an English-speaking class. Spanish-speaking children are bused in from six other elementary schools to complete the Spanish-speaking classes. Mathematics classes and most other classes are carried out almost entirely in the specified language (English or Spanish). Many children in the English-speaking classes also speak Spanish, and some children in the Spanish-speaking classes also speak at least some English. Parents can choose which class their child attends, but the child must demonstrate sufficient competency in English to be placed in the English-speaking class. Roughly a quarter of the first graders had attended neither kindergarten nor nursery school before first grade.

All first graders who had entered the school by mid-December were included in the study sample. The Spanish-speaking first-grade class varied in size from a low of 17 children at the end of September (children continue to enter school over the first several weeks of school) to a high of 28 children. For this study, the sample was the 17 children who were in the class from mid-December through June. The English-speaking class varied in size from 24 to 28 children, with 33 different children present at some time. The sample for the end-of-the-year interviews was the 20 children who were in the class from mid-December through June and were available for end-of-year interviewing.

### *In-Class Place Value, Addition, and Subtraction Learning*

During the teaching experiment, various data were gathered concerning children's learning of place-value concepts, two-digit addition, and two-digit subtraction. Notes were made about various aspects of children's learning and about errors children made, including analyses of homework and classwork. Classwork was collected on some days. One-item quizzes were given in the English-speaking class. The regular classroom teachers reported issues they thought important. These data are summarized in the beginning of the results section.

### *End-of-the-Year Interviews*

Individual interviews were carried out in late May and early June with all sample children. Interview items were selected mainly from other published studies to obtain data on our children that could be compared with the data reported in these other studies. The interviews contained a large number of items. Therefore, some items were given to all children, and some were given only to a subsample of children drawn from across the whole achievement range of the class. If a child became tired during any interview, the interview was completed at another time.

*Triad tasks.* Triad questions examined which of the various relationships in Figure 2 had been learned by children for the ten-sticks and dots quantities. On different items, for a given ten-sticks and ones quantity, children were asked to say how many in English (or Spanish) words and in tens-and-ones words and to make written numerals for that many ten-sticks and dots. They were asked to make numerals and ten-sticks and dots for English (Spanish) number words and for tens-and-ones words and to make ten-sticks and dots for numerals.

The following cardinal-tens task, based on a task devised by Hiebert and Wearne (1992), was used to assess whether children could conceptualize groups of ten in a two-digit number: "There are 53 first graders at Esperanza School. How many teams of 10 children could you make with these 53 children? Why?"

An item from Miura et al.'s (1993) task for assessing cognitive representation of number (the child's conceptual structure for a two-digit number) was given. Children were shown a two-digit number (42) written on a card. They were asked to make that number from a pile of base-ten blocks (1 × 10 longs and unit blocks). Our children had not previously made two-digit numbers with base-ten blocks.

*Place-value understanding*. Four tasks were given to assess the children's understanding of the quantity referenced by the tens digit. The first was the Kamii task (Kamii, C., 1985, 1989; Kamii, M. 1982). Children are shown the number 16 written on a card. They are asked to make that many chips from a pile of chips. The interviewer then gestures to the 6 and asks, "What does this part mean? Show me with the chips what this part means." Children almost invariably count out 6 chips from their pile of 16 chips. The interviewer then gestures to the 1 and asks, "What about this part? Show me with the chips what this part means." Many children show 1 chip rather than indicate the remaining 10 chips (or any 10 chips).

The three other digit-reference tasks were taken from the assessment of place-value understanding of U.S., European, and East Asian children given by Miura et al. (1993). One was the Ross task (Ross, 1986, 1989), which adds a misleading perception to the Kamii task. A child is shown 13 objects and three cups and is asked to put 4 objects in each cup. Then the child is asked to count how many objects there are and is shown a card with the numeral 13. The child is asked to show with the objects the meaning of each digit. Here, the 1 object left outside a cup could be mistakenly associated with the 1 in the tens position, and the 3 could be taken to mean the three cups. A second task was a noncanonical display of tens and ones quantities; the number 42 was shown as 3 tens and 12 ones rather than as 4 tens and 2 ones. Miura et al. did this task with base-ten blocks, but we did it with ten-sticks and dots to increase comparability with our other triad tasks. Our children had never seen a ten-sticks-and-dots display like this. The third digit-reference task (a) asked children to show two-digit numbers in base-ten blocks and (b) asked which number was the tens and which was the ones. The former part of the task was assessed in two word problems given with base-ten blocks. The latter part of the task was assessed in the triad tasks previously described.

*Two-digit addition and subtraction with regrouping*. Standard vertical two-digit addition and subtraction numeral problems were given to all children. They were asked to solve these with ten-sticks and dots, except for the subtraction problem for the Spanish class. That class had not done subtraction requiring trading with ten-sticks and dots. These children were asked to use base-ten blocks with units marked on the tens to solve these problems.

Word problems with two-digit numbers were given with base-ten blocks to assess the children's abilities to generalize their ten-sticks-and-dots methods to a new tool. Children were given one addition problem ("Roberto had 27 coins in his bank. Then he put 36 more coins in his bank. How many coins does Roberto have in his bank now?") and one subtraction problem ("There were 74 children on the school bus; 38 children got off the bus. How many children were left on the bus?").

## RESULTS

### *Learning in the Classroom*

*Tens-and-ones words and English or Spanish words*. The research and the classroom teachers reported that children found the regular tens-and-ones words easier to learn than the standard English or Spanish number words. This was especially true for the least advanced children. Children did not seem to confuse the two kinds of words; no construction from one kind was carried into the other kind of words. The major difficulty in learning the number words was learning the list of tens to 100. In Spanish, *sesenta* and *setenta* were special sources of difficulty. In English the phonological similarity in some teens and decade words (e.g., *thirteen* and *thirty*, *eighteen* and *eighty*) sometimes led children to make errors during class activities and seemed to interfere with learning for some children.

*Two-digit-addition learning*. There was considerable variability in how children drew the ten-sticks and dots, enclosed 10 dots, and showed the answer (see the four methods at the top of Figure 3). Watching children solve a problem also revealed other differences in the children's methods. For finding the total, some children counted ones first, and others counted tens first. For finding how many tens, some children counted all, some counted on from the first number of tens, and some children used known facts to add the tens (e.g., "3 tens and 2 tens make 5 tens"). Some children integrated the new circled ten into a sequence count of the total, and others counted it as another ten in their count of the tens. Throughout the classroom-learning period, most children in the Spanish-speaking class demonstrated sequence-ten strategies. Children in the English-speaking class were more evenly split between sequence-ten and separate-ten strategies, but more demonstrated the latter.

In the English-speaking class, the numbers of children adding correctly on one-item tests given at the beginning of the class following each of the four days of teaching were 8, 13, 12, and 15 out of 24 to 26 children. On the final test, four of the nine errors were execution (miscounting) rather than conceptual errors. Of the more substantial errors, two children reversed the answer when writing it; three were more substantial conceptual errors. Two children still making substantial conceptual errors were not present on that day. The major error that disappeared over these 4 days was ignoring the extra new enclosed ten (not counting it in the total). Other types of errors remained relatively flat: About 5 or 6 children made minor execution errors each day, and 2 to 4 children made fundamental grouping and quantity-representation errors (but some children making such errors were absent each day). Only 5 children were perfect on all four tests; the children making execution errors varied from day to day.

Children did not spontaneously write a carry mark or write and then correct a number with too many ones (e.g., write 714 and change it to 84 after enclosing the dots). Both of these possibilities were discussed in class. By the end of the second session, most children preferred variations that annexed the enclosed 10 dots to a count of the tens or to a sequence-tens count; these children did not record a 1 anywhere. These preferred methods are counting precursors to the European method of mental addition in which the user adds from left to right and looks ahead to see if he or she will have another in a given column before writing the total for that column. Written 1s did occasionally appear in class later, as children learned them from parents and siblings.

Children in the Spanish-speaking class generated and discussed various methods of recording the extra ten in the numeral addition problem. The worksheet in Figure 4 shows several of these and also shows several different kinds of linking enabled by the drawn ten-sticks and dots. This worksheet is an early form that still showed all the ones in each ten. Children drew lines through each column of 10 ones to make 1 ten. The addends and the total have been labeled (with some misspellings) with the entities given in the word problem the children generated for that addition combination. The children always contextualized their problems. In the problem on the top left, the child drew an arrow to move the enclosed 10 dots over with the other tens; this reduces the chance that it will not be counted as a ten. In the problem on the bottom left, the child carried out a correct numeral method. The sum of the ones was written, followed by the tens in each addend (14 + 30 + 20 = 64). In the enclosed-dots part of that problem, the new ten is explicitly indicated by a 10. For the problem on the bottom right, the child wrote a sequence counting method that avoids making another ten by adding the top ones onto the total of the tens (80 + 9 = 89). The child then wrote the counting-on of the bottom ones from that total (90, 91, 92,…, 97). A numeral 1 was written above the tens column in the numeric problem to show this counting over a decade, and the sum 97 was written both below the sticks and dots and below the numeric problem. In the Spanish class, other written methods of recording the extra ten that arose were to write a 10 below the second addend or to write a small 1 (ten) on the line below the tens.

### *Two-Digit-Subtraction Learning in the English-Speaking Class*

Many children made rapid progress in subtracting correctly. All children opened a ten by drawing the 10 ones (see two examples at the bottom of Figure 3). Correctly taking away the tens was particularly easy: After only two sessions, 17 of 22 children correctly crossed out the correct number of ten-sticks on the quizzes at the beginning of class. Total correct responding rose more slowly over the first five quizzes: from 4 of 20 totally correct to 6 of 20, 10 of 21, 9 of 20, and 12 of 20. Four of the 8 children with errors on the last quiz made only minor execution errors.

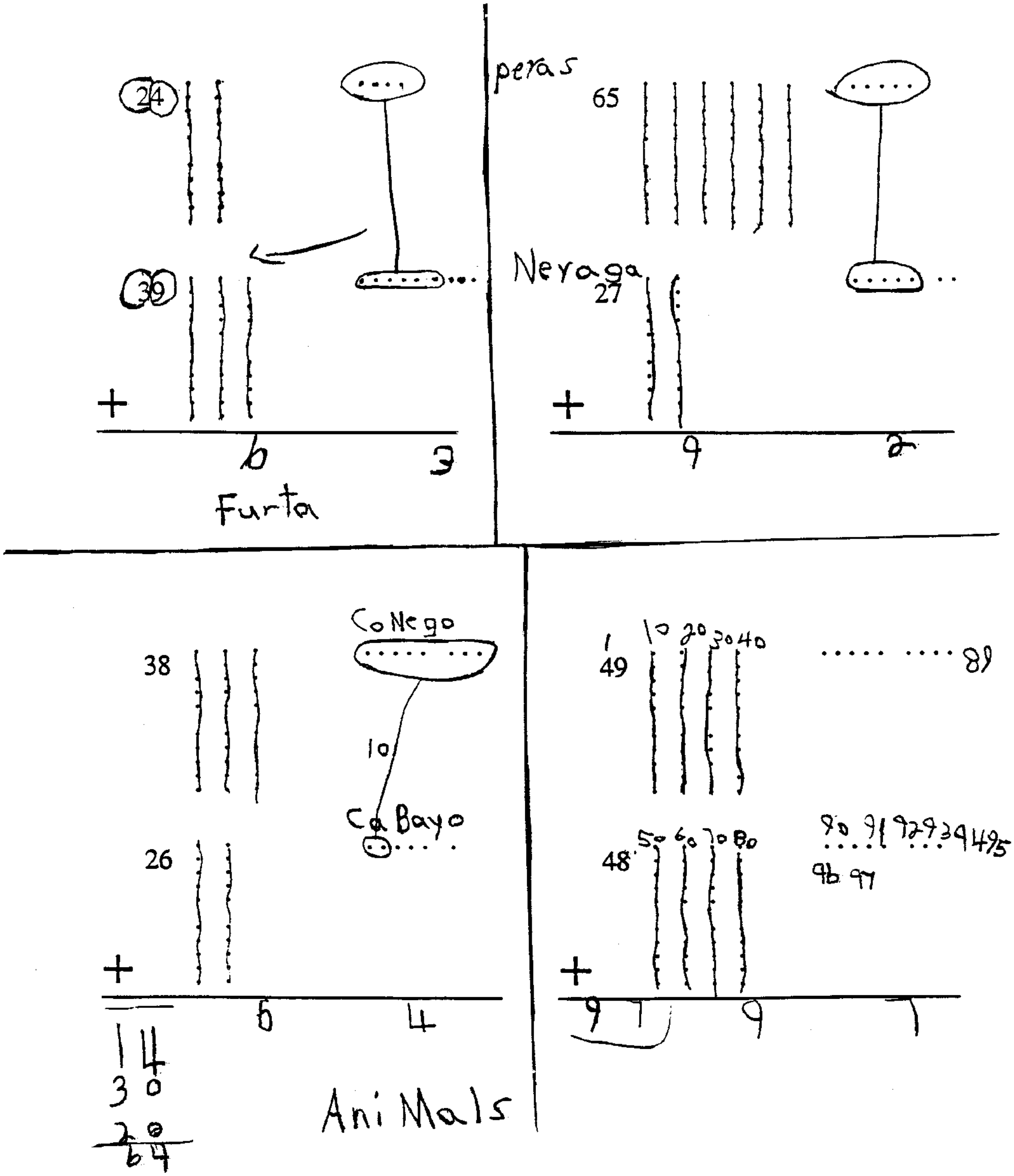


*Figure 4*. Different addition methods used early in the Spanish-speaking class.

The remaining 4 children were still making fundamental quantity-representation or regrouping errors; 2 other children absent that day were also still making fundamental errors. All these children were tutored children. Three of these children began to improve significantly after a few tutoring sessions. The remaining 3 children were absent so frequently during the tutoring period that they did not receive significant amounts of tutoring and did not improve on subtraction.

Performance on subtraction dropped somewhat when addition and subtraction were mixed. However, by the third class with mixed operations, many children were differentiating their addition and subtraction methods. On the last 4 days of mixed quizzes, from 14 to 16 children (out of the 18 to 23 children present) subtracted perfectly with ten-sticks and dots or had begun to do all or part of the subtraction mentally without making ten-sticks or dots (3 children).

Children made many more different kinds of errors in subtraction than in addition. Practically every error we could think of was made by some child at some time. In addition problems, most errors were in regrouping, and these were of only a few kinds. In contrast, the subtraction regrouping (opening a ten) seemed to be relatively easy to understand and to carry out with an overall correct approach. However, it was subject to many minor execution errors. Children almost always opened a ten and crossed out the correct number of ones. But miscounting occasionally occurred at each possible counting step, and children sometimes counted only part of the tens or part of the ones. The latter was quite common because the ones for the answer were in two separate places: the new, opened, encircled ones that had not been crossed out and the ones in the original number. Children sometimes reversed the tens and ones in the answer, occasionally indicating conceptual confusion. But the reversal more often occurred when a child wrote the ones first and then wrote the tens after (to the right of) the ones (following the usual order of writing). Methods varied across children. Some children took away tens first, and others took away ones first. Some children sequence-counted the answer and wrote the whole two-digit answer. Others counted and wrote the tens and ones separately, with some children writing each first.

### *End-of-the-Year Interview Results*

The studies from which the end-of-the-year interview tasks were taken used samples that included considerable numbers of middle-class children. On the basis of this class difference, our sample of predominantly poor (free lunch) children would be expected to do less well.

*Triad tasks.* All children in both classes could do the tasks assessing all six relations for the inner separate-tens-and-ones triad. For the unitary conception, one child in each class could not count to 100 by ones. These children, and an additional child from the English-speaking class, could not count to 100 by tens. For smaller two-digit numbers, these three children did do tasks demonstrating the relations in the unitary and sequence-tens triads. All other children did the sequence-tens-and-ones triad tasks correctly. Most children in the Spanish-speaking class did these directly by counting by tens. Many children in the English-speaking class demonstrated the separate-tens-and-ones triad relations and then translated this result to English number words (e.g., counted "1, 2, 3, 4 tens and 1, 2 ones so that's forty-two").

Many children were able to demonstrate triad relations in the cardinal ten-structured task ("There are 53 first graders at Esperanza School. How many teams of 10 can be made?"). Almost all the Spanish-speaking children (94%) and half the English-speaking children answered correctly. Almost all of these children (92%) knew the answer rapidly without counting or drawing. The second and third graders in the project school had been interviewed on this task in the spring before the project began. The Spanish-speaking first graders did considerably better than had these older children receiving traditional textbook instruction (94% versus 32% and 50%), and the English-speaking first graders did better than the second graders and as well as the third graders. The Spanish-speaking class did considerably better than the Hiebert and Wearne (1992) first graders receiving traditional instruction or the alternative instruction (story-based experiences with base-ten blocks used to build up tens-groupings and children's computation methods; 94% versus 50% and 63%, respectively), and the English-speaking class did about the same as these groups (J. Hiebert, personal communication, 8 February 1996). Our children also did considerably better than children using *Everyday Mathematics* (EM), a reform curriculum (Bell & Bell, 1990): the mean for our two classes was 72% compared with 34% for the overall EM sample containing a range of schools and 12% for EM Chicago schools (a sample more comparable to ours) (Drueck, Fuson, & Carroll, 1997).

On various tasks that assess whether children are thinking unitarily or with tens and ones, our first graders from both classes predominantly demonstrated tens-and-ones thinking. Their performance thus looked more like that of East Asian children than of U.S. children, who predominantly demonstrate unitary or concatenated single-digit conceptions (Miura et al., 1988). The East Asian children were tested in the first half of the year, whereas ours were tested at the end of the year, so ours are still behind East Asian children in the timing of when they use tens and ones. On Miura's task of the cognitive representation of numbers, 88% of our children made a ten-structured 42 using 4 tens blocks and 2 units blocks compared with a mean of 89% of children from the People's Republic of China, Japan, and Korea making a ten-structured display (a mean of 10% of these 89% children made a noncanonical-ten version that had some tens and more than 9 ones). When asked to make a different block presentation for 42, only 7% of our children made a unitary presentation; a mean of 53% of the East Asian first graders made a unitary presentation. Most of the 74% of our children making a correct second presentation made a noncanonical-ten arrangement in which some ones were arranged in groups of ten.

*Place-value understanding*. On the Kamii task, 63% of our first graders immediately said that the 1 in 16 was 10 chips. This is considerably higher than the 42% of M. Kamii's (1982) 9-year-olds who were correct and than the 32% of the second and third graders in our project school before the project began. Our 63% is about the same as the 60% of C. Kamii's (1985) affluent suburban sixth graders. Our 63% is also considerably higher than the 20% making such a response in the overall sample of children using EM and the 9% in the Chicago sample using that reform curriculum (Drueck, Fuson, & Carroll, 1997).

Children were given three place-value understanding tasks that assess children's quantity meanings for each digit in a two-digit number. All tasks and comparison samples are from Miura et al. (1993). Our first graders did as well as East Asian children tested in the first half of the year and much better than U.S. first graders from a selective academically rigorous school with monolingual middle- and upper-middle-class children (that sample and our children were tested at the end of the year).

First, on Ross's (1986) perceptually misleading digit-correspondence task, both of our classes (Spanish 80%, English 55%) did much better than Miura et al.'s (1993) children from the United States (21%). The performance of our English-speaking class (55%) was equivalent to that of Miura et al.'s children from Japan (46%), and our Spanish-speaking class (80%) performed considerably better than the Japanese children, although not as well as the Korean first graders (96%).

Second, all our children indicated ones' and tens' positions correctly on the triad tasks previously reported, and all children correctly represented numbers with base-ten blocks in solving a two-digit addition or subtraction word problem. In contrast, on the item combining these tasks in the Miura et al. (1993) study, only 33% of their U.S. children were correct. The East Asian children averaged 84% correct on this task.

Third, the noncanonical task was given to our children using their familiar ten-sticks and dots rather than the base-ten blocks used by Miura et al. (1993). Our children had not seen items like this before. Therefore, it is a test of their abilities to use their conceptions of the ten-sticks in a new situation, but it is not strictly comparable to the data from the other groups. When asked to write how much 3 ten-sticks and 12 dots were, 82% of our Spanish-speaking children and 55% of our English-speaking children correctly said 42. This compares with 25% of Miura et al.'s U.S. children and 52% of their East Asian children answering correctly with base-ten blocks. More (by a ratio of 3 to 1) of our Spanish-speaking class demonstrated a sequence-tens-and-ones conception than a separate-tens-and-ones conception (i.e., they counted by tens [10, 20, 30, 31,…, 42] rather than counting the tens [1, 2, 3 tens and 1, 2,…, 12 ones is thirty and twelve is 42]), and more (by a ratio of 6 to 1) of our English-speaking class did the reverse.

*Two-digit addition and subtraction with regrouping*. On the two-digit addition problem (48 + 36) using ten-sticks and dots, 90% of our children's solutions were correct. The children who solved the problem correctly and one additional child explained that they were making another ten and indicated their enclosed dots as doing this. Most of the Spanish-speaking class found the total by sequence-tens-and-ones counting rather than by counting the tens (a ratio of 11 to 1), and more of the English-speaking class found the total by separate tens-and-ones-counting than by sequence counting (a ratio of 4 to 1). Our 90% correct compares quite favorably with the percentage of first graders who become able to solve such problems without explicit classroom opportunities to do so: 35% for conceptually instructed children and 25% for traditionally instructed children (Hiebert & Wearne, 1992).

On the two-digit subtraction problem using ten-sticks and dots, all the children in the English-speaking class correctly opened a ten-stick by drawing 10 enclosed dots and correctly took away the required ten-sticks and dots. Of these, 70% then wrote the correct answer. The errors were typical of those made during class and discussed earlier; none of them reflected the concatenated single-digit conception of numbers resulting in the typical errors made by U.S. children. Subtraction problems with trading are not usually taught in first grade, and they seem much more difficult for children to solve if they do not have an opportunity to do so in the classroom. For example, in the Hiebert and Wearne (1992) study, only 6% of the conceptually instructed first graders could solve such problems, and none of the traditionally instructed children did so.

The Spanish-speaking first graders had not had opportunities in class to solve subtraction problems requiring trading. On a numeral problem given with base-ten blocks that had units marked on the ten-bars, all children made the number as tens and ones and took away the tens. All but one child took away ones from a ten-bar, and 88% of the answers were correct.

The word-problem tasks with base-ten blocks were given to assess whether the ten-structured conceptions built by our children would generalize to these unfamiliar tools. On the addition problem, 100% of the Spanish-speaking children were correct, and 90% added by sequence-tens counting the total. Of the English-speaking children, 75% were correct; 84% of these traded ten units to make another ten bar, and most of these counted the tens and ones separately.

On the subtraction problem, all children showed correct digit correspondence by making 74 with 7 ten-bars and 4 units and trying to take away 3 ten-bars and 8 units. All but two children had a correct strategy for solving this problem and did see the ten-bars both as 1 ten (when making the 74) and as 10 ones (when taking away some or all of the 8 ones from it). Half carried out their strategy correctly, and the rest made some error in executing their strategy.

We are not aware of comparable data to assess the relative competence of our children on these word-problem tasks. The two data sets we know of gave first graders only paper and pencil and no concrete materials. Stigler, Lee, and Stevenson (1990) reported that Japanese, Taiwanese, and U.S. first graders were 29%, 25%, and 13% correct, respectively, on a word problem for adding 26 + 19. Hiebert and Wearne (1992) reported that 35% of their conceptually instructed first graders and 25% of the traditionally instructed first graders were correct on an addition word problem with trades. With base-ten blocks, our proportion correct was 89% on a word problem of that type. This is similar to the 90% correct on the numerical two-digit addition problem for our children using ten-sticks and dots drawings. Our children could have used ten-sticks and dots in the Stigler et al. task, so these drawings are a powerful tool that puts more difficult problems within the reach of first graders.

## DISCUSSION

It did prove to be possible to support most children's construction of most elements of the conceptual structures in Figure 2 using ten-stick and dot quantity drawings. Furthermore, on a range of unfamiliar tasks, many children showed a robust preference for ten-structured conceptions, performing like children in China, Japan, and Korea rather than like age-mates in the United States or like children in higher grades in the United States. Most children were also able to carry out a ten-structured solution to two-digit addition and subtraction problems and to explain their regrouping. This is considerably above what first graders in the United States ordinarily have an opportunity to learn because such problems with trades are usually not included in first-grade textbooks or are in the final chapter, which many teachers do not reach (Fuson, 1992; Fuson et al., 1988). Performance was considerably above that reported for U.S. children receiving traditional and reform instruction and was above that reported for Japanese and Taiwanese first graders on some tasks. This superiority is partly because the children's conceptual tool, the ten-sticks and dots, could be drawn on paper and thus could be used on homework or in an assessment whenever pencil and paper were available.

Some reform approaches to primary mathematics have been successful in enabling many children to invent accurate and understood methods of two-digit addition and subtraction. But some children in these projects continue to use unitary methods into second, third, and even fourth grade (Cognitively Guided Instruction: Steinberg, Carpenter, & Fennema, 1994; *Everyday Mathematics*: Drueck, 1996; Purdue Problem Centered Project: Cobb, 1995; Lo, Wheatley, & Smith, 1994). Our teacher-orchestrated activities to help children construct sequence-tens and separate-tens conceptions and then to use one or the other in two-digit addition or subtraction were quite successful: No first grader used a unitary method. This suggests that it might be very helpful for teachers to carry out such activities with whatever quantity conceptual supports are used in their classrooms or to add ten-sticks and dots activities as recordings of any quantities that are used.

The children in the Spanish-speaking class showed in several tasks a preference for counting by sequence-tens rather than counting by separate-tens. This preference facilitated their solutions with the unfamiliar media of noncanonical sticks and dots and of the base-ten blocks because they did not have to make another ten explicitly: counting by tens and then counting the ones took them up over the next decade to get the answer. In contrast, more children in the English-speaking class demonstrated separate-tens-and-ones conceptions in which they explicitly had to make another ten by grouping or by adding or they had to break a ten. In some new situations, fewer of them were able to do this accurately.

Although the sequence-tens-and-ones conception affords the easier solutions not requiring making another ten, it is not clear that this conception is more powerful or that it will remain more powerful for three-digit and four-digit addition and subtraction. Sequence counting becomes more difficult and burdensome with more places, whereas separate-tens-and-ones solution methods generalize easily to more places and carry little additional memory load while the child is carrying out such a solution (see Fuson, 1990, for a review of this evidence). Furthermore, some of the difficulties of the English-speaking children might have been eased if more focus had been placed on these issues in class. Some children were still struggling with the left and right locations of the tens and the ones. This was complicated by the fact that some counted and wrote the ones first and then tended to write the tens on the right, following the left-to-right reading and writing order. Use of the T-frame labeled with tens and ones, as in the Spanish-speaking class, or use of the decade and ones supports from that class (e.g., showing the "invisible" zero under the ones digit) might have helped these children. Giving experiences with noncanonical tens and ones quantities (i.e., > 9 ones) in ten-sticks and dots or other forms would also be helpful. Such experiences would enable children to understand that their interpretations of their written 312 as 3 tens and 12 ones is not necessarily shared by everyone; children could then decide how such quantities need to be written so that anyone could interpret them accurately. Fuson et al. (1992) reported such an extended discussion by second graders using base-ten blocks. This kind of understanding may have been facilitated for the Spanish-speaking class by their work with three-digit numbers.

The difference between the two classes in preferred conceptions illustrates how instructional emphases in the uses of a conceptual tool and the uses of different tools can support different conceptual constructions. The research teacher for the English-speaking class was focusing on the initial development of activities to support separate-tens-and-ones conceptions, and consequently the activities to support sequence-tens learning fell into the background. The research teacher for the Spanish-speaking class used the separate tens-and-ones activities but also initiated many sequence-tens activities. To help children construct all the conceptions, teachers must create a balance of supports and of activities that stimulate each conception.

It is possible that the Spanish words for 16 through 19 facilitate, at least somewhat, the Spanish-speaking children's comprehension of these quantities as "a ten and some ones." We have no data directly concerning this issue except for one incident that occurred in the final interview with a Spanish-speaking child. This incident indicates both that the words themselves *are not sufficient* for this understanding (they are only *potentially* meaningful referrers) and that they *can facilitate* this view (they *are* potentially *meaningful* referrers). On the Kamii task, one child who had said that the 1 in 16 meant one chip, suddenly, at the end of the interview, brightened with the classic lightbulb facial expression and said, "Diez y seis [the sequence word for 16] es un diez y seis unos." She demonstrated this with the chips by showing 10 chips and then 6 chips. Thus, she restated the meaning of the sequence word *diez y seis* (ten and six) as the tens-and-ones words used in class, *un diez y seis unos* (one ten and six ones), to express the quantity of the written numeral 16.

We have no definitive data on the conceptual effects of the use of tens-and-ones words because it is difficult to tease apart the various ten-structured supports we used. The tens-and-ones words were easier to learn than the standard English and Spanish number words to 100; therefore, children could participate in classroom activities while still learning the standard number words. This seemed important in the classroom. The tens-and-ones words also facilitated clear classroom discourse about the quantitative meaning of the written numerals, as in the foregoing interview incident. They may also have facilitated turning the children's attention toward quantities as tens and ones and therefore facilitated children's construction of ten-structured quantities. The tens-and-ones words were crucial in explaining multidigit solution methods, and they served as a powerful semantic critic that often enabled children to correct an error (e.g., "Are those tens?"). Because the tens-and-ones words were less familiar than the English and Spanish words, the research and classroom teachers sometimes forgot to use tens-and-ones words in new contexts. However, the goal to use them did serve many times to remind these teachers to facilitate explicit discourse about tens and ones.

We are not claiming that most children had complete, flexible, and generalizable understanding and use of all the relations in Figure 2. Various limitations and difficulties with particular relations were exhibited on the tasks. In particular, children could demonstrate a given relation while not necessarily being able to use, or knowing to use, that relation in a more complex task or in an unfamiliar situation. The ten-sticks-and-dots representation did allow children to carry out various sequence-tens or separate-tens count-all or count-on methods for addition and take-away-and-count-the-rest ten-structured methods for subtraction. For many, the construction of numeral and mental two-digit methods remained a major task for second grade. But the children did have a robust start, and they were advanced for their grade.

Several kinds of data indicate that most children were treating the ten-sticks and dots as quantities of tens and ones to be put together or taken away: They used different methods. Their drawings of quantities with ten-sticks and dots looked different. They enclosed or opened quantities to make a ten differently, and they made recordings to denote making another ten in different ways. All children explained what their enclosing in addition or opening in subtraction meant. They were not learning a traditional algorithm with numerals disconnected from quantities or rote methods of using the drawn quantities. The ten-sticks and dots were conceptual tools children used for solving problems. They were flexible tools that could be used in different ways by different children.

Teaching as assisting children to construct personal meanings for the conceptual tools used in the classroom included orchestrating whole-class conceptual and practice activities; leading reflective activities for individuals, small groups, and the whole class; providing direct assistance to individual children; and organizing additional help by peers and adults within and outside class for those who needed it. In the multidigit domain, the conceptual web is very complex. Children begin with different initial knowledge and proceed with different learning rates and different opportunities to learn (because of variations in school attendance and learning support at home). Therefore, we expected many children to make partial errors or omissions that with feedback and support for the necessary learning would gradually come to be corrected as the children had opportunities to construct more adequate understandings. This is what we observed in the classroom and in the tutoring of children in the English-speaking class. Hiebert and Wearne (1996) also reported finding many different individual learning paths in place value and multidigit addition and subtraction understanding rather than a simple linear progression through which a teacher might try to help all children.

We want to emphasize how complex this teaching-learning task is. In Vygotskiian terms (1934/1962, 1978, 1934/1986), the learning zones (zones of proximal development) and individual constructive paths of children in our classes varied so much that it is difficult for a teacher to meet all the needs for assistance at the same time.

It is impossible to do it alone or all in the same way. In our view, the teacher must organize the required assistance from classmates, from the family at home, and from available adults in the school (teacher's aides, parent volunteers) as well as from the teacher. In the simpler domain of single-digit addition and subtraction, children enter school with more knowledge, and the constructive tasks are much simpler and require less new social-cultural semiotic understanding. The need for teacher assistance is consequently much less. However, even in the single-digit domain, some children still require assistance initially (if they enter lacking requisite knowledge, as did some of our children), and some require assistance to move on to more advanced methods. More learning assistance is required as the mathematical domain increases in complexity; as more social-cultural semiotic knowledge is required; and as features of this knowledge suggest incorrect meanings, such as two-digit numerals for the decade conception (e.g., the common fifty-three as 503 error).

It is important for such assistance to begin with finding out where the child is and then to help the child with his or her method. Yackel (1995) pointed out confusions that can result when a teacher attempts to impose a specific method on a child, and Cobb (1995) illustrated some similar difficulties when peers focus heavily on their own methods. We have found that the peer assistance that naturally exists in classrooms frequently consists initially of telling the answer or "doing-for," using the "helping" child's own method. However, if the teacher emphasizes that helping means to help another do it his or her way and monitors and supports effective helping, at least some first graders can learn to assist peers quite effectively (e.g., Fuson & Smith, 1995; Fuson & Smith, 1997).

The power of the individual tutoring sessions highlights several important points. At least a third of the English-speaking class seemed unable, in the whole-class situation with all of its distracting perceptual pulls, to concentrate in a sufficiently sustained manner to make important connections and to compose various pieces of knowledge into coherent and accurate action plans. Individual tutoring sessions with an adult readily revealed any missing or weak elements. And, somewhat surprising to us, these could be learned fairly quickly and used consistently within the complex whole in which they were embedded if the teacher just supported attention to these elements and led the construction of the required relations. Children varied in the elements they were missing, so this tutoring needed to be individual for assessment reasons as well as for attentional and affective reasons. Our experience suggests that it is the attentional demands for making requisite relations among multiple aspects (words, written numerals, and quantities) and for switching between two kinds of entities (tens and ones) within these aspects that put these mathematical topics out of the reach of some to many first graders in the busy perceptual environment of the classroom. However, these concepts are accessible in one-to-one tutoring sessions. For the lower third of the class, a stronger emphasis in the classroom on the inner triad of tens and ones, in which the patterns are simplest and few irregularities exist, might serve as a centering force and allow children to build up the other triads, with all their irregularities, linked to this central core.

An alternative approach is to work heavily on improving the attentional capabilities of children. The classroom teacher of the Spanish-speaking class did so in various ways (Lo Cicero & Cora, 1996), and more children consistently attended well to discussions in that classroom than in the English-speaking classroom. Assistance in the Spanish-speaking classroom was also frequently given at the chalkboard by other children in a general classroom culture of helping.

The results reported here clearly indicate that all U.S. children can do enormously better than they ordinarily do in primary school mathematics. Furthermore, the widely reported gap in performance and understanding between East Asian children and children in the United States can be narrowed or eliminated, even in poor inner-city schools. Doing so requires a substantially more ambitious first-grade curriculum and active teaching that supports the children's construction of a web of multiunit conceptions in which number words and written number marks (numerals) are related to ten-structured quantities. Drawn quantities, instead of objects, can serve as meaningful ten-structured quantities that support reflection, communication, assistance, and teacher assessment of children's thinking.

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## Authors

**Karen C. Fuson**, School of Education and Social Policy, Northwestern University, 2115 N. Campus Drive, Evanston, IL 60208-2610; fuson@nwu.edu

**Steven T. Smith**, Northwestern University, School of Education and Social Policy, Northwestern University, 2115 N. Campus Drive, Evanston, IL 60208-2610

**Ana Maria Lo Cicero**, Northwestern University, School of Education and Social Policy, Northwestern University, 2115 N. Campus Drive, Evanston, IL 60208-2610