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Supporting multiple 2-digit conceptual structures and calculation methods in the classroom: issues of conceptual supports, instructional design, and language

Karen C. Fuson and Steven T. Smith
School of Education and Social Policy, Northwestern Univerrsity

1 Introduction

In this chapter we will present theoretical descriptions of children's conceptual structures for 2-digit numbers and examine issues concerning how to support children in learning and using these conceptual structures in 2-digit addition and subtraction. We first briefly overview some aspects of our teaching approach in our current *Children's Math Worlds* project. We then summarize the UDSSI Triad Model of five conceptual structures for 2-digit numbers used by children who speak European languages. Next we describe an initial portion of our local instructional theory for helping children construct the three most advanced conceptual structures. Then we describe and discuss methods of 2-digit addition and subtraction and how these relate to problem situational structures and to 2-digit conceptual structures. We discuss classes of conceptual supports for 2-digit numbers and calculation, relationships between solution methods and conceptual supports, and six issues concerning implementing vertical mathematization and reflection in the classroom. We then briefly consider issues surrounding mental calculation.

We will use throughout the chapter the term 'method' rather than 'strategy' or 'procedure' for the way in which a child solves a 2-digit problem. 'Procedure' has for some readers a negative connotation of a rote method done without understanding. 'Strategy' implies some level of thoughtfulness and a choice of a method which may not be present for a given solution. We therefore prefer to use 'method' as a neutral term between these two extremes, and append adjectives if necessary for further definition.

Overview of Children's Math Worlds project

Children's Math Worlds is a project that is developing a mathematics curriculum by working initially and primarily in both English-speaking and Spanish-speaking urban Latino classrooms. We work simultaneously to develop understandings and models of children's conceptions of single-digit addition and subtraction, multidigit addition and subtraction, and word problem solving and to design effective teaching/learning activities that are based on children's understandings and are implementable in urban classrooms. The focus here is on our work in the multidigit domain; see Fuson, Hudson, and Ron (1996) for a summary of word problem work and Fuson, Perry, and Ron (1996) for an overview of the single-digit work.

Typically in the United States children are taught multidigit addition and subtraction without sufficient use of physical materials that help children construct concepts of multidigit numbers as consisting of groups of hundreds, tens, and ones. Instead many children view multidigit numbers as single digits placed beside each other (concatenated single digits); this view leads children to make many errors, especially in subtraction where they typically solve 72 – 28 as 56. In the *Children's Math Worlds* Project, we use various kinds of materials to help children construct conceptual understandings of numbers that they can use in computation. Because of large numbers of children who enter with little background in urban schools, and because of the long time it takes many children to construct robust conceptual multidigit structures, we focus heavily on materials that can help all children build methods that are generalizable to several digits. However, we also from the beginning of the multidigit work emphasize children's invention of mental (and sometimes also finger) methods for solving various problems and continue this focus on invention and exploration of different methods. Typically in our classrooms the top children invent a range of methods, middle children use quantities and then move to a written numerical method (often the traditional algorithms) that they can explain and understand, and lower children struggle to carry out correct methods using ten-structured quantities. With help, most of the lower children can come to general numeric methods they can explain, but some of them need to continue to use drawn quantities for long periods of time.

Nanci's Method

$49 + 25 = 6$

Four tens and two tens (writes 6).
(Looks at the ones; erases the 6.)
I can make another ten, and then you count the ones (fingers count 5 on to 9), writes 74.

Later she invents a way to record the new ten:

$$49 + \overset{3}{\cancel{2}}5 = 74$$

Cinthia's Method

$25 + 47$

I took three from the five and put it with the seven. Then I counted two plus four is six. Then there is another ten, so seven tens, and there are two left, seventy-two.

Later she invents

$$48 + 2\overset{5}{\cancel{7}} = 75$$

Viviana's Method

$48 + 23$

Forty and two tens makes sixty.
Eight in my mind. 68, 9, 10, 11, 71.

Martha and Rufina's Methods

$$\overset{1}{\underline{3}}7 + \underline{2}6 = 63$$

$$48 + 16 = 64$$

$$\begin{array}{r} 1 \\ 25 \\ +\ 49 \\ \hline \end{array}$$

Jorge's Method

$56 + 27 =$

I know these are tens.
50, 60, 70. Then I counted 7 (7 fingers up): 71, 72, 73, 74, 75, 76, 77. Then I counted 6 more (6 fingers up): 78, 79, 80, 81, 82, 83.

Karina's Method

$37 + 56$

Eight. Eighty seven
(counts on fingers, 6 fingers)
8, 9, 10, 90, 93

Methods of Marking Tens and Ones

TO TO
$34 + 19 =$ __ __

$4|8 + 1|6 =$ __ __

$\underline{4}7 + \underline{2}8 =$ __ __

$25 + 47 =$ __ __

figure 1: mental and written numeric 2-digit addition methods

To overview the early phase in multidigit addition, Figure 1 displays a few of the methods first graders near the end of the year invented when given horizontally presented tens and ones and challenged to see if they could find ways to solve them without drawing all the tens and ones. Notice that all the methods involve shifts between external and internal compositions of tens and ones. All at the very least involve written numbers serving as external memories that enable children to point out and focus on parts and then later return to other parts without 'forgetting' them, as they might for orally presented problems. Jorge (56 + 27) points to the 5 and the 2, emphasizes that they are tens, can and does count on tens from 50 keeping track internally (50, 60, 70), then uses 7 external finger counters to count on ones and then 6 more fingers to count on the rest of the ones in the problem. Nanci (49 + 25) composes 4 tens and 2 tens as 6 tens (internal fact), writes it (external token), looks ahead at the 9 + 5 ones, erases the 6 and says 'I can make another ten', increments the six tens to seven tens internally and then writes the 7, then uses fingers (external ones tokens) to count 5 onto a mental 9 ones. She also knows to avoid using the ten again when counting on 9 + 5 (because she already incremented her tens before finding out exactly how many ones there were). Cinthia (25 + 47) seems to make visual use of the written numbers, allowing her to take '3 from the 5 ones and *put* it with the 7 to make 'another ten', and still retain that there are '2 left' (from the 5). Knowing that 3 was needed to make ten, and being able to take 3 from 5 and know what is left, appear to happen on the fact level. Karina (37 + 56) also makes use of breaking the sum of ones into ten and ones left, but does quite a range of other methods as well (not shown in Figure 1). Viviana, except for use of the written number problem (48 + 23), seems to count tens and ones entirely internally: 'Forty and 2 tens makes sixty (note mixture of sequence tens and separate tens), 8 in my mind. 68 (mentally adjoining 8 ones), 9, 10, 11, (converting 11 into ten increment from sixty to seventy and one more) 71.'

Not only can different children work at different levels in the same classroom using external quantities, external tokens of them (fingers, written numbers, drawn tens and ones), and internal versions of quantities and words, but even advanced children find it helpful to weave methods across internal and external countable tens and ones, possibly to distribute processing burdens. The variation, even beyond affording accommodations to individual needs, seems to be helpful in stimulating thinking. Some children in this first-grade classroom preferred to continue working exclusively with quantities. Although we expose children to the challenge of composing tens and ones by media other than external tens and ones, we also let children do whatever they need to do to solve problems.

A major issue at this stage for many children was differentiating and remembering which of the numbers were tens and which were ones. Because we gave problems horizontally to force children to attend to this differentiation, children invented varied and elaborate scaffoldings to mark which were tens and ones. They underlined tens, drew loops and lines to connect the tens, drew separating lines between tens and ones, and labeled tens and ones. Rapid correct tens and ones interpretations are crucial to any internalization of multi-step 2-digit operations. Children have to chunk partial results and so have to know what to chunk with what. If they have to spend much attention on what goes with what, they can easily overload memory, lose track of what they are doing, and forget the numbers involved in the situation or their already obtained partial results.

3 Analysis of the mathematical domain

3.1 A model of conceptual structures used in the domain

Earlier literature identified three correct conceptions used by children in the United States: a unitary conception in which children count a 2-digit quantity by ones, a sequence conception in which they count by tens and then by ones, and a separate tens and ones conception in which the units of ten and the units of one are counted separately (see Fuson, 1990a, for a review of this literature). For example, if counting 3 bars each made from 10 unifix cubes and 2 extra cubes, children using a unitary conception would count all 32 of the unifix cubes (1, 2, 3, ..., 32), children using a sequence-tens conception would count '10, 20, 30, 31, 32,' and children using a separate-tens conception would count '1, 2, 3 tens and 1, 2 ones. 32.'

Children also use a concatenated single-digit conception in which the 2-digit number is thought of as two separate single-digit numbers. Because any single-digit number can be added to or subtracted from any other, this meaning cannot direct or constrain addition or subtraction methods. It leads to many well-documented errors (e.g. see VanLehn, 1986, for a discussion and examples). This concatenated single-digit meaning arises when insufficient opportunities are given to children to link accurate multi-digit quantity meanings to the written numerals in use in adding and subtracting.

In Fuson et al. (in press) and in Fuson, Smith, and Lo Cicero (in press), we extended this earlier work to a UDSSI Triad Model named for the five correct conceptions described in the model: unitary, decade, sequence-tens, separate-tens, and integrated conceptions. The UDSSI Triad Model is shown in the main part of Figure 2 (taken from Fuson, Smith and Lo Cicero, in press). Our view of these conceptions is that they involve a triad of relationships between quantities, number words, and written number marks. With single-digit numbers, there are three 2-way links in the triangle formed by these quantities, words, and marks (see the top left corner of Figure 2). Each 1-way link describes the numerical aspect initially seen or heard and the aspect that is linked to it; for example, I hear 'five' and think/see/can write 5 (bottom left-to-right arrow) or I see five birds and think/can say 'five' (left arrow from top to bottom). The user of the concatenated single-digit conception (top left) constructs these six relations for each of the pairs of single digits in a 2-digit number.

Children's early conceptual structures are shown on the outside of the big triangle in Figure 2. All children begin with a unitary conception that is a simple extension from the unitary triad for single-digit numbers. With this conception, the separate number words (e.g. twenty six) and the two digits (e.g. 26) do not have separate quantity referents. The whole number word (e.g. sixteen) or whole numeral (16) refers to the whole quantity. With time and experience, the first digit takes on a meaning as a decade in the decade and ones conception, and the second digit takes on a meaning as the extra ones in a decade. The number marks for this decade conception can be better understood if one thinks of the ones as written on top of the decade quantity (the arrow in Figure 2 shows the 3 going on top of the 0 in 50). This conception of a 2-digit quantity as a decade and some ones was identified by Murray and Olivier (1989). It leads some children to write number marks as they sound: as 50 and then a 3, so 503.

The sequence-tens and ones conception develops out of the decade conception as children become able to count by tens and to form conceptual units that are groups of ten single units (these may arise independently). Initially with the sequence-tens conception, there is no immediate knowing that there are five tens in fifty, though a user of this conception could find out by counting '10, 20, 30, 40, 50' while keeping track of the five counts.

Some children have experiences in which they come to think of a 2-digit quantity as composed of two kinds of units: units of ten and units of one. When adding or subtracting 2-digit numbers in this way of thinking, children count, add, or subtract the units of ten and then count, add, or subtract the units of one (or vice versa), leading to our designation of this way of thinking as the separate-tens and ones conception. In Figure 2, we show these units of ten as a single line to stress their (ten)-unitness, but the user of these units understands that each ten is composed of ten ones, and can switch to thinking of ten ones if that becomes useful.

Children's construction of the sequence-tens and separate-tens conceptions seems to depend heavily on their learning environment, though individuals in the same classroom may construct one or the other of these first. Which is first may partly depend on whether a child focuses on the words, which facilitate the sequence-tens conception, or on the written numerals, which facilitate the separate-tens conception.

Children may eventually construct both the sequence-tens and separate-tens conceptions and relate them to each other in an integrated sequence-separate conception (these connections are shown in Figure 2 as the double arrows). Children connect fifty and five tens, and the written marks 53 can take on either quantity meaning (fifty-three or five tens three ones).

Although we had originally conceptualized each of the 2-digit conceptual structures as a triad of six relations, it later became clear that only the separate-tens and ones conception has direct links between quantities and marks, and then only where the quantities of tens and ones are small enough to be subitized (immediately seen as a certain number of units) or are in a pattern. The other three conceptions must relate quantities to written marks via the number words by counting. Therefore the link between quantities and marks is not drawn in Figure 2 for these conceptions.

3.2 Learning to construct and operate on 2-digit quantities in Children's Math Worlds

There is a widespread cultural activity in which young children invest considerable efforts on their own, before entering school, that can serve as the point of departure for activities helping children to appropriate the grouping and counting processes at the core of the base-ten number system and multi unit arithmetic: Children try to extend their counting sequences. They initially try to do so by directly extending the chaining process by which they have memorized the first ten number names, and then they make use of what regularities are accessible to them. The number names give a pattern of x-ty 1 through x-ty 9 chunks (e.g. 21 to 29), then a shift to some new 'x-ty' word, to which a further x-ty 1 through x-ty 9 chunk can be appended. However, the English number names do not clearly signal the order of the decade words. 'Forty' is not obvious as a verbal abbreviation of 4 tens, and the rest of the first 5 decade names are either irregular (the teens) or offer even more obscure references ('twen' for 2, 'thir' for 3, 'fif' for 5). Children are reduced to attempting to memorize which 'ty' word comes next, amidst the interference of intervening x-ty 1 through x-ty 9 cycles. The errors children typically make reflect precisely these chunks, but with a confused decade list, e.g. 1 to 29, 50, 51 to 59, 30, 31 to 39, 20, 21 to 29, 40, etc. (Fuson, Richards and Briars, 1982). Cross-sectional data indicate that it takes children in the United States on the average about one and a half years to learn how the decades themselves are ordered (Fuson, Richards and Briars, 1982). Further, this learning is frequently interactively constructed as if it were a simple memorization task. The child counts one through twenty-niiiiiiiine, and pauses, searching for the next 'ty' word. The adult or child audience either immediately supplies the correct word to fill the pause or waits for the counter to make a guess and corrects if necessary. The child then marches rhythmically through the next x-ty one through x-ty nine chunk, to the next memory search for the next decade word.

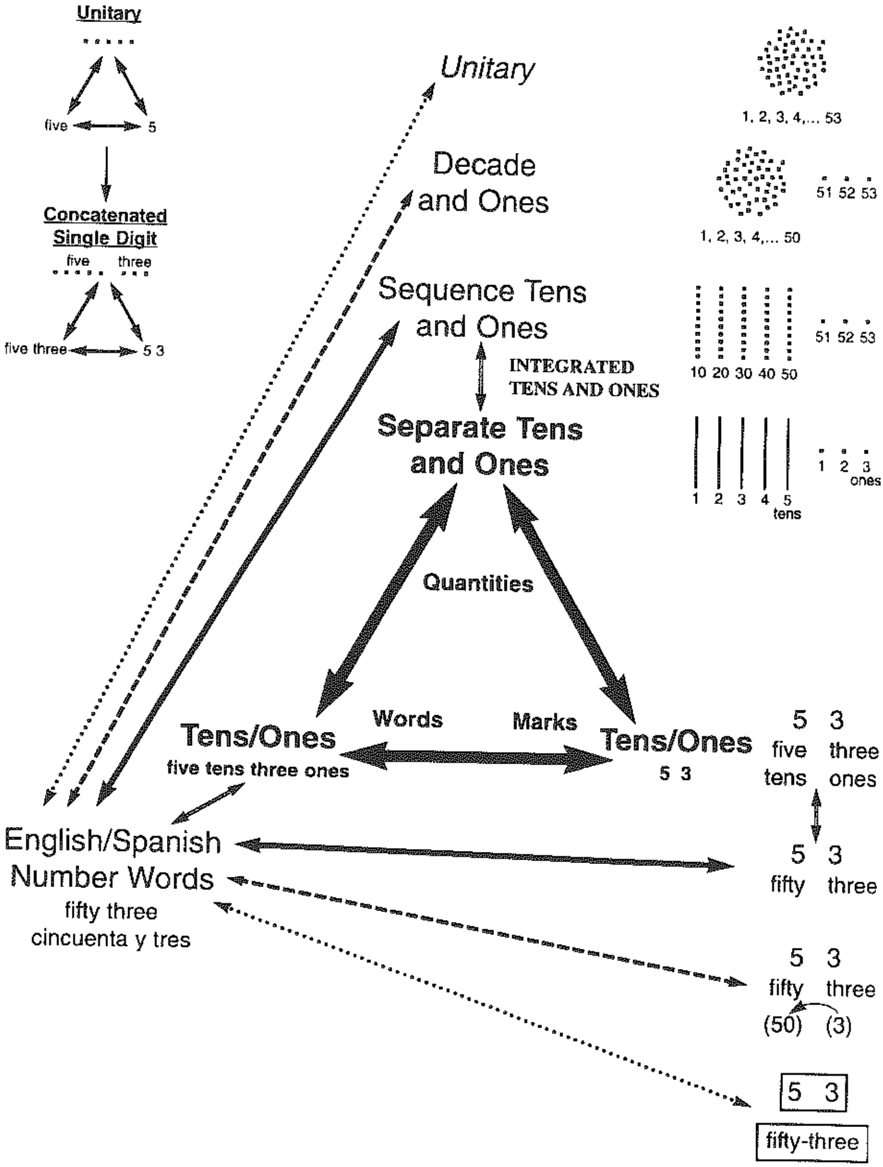


figure 2: a developmental sequence of conceptual structures for two-digit numbers: the UDSSI triad model

3.3 Helping children to build a generative core of base-ten quantity, number word, and number marks interrelationships

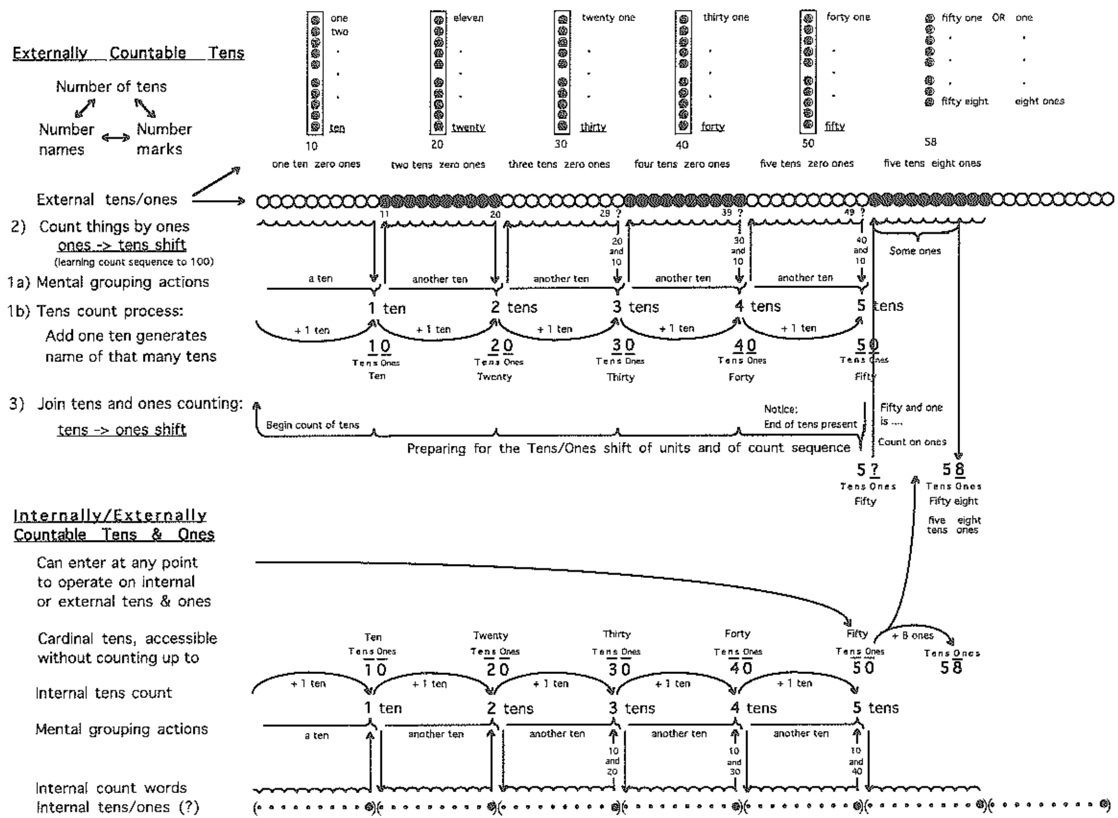
We have found that many urban first graders, some second graders, and a few third graders cannot count to 100. We therefore have had to design activities to help all children learn to count to 100. We begin by responding to children's search for the next decade word, not in the framework of memorization, but by showing them that they can learn to figure out what the next decade word is going to be. We try to make the relationships at each ten between the number of groups of tens (e.g. three tens), the name for how many things are in those tens (e.g. thirty), and the written number marks (30) accessible for learning and understanding. We simplify children's access to this network by centering the task as seeing and counting groups of ten ('one ten, two tens, three tens'). This is a simple extension of ordinary counting, and may even be, at least initially for some children, ordinary counting (i.e., the 'tens' are not yet units of ten or perhaps not even ten ones for them, initially). This count *of* tens then is linked to written number marks as telling how many tens (<u>1</u>0, <u>2</u>0, <u>3</u>0) and to number words (by counting all the things inside the tens to find out how many are in that many tens).

Figure 3 is simultaneously a model of the conceptual structures children need to construct in learning to count to 100 by tens and by ones using decade, sequence-tens, separate-tens, and integrated tens conceptual structures and a model of the teaching activities we design to help them do so. Children enter the network at 1a (using a mental grouping action to focus on the ten). The teacher then introduces the possibility of using the count of tens groups to figure out how to write and name the number (1b in Figure 3). We will describe that process below. The bottom half of Figure 3 is the later internalization of all or parts of the initially external counting activity, though, of course, many aspects of the top model are internal conceptual activities from the beginning of conceptual counting. Figure 3 shows at the top the ten groupings we used in the *Children's Math Worlds* project. These are strips of cardboard each showing ten pennies; on the back is one dime (the U.S. 10¢ coin). Below the penny strips is shown the beadstring used in the Dutch program. The activities we use in the classroom are designed to help children build sequence-ten and separate-tens conceptions and relate them to each other.

3.4 Counting by tens, counting the tens, number marks, and number names

We begin by building up tens grouping experience and exploring the relationship between tens groupings and count words. We ask, for example, 'How many groups of ten can you make from 30 pumpkin seeds?' and conversely, 'If you have 3 groups of ten pumpkin seeds, what number of pumpkin seeds do you have?' We finally discuss 'Is 3 tens the same number as 30?' Children are initially divided in their opinions on that question: The 'same number as' concept is as much under construction as it is a tool in early grouping discussions (Baroody and Ginsburg, 1985; Fuson, 1988), and counting as a criterion is vulnerable to a lack of counting expertise.

But systematically iterating the 'x tens -> what number?' question quickly builds up counting expertise and establishes the core of base-ten quantity, word, and numeral interrelationships (see 1b in Figure 3, 'the tens count process' for a model of this process). A specific account of one way of building up this core may be helpful. The teacher puts up 1 group of ten (we use strips of 10 pennies, but any objects grouped by ten will do), counts the pennies by ones with the class, discusses the writing of '10' below the penny strip in terms of its meaning as 'one ten' and (pointing to the zero '1<u>0</u>') no ones extra.' Here, the links of the words to the written numerals and their left-right positions are crucial and are emphasized by gestures. The teacher then puts up another ten and asks 'How many tens now', eliciting the answer from the children and then modeling getting the answer by counting the tens, '1 ten, 2 tens', and again writes that answer, that number of tens counted (<u>2</u>0), below the penny strip (again connecting the left-right positions to the number of tens and the number of ones). The goal of this part of the activity is to help children learn that if they can count the number of tens, they can write the correct 2-digit numeral: the 2-digit numeral *means* the number of tens. Finally the teacher asks, '2 tens is what number?' At this point the number name 'twenty' can be cued by a range of meaningful sources. More knowledge is pointing to it than just one (maybe memorized, maybe not) link in a verbal chain (19, 20). Some children may already have grasped the gist of the initial 'same number' discussion, or at least be cued by it. Some will have written numeral -> number name knowledge ('20' -> 'twenty') and are cued by that. Some first graders know that 'ten and ten makes twenty' and volunteer that. Some will have rapidly counted by ones while the teacher was posing the question and volunteer 'twenty'. This is a process of social elicitation: Some in this cultural pool invariably use one or another of these cues. The teacher can further establish this linkage by leading a choral count of the individual pennies to 'check' if 2 tens is 'really' twenty (most first graders can count to 20). Finally, a chaining mechanism of tens is evoked to establish a counting sequence. The count of tens built up so far is reiterated '1 ten, 2 tens' (while pointing to each strip) and also 'ten, twenty' (also while pointing to each strip). But this sequence of counting by tens now inherits the range of links that effectively cue it for that individual.


Externally Countable Tens
Number of tens
Number names
Number marks
External tens/ones
2) Count things by ones
ones -> tens shift
(learning count sequence to 100)
1a) Mental grouping actions
1b) Tens count process:
Add one ten generates name of that many tens
3) Join tens and ones counting:
tens -> ones shift
one two
ten
10
one ten zero ones
eleven
twenty
20
two tens zero ones
twenty one
thirty
30
three tens zero ones
thirty one
forty
40
four tens zero ones
forty one
fifty
50
five tens zero ones
fifty one OR one
fifty eight
eight ones
58
five tens eight ones
11
20
29 ?
39 ?
49 ?
a ten
another ten
20 and 10
30 and 10
40 and 10
Some ones
1 ten
2 tens
3 tens
4 tens
5 tens
+ 1 ten
10 Tens Ones Ten
20 Tens Ones Twenty
30 Tens Ones Thirty
40 Tens Ones Forty
50 Tens Ones Fifty
Begin count of tens
Notice:
End of tens present
Fifty and one is ....
Count on ones
Preparing for the Tens/Ones shift of units and of count sequence
5 ? Tens Ones Fifty
58 Tens Ones Fifty eight
five tens eight ones
Internally/Externally Countable Tens & Ones
Can enter at any point to operate on internal or external tens & ones
Cardinal tens, accessible without counting up to
Internal tens count
Mental grouping actions
Internal count words
Internal tens/ones (?)
Ten Tens Ones 10
Twenty Tens Ones 20
Thirty Tens Ones 30
Forty Tens Ones 40
Fifty Tens Ones 50
+ 8 ones
Tens Ones 58
10 and 20
10 and 30
10 and 40

For each remaining decade to 100, the mappings between the number of tens, number marks, and number words can then be elicited from the children themselves, almost without error, by adding 1 ten, then reiterating the questions referred to above: How many tens now? How do we write x tens? X tens is what number? With each added ten, the accuracy with which the number name is inferred increases (Smith, 1994). Over several such counts, most children learn to figure out the number name by this counting of the number of tens. A generative activity replaces memorization. Over a few days, many children learn to count tens both ways (count the tens and count by tens) and to map reversibly between each combination of the number of tens, written numerals, the number of ones, and number names. They can also quickly learn to add tens quantities (e.g. 60 + 30) because they can now construct and count them (if they have physical tens groupings with which they can work or which they can draw).

The teacher orchestrates further performances of various counting activities using the generative-tens model. Some children initially participate only with tens and ones words because those are the simplest. Increasingly, children come to fill in more and more parts of their own web of knowledge within counting activities. All the complex links involved are made by the teacher in different ways in different activities. The focus is continuously on helping children to link the number of tens, the written number marks, the number names (how many things in all), and the number of ones, and to negotiate the ones/tens and the tens/ones counting shifts. This focus continues throughout the 2-digit addition and subtraction activities, which are viewed as settings within which children can build up and use their 2-digit web of knowledge.

Counting things by ones to 100

For children to learn to use the tens count to support counting to 100 by ones (step 2 in Figure 3: the ones-to-tens shift) requires a further layer of interactive attentional direction, directly corresponding to an extra layer of complexity in the activity. Figure 3 portrays the extra layer of attentional directives involved in shifting from counting ones to using the grouping and incrementing tens process discussed above. This layer of attention is again (as with the tens count) established interactively in the classroom, via questions. The point is to help children replace pausing and searching memory for the next ten with pausing and framing a question. If a child is at twenty-nine, for example, and pauses, what frames the question is realizing that s/he now has counted up to twenty *and ten*, or another ten. This is the ones-to-tens shift. Once s/he asks the question, how many tens now?, s/he has shifted into the tens count process, discussed above and shown in 1a and 1b. The use of external tens groupings (see examples at the top level of Figure 3) is quite important here because the child can then simply count the number of tens so far, including the new ten. With more experience, a child may anticipate this process and abbreviate it, keeping track of the tens already counted, and knowing that one more ten is to be added: '2 tens, and one more ten makes three tens, thirty.' As that anticipation becomes consistent, so that a child is looking forward to the next ten (thirty) while counting up towards it in the twenties (see 3 in Figure 3), the counting sequence is becoming automatized, but now in the correct order, and with countable tens groupings embedded in it.

Through such activities, these countable tens are simultaneously building up rapid reversible mappings between quantity, number word, and written numeral uses, as sketched above. A number of tens can be produced in any of these forms (and added as well), given any other. Tens can be counted and produced as separate groupings as well as in sequence form.

3.6 Join tens and ones counting

But tens *and* ones cannot yet be counted or produced jointly to find any 2-digit number without a further layer of attentional directives. We have found that children can map a range of ten relationships, and be able to discriminate tens and ones, without being able to count them together. The tens count has a momentum once it gets going, and kids simply continue that tens count onto any ones present (Smith 1994; Fuson and Smith, 1995). Figure 3, at the 'Join tens and ones counting' level (step 3), portrays the layer of attentional directives involved. Children need to anticipate that some extra ones as well as some tens are to be counted, and maintain that anticipation sufficiently to monitor when all the tens have been counted, and then stop counting tens. They then, or earlier while tens-counting, need to prepare for the tens/ones shift of units (from units/groups of tens to units of ones) and of the counting sequence (from counting the tens or counting by tens to counting by ones from one or from the decade word). If they were counting by tens (ten, twenty, thirty, forty, fifty), they need to know that 'fifty and one more is fifty-one'. That is, they need to understand at the ones level a relationship of 'another entity' is '+1' is 'the next count word' within the unitary/decade count sequence. The counting by ones sequence also has to be familiar enough that children can easily negotiate this while remembering their tens count of 'fifty'; kindergarten children sometimes forgot their tens count result when initially attempting the tens/ones shift (Smith, 1994). If children were counting the tens ('one, two, three, four, five tens'), the tens/ones shift is easier: They just start counting from one again. But the cardinal joining of the tens and ones at the end of both the tens counts and the ones counts is simpler for the sequence-tens than for the separate-tens words because 'fifty eight' carries a joining from the unitary and decade conceptual structures. Managing everything in either tens/ones shift is demanding, and it takes most kids repeated efforts to do so consistently.

Again, this layer of attentional directives is established interactively, via questions, but a very simple question usually suffices to enable children to self-correct their errors: When a child misses the shift and counts ones as tens, a helper asks, 'Are those tens?'. This is often enough to engineer a shift to counting ones at that point, though multiple attempts across multiple sessions are often required to establish it consistently (see Smith, 1994, for a study of kindergartners negotiating this shift and Fuson and Smith, 1995, for a case study of first-grade peer tutoring with adult help). For some children, more scaffolding may be required, e. g., 'What are they?' (ones) 'So fifty and one makes...' (children can usually then continue the rest of the way to the next tens-ones shift).

With the ability to count jointly separate external tens and ones groupings into a whole number, or produce separate tens and ones from a verbal or written whole number, children have a minimal core of processes sufficient for adding 2-digit quantities with regrouping as well as other uses. We also help children develop right away other tens counting and adding processes to explore and use in multi unit arithmetic (this is often labeled 'mental addition': we will address this below). In particular, we develop counting on from 2-digit numbers as a basis for a broader development of 2-digit addition and later subtraction. Children keep track of such counting on with fingers or other means. Thus, we try to make available as rapidly as possible the whole range of solution methods to be discussed later by asking children to try to solve some problems without drawing all of the quantities, while of course allowing those who really feel that they need to do so to draw them.

Another naturally-occurring cultural activity begins to elicit internal tens and ones models: coin counting. Counting dimes and pennies extends counting-tens-and-ones experience to an activity in which children must overcome the perceptual influence of a dime as 'one' thing rather than as a visible ten (see Fuson and Smith, 1995, for a case study of such difficulties). Adding nickels to dimes and penny situations requires a child to construct a method for tracking the ones to be counted on: a nickel (5¢) does not offer 5 ones to count externally, but children can use 5 fingers or learn to count on by 5's. Another aspect of counting on, being able to start at any point in the counting sequence, is developed with other kinds of problems (e.g. A soda costs 48¢, but you also have to pay 4¢ tax. How much do you have to pay?). Being able to count on from any 2-digit number can be learned conceptually in the same manner as single-digit counting on (see Fuson, 1988, for a summary). The main constraint here is an insufficiently learned count-to-100 sequence. Such problems also give practice in counting over a decade word, which some children need.

2-digit addition

With experience building up in parallel in both counting external tens and ones and counting on that involves internalized abilities to start counting at any point in the sequence (without counting up to it) and the construction of tracking methods (e.g. fingers) in lieu of tens or of ones to count, children can construct a wide range of methods for 2-digit addition. However, some need to continue working with external tens and ones for a long time. Indeed, 2-digit addition with regrouping actually serves to consolidate competence at counting external tens and ones. Children who are still constructing the sequence-tens words can use tens and ones words and counts while in other activities working on their counting by tens.

Two-digit addition in this mode minimally requires constructing two sets of tens and ones, then counting the tens and ones together. Children don't even have to consciously regroup if they count the tens first: They can continue counting ones in the sequence across decades. Some first-grade children may at this point still be confusing which are tens and which are ones in 2-digit numbers (due to left-right confusion) or occasionally count ones as tens. The examples given earlier demonstrate the range of methods some children can construct while others are still struggling with basic place-value and 2-digit concepts. We view 2-digit addition problems in which the ones exceed ten (e.g. 38 + 26) as excellent activities within which children can continue to construct and use ten-structured 2-digit conceptions of numbers.

4 Two-digit addition, subtraction and unknown addend methods: Issues concerning problems, instructional sequences, conceptual supports, and number words

4.1 2-digit addition and subtraction methods and their developmental relationship to problem situation structure

Methods used by children to solve 2-digit addition, subtraction, and unknown addend problems are shown in Table 1; these methods were used by children in four projects that emphasized learning mathematics with understanding (Fuson et al., in press). Table 1 has been adapted from Fuson et al. (in press) to label in bold the methods identified by Beishuizen (Beishuizen, 1993, this volume; Klein, Beishuizen and Treffers, in press) that are used by Dutch children receiving instruction using traditional textbooks, using a Realistic approach with an empty number line, or using a Gradual approach with an empty number line. See Beishuizen (this volume) for more detailed descriptions of the methods in bold.

Begin-With-One-Number Methods: Increase or Decrease By Tens and Ones		
38 + 26 = ☐	64 - 26 = ☐	38 + ☐ = 64
Count on/add on tens then ones **AD/N10** [58+2=60; 60+4=64]	Count down/subtract tens then ones **SUB/N10** [44-4=40; 40-2=38]	Count up/add up tens, then ones **AOT/N10**
38,48,58,59,60,61,62,63,64 38 + 20 —> 58 + 6 —> 64	64,54,44,43,42,41,40,39,38 60 - 20 —> 44 - 6 —> 38 or 64 - 26	like count on, keep track: count up 26 like add on, keep track: add up 26
Overshoot and come back[a] **AD/N10C** 38 + 30 —> 68 - 4 —> 64	Overshoot and come back[a] 44 **SUB/N10C** 38 64 - 30 —> 34 + 4 —> 38	Overshoot and come back[a] **AOT/N10C** like addition, keep track: add up 26
Count on/add on to make a ten, count on/add on tens then rest of ones **AD/A10** [40+24=64]	Count down/subtract to make a ten, count down/subtract tens then rest of ones **SUB/A10**	Count up/add up to make a ten, count up/add up tens then rest of ones **AOT/A10**
38,39,40,50,60,61,62,63,64 38 + 2 —> 40 + 20 —> 60 + 4 —> 64	64,63,62,61,60,50,40,39,38 64 - 4 —> 60 - 20 —> 40 - 2 —> 38	like count on; keep track: counted up 26 like add on; keep track: added up 26
Mixed Methods: Add or Subtract Tens, Make Sequence Number With Original Ones, Add/Subtract Other Ones		
Count on/add on tens, add ones, count on/add on other ones **AD/10s**	Count down/subtract tens, add original ones, count down/subtract other ones **SUB/10s**	Count up/add up tens, add original ones, count up/add up other ones **AOT/10s**
30,40,50,58,59,60,61,62,63,64 30 + 20 —> 50 + 8 —> 58 + 6 —> 64	60,50,40,44[b],43,42,41,40,39,38 60 - 20 —> 40 + 4[b] —> 44 - 6 —> 38	30,40,50+8[b]—>58,59,60,61,62,63,64;26 30 + 20 —> 50 + 8[b] —> 58 + 6 —> 64; 26

Change-Both-Numbers Methods

Move some from one number to the other to make a tens number[c] (maintaining the total) **AD/A10**	Make subtracted number a tens number, change other to maintain difference	Make initial number a tens number, change other to maintain difference
38 + 2, 26 - 2 —> 40 + 24 —> 64	26 + 4, 64 + 4[d] —> 68 - 30 —> 38 Latin America, Europe: add a ten to both numbers $\begin{array}{r} 6^{1}4 \\ -\,{}^{1}\underline{2\,6} \\ 3\,8 \end{array}$	38 + 2, 64 + 2[d] —> 40 up to 66 —> 26

38 + 26	64 - 26

Decompose-Tens-and-Ones Methods: Add or subtract everywhere, then regroup

Add tens, add ones, **ADD/1010** make 1 ten from 10 ones

$$\begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 5\,0 \\ \underline{1\,4} \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 5\,14 \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ \cancel{5}\,4 \\ 6 \\ 6\,4 \end{array}$$

or erase 5

Add ones, add tens make 1 ten from 10 ones

$$\begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 1\,4 \\ \underline{5\,0} \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 5\,14 \\ 6\,4 \end{array}$$

Subtract tens, subtract ones, combine totals **SUB/1010**
This requires some notion of negative numbers of owing, or ones "in the hole"

$$\begin{array}{r} 6\,4 \\ \underline{-\,2\,6} \\ 4\,{-2} \\ 3\,8 \end{array}$$

so 40 - 2 = 38

Subtract ones, subtract tens, combine totals
same as preceding method

4 - 6 = 2 is difficult
so 4 - 6 = 2 —> 42
is a typical error

Decompose-Tens-and-Ones Methods: Regroup, then add or subtract everywhere

Look to see if total ones ≥ 10, record or remember; then make 1 ten from 10 ones, add tens, add ones or make 1 ten from 10 ones, add ones, add tens.

$$\begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 6\,4 \end{array} \qquad \begin{array}{r} 4 \\ \cancel{3}\,8 \\ \underline{+2\,6} \\ 6\,4 \end{array} \qquad \begin{array}{r} 1 \\ 3\,8 \\ \underline{+2\,6} \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 6\,4 \end{array}$$

Make 10 ones from 1 ten [open a ten], then subtract tens, subtract ones or subtract ones, subtract tens

$$\begin{array}{r} 5\,{}^{1}4 \\ \cancel{6}\,\cancel{4} \\ \underline{-\,2\,6} \\ 3\,8 \end{array} \qquad \begin{array}{r} 5\,{}^{1}0 \\ \cancel{6}\,4 \\ \underline{-\,2\,6} \\ 3\,8 \end{array} \qquad \begin{array}{r} 5 \\ \cancel{6}\,{}^{1}4 \\ \underline{-\,2\,6} \\ 3\,8 \end{array} \qquad \begin{array}{r} 64 \longrightarrow 5\,0 \\ \underline{-\,2\,0} \\ 3\,0 \end{array} \quad \begin{array}{r} 1\,4 \\ \underline{-\,6} \\ 8 \end{array} \quad \begin{array}{r} 6\,4 \\ \underline{-\,2\,6} \\ \longrightarrow 3\,8 \end{array}$$

Decompose-Tens-and-Ones Methods: Alternate adding/subtracting and regrouping			
Alternate adding and making another ten:		Alternate subtracting and opening a ten:	
Add tens, look to see if there is another ten, add ones	Add ones, make 1 ten from 10 ones, add tens	Subtract tens, open a ten, subtract ones	Subtract ones, open a ten, subtract tens
38 or 38 look +26 erase +26 before ~~5~~ 64 writing 64 tens	these look just like regroup, add ones, add tens methods above	14 6~~4~~ or 64 -26 erase -26 ~~4~~ ← 38 38 open before writing tens	64 58 -26 -20 -2 38 [we have not seen this]

[a]The reverse of this method is also used occasionally: increase the first number to make an easy addition/subtraction, add/subtract, and decrease the answer to compensate (38 becomes 40 + 26 = 66 - 2 = 64 or 64 becomes 66 - 26 = 40 - 2 = 38).

[b]Forgetting to add back in the original ones (the 4 from 64 or the 8 in 38) or subtracting them are (in a subtraction problem) frequent errors. The ones from the 26 sometimes are subtracted first and then the ones from the 64 are added back in; forgetting to add the 4 or subtracting it are also frequent errors.

[c]This is a different way to think of the method just above (sequence add on to make a ten) in which the rest of the number is added at once instead of in two steps as tens and ones.

[d]This step is difficult; the number is often subtracted rather than added, confusing what must be kept constant in addition (the total) and in subtraction (the difference).

Note. All methods in the table are for problems requiring regrouping (making another ten from ten ones or opening of ten to make 10 ones). This is done explicitly or by counting or adding/subtracting over a ten. Problems without regrouping are much simpler. All separate-tens-and-ones methods may also be written horizontally. Unknown addend methods for separate-tens and ones can be done by writing the addition problem with the second number empty and then adding up to the total to find that number. Single-digit subtraction for separate-tens and ones methods may be done as an unknown addend method (forward count up/add up methods). This table has been adapted from Fuson et al. (in press).

table 1: 2-digit addition, subtraction, and unknown addend methods using sequence-tens and/or separate-tens

Table 1 identifies three kinds of 2-digit methods: addition methods in which two 2-digit numbers are combined to make a total, subtraction (take-away) methods in which a 2-digit number is taken away from a larger 2-digit number, and adding-on unknown-addend methods in which the unknown addend is found by adding on from the known addend to get the known total (the number added on is the unknown addend). A fourth method can be used: taking-away unknown-addend methods in which the unknown addend is taken away from the known total to reach the known addend (the number taken away is the unknown addend). We did not include this method in Table 1 because it was rarely used by children in our projects (most children used adding-on unknown-addend methods instead).

The four classes of methods in Table 1 are taken from the literature about methods children use to solve single-digit word problems (see reviews of this literature in Fuson, 1992a, 1992b, 1994, where kinds of word problems are related to kinds of solution methods). The four classes of single-digit methods move through three (or four, depending upon details of classification) developmental levels of increasing abstraction and abbreviation. Children begin by directly modeling the problem situation with objects; they count out objects to show each number in the problem. They later begin to abbreviate initial modeling steps by embedding addends within totals, and they use the number words themselves to show numbers in the problem (counting on, counting back, counting up to, counting down to). Even later they chunk small numbers within other numbers to use derived facts (e.g. $6 + 7 = 6 + 6 + 1 = 12 + 1 = 13$). Finally, they may know number triplets so that they can immediately generate an answer.

In the first stage, children's solution methods directly follow the problem situation. At the number-word solution level, many children frequently solve a problem using a method that directly models the problem situation (e.g. counting up to for a Change-Add-To unknown change problem). However, some children begin to free themselves from the problem structure and select problem solution methods that differ from the problem situation (e.g. using counting up to for a Change-Take-From unknown result problem that formerly was solved by taking away). This freedom can be facilitated by instruction that discusses such alternatives. Until this independence of solution method from problem situation occurs, the solution methods in Table 1 also tend to describe the structure of the underlying problem situation that is solved by that method.

Many traditional methods of instruction have assumed that children have only two classes of methods: addition and subtraction. Children were to solve problems by deciding which class of solution methods to use (i.e., whether to add or to subtract). They usually were supposed to write a number sentence showing this method (e.g. $14 - 8 = ?$) and then carry out the operation shown in the number sentence. Research in the 1970's and 1980's (see Carpenter, Hiebert and Moser, 1983, or the above references for reviews) indicated that some children instead used number sentences to show the problem situation (e.g. 8 + ? = 14). Children forced to write a solution sentence that differed from the problem situation often solved the problem first and then wrote the solution sentence. Thus, word problem solving for children goes through at least four distinct levels of conceptualization (Fuson, Hudson and Ron, 1996). A solver first forms a *situation conception*: an initial conception of the problem situation in the world (I have some apples of which I eat some). The mathematical elements are then focused on to construct the *mathematized situation conception* (e.g. 14 take away 8 to make how many?). The unknown is then focused on to construct the *solution method conception* (e.g. 8 plus how many will give me 14?). That solution conception is then carried out by particular solution actions (e.g. counting from 8 up to 14 with fingers).

Problem situations involving 2-digit quantities can be solved by unitary methods just like the single-digit methods. Children's external models for the mathematized situation conception, the solution method conception, or the solution method itself may be at any of the developmental levels, though the final level of known fact is rare for 2-digit numbers except for special combinations such as 50 + ? = 100. However, many children also begin to develop conceptual structures for 2-digit numbers that enable them to carry out solution methods involving counting or adding/subtracting groups of ten entities. These more complex 2-digit solution methods fall into the same four classes of methods identified for single-digit numbers and outlined above with respect to Table 1. Because children construct these conceptual structures using groups of ten only after they have reached at least the second single-digit stage of counting on/counting up, they may already have some freedom from the problem structure in selecting a solution method. However, this issue of the extent of the freedom children can exercise in their choice of a solution method for given problem situations (what Beishuizen, this volume, called 'mismatches') has not been researched nearly as much as for single-digit numbers. Because the counting down methods tend to be so difficult, the counting down to methods may be even more so. Therefore in Table 1, we only emphasize counting up to, because it has been found to be simpler than counting down for single-digit subtraction (e.g. see literature reviewed in Fuson, 1992a, 1992b). Using for 2-digit numbers the whole range of word problem types (see details in Fuson, 1992a, 1992b) is one way to stimulate a wider range of 2-digit solution methods. Problems asking children to find the difference may be especially productive because different children interpret such problems in different ways (e.g. Beishuizen, this volume; Hiebert et al., 1996). Some work by Beishuizen (this volume), Van Eyck (1995), and van Lieshout (this volume) did indicate considerable dependence of solution method on problem structure. But little other work has been reported. Especially needed is work concerning various kinds of instructional supports on this issue.

Type of Embodiment	Embodiment Medium: Objects	Embodiment Medium: Drawn Objects	Embodiment Medium: Drawn Numbers and Objects
SIZE: Tens & Ones Cumulative length	3 decimeters 5 centimeters Beadstring		Meter stick Number line Open number line 35 Thermometer
Cumulative area (folded length)	Gold bars Bead Frames: ten rows of ten movable beads		100 number grid: 10 rows or columns of ten numbers each
Number of groups	Base-ten blocks or unifix cubes Penny strips		10 20 30 31 32 33 34 35 1 2 3 4 5 ones OR 1 2 3 tens 10 10 10 1 1 1 1 1 10 10 10 1 1 1 1 1 10 20 30 31 32 33 34 35
DECADE:	Any of the objects above can be thought of as decade objects: 30 + 5 where both are counted by ones.		Montessori cards with objects 3 0 5 → 3 5 plain 3 0 5 → 3 5
PLACEVALUE:	Count fingers as ones Count fingers as tens Abacus Poker chip computer (Tens: ooo; Ones: ooooo)		

table 2: classes of object and drawn conceptual supports for 2-digit numbers

There is a very complex intertwining of solution method, 2-digit conceptual structure, conceptual support introduced in the classroom, and number words. Many variations are possible. Before we discuss this intertwining further, we will turn to a brief classification of potentially meaningful conceptual supports for 2-digit numbers and addition and subtraction methods with such numbers.

Classes of conceptual supports for 2-digit numbers and calculation

Table 2 shows different classes of conceptual supports that can be used in classrooms to help children construct conceptual structures for 2-digit numbers. The major subclasses are size conceptual supports that show tens and ones, decade conceptual supports that show the decade and the ones, and place-value conceptual supports that show 1-digit numbers in different left-right locations (or taken from different left-right locations for finger ones and finger tens). Within each major subclass are shown three possible conceptual support media: objects, drawn objects, and numbers with drawn objects. Within the size conceptual supports are three subclasses. Considerable research has focused on differences in children's understanding according to these subclasses. In general, Dutch researchers (e.g. Beishuizen, Gravemeijer, Treffers) have tended to make sharper distinctions between the effects of these subclasses than have U.S. researchers (see chapter with Discussions at the conference).

Size conceptual supports can show tens and ones by cumulative length, by cumulative area (a sort of folded area with adjacent ten-lengths), and by the number of groups of tens and ones. Recent Dutch approaches have used cumulative length and cumulative area conceptual supports: the bead string, the open number line, and the 100 number grid. CGI (Carpenter, this volume; Carpenter et al., 1995) and the Hiebert and Wearne project (Hiebert and Wearne, 1992, 1996) used number-of-groups conceptual supports: base-ten blocks or Unifix cubes in groups of ten. Cobb projects have used the 100 number grid (Cobb, 1995) and number-of-groups conceptual supports: Unifix cubes in groups of ten and drawings of packages of ten candies and single candies (e.g. Cobb et al., in press). The South African project has used place-value conceptual supports (bead frames to build sequence-tens), and some classes have used Montessori cards without object drawings (Fuson et al., in press). In our Children's Math Worlds project, we have at various times used cumulative length conceptual supports (base-ten blocks as lengths; thermometers), number-of-groups conceptual supports (base-ten blocks, drawn blocks, and drawn blocks with numbers; penny/dime strips, drawn penny strips and pennies, drawn dimes and pennies with numbers; collections drawn as dots and ten-sticks of dots), decade conceptual supports (Montessori cards with objects), and place-value conceptual supports (finger ones and finger tens: using fingers to count by ones or count by tens).

Each of the types of conceptual support highlights particular aspects of the UDSSI Triad Model of conceptual structures for 2-digit numbers. The size cumulative length conceptual supports parallel the form of the unitary sequence of number words. As the tens within these conceptual supports come to be noticed and used by children (e.g. the alternating colors of tens beads; large 10, 20, 30, etc. markings on the thermometer), these conceptual supports can be used to construct and use the sequence-tens and ones conception. The cumulative area models foreground by their rows of ten the sequence-tens within a unitary sequence, but children can use these models at either level (unitary or sequence-tens). However, moving from unitary to sequence-tens methods on the 100 number grid, i.e., coming to see vertical jumps within the 100 number grid as a shortcut for counting all ten single jumps (as increasing or decreasing by ten in one jump) may be quite slow or be done without understanding (Cobb, 1995; Fuson, 1996). Drawing ten more squares after 36, and then 10 more squares, etc. rather than just counting them (or drawing on the number grid) might help children see the part of the ten up to the next decade and the rest of the ten within that decade. The size conceptual supports showing the number of groups of ten and the number of groups of one present the meaning of the 2-digit place-value numerals. The decade conceptual supports show the meaning of a 2-digit numeral as a decade plus some ones (30 + 8 = 38). The place-value conceptual supports look like the place-value numerals in that the tens and the ones look identical and are differentiated only by location. Some place-value conceptual supports use color as well as location to differentiate tens and ones. Because these introduce an extraneous meaningless feature that can be used by children instead of left-right position, color seems counter-productive. It seems better to use chips all of one color so that the poker chip computer looks maximally like written 2-digit numbers. However, because many children are counting on and counting up by the time they reach 2-digit computation, the need for objects to calculate sums or differences of tens or of ones does not seem imperative. Children could use fingers for such calculations (what we labeled finger ones and finger tens in Table 2). Therefore, a chip computer might only be used briefly to show the idea of place value (that numbers in positions look the same but they name different size groups). Even for children who know their sums and differences to 18, conceptual supports can be helpful in deciding how to combine, separate, or compare the tens and ones in multidigit numbers.

4.3 Relationships between solution methods and conceptual supports

The Dutch empty number line facilitates sequence counting solutions in which children begin with one number and move up or down the number-word sequence. The number-of-groups conceptual supports can be used for such sequence solutions (such solutions appear in our *Children's Math Worlds* classrooms and in CGI classrooms where such supports are used), but the number-of-groups supports especially facilitate decomposition methods in which groups of tens are combined separately from groups of ones. However, these grouping methods can be carried out using sequence-tens or decade words (e.g. fifty plus thirty is eighty) or separate-tens words (e.g. five tens plus three tens is eight tens). Many children in each project did use the methods consistent with their instructional supports.

However, within every project, some children used methods that were not so strongly facilitated by their instructional supports. This is probably due to several factors. First, each instructional support can be used for most methods even if one method is most obvious with that support (this position is not held by all conference participants; see discussion); therefore children can invent new methods even if they have not been discussed in a class. The empty number line or 100 grid can be used to do decomposition methods: A child could draw 30, then 20, then 8, then 6 to make 64 on the number line or on a 100 grid. Children can use number-of-groups conceptual supports to do methods that begin with one number: They can make the first quantity and then add on or take away and count as they do so. The methods using tens (A10) are especially clear with number-of-groups conceptual supports. The empty number line actually is no more facilitative of the N10 count/add on/up or back methods than the number-of-groups conceptual supports except for a general unitary up/down sense. With either kind of conceptual support, children must have some kind of learning experience to see and learn the regular pattern involved in counting on from 38 by tens (38, 48, 58). The number-of-groups conceptual support does afford the interpretation of 3 tens 8 ones plus 1 more ten is 4 tens 8 ones as well as a sequence word interpretation.

Second, a focus either on number words or on the written number marks may lead a child to a particular class of methods. Some children seem to be pulled by the number words and thus think predominantly (or at least initially) with sequence conceptions and use sequence-tens. Other children seem to be pulled by the number marks and think in terms of the groups of tens and single ones. Whether these preferences reflect more general dispositions toward oral versus visual thinking is not clear because so few data exist concerning these individual differences in uses of methods.

Third, the method used might vary with how well children can count to 100 by tens and by tens and ones. For children who cannot count to 100, using tens words and separate-tens conceptions initially is easier because one cannot begin to construct sequence-tens, or even the whole unitary sequence, until one knows the counting words of that sequence. Furthermore, we know that the counting sequence to 18 must be quite automatized for children to begin to count on/up within it. Therefore, it seems sensible that the count by tens list and the count by tens list embedded within the unitary count to 100 must be well automatized for children to use it in solution methods. This may be one reason why weaker Dutch pupils use the separate 1010 methods instead of the sequence N10 or A10 methods.

Fourth, the 2-digit conceptual web is very complex, and children have to construct it piece by piece. Early successes on the sequence path or the separate path may start a child down that path. Brief comments by a peer, sibling, or parent may be enough to facilitate an initial path. In a classroom where activities are designed to help children construct both sequence-tens and separate-tens, children still cannot do both simultaneously and so begin one path first.

Fifth, the sequence of problems given may affect the solution paths taken by individual children. The South African Problem Centered Mathematics Project found that teachers who gave many 2-digit addition problems before giving 2-digit subtraction problems had many more children who did incorrect mixed methods by incorrectly generalizing from the addition method: Add the decades and add both ones became subtract the decades and subtract both ones. In Beishuizen (1993) more of the weaker pupils might have used 1010 because 2-digit problems with no trades (regroupings, borrow/carries) were given for a long time before problems with trades appeared. The 1010 method is particularly easy for problems with no trades, but it requires thinking about the directionality of the subtraction of the ones to work for problems with trades.

Sixth, the number words used by children might facilitate one kind of method more than another. We have already discussed differences between using sequence words and tens words. Although each kind of word can be used with the opposite method, they do match the begin-with-one-number and decomposition methods better. Most classrooms do not emphasize the tens and ones words as much as we do in *Children's Math Worlds,* so many European number words would seem to suggest decade or sequence solution methods. However, Dutch and German reverse the decade and ones words for all number words between 10 and 100, not just for the teen words as in English. Hearing (internally or externally) a problem as 'eight and thirty plus six and twenty' seems to us to emphasize the separateness of the decade and ones portion of the number because the split is so obvious: The problem sounds as if you need to add four separate numbers. In contrast, hearing the same problem as 'thirty eight plus twenty six' sounds more like adding just two numbers. Therefore, Dutch and German words may predispose children to use decomposition methods, and this may be especially true for weaker children whose sequence-tens conceptions may be weaker and therefore less able to overcome the suggestion of the words themselves. English words (and others like them in which the decade and ones portions elide together to sound more like a single number) may support more those methods that begin with one number. This effect of number words may be one reason that Dutch researchers have used a sequence conceptual support (the empty number line) and U.S. researchers have used number-of-groups conceptual supports: Each is trying to support children to construct that which is less clear in the child's number words (participants at the conference had differing views on this suggestion, see chapter with Discussions). This contrast between methods suggested by the words and those suggested by the conceptual support then may be a reason that children in both countries invent and use both kinds of methods. It may, of course, be even more complex than this. The reversals in German and Dutch may make it easier for children to see or to say the pattern in jumping by tens: 'Eight and thirty, eight and forty, eight and fifty, eight and sixty' seems easier than 'thirty eight, forty eight, fifty eight, sixty eight' both conceptually and procedurally (you can elongate the 'eight' while thinking of the next decade word). The different intuitions at the conference about these issues of number words within the researchers from a given country may also indicate that individual differences exist in children in what the number words do and do not facilitate.

Vertical mathematization, reflection, and children's conceptual advancement

The Gravemeijer paper (this volume) summarizes the work in The Netherlands concerning vertical mathematization and reflective cycles in which models of situations become models for mathematical reasoning about methods and numbers. Our own work has rested on similar assumptions and was described briefly earlier. Here we would like to stress six aspects of such cycles at work in the classroom that can greatly facilitate children's movement through developmental levels. These aspects are especially necessary for the less advanced children in a class. Research conducted by one of us (Fuson) on a longitudinal study of one of the U.S. reform curricula, the *Everyday Mathematics* curriculum from the University of Chicago, has indicated that these are problematic aspects that need to be emphasized in a curriculum if teachers are to do them.

First, the models chosen by a curriculum or by teachers using a teaching approach such as CGI must be used in the classroom in such a way that they enable children to construct a model of the mathematical domain (in this discussion, 2-digit addition and subtraction). The *Everyday Mathematics* curriculum emphasizes the hundreds grid, and most teachers we visited had such a grid in the classroom. But the ways in which the grid was used did not enable some to many children in a given classroom to use it as a meaningful model of counting, adding, or subtracting tens and ones. Many children did not see the tens on the hundreds grid, especially when counting on from non-decade numbers such as 38. Second graders in Cobb (1995) also could not use the hundreds grid until they had constructed tens-units with some other model. The way the Dutch textbooks used the hundreds grid (reported in Beishuizen, 1993) seems particularly problematic. One square was darkened to show a quantity rather than darkening all of the squares up through that square; this is a counting rather than a cardinal quantity model.

Second, classroom activities need to be designed that move *all* children from unitary conceptions to sequence-tens and separate-tens (number of tens) conceptions.

Although, as Carpenter in his paper (this volume) points out, a class may collectively have a great deal of tens knowledge, the task of the teacher is to help every child have and use such knowledge. The *Everyday Mathematics* curriculum contains counting activities to practice counting by ones, tens, and fives (also other numbers) and to practice writing large numbers. But it does not contain a sustained sequence of activities that help children count or combine or separate tens and ones quantities. Consequently, the more advanced children in a class do invent mental methods, but the less advanced children have no methods except sometimes the standard algorithm they have learned somewhere (sometimes from teachers just before standardized tests). They use the latter with no connections to tens or ones quantities, and make many of the typical errors, especially in subtraction (Murphy, 1997). In the revised Dutch empty number line approach, there were explicit activities with the beadstring and empty number line to support construction of sequence-tens. Most CGI teachers and the teachers in the earlier Cobb projects (e.g. Cobb, Wood and Yackel, 1993; Cobb and Bauersfeld, 1995) did use models of tens and ones (often Unifix cubes stored in columns of ten), but few seem to have used systematic activities with such models to facilitate all children's construction of the generative tens conceptual structures. Some children in third grade (Lo, Wheatley and Smith, 1994) and in the fourth grade (Steinberg, Carpenter, and Fennema, 1994) were still using unitary methods. To us, this seems unnecessary and unacceptable. Having a period of exploration with models-of in first grade as outlined in Carpenter (this volume) seems fine as long as less advanced children are helped in some way to advance. But all second graders, even in urban schools, can come to use addition and subtraction methods using tens by several months into the school year if they have activities to help them construct the conceptual prerequisites for such methods (Fuson, 1996; Fuson, Smith and Lo Cicero, 1996).

A third aspect of using models-of that can support their use and reflection on such use is using drawn methods rather than physical objects. With drawn methods, a record of the whole problem-solving process is available after problem solving. This permits the teacher to examine children's methods and look for children's errors after a class is over. Such monitoring can provide daily feedback loops that permit teachers to select students to demonstrate particular methods or choose errors that would provide useful discussion for many children in the class. If students work at the board or on individual chalk boards (these are methods used frequently in the *Children's Math Worlds* project), their solution is available for reflection by their classmates. No waiting is necessary while children put their method on the board, a management consideration of importance in many schools. Working at the board seems to facilitate children helping each other more than working in a smaller scale on a piece of paper. It is also easier for a teacher to observe such helping. The empty number line is such a drawn method, and in Dutch classrooms one author has visited, seems to facilitate reflection and discussion much as our drawn ten-sticks and ones do. For the less advanced children, having their method physically present also seems to facilitate their explanation of their method. They can rely on gesture as well as on words, and the drawing helps them remember and sequence the steps of their explanation or learn to do so if they require help from the teacher or a peer.

Fourth, drawn methods can facilitate the linking of the model of the tens and ones to written numerals. For larger numbers, the goal of using models-of is so that the written mathematical symbols take on the mathematical meanings of the models-of. This process is facilitated greatly if the models-of are linked to the written mathematical symbols (Burghardt and Fuson, 1996; Fuson and Briars, 1990; Fuson, Fraivillig and Burghardt, 1992). Such linking can also be done in reverse by reading a numerical record of a method using models-of language. In Fuson (1986) reverse linking by asking children to 'think about the blocks' (they had used base-ten blocks to build their understanding of their subtration method) was sufficient for children to self-correct subtraction errors in problems with zeroes in the top number.

Fifth, 2-digit numbers within addition and subtraction problems need to be read as decade words or tens and ones words and not as concatenated single digits (e.g. 38 + 26 said as 'thirty plus twenty' or 'three tens and two tens' not as 'three plus two and eight plus six'). Many *Everyday Mathematics* teachers at least sometimes use concatenated single-digit language ('three plus two is five' for 38 + 26) themselves, and more allowed children to do so. We have found that using decade words and also saying the number of tens is necessary to keep all children in a class with the discussion. Initially, when many children are working on constructing a generative tens conceptual structure, some are thinking only with decade words (ordinary English or Spanish counting words), and others are thinking only with number-of-tens words (frequently these include the least advanced children, who do not yet know the decade words). Using both of these kinds of words can help each of these kinds of children understand any discussion, and later it helps children begin to construct the other related meaning. Coming to use and think with both kinds of words also gives children flexibility in understanding various kinds of addition or subtraction methods.

A sixth aspect that ties together all of these issues concerning vertical mathematization and reflection is the role of the teacher in relating a given child's described solution method to more advanced and to more primitive methods so that children at different levels of mathematization can understand that method. It is not necessary that a teacher do this for every method given by a student, but doing it frequently can help. For example, in the first-grade example from early in the year in Carpenter (this volume), the teacher was trying to help the child advance by asking her to describe her blocks method without using the blocks. Such experiences can help children to move from an objects method to an oral method. An important and natural step after such a verbal description without blocks present would have been to do the description again showing the oral actions with the blocks. This would have made the oral description accessible to most students. In the third-grade example in that paper, the reverse happened. The student said that she did not need blocks to describe her method, but the teacher insisted that the first description of the method be with blocks. This allowed the less advanced children to follow that method. Following that blocks description with an oral description not linked to the blocks might then have helped some of the listeners be better able to move away a bit from the blocks to using words. The third related method then moved to using the words as the objects, and fingers were used to keep track of the tens counted on and then of the ones counted on. Juxtaposing these methods in this fashion can help children see the relations between them and move ahead a level.

In our observations of *Everyday Mathematics* classes, teachers rarely carried out such supports for vertical mathematization, and the curriculum did not support them to do so. Most methods were described only orally. The methods that used objects were rarely acted out or described in detail; a brief 'I counted' or 'I used the hundreds grid' was accepted by the teacher. Thus, other children did not get to see concrete methods carried out, and oral methods were not related to quantities for those children at a lower level. Consequently, some to many children were not able to follow such descriptions (Murphy, 1997). A few teachers did record on the board in numerals whatever method a child described. This served the fourth aspect above of linking a method to written numerals and also helped memory because the whole method was there to be reflected upon after the description was completed. If descriptions of solution methods are to serve to do more than emphasize the individuality of methods and give children practice in describing their method (both worthwhile but limited ends), teachers need to link them to other methods within a vertical mathematization learning trajectory.

4.5 Developmental levels and 2-digit solution methods

The Gravemeijer paper (this volume) describes very well the goal of vertical mathematization as directing the design of instructional sequences. For 2-digit numbers, we see two concurrent kinds of vertical mathematization that specify the movement of individual students from using models-of a meaningful quantity context to using models-for mathematical reasoning. The first is similar to experiential levels for single digits: moving from the use of objects presenting quantities to the use of counting words presenting quantities (and for some problems to the use of recomposition change-both-number methods) to the eventual use of addition and subtraction facts at least for some parts of some problems and the use only of written numbers (and perhaps fingers) to record some method of 2-digit calculation. The second moves through the conceptual structures for 2-digit numbers: from a unitary conception to a decade conception to the sequence-tens or the separate-tens conception and eventually to an integrated-tens and ones conception that relates sequence-tens and separate-tens. These two involve different kinds of mathematical advancements by children. The first uses the models-of to models-for distinction. The second involves conceptual structures that become mathematically more sophisticated and perhaps should be given a different label; for convenience we will continue to call both of these kinds of vertical mathematizations.

The Dutch instructional sequence using the bead string and on to the empty number line does accomplish both of these kinds of vertical mathematization from objects up through drawn length and number methods using sequence-tens. The three US projects represented here at the conference all used size conceptual supports that presented the number of groups of tens and of ones, and they all accomplished at least parts of both vertical mathematizations. *Children's Math Worlds* used in different years two different instructional sequences: one went from grouped objects to drawn ten-sticks and dots with or without numbers to written numerical methods, and the other went from use of penny/dime strips to drawn ten-sticks and dots with or without numbers (and for some classes also to drawn coins) and from there to numerical methods, with mental methods used throughout for some classes. The CGI sequence reported by Carpenter (this volume) used base-ten blocks or Unifix tens and moved to number-word solutions without drawings. The Cobb and Yackel projects used at various times Unifix cubes stored in columns of ten, hundreds grids, drawn number balances, drawings of ten candies and single candies, and a computer program that allowed operating on (decomposing and composing) such drawings. Children moved at their own pace from using these conceptual supports to oral and number methods without such supports.

The solution methods in Table 1 have an interesting relationship to the developmental sequence of single-digit solution methods described earlier. The methods that decompose a 2-digit number into its tens and ones and then add or subtract those tens and ones are initially most like the single-digit Level 1 object methods. The adding and subtracting of tens and ones objects are similar to those used for single-digit numbers, and the tens and the ones are each counted by single-digit numbers. The new and difficult aspects of these 2-digit methods are knowing throughout the solution which are tens and which are ones and understanding how to deal with any needed trading (needing to make another ten from the ones or opening a ten to get more ones to subtract). The 2-digit methods that begin with one number are quite like the sequence number-word single-digit methods that count up or down the sequence. The 2-digit methods that change both numbers are like the single-digit Level 3 derived fact methods.

However, the 2-digit methods each also follow the single-digit developmental levels within themselves. The decomposition-into-tens-and-ones methods move from counting objects to counting on, down, or up or using known facts to find the total or difference of the tens or of the ones. The begin-with-one-number methods often start with concrete objects before they become sequence counting methods.

The methods that change both numbers may initially be done with concrete objects and then as sequence methods. Both for that reason, and because many children are at higher single-digit levels by the time they are working on 2-digit addition and subtraction, we would not expect the 2-digit methods to exhibit strong level effects among themselves. Rather, as discussed above, they are subject to several different kinds of instructional and individual influences.

Another aspect of 2-digit addition and subtraction methods that makes them different from single-digit addition and subtraction methods is that they are more complex multi-step methods that stretch memory very considerably. For this reason, it is frequently very useful to record results of some steps in the method. Numbers can be used to record such steps. Therefore numbers take on considerable importance, and numbers may be used to scaffold a multi-step method. Even when children are not allowed to record intermediate steps, the numbers facilitate computation, as indicated by the superiority of mental computation with problems with numbers visible rather than just presented orally (Reys et al., 1995; Reys, 1984).

5 Why mental computation?

Mental computation is stressed in some countries and by some researchers. The Dutch curriculum places considerable emphasis on mental computation for 2-digit numbers, delaying written methods until third grade. Reys et al. (1995) conclude their paper on mental computation in Japan by asserting:

> 'Finally, mental computation (when defined as self-developed strategies based on conceptual knowledge) should be a central focus of a computation curriculum. Whereas all agree on the importance of mental computation, surprisingly little is known about it in most countries. (p. 324).'

We think that it is very important to consider carefully the possible roles of mental computation in children's learning. If mental computation means moving directly from a problem presented orally or with written numbers to a method done completely internally, it seems clear that such mental computation should follow, and not precede, children's solution of such problems using some kind of quantity referent (objects or a familiar situation) for the written numbers in the problem. Such a quantity referent is necessary for children to have and use a meaning for the numbers with which they carry out a computation (e.g. for 2-digit numbers, for decimals, for fractions). Much research in many mathematical domains (e.g. see the reviews in Grouws, 1992) indicates that students must first construct meanings in these ways. Only after experience using the external referents do students construct robust enough internal conceptions to use these for mental computation. The recent research on the use of the beadstring and drawn open number line clearly recognizes that children need quantity referents initially.

Mental computation of carefully selected and sequenced problems can play an important role in helping children construct concepts for that mathematical domain and perhaps provide an impetus for vertical mathematization of concepts or of methods. As discussed earlier, in the *Children's Math Worlds* curriculum, we now use a sequence of different kinds of 2-digit problems (e.g. 40 + 10 initially, then 40 + 30, then 40 + 7, then 47 + 5, and then 40 + 36). These allow children to construct the major connections in Figure 2. The problems are given initially with external quantities (penny strips and pennies), and children are encouraged from the beginning to solve them mentally if they can. During the rest of the year, such mental computation questions are asked, and gradually more children become capable of solving them. But the major function of such work is to facilitate children's construction of the whole generative tens conceptual structure. We do eventually ask children to solve mentally the more difficult problems such as 38 + 26, but it is expected that only a few children will be able to do so initially. Most children need to use drawn objects and numbers to solve such problems initially.

We think that it is important that our present sequence of problems does not contain problems with no trades such as 32 + 36 or 46 - 25. In subtraction, such problems without trades can suggest the incorrect subtraction method in which the smaller top ones are subtracted from the larger bottom ones. Giving problems requiring trades immediately contributes to children's construction of tens-ones shifts within sequence-tens and/or separate-tens conceptions because children have to confront the issue of making another ten or opening a ten. These issues are easy to solve with drawn quantities.

It seems to us that an unanalyzed stress on mental computation is partially a result of traditional curricula in which children learned written number calculation with little understanding. Mental calculation in such cases meant either a child seeing that written method in his/her head or using a method the child had invented. In the latter case, such invented methods had a better chance of being conceptual than the taught algorithm. Therefore, mental computation was one way to encourage such inventions. For example, Reys et al. (1995) found that many Japanese children reported that they had invented the mental computation methods they used that were not just seeing written calculations. However, in classrooms in which all methods are based on understanding and no method is compulsory, mental computation no longer confers the advantage of more understanding. We certainly advocate 'self-developed strategies based on conceptual knowledge' but strategies do not have to be mental to meet this criterion.

Use of mental estimation is also related to mental calculation. Estimation is related to the elusive to characterize but desirable 'number sense'. Both estimation and number sense are desirable, but they involve different processes than exact mental computation. They both require a strategic analysis of the numbers in the problem, the result of which then directs the consequent process. These both are useful in real life and in checking (and sometimes directing) problem solving. They each require new ways to use a generative tens conceptual structure, and so help children extend their understanding. But these are both quite different from exact mental computation.

The roles mental calculation should play in the curriculum depend considerably upon what mental computation is considered to be. It might be considered to be the solution of problems presented orally without the use of any external action or object. This is a stringent definition that eliminates the use of the known numbers even as memory supports. The consistent and in many cases very large reductions in correct answers between problems presented orally and those presented in numbers for Japanese students (Reys et al., 1995) and for U.S. students (Reys, 1984) indicates how much the written numbers facilitate thinking about a solution.

Mental calculation might be the kind of calculation expected for written numeral problems. One bilingual staff member on the *Children's Math Worlds* project was taught in Argentina calculation methods for addition, subtraction, multiplication, and division in which every intermediate step was done mentally and only the answer was written (the answer could be written digit by digit as these were produced). However, as in the United States with traditional written algorithms, some children secretly used fingers to calculate.

When no intermediate step can be written, methods that carry along the answer-in-progress have an advantage because the steps already done are less likely to be forgotten. The methods that begin with one number do this. The methods (N10) that add/subtract the tens then the ones (or vice versa) are also particularly easy because each of those numbers is sitting there in the given problem number as a memory support. The other two kinds of begin-with-one-number methods require more remembering because the amount overshot (in N10C) must be remembered or the part of the ones not added or subtracted initially must be remembered (in A10). However, if intermediate steps can be written, these methods lose their advantage. Especially for less advanced children, an early stress on mental calculation seems like an unnecessary and perhaps even unfair choice. The focus in the Netherlands on 'mental calculation' in the second grade is actually more of an emphasis on using 'handy' numbers or methods that take advantage of the specific numbers in a given problem. The calculation does not have to be mental but may use objects or drawn supports such as the empty number line.

6 Conclusion

There now exist several examples of instructional sequences that support learning trajectories of children through vertical mathematization of 2-digit addition and subtraction methods while also affording a range of different solution methods carried out by different children. As more and more such classrooms exist, and as teachers attempt to use new curricular materials in these areas, we should be able to learn more about maximal learning trajectories and about how to support all children successfully to methods using tens. We also may learn more about the following important issues: How can discussions be used most profitably for the benefit of all children in the class? How much understanding of each method should all children have? Can understanding follow as well as precede learning a method? What are the relationships between autonomy and being helped to learn a method?

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Correspondence concerning this chapter should be addressed to Karen C. Fuson, School of Education and Social Policy, Northwestern University, 2003 Sheridan Road , Evanston, Illinois 60208 USA. Electronic mail may be sent via Internet to fuson@ils.nwu.edu

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