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Levels in Conceptualizing and Solving Addition and Subtraction Compare Word Problems

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This article presents an analysis of conceptual and linguistic complexities of matching situations expressed as word problems and describes possible ways of conceptualizing and solving such problems. Data from first and second graders suggest a progression of four levels in conceptualizing and solving these problems. In the first—Relational—level, children can answer "Who has more/less?" but not "How much more/less?" In the second—Language Cue—level, children are more likely to solve problems with action, Equalizing language ("If he gets 2 more cats, he will have as many cats as dogs") than with static, Compare language ("He has 2 more dogs than cats"). They are especially likely to solve problems in which finding the unknown compared quantity is directed by keywords in the relational sentence. At the third—Understand Matching Situations—level, children find Inconsistent problems (those in which the relational sentence is opposite to the needed solution action) considerably more difficult than other types. Children overwhelmingly solve problems in which one compared quantity is unknown by using an Equalizing approach in which the Extra quantity is added to or taken from the other known quantity. They predominantly solve problems in which the difference between two known compared quantities is unknown by using a Matching conception in which the Small quantity

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is taken from the Big quantity. At the fourth—Solve Inconsistent—level, children come to be able to solve Inconsistent problems, primarily by using Equalizing conceptions in which the relation given in the relational sentence is reversed.

Children's solving of addition and subtraction word problems has received considerable attention in the last 15 years (for early reviews of the literature, see Carpenter & Moser, 1983; Riley, Greeno, & Heller, 1983; for more recent reviews, see Fuson, 1988, 1992a, 1992b). One major situational type of such problems has been more difficult for children to solve, has received less research attention, and has been modeled less successfully in computer models of children's word problem solving (e.g., Riley & Greeno, 1988). In this problem type, two quantities are compared with each other to find out how much smaller or larger one is than the other.

Comparison situations are difficult because they are static, they are relational, the third quantity does not exist separately in the situation but must be derived from the comparison operation, the relation can be phrased in several different ways (some of which are linguistically complex), and the relational sentence contains information both about *which* quantity is more or less and about *how much* more or less. The static nature of the situation does not suggest adding or subtracting operations, so moving from understanding the situation to forming a numerical solution method may be problematic for novice problem solvers. Some children initially have considerable difficulty even seeing the third derived Extra quantity in the situation (Ron & Fuson, 1996). Children must decide which quantity is more and which is less from the relational sentence and then keep these quantities distinguished during problem solving, when there may be little in the situational context to differentiate them. All of these characteristics mean that such problems are particularly subject to effects of what Reusser (1990) and Staub and Reusser (1992) called *presentational structure:* surface features of problems such as linguistic form and order of numbers.

In this article, we examine relations among two major linguistic forms of such problems (*Compare* and *Equalize*), two phrasings of the relational statements (*additive* and *subtractive*), and the three quantities in the situation that can be the unknown (*Big, Small,* and *Extra*). First, we describe these linguistic forms and phrasings for each unknown quantity. Then, we outline alternative ways to mathematize (move from the word problem toward the solution of) these various word problems. These analyses lead to some predictions about relative problem difficulty of various problems. Data are then presented to address three issues: effects of linguistic phrasings and unknowns, age or experience differences in such effects, and errors and number sentences as windows on children's mathematizing of such word problems. Predictions of two computer word-problem-solving models concerning such effects and concerning children's errors are evaluated, and alternative developmental levels in solving comparison word problems are proposed.

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In the word-problem literature, such problems have usually been termed *Compare* problems. However, the word *compare* does not have a mathematical meaning specific to these problem situations. Furthermore, when used in phrases such as "compare and contrast," *compare* means to focus on the similarities in the comparison. In this sense, these problems might be more aptly termed *Contrast* problems. *Compare* is also used to describe multiplicative compare problems (e.g., Greer, 1992). To distinguish the addition or subtraction comparison situations, we will, therefore, use the more specific term *Matching* to describe all such underlying situations. *Matching* is the underlying action in all such situations, even when it is carried out by developmentally later addition or subtraction. We reserve the term *compare* for Matching problems given in the usual language of earlier research: "How many more/less?" (Compare More for uses of *more* and Compare Less[1] for uses of *less*).

CONCEPTUAL AND LINGUISTIC COMPLEXITIES OF MATCHING SITUATIONS

In a real-world matching situation, only two quantities exist: One set of entities is a Big(ger) quantity, and the other set of entities is a Small(er) quantity. At issue is how much bigger or smaller one quantity is than the other. These quantities somehow need be matched to find out how many extra entities the Big quantity has (or how many entities the Small quantity is lacking). These extra entities are the third quantity—the Extra quantity—in the situation. In the real-world situation, the Extra quantity is embedded within the Big quantity and does not appear separately from the Big quantity. This Extra quantity is only found by a matching process that partitions the Big quantity into two quantities: a quantity equivalent to the Small quantity and the leftover Extra quantity.

Mathematically, the Extra quantity is an absolute difference: $|\text{Big} - \text{Small}|$ or $|\text{Small} - \text{Big}|$. The order in which the Big and Small quantities are matched does not matter. Because *Difference* also describes the answer in a subtraction problem and matching problems are not always solved by subtraction, we use the situational term *Extra* rather than the mathematical term *Difference* or *Absolute Difference* for this third derived quantity. *Extra* will also remind the reader of the derived nature of this quantity.

The relation between the Big and Small quantities can be described linguistically in several different ways. In the top section, Table 1 gives the four phrasings focused

[1]Most word problems for young children ignore the mass or count character of the compared quantities and use the word *less* rather than *fewer* for counted discrete quantities because children are assumed to be less familiar with the word *fewer*. *Fewer* is rarely used (less than one occurrence per million words in the Thorndike/Large word-frequency count), whereas *less* is much more common (more than 100 occurrences per million). We use *less* throughout this paper.

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TABLE 1
Wording and Mathematizing Alternatives for Word Problems With a Matching Structure

Additive Wording	*Subtractive Wording*
Unknown Extra	
Bob has 7 cats. He has 9 dogs. Compare More: How many *more dogs* than *cats* does Bob have? Equalize +: How many *(more) cats* does he have to *get* to have as many *cats* as *dogs?* 9 M 7 = ___ (Match) 7 + ___ = 9 (Equalize)	Bob has 7 cats. He has 9 dogs. Compare Less: How many *less cats* than *dogs* does Bob have? Equalize –: How many *dogs* does he have to *give away* to have as many *dogs* as *cats?* 9 M 7 = ___ (Match) 9 – ___ = 7 (Equalize)
Consistent	
Unknown Big	*Unknown Small*
Bob has 7 cats. Compare More: He has 2 *more dogs* than *cats.* How many dogs does Bob have? Equalize +: If he *gets* 2 *(more) cats,* he will have as many *cats* as *dogs.* How many dogs does Bob have? 7 + 2 = ___ (Equalize) ___ M 7 = 2 (Match)	Bob has 9 dogs. Compare Less: He has 2 *less cats* than *dogs.* How many cats does Bob have? Equalize –: If he *gives away* 2 *dogs,* he will have as many *dogs* as *cats.* How many cats does Bob have? 9 – 2 = ___ (Equalize) 9 M ___ = 2 (Match)
Inconsistent	
Unknown Small	*Unknown Big*
Bob has 9 dogs. Compare More: He has 2 *more dogs* than *cats.* How many cats does Bob have? Equalize +: If he *gets* 2 *(more) cats,* he will have as many *cats* as *dogs.* How many cats does Bob have? 9 – 2 = ___ (Equalize Reverse Relation) 9 M ___ = 2 (Match) Equalize Use Relation: ___ + 2 = 9	Bob has 7 cats. Compare Less: He has 2 *less cats* than *dogs.* How many dogs does Bob have? Equalize –: If he *gives away* 2 *dogs,* he will have as many *dogs* as *cats.* How many dogs does Bob have? 7 + 2 = ___ (Equalize Reverse Relation) ___ M 7 = 2 (Match) Equalize Use Relation: ___ – 2 = 7

Note. M = matching.

on in this article. All four phrasings express both the order relation (greater than or less than) and the size of the Extra. The Compare More or Compare Less language does this in a static way. The Equalize + or Equalize – language does so in an active way, suggesting adding to the Small to make the Big (Equalize +) or taking from the Big to make the Small (Equalize –). With the Compare (More or Less) pairs, the subject or object and the order relation (greater than or less than) are reversed.

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With the Equalize (+ or –) pairs, the object of the action and the direction of the action are reversed.

Each Equalize sentence suggests a solution action that could be carried out to find the Extra, and the Compare sentences do not. The suggested Equalizing solution action is given in equation form in the table. These equations form a mathematizing of the Equalize situation that ignores irrelevant features and focuses on the mathematically important features. The 7 + ___ = 9 sentence might express a mathematized situation conception,[2] something such as "7 cats plus how many cats is equal to the 9 dogs Bob has?" To solve this sentence or conception, the solver would have to focus on the unknown and choose a solution method for finding this unknown (make a solution procedure conception for the problem). For example, put up 7 fingers, then put up more until 9 fingers are up, and count the extra fingers put up. Or count from 7 to 9, putting up fingers to count how many more. For Equalize problems, making the mathematized situation conception and the solution procedure conception follows readily from the problem language.

The Compare sentences require (a) an understanding that some kind of matching process is necessary to find the required "how much more" or "how much less" and (b) the ability to carry out some version of a matching process. We express this mathematizing by 9 M 7 = ___, where M stands for matching. The language cues this matching less directly than the Equalizing sentences do their actions, and so Compare wordings might be expected to be more difficult than Equalize wordings early in problem solving.

Interactions Between the Unknown and Linguistic Phrasings

Each of the quantities (Big, Small) can also be unknown. For each of these unknowns, four possible wordings of the relational sentence are also possible: Compare More, Compare Less, Equalize +, Equalize –. Unlike the unknown Extra problems, which mathematize from the problem language fairly directly, the unknown Big or Small problems mathematize directly only for one of the two possible phrasings in each Compare More versus Compare Less or Equalize + versus Equalize – pair. These more direct wordings are given first in Table 1 (in the middle section) and are termed *Consistent* problems, after Briars and Larkin (1984). The other wordings for each pair are given in the bottom section; they are

[2]The language in these informal paraphrases of mathematized situation conceptions here and later violate normal algebraic equation conceptions, especially for the Equalizing conceptions. Because the situation for the Equalizing language involves a number of cats that is made equal to the number of dogs, the informal situation meaning of the equation is cats on one side and dogs on the other. In formal mathematical language, the units for an equation have to be comparable. Therefore, the formal algebraic phrasing of the sentence 7 + ___ = 9 would have to be something more cumbersome and precise, such as "7 cats plus how many cats equals the same number (9) as the number of dogs Bob has?"

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termed *Inconsistent* because the problem language is opposite the solution action required to find the unknown quantity (this is discussed more fully later).

The Equalize mathematizations for the Consistent problems follow from the problem language and are given in equation form. For the unknown Big problem, this might be thought of as something such as "7 cats plus getting 2 cats is how many dogs?" Focusing on the unknown and deciding on the solution action (making a solution procedure conception) are very easy because the unknown is the result of adding 2 to 7. A solver only has to choose some method of starting with 7 and getting 2 more.

The Consistent Compare problems can also be mathematized into an Equalize form. To do this, the solver begins with the known quantity (e.g., 7 cats) and uses the information in the relational sentence to equalize that quantity (there are "2 more dogs than cats" so dogs are more, so add the 2 to the 7 cats to get the dogs). The known quantity is added to or subtracted from to make it equal to the unknown quantity.

The problem wordings in Table 1 are separated into additive and subtractive columns, according to whether the Equalize action is additive or subtractive. Compare More wordings are also considered additive because "more" suggests adding, and Compare Less wordings are considered to be subtractive. The Equalize words are more strongly directive of additive or subtractive solution actions than are the Compare words and thus might be advantageous early in problem solving.

The Inconsistent problems cannot be mathematized in the same way as the Consistent problems. If one begins with the known quantity (e.g., 9 dogs), the relational sentence must be reversed to give an accurate mathematical conception of the situation. From the Equalize + sentence, "If he gets 2 cats, he will have as many cats as dogs," one needs to make the opposite Equalize – sentence, "If he gives away 2 dogs, he will have as many dogs as cats" to get $9 - 2 = \_\_\_$. The Compare More sentence says "2 more dogs than cats," so the 9 dogs are already more than the cats and must be equalized by decreasing 2 (making 2 less) in order to get the correct number of cats. These mathematizations are called *Equalize Reverse Relation* in the table. They begin with the known quantity.

An alternative choice for Inconsistent problems is to maintain the relation given in the relational sentence. These mathematizations are given as *Equalize Use Relation* sentences. They both begin with the unknown quantity.

Both Consistent and Inconsistent problems can also be mathematized by thinking of the underlying matching situation between the Big and Small quantities. These are given as Match sentences. They involve an unknown quantity as one of the matched quantities but the Extra quantity is known. In order to move from a Match conception of the situation to a solution action, the solver will have to know which quantity is Big and which is Small. The Extra 2 will have to be added to the Small or subtracted from the Big. Which quantity is Big and which is Small cannot

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be determined from the Match conception (see Table 1); it must be determined from the relational sentence.

The Compare and Equalize relational sentences have important differences in how easy it is to determine the Big and Small quantity. The Compare relational sentences directly state "*x* is (*n*) more than *y*" or "*x* is (*n*) less than *y*." Therefore, ascertaining which quantity is Big (more) or Small (less) is easy: It requires only ignoring the Extra number in the sentence. It is more difficult to decide which is the Big and Small quantity from the Equalize relational sentence. One has to reverse the relational sentence to decide which is Big and which is Small. For example, "If he gets 2 more cats, he will have as many cats as dogs," does not mean that there are more cats (that cats is the Big quantity); there are actually less cats than dogs (cats is the Small quantity) because one has to add to the cats to get the number of dogs. Therefore, moving from an Equalize wording to use a Match mathematization would be more difficult than doing so for a Compare wording.

Children also can solve any of the problem wordings by a problem language "keyword" method in which they do not even attempt to understand the situation given in the problem. Instead, they simply focus on the two numbers given in the problem and on some keywords that direct addition or subtraction. Thus, any problem with *more,* or *gets,* or *finds* would be added, and any problem with *less,* or *gives away,* or *loses,* or *eats* would be subtracted. Such a keyword strategy will be correct for Consistent problems, incorrect for Inconsistent problems, and correct for unknown Extra problems with subtractive language.

Predictions of Problem Difficulty

To date there has not been systematic evidence on all of the unknowns and language forms in Table 1. However, two computer models of children's solving of word problems did make predictions about Compare problem wordings. The computer model developed by Briars and Larkin (1984) predicts the following order of solution: unknown Extra before Consistent before Inconsistent (which they termed *conflict* problems). The unknown Extra problems are hypothesized to be solved early because objects can be made for the compared quantities and then matched. The difficulty of the Inconsistent forms is the same as discussed earlier: a conflict between the action directed by the comparison term *more or less* and the operation necessary. The computer model developed in successive versions by Riley, Greeno, Heller, and Kintsch (Kintsch & Greeno, 1985; Riley & Greeno, 1988; Riley et al., 1983) predicts the same order of difficulty as the Briars and Larkin model, but for somewhat different reasons. In the Riley, Greeno, Heller, and Kintsch model, the unknown Extra problem is at their Level 1 because the two known-quantity compared sets can be constructed and the difference can be ascertained by counting or matching. The Consistent problems are at Level 2 because they require building

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the big set by combining or taking away the two known quantities. The Inconsistent problems are at Level 3 because these are postulated to require additional part–whole knowledge in order to find a subset of a set that is compared with another set.

Our analysis of Matching wordings and of their affordances for mathematizing in different ways as summarized in Table 1 leads to somewhat different predictions. First, the differences in how the Compare and the Equalize wordings express the underlying relation suggest differences in solution difficulty for these problems. All of the Equalize wordings cue a solution procedure more directly than do the Compare wordings. The Equalize wordings also make it more difficult to ignore the comparing part of the sentence because it is not possible simply to drop the single word *more* or *less*. Especially when children first meet matching problems or for children who are not especially strong in standard English, two predictions seem sensible. Prediction 1 is that Equalize forms will be easier than Compare forms for the problem types in which the Equalize sentence directs the correct solution procedure (Consistent and Extra). Prediction 2 is that Equalize forms will be more difficult than the Compare forms for the Inconsistent wordings because Equalize sentences are more directive of the wrong action.

Second, the analysis summarized in Table 1 suggests that problem difficulty will vary according to the mathematizations used by the solver. Children who use Match conceptions for all problem types may find unknown Extra problems easiest because both matched quantities are known. It seems likely that only by working with such situations for a while can children form a robust enough matching situational conception to deal with having one of the quantities unknown, as in Consistent and Inconsistent problems. Because solving such problems requires deciding which is the Big quantity and which is the Small, and this is easier from the Compare sentences, Compare problems might be easier than Equalize problems just as Matching-conception children begin to be able to solve these.

Children who use Equalize conceptions for all problem types would seem to be able to solve Consistent problems before either unknown Extra or Inconsistent problems. This is because the Consistent Equalize conceptions add or subtract the two numbers given in the situation, and children are able to do this very early, usually in kindergarten. The unknown Extra Equalize forms have the unknown number added on or subtracted from a known quantity. Children can begin to solve such situations relatively early, usually sometime during first grade. At this time, these two problem forms would be equivalent in difficulty. The Inconsistent problems would be solved last because children either must reverse the relation in the sentence (to use Equalize Reverse Relation conceptions) or must solve unknown start situations (if they use Equalize Use Relation conceptions). Both of these are difficult (for a review of levels of developmental sequence of solution procedures, see Fuson, 1992a, 1992b).

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An alternative prediction is that children will use different mathematizations on different problems. The Match conception is particularly easy for unknown Extra problems, and the Equalize conception is particularly easy for Consistent problems. In this case, there might not be differences between these two problem types within Compare wordings, at least.

Predicted Errors

For Compare problems, both the Riley, Greeno, Heller, and Kintsch (Kintsch & Greeno, 1985; Riley & Greeno, 1988; Riley et al., 1983) and the Briars and Larkin (1984) models predict that children who do not get the Inconsistent problem correct will give the known Extra as the answer. This error is hypothesized to stem from children ignoring the word *more* or *less* in the comparing sentence and therefore reading the Extra quantity as belonging to the subject of that sentence (i.e., hearing "he has 2 more dogs than cats" as "he has 2 dogs"). The Briars and Larkin model predicts that, on unknown Extra problems, children not solving the problem will give the known quantity consistent with the comparing word (will give the Big quantity for More problems and the Small quantity for the Less problems), again because they omit the word *more* or *less* in their interpretation of the question. The Riley, Greeno, Heller, and Kintsch model makes no prediction for these problems. On the Inconsistent problems, the Briars and Larkin model predicts either giving the Extra quantity (with the same comprehension error as for the Consistent problem) or using the opposite operation, as discussed earlier for a keyword strategy.

These models do not make predictions concerning errors children will make on Equalize problems. However, Equalize relational statements do not turn into simple descriptions of quantity by dropping the word *more* or *less* as do More or Less relational statements, and the Equalize relational statements have as subjects the compared quantity that is not the subject in the More or Less relational statement. Therefore, the proceeding arguments do not seem to extend easily to Equalize forms.

We have discussed how Inconsistent problems suggest doing the operation opposite to that required, and thus, a solver needs strong attention to which quantity is which in order not to switch the quantities and form an incorrect conception. Such a tendency may be increased by the similarity of English phrasings that have opposite meanings, for example, "2 less cats" and "cats less 2" (cats – 2) and "9 take away 2" versus "take 2 from 9." Issues of directionality in subtraction are especially complex. It seems possible that some switching of the two quantities would occur during problem solving, leading to a few errors of this type.

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THE EMPIRICAL WORK

Data were collected using two different first- and second-grade samples. Both the timing of data collection (beginning of the year versus midyear) and the socioeconomic status (SES) levels of the samples (the beginning-of-the-year sample had considerably more low-SES children than did the midyear sample) indicated that the beginning-of-the-year sample would be developmentally earlier than the midyear sample. The predictions concerning the earlier supportive nature of Equalize versus Compare problems were examined with the developmentally earlier sample; only the "more" and "add to" problems were given in this study because the children were also given a range of other problems. The midyear sample received a complete design of all 12 possible Compare problems resulting from varying the language form (Compare vs. Equalize), the three quantities as the unknown (Big, Small, and Extra), and the additive (Compare More and Equalize +) and subtractive (Compare Less and Equalize –) phrasing of the relational sentence. The use by some second graders of written equations or other forms of all three numbers in a problem allowed an initial exploration of mathematizations of the situation.

Method

Participants. The beginning-of-the-year participants were first ($n = 122$) and second ($n = 102$) graders in two schools in a small city on the northern border of Chicago. There were six classes of first graders, three at each school, and five classes of second graders, three at one school and two at the other school. This sample was racially and economically heterogeneous. One fourth of the school population met the criteria for federal school lunches, another fifth were from blue-collar homes, and there was a substantial upper-middle-class population with highly educated parents.

The first graders had not had experience with any of the word problems. Many of the second graders had participated as first graders in instructional studies focused on moving children from the direct-modeling counting-all and take-away solutions for addition and subtraction expressions (e.g., 8 + 6 and 15 – 7) to sequence counting on and counting up using one-handed finger patterns to keep track (Fuson & Secada, 1986; Fuson & Willis, 1988). On one day they had used word problem situations (Compare More or Compare Less and Equalize + for unknown Extra problems) to give forward meaning to written subtraction sentences (such as 15 – 7) but had not solved other types of Compare problems or worked extensively with unknown Extra problems. Because counting on is a solution procedure only available at Level 2 in the computer models, this instruction might have moved some children from Level 1 to Level 2 earlier than they might have moved themselves. However, most beginning-of-the-year second graders are beyond Level

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1 without instruction, so these children were not different in the accessibility of problems to solution.

The midyear participants were 93 first and 119 second graders from 12 classes in a school in a small city in New Jersey. The students were from middle-class and upper-middle-class backgrounds. Teachers reported that the children had not had experience with the problem types used in this study.

Test and procedure. Problems were given to the beginning-of-the-year sample in the 2nd week of school. All children were given a 24-problem test over 4 days. Six problems were given on each day because we were afraid that the first graders at the beginning of the year would not be able to attend to more problems than that. The test contained three Compare More problems and three Equalize + problems (an unknown Extra, a Consistent, and an Inconsistent problem). These problems were intermixed with other problem types; the results on those problems are not germane and so are not reported here. The first-grade test contained sums less than or equal to 10, and the second-grade test contained sums between 10 and 18. No doubles or combinations with 1 were used because these are typically learned as memorized facts before other combinations and thus might facilitate problem solution. The Compare More and Equalize + problems were of the form used in earlier research in which two people or animals each had a different quantity of the same thing.

For the midyear sample, in January, children were given nine word problems on each of 2 days. All three numbers for each problem were less than or equal to 10, and no doubles (a + a) or problems using a 1 were included. The 18 problems included all 12 of the Compare and Equalize forms given in Table 1. These all concerned various animals owned by Farmer Bob (Table 1 shows only dogs and cats to permit simpler comparisons across problem forms). To provide variety, three noncanonical Change and three Combine problems were also included.

The procedure was the same for both samples. Problems were typed on the child's page. The teacher read each problem aloud to the whole class as many times as children wished. Children could use their fingers, draw on their papers, or add or subtract mentally. Children were not given blocks or other counting objects because most second graders did not need them, and we wanted to keep testing conditions similar across grades.

Scoring. It was anticipated that the beginning-of-the-year first graders might be less accurate computationally than the other children. Therefore, both correct answers and correct strategies were evaluated. Variation across problem forms was similar for the strategy and answer scores, although strategy scores were higher; therefore, only answer scores are reported here. There were only small differences

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between correct answer and correct strategy scores for the midyear sample, so only the correct answer scores are reported here. All scorers and test givers were blind to the predictions of the study.

Results and Discussion

Problem difficulty. The directive solution actions of the Equalize forms were helpful for novice problem solvers, but more developmentally advanced children did as well on the Compare forms (see Table 2). The Equalize forms have about 20% higher correct solutions than the Compare forms but only for our beginning-of-the-year first graders on unknown Extra and Consistent problems (Inconsistent problems were at floor). The Equalize Consistent problem was particularly easy; it was solved by over half of the children. The Consistent problems were easiest for both the Compare and Equalize wordings. In the Riley and Greeno (1988) sample, Consistent problems were also somewhat easier than unknown Extra problems for Subtractive but not for Additive wordings.

For our first-grade midyear and second-grade beginning-of-the-year samples, the predicted interaction between Compare or Equalize wordings and Inconsistent versus other unknowns was present but very attenuated: The Equalize unknown Extra and Consistent problems were a bit easier than the Compare wordings, and the Equalize Inconsistent problems were a bit more difficult. The only difference for the most advanced sample (second-grade midyear) was in the most difficult problems: Equalize Inconsistent problems were easier than Compare Inconsistent problems, which is the opposite of the predicted difference for those problems. Thus, it might have been easier to reverse the action Equalize sentences than the static Compare sentences.

Except for our youngest children (beginning-of-the-year first graders), unknown Extra and Consistent problems were roughly equivalent in accuracy, and these were considerably easier than the Inconsistent problems in almost all cases. Inconsistent problems seem to remain very difficult, even when children can solve the other two types. Within-child analysis indicated that solving the Consistent and unknown Extra problems was somewhat independent: Across the various problem wordings and samples, between 15% and 33% of the children solved one of these problems correctly but not the other. At no age did children show the computer model predicted difficulties: Extra (Level 1) > Consistent (Level 2) > Inconsistent (Level 3).

For certain ages and problem forms, problems using Less were more difficult than problems using More. Inconsistent Less problems were more difficult than Inconsistent More problems for most ages and samples. Younger children (our midyear first graders and Riley and Greeno's, 1988, end-of-the-year first graders) solved somewhat more unknown Extra problems with More than with Less

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TABLE 2

Percentage of Correct Answers by Problem Wording

Sample and Time of Year	Additive Wording						Subtractive Wording					
	Compare More			Equalize +			Compare Less			Equalize –		
	Extra uExtra	*Con uBig*	*Incon uSmall*	*Extra uExtra*	*Con uBig*	*Incon uSmall*	*Extra uExtra*	*Con uSmall*	*Incon uBig*	*Extra uExtra*	*Con uSmall*	*Incon uBig*
First grade												
Beginning of year	7	30	5	31	57	6						
R & G, end of year	33	33	11				17	28	22			
Midyear	78	70	49	87	77	44	65	41[a]	35	80	78	32
Second grade												
Beginning of year[b]	52	53	46	65	66	41						
R & G, end of year	65	60	35				65	80	15			
Midyear	90	89	62	94	85	86	97	88[a]	41	90	95	65

Note. Con = Consistent; Incon = Inconsistent; u = unknown; R & G = Riley & Greeno (1988) end-of-year results with no objects available to the children.
[a]This problem used *fewer* rather than *less*. [b]This sample had numbers with sums between 12 and 18; all other samples had numbers with sums of 10 or less.

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wording. Thus, for young children on unknown Extra problems, familiarity with using the term *more* to suggest a matching situation is more important than any language cuing function of the word *less* suggesting subtraction. The only place that such a cuing function of *less* seemed to be operative was on Consistent and Inconsistent problems for the Riley and Greeno second-grade sample: The Consistent Less problems exceeded the Consistent More problems, but the Inconsistent Less problems (where the cue to subtract would have been wrong) were worse than the Inconsistent More problems.

A mistake was made on our midyear Consistent problems. The word *fewer* rather than *less* was inadvertently used; *less* was used on the other two unknown problems. Many of the first graders seemed to respond to this problem as if *fewer* meant *more*, solving it at the same rate as the Inconsistent More problem.

Inconsistent additive forms were easier than the Inconsistent subtractive forms. The Inconsistent Less problem was the most difficult problem of all. Thus, children seem to have more difficulty undoing or reversing subtractive than additive relations or operations. This is consistent with results for Compare More or Compare Less forms reported by Fuson and Willis (1986) for children and by Lewis and Mayer (1987) and Lewis (1989) for college students.

Incorrect answers. The percentage of children giving a known quantity or using the opposite operation is given in Table 3. For most problems, there was a major difference between grades for both samples: Relatively more first graders gave a number from the problem as the answer and relatively more second graders used the opposite operation.

On Compare problems, errors fit the Briars and Larkin (1984) predictions fairly well on some problems and less well on others. On Compare More problems, fairly high percentages of children did give the Extra on Consistent problems (Riley & Greeno, 1988, also predicted this Extra error) and, on both More and Less unknown Extra problems, did give the known number consistent with ignoring the words more or less in the comparing sentence. On Inconsistent problems with both More and Less, a considerable number of children used the opposite operation, suggesting that they employed a keyword approach or interchanged the two compared quantities.

On most problems, however, some child gave one of the three possible kinds of wrong answers, and on some problems, as many children gave other kinds of answers as gave the predicted errors. Consistent problems using Less had a substantial proportion of opposite operation errors. This was the sample that heard the word *fewer* rather than *less*, so a considerable number of children seem to think that *fewer* means *more*. On Inconsistent problems, a substantial proportion of the beginning-of-the-year first graders gave more known Big quantity answers than the

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TABLE 3
Percentage of Children Giving a Known Quantity or Using the Opposite Operation

Sample	Problem Type								
	Compare More								
	Unknown Extra			*Consistent (Unknown Big)*			*Inconsistent (Unknown Small)*		
	Big[a]	*Small*	*OppOp*	*Extra*[a,b]	*Small*	*OppOp*	*Extra*[a]	*Big*	*OppOp*[a]
Beginning of year									
First grade	34	9	13	20	10	4	8	22	32
Second grade	18	2	10	15	2	4	9	4	3
Midyear									
First grade	12	—	5	18	2	2	5	9	30
Second grade	—	—	6	4	—	6	2	2	32
	Equalize +								
	Unknown Extra			*Consistent (Unknown Big)*			*Inconsistent (Unknown Small)*		
	Big	*Small*	*OppOp*	*Extra*	*Small*	*OppOp*	*Extra*	*Big*	*OppOp*
Beginning of year									
First grade	18	10	10	3	6	3	1	19	33
Second grade	6	1	7	1	5	10	10	6	21
Midyear									
First grade	3	2	2	3	8	6	8	33	12
Second grade	—	—	3	1	2	10	1	3	11
	Compare Less								
	Unknown Extra			*Consistent (Unknown Small)*			*Inconsistent (Unknown Big)*		
	Big	*Small*[a]	*OppOp*	*Extra*[a,b]	*Small*	*OppOp*	*Extra*[a]	*Big*	*OppOp*[a]
Midyear									
First grade	1	17	1	18	3	32	8	5	41
Second grade	—	—	3	—	—	8	—	—	57
	Equalize –								
	Unknown Extra			*Consistent (Unknown Small)*			*Inconsistent (Unknown Big)*		
	Big	*Small*	*OppOp*	*Extra*	*Small*	*OppOp*	*Extra*	*Big*	*OppOp*
Midyear									
First grade	3	2	1	4	4	3	2	37	23
Second grade	—	3	4	—	—	5	—	7	27

Note. OppOp = using the operation opposite to the relation or action in the problem situation wording.
[a]Answer predicted by Briars & Larkin (1984). [b]Answer predicted by Riley & Greeno (1988).

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predicted Extra quantity answers, perhaps indicating a confusion between the two quantities in the situation (Big and Small).

Error patterns on the Equalize + problems were similar to those on Compare More problems for the unknown Extra problem. The giving of the Big number might have resulted from a truncation of the equalize question from "How many cats does he have to get to have as many cats as dogs?" to "How many cats are as many cats as dogs?" On the Consistent Equalize + problems, few children gave the Extra quantity, as predicted, because it is more difficult to misunderstand the question by just ignoring the single word *more* as in the Compare More problem.

On both Equalize Inconsistent problems, there was an equal split between giving the known Big or Small known quantity and using the opposite operation. The latter seems likely to arise from using a keyword approach. Giving the known quantity may have been a response to a variant of the actual Equalize question. The Inconsistent problems state the known quantity in the first sentence, then give an if–then sentence (e.g., "If he gets 2 more cats, he will have as many cats as dogs"). They then ask the value of the unknown quantity *before* the action in the if–then question ("How many cats does he have?"). This time, however, is not explicitly defined: The question could be taken to ask how many cats after the equalizing operation is carried out, which is the known quantity. Only a few second graders made this error, so older children must be better able to remember the initial situation while also carrying out the hypothetical equalizing operation in order to disambiguate the problem question.

On the Inconsistent problems, the midyear sample used the opposite operation more often on the Compare problems than on the Equalize problems and more often on the subtractive language forms than on the additive language forms. Thus, the Compare and subtractive forms seem especially susceptible to keyword methods.

Children in the midyear sample used the opposite operation more on additive unknown Extra problems than on subtractive versions. Thus, at least some children seem to have used keyword approaches in which they added because of the words *more* or *get.*

Problem mathematizations indicated by number sentences. A number of second graders in the midyear sample wrote horizontal or vertical number sentences for the problems. Most such sentences contained three numbers (the two known numbers and the answer that was found), and the answer frequently was not differentiated from the other numbers. Thus, it is not clear whether (a) these numerical sentences initially presented the situation to a child who then filled in the unknown number (e.g., wrote 7 + ___ = 9 and then wrote in the 2); (b) the sentence functioned this way initially, but because the numbers were so small, the child filled in the answer in the act of writing the sentence; or (c) the child already knew the answer, and the sentence was a demonstration that the answer was correct.

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The sentences used are given by problem type in Table 4. In addition to the regular sentences shown in the table, for each problem type, from 1 to 5 children wrote invented comparison notations that showed the two compared quantities with the extra written off to the side: 7 ↔ 9 2 or 7:9 2. These were Match problem conceptions.

The predominant conception for the unknown Extra problems was a Match sentence Big – Small = Extra. Compare More and Compare Less problems had about the same number of such sentences, indicating that the word *more* did not have much addition effect by second grade. There were substantially fewer such sentences for the Equalize + problem than for the Equalize – problem (12 vs. 30), suggesting that the adding-to language was not consistent with a Match conception of the situation. Some children did write Equalizing sentences, with more doing so for Equalize wordings than for Compare wordings. Most of these sentences used the relation given in the wording, but 3 children reversed the relation to write a subtraction sentence for the Equalize + problem.

For the subtractive language problems, Match sentences are the same as keyword sentences. However, as many children wrote Match sentences for the More as for the Less problems. Therefore, by midyear in second grade, few Match sentences for Less problems seem to be only serendipitously correct keyword forms, although this certainly is possible for younger or less linguistically or mathematically advanced children.

A number of children wrote Match sentences in which the subtraction of the given quantities was reversed (e.g., 7 – 9 = 2), but the correct difference was given. These were written on the More and Equalize – problems (8 and 11 children) but not on Less and Equalize + problems. The order of the larger number (first or second) used in the first two sentences of these problems seems to be responsible. On the More and Equalize – problems, the Small quantity was given in the first sentence (suggesting 7 – 9); the Small quantity was given in the second sentence of Less and Equalize + problems (suggesting 9 – 7). Such reversals are not too surprising, given the reversals in common English phrases for subtraction (e.g., "9 take away 7" but "7 from 9"). Even high-achieving second graders wrestle with the order of saying and writing subtraction (Burghardt & Fuson, 1994).

Almost all correct sentences for Consistent problems across all four problem types were Equalizing sentences that began with the known compared quantity and used the comparing word or action in the relational sentence. Most of the correct sentences for the Inconsistent problems also were Equalizing sentences. Most of these began with the known number and reversed the relation, thus giving the unknown as the result of the left side of the sentence. A few sentences began with an unknown number and used the relation given; all but one of these were on the Equalize problems. Perhaps the explicit action relational sentence in these problems strongly suggested a solution form and also enabled children to solve that sentence type with the unknown at the beginning. The Inconsistent problems also had a

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TABLE 4
Number of Midyear Second Graders Writing Particular Number Sentences for Matching Situations

Wording	*Unknown*					
Compare More	Unknown Extra		Consistent (Unknown Big)		Inconsistent (Unknown Small)	
	M $(9 - 7 = \underline{2})$	25	EqUseRel $(7 + 2 = \underline{9})$	40	EqRevRel $(9 - 2 = \underline{7})$	12
	EqUseRel $(7 + \underline{2} = 9)$	2			KIncAns $(9 + 2 = 11)$	13
	KCorAns $(7 + 9 = \underline{2})$	1				
	KIncAns $(7 + 9 = 16)$	7				
Equalize +	Unknown Extra		Consistent (Unknown Big)		Inconsistent (Unknown Small)	
	M $(9 - 7 = \underline{2})$	12	EqUseRel $(7 + 2 = \underline{9})$	21	EqRevRel $(9 - 2 = \underline{7})$	25
	EqUseRel $(7 + \underline{2} = 9)$	3	EqRevRel $(\underline{9} - 2 = 7)$	1	EqUseRel $(\underline{7} + 2 = 9)$	2
	EqRevRel $(9 - \underline{2} = 7)$	3			EqUseRel + SE $(2 + \underline{7} = 9)$	2
	KCorAns $(9 + 7 = \underline{2})$	1			KCorAns $(9 + 2 = \underline{7})$	1
					KIncAns $(9 + 2 = 11)$	9
Compare Less	Unknown Extra		Consistent (Unknown Small)		Inconsistent (Unknown Big)	
	M or KCorAns $(9 - 7 = \underline{2})$	24	EqUseRel $(9 - 2 = \underline{7})$	23	EqRevRel $(7 + 2 = \underline{9})$	8
	EqUseRel $(9 - \underline{2} = 7)$	1			EqRevRel + SE $(2 + 7 = \underline{9})$	1
					EqUseRel $(\underline{9} - 2 = 7)$	1
					KIncAns $(7 - 2 = 5)$	27
Equalize –	Unknown Extra		Consistent (Unknown Small)		Inconsistent (Unknown Big)	
	M or KCorAns $(9 - 7 = \underline{2})$	30	EqUseRel $(9 - 2 = \underline{7})$	36	EqRevRel $(7 + 2 = \underline{9})$	9
	EqUseRel $(9 - \underline{2} = 7)$	11	EqRevRel $(\underline{7} + 2 = 9)$	1	EqUseRel $(\underline{9} - 2 = 7)$	5
					KIncAns $(7 - 2 = 5)$	12

Note. All examples are for a problem in which Big = 9, Small = 7, and Extra = 2. The underlined number is the unknown. The sentence abbreviations are as follows: M = Match; EqUseRel = Equalize use relation; EqRevRel = Equalize reverse relation; KCorAns = Keyword Correct Answer (operating on the two given numbers using the relational word or action but giving the correct answer for the problem situation even though the sentence is not accurate); KIncAns = Keyword Incorrect Answer (operating on the two given numbers using the relational word or action and giving the incorrect answer as dictated by the sentence); SE = subset equivalence (reversing the two addend numbers). EqUseRel begins with the known quantity for Consistent problems and with the unknown quantity for Inconsistent problems. EqRevRel does the opposite.

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considerable number of keyword incorrect sentences that used the opposite operation. Compare Less problems had an especially large number of such sentences.

Thus, over all of these problems, children who wrote solution sentences predominantly wrote sentences that used the two known quantities on the left of the equation. For unknown Extra problems, they predominantly wrote Match sentences. For Consistent and Inconsistent problems, correct sentences were almost all of Equalize form, even for the Compare wordings. For all problem subtypes, from 1 to 5 children also used an invented method that was not a number sentence to show a Match conception. Future research might explore how these sentences were used in problem solving and whether they were used to show the situation, the solution method, or both. Asking all children to use some kind of problem-solving recording, rather than analyzing only spontaneously written recordings, would also ascertain whether the distribution of problem approaches would be the same as in this report.

GENERAL DISCUSSION

These results suggest both a progression of levels in the conceptualization and solving of matching situations and considerable variability (especially initially) in solving a given situation when the problem language is varied. In the first—Relational—level, children can answer "Who has more/less?" but cannot find out how much more or less is represented. Children at this level may give numerical answers by choosing a (random) number, choosing a problem number randomly, or ignoring the word *more* or *less* in the relational sentence.

In the second—Language Cue—level, children are heavily dependent on language cues in the problem wordings. They are more likely to solve the Consistent problems than the unknown Extra problems and Inconsistent problems. Some of these early Consistent solutions are false positive solutions that do not truly reflect understanding and use of the underlying problem situation but only a use of key words in the problem language. Equalize wordings are easier than the static Compare More or Compare Less wordings for both Consistent and unknown Extra problems.

In the third—Understand Matching Situations—level, children solve the straightforward matching situation problems (Consistent and unknown Extra) but not the tricky Inconsistent problems. Equalize (versus Compare) wording is less influential because children can now cope with standard More or Less wordings. Children overwhelmingly solve Consistent problems by an Equalizing conception in which the Extra quantity is added to or taken from the known Big or Small quantity. They predominantly solve unknown Extra problems by a Match conception in which the Small quantity is taken from the Big quantity. For such problems, some children use an Equalizing conception in which the relational sentence directs

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the unknown Extra to be added to the Small or taken from the Big. Perhaps because children predominantly use different conceptions for Consistent and for unknown Extra problems, some children may solve one of these problems without solving the other. The additive Inconsistent problems are easier than the subtractive Inconsistent problems, suggesting that it is easier to reverse an addition operation than a subtraction operation.

At the fourth—Solve Inconsistent—level, children come to be able to solve Inconsistent problems, primarily by using Equalizing conceptions in which the relation given in the relational sentence is reversed. For problems with Equalize wording, a few children instead use Equalizing conceptions that use the given relation but yield an unknown Start sentence that must be solved. Two studies report similar results with German and Belgian children (these became available to us after our situation and strategy analyses were completed and our data were gathered). Stern (1989) reported that German elementary school children wrote many more sentences beginning with the known number and reversing the relation than began with an unknown number and used the given relation. Stern (1993) identified knowledge about the relational sentence that seems to be prerequisite to solving Inconsistent problems. She found that children who solve Inconsistent problems could choose both forms of a relational sentence as correct, whereas children not solving such problems asserted that only one of the two forms of the relational sentence was correct. Verschaffel (1994) reported retelling data that indicated that successful Belgian solvers of Inconsistent problems began with the known set and reversed the relational sentence: 61% of the 73% correct retellings had the relational sentence reversed, and a considerable proportion of children who retold the problem as it was given first began a reversed relational sentence and then corrected themselves.

These four levels are consistent with results reported by Fuson and Willis (1986) for first- and second-grade classes classified by achievement in a study that used a complete design of all 12 Compare and Equalize problems but used the word *fewer* rather than the word *less.* Interpretation of results for some lower achieving or younger classes was complicated in that study by performance on the *fewer* problems that seemed to have resulted from children's understanding *fewer* to mean *more.* The second level is also confirmed for German first graders by Stern and Lehrndorfer (1992), who found with More or Less problems that Consistent was greater than unknown Extra, which was equal to Inconsistent, with performance on the latter two low.

All of our proposed solution methods depend on understanding which quantity is bigger and which is smaller (who has more and who has less). Helping solvers decide which quantity is bigger would therefore seem to facilitate performance. Some recent evidence indicates that this is true for children and adults. Stern and Lehrndorfer (1992) reported positive effects of problem manipulations that increased children's understanding of who has more. With German first graders, they

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used a special context condition that supported understanding of which person had more (the older sibling had several different kinds of more or larger quantities). Children did better on unknown Extra and Inconsistent problems than in the ordinary control condition. Another condition of theirs indicated that children can solve unknown Extra problems by becoming more aware that someone has more, even if they do not correctly determine who has more. In this condition, children heard similar stories about siblings who had several different inequitable entities, but the entity in the relational question was the reverse of all of the other entities and of the siblings' ages. More children solved the unknown Extra problem than in the control condition, but other problems did not improve. An early version of a Match conception in which the Big and Small quantities are not distinguished would be sufficient to solve an unknown Extra problem because children would subtract the smaller from the larger number. Lewis (1989) reported with college students that teaching them a strategy in which they had to decide which quantity was larger facilitated performance on Inconsistent forms.

Our results with children concerning solutions to Inconsistent problems are consistent with the results reported by Mayer and colleagues (Hegarty, Mayer, & Green, 1992; Lewis & Mayer, 1987) for college students: Many reverse the relational sentence to solve such problems. Thus, even these older students find it easier to begin with the known quantity and reverse the relational sentence (use an unknown result form) than to keep the relational sentence and begin the solution conception with an unknown quantity. Mayer and colleagues conclude that their results are due to a preference for problems in the Consistent form. Our analysis suggests that this preference is for beginning the solution with the known quantity, perhaps because doing so facilitates the arithmetical solving required. The language in Consistent problems allows one to begin with the known quantity and use the given relation without reversing it. Inconsistent problems require beginning with an unknown quantity (if one uses the given relation) or reversing the given relation (in order to begin with a known quantity). If people had a preference for beginning solutions with an unknown number, Inconsistent problems would be easier. Hegarty et al. also concluded that college students getting Inconsistent problems incorrect did not form a conception of the problem situation but used a keyword approach. Thus, this primitive keyword method does not necessarily disappear with age or experience.

A related difficulty in the domain of multiplication that also remains for a long time, even for college algebra students and well-educated adults, is the well-documented professor–student problem error. Many people directly translate the English sentence, "There are six times as many students as professors," into the equation $6S = P$ instead of equalizing the relation to form the correct solution equation $6P = S$ or $S/6 = P$ (e.g., Kaput & Clement, 1979; Lochhead, 1980). As with the addition and subtraction Compare More or Compare Less wordings, one must think about which quantity is larger and write the equalizing equation to reflect that relation

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(think "There are many more students than professors, so I must multiply the number of professors by six to make that product equal the number of students"). Our results, and the underlying analysis of Compare and Equalizing language, suggest that a major source of difficulty in dealing with additive and with multiplicative compare problems is two confusions made by solvers of such problems. One is a confusion between writing a problem situation equation and writing a solution procedure equation, and the other is a confusion between the underlying compare situation and the equalizing solution that will solve that compare situation. When students write 6S = P for the student and professor problem (or 7 + 3 = 10 for "María has 7. María has 3 more than Juan. How many does Juan have?"), they are recording the problem situation. There are 6 students for each professor (6S:P), and María does have 7 and does have 3 more than (7 + 3) Juan. But the equation form for a compare situation does not show the compared situation as it exists in the world. The equation form shows quantities that have been made equal, that have been equalized. The equation 6P = S shows the opposite of the actual relation in the situation (6 students per professor); the quantities P and S have been equalized by using on P the opposite of the relation in the world. Similarly, M – 3 = J shows how to find Juan's quantity from María's quantity; the "3 more than" relation must be reversed. For both additive and multiplicative Compare wordings, an equation shows what must be done to the numbers to find the unknown: It is a solution procedure equation that equalizes (undoes) the relation in the situation. Deciding which quantity is Big and which is Small and then writing an equation to turn one of these quantities into the other (i.e., equalizing it) will form a correct equation.

These two confusions, between the situation–solution function or (goal) of writing an equation and between the situation–solution nature of the Compare and Equalizing language, relate to and extend several accounts of the source of difficulty in the professor–student problem (for a recent review and new evidence, see MacGregor & Stacey, 1993). Syntactic translation, Clement's (1982) *static comparison,* Kaput's (1985) *imagistic strategy* and Kaput's (1987) *power of natural language,* and the empirical results and *cognitive models* of MacGregor and Stacey (1993) all seem to result from (a) the (perhaps unconscious), goal of writing a situation equation rather than a solution equation accompanied by (b) a lack of understanding (or use of knowledge) that the Compare language is only situation language. Different kinds of language express the situation (Compare language), a solution method linked to the situation (Equalizing language), and a solution method (the equation language used by MacGregor & Stacey, 1993). The difficulty of the Compare language is that it inherently describes the relation in the situation. Furthermore, because this relation can always be described in two ways, one way will always be opposite to the needed solution method, so difficult versions of any situation can always be posed. The Equalizing language is perfect in-between language: It describes what happens to the numbers but does so within the situation ("If you hired six times as many professors as there are now, there would be as

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many professors as students"). Trying to say or write such Equalizing relational sentences might be an effective way for older students to solve Inconsistent Compare language problems, because the Equalizing language parallels equation forms more readily than does Compare language.

More generally, failure to distinguish between a mathematization of the problem situation (a mathematical situation conception) and the solution method (a solution procedure conception) complicates research discussions about word problem solving as well as individual efforts at problem solving. For example, in the studies by Mayer and colleagues (Hegarty et al., 1992; Lewis & Mayer, 1987) discussed earlier, problems are classified as addition or subtraction problems only according to the solution method required and not also according to the problem language. In Inconsistent problems, these will be opposite. Because a given problem can be solved in various ways (e.g., by a forward unknown addend approach as well as by a subtractive take-away approach) and can be worded in various ways, classification by both is helpful. Specifying the unknown separates problems into those requiring a solution method adding two quantities (unknown Big problems) and those requiring a solution method finding an unknown addend (unknown Small or unknown Extra), but it does not specify the solution conception used by a given problem solver. The problem language must be specified to differentiate Consistent from Inconsistent problems.

A similar situation–solution confusion seems to be involved in the use of a part–whole schema to solve Inconsistent problems by the Riley, Greeno, Heller, and Kintsch (Kintsch & Greeno, 1985; Riley & Greeno, 1988; Riley et al., 1983) word-problem-solving model and in uses by others of this part–whole schema. This part–whole schema is discussed as if it is related to the part–whole schema used for Combine word problems (e.g., "How many do Joe and Susan have altogether?"); that is, it is situational. However, the knowledge required in the Compare Inconsistent problems is that of advanced numerical relations needed in the solution method, not in the original construction of the matching situation conception (for reviews of levels in numerical solution methods and their use in word problem solving, see Fuson, 1992a, 1992b, 1994).

A recent article by Fan, Mueller, and Marini (1994) is relevant both to the issue of levels of solving and to the confusion of situation and solution conceptions. Fan et al. identified strategies first graders used in solving with blocks unknown Extra problems using Compare More, Equalize +, and Won't Get (as in "How many birds won't get a worm?"; Hudson, 1983) wordings. They found that the wording affected the strategy: Children for Equalize + wordings predominantly added on blocks one by one to the Small blocks pile (or row) to make it equal to the Big blocks pile; for Won't Get wordings, predominantly matched the two piles of blocks in rows and counted or took away the Extra blocks; and for Compare More wordings, did a mixture of strategies including what they called a Part–Whole strategy (defined as describing the solution method as addition or subtraction without counting of

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individual blocks). These interesting results indicate effects of problem wording for the simple unknown Extra problem type. However, these strategies also reflect different levels of solution methods, so problem wordings may also afford display of such solution method levels. Matching is a Level 1 direct modeling strategy, adding on is a Level 1½ strategy (the first addend is truncated but not necessarily embedded within the larger number), and part–whole is a Level 3 or 4 strategy expressed as addition or subtraction. Differentiating solution methods by developmental level would be helpful in future research on matching situations. To return to the original situation versus solution distinction, using the term *part–whole* for a solution strategy in the context of the Riley et al. (1983) model, as in the Fan et al. article, seems unnecessarily confusing. Part–whole is used in many different ways in the early number and arithmetic literature; using a more descriptive term such as *addition* or *subtraction* for the solution method not involving counting would be preferable.

A final example of confusion between the situation conception and the solution conception (and some confusion concerning levels of problem solving in matching situations) occurs with the use of Okamoto's (1994) application of Case's (1992) central numerical structure to word problem solving, and Fan et al.'s (1994) discussion of it with respect to their first-grade data. Fan et al. do make these situation–solution distinctions in some places but conclude that "the coordination of mental number lines" (p. 355) is the crucial component in solving such problems. Such coordination of mental number lines is a later development in Case's numerical structure. There was no evidence in Fan et al.'s data that children used mental number lines at all; their evidence instead spoke eloquently to children's use of blocks to model the situation given in the problem, even when they had no blocks available (they instead then imagined and counted them). Children clearly first modeled the situation with blocks (i.e., mathematized the problem with blocks) and then acted on that mathematization to find the answer (shifted to using it as a solution conception). Any "mental number line" would have been a numerical version of the matching problem situation, that is, a later developmental solution method involving counting or adding up or down a number list (using the counting sequence as a representational tool where the words in the number list are the objects added or subtracted). The coordination in this study came from the matching nature of the situation, not from independent mental operations of the children. Thus, it would be more accurate to describe children as using a "number list" rather than a "number line" because a list uses discrete objects (such as the counted blocks or the words themselves as objects), and a number line contains continuous measures (lengths). The source of the coordination in the underlying matching situation with discrete objects also should be made clear. Furthermore, it would be useful to differentiate the use of the list as a tool to count objects versus its use as a mental tool in which the number words become the objects because these are different

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developmental stages. Without these distinctions, the use of Case's central numerical structure (successive unitizing and coordination of number lists) in research discourse will get as confused as the use of the term *part–whole schema.*

Solving word problems is a demanding task that requires a great deal of special linguistic and real-world knowledge and also requires actually trying to solve the problem (as opposed to using a deficient approach such as the keyword approach). Most such word problem solving is carried out in the complex social situation that is a classroom, so it is subject to various kinds of attentional lapses or intrusions. This has implications both for trying to model children's word problem solving and for instruction. Using only individual interview data in which children are asked to model the problem situation one sentence at a time with objects considerably oversimplifies a child's actual problem-solving task in the classroom. However, cues outside the problem may be available in the classroom (e.g., children saying answers or giving intermediate problem information such as who has more); such cues may support or interfere with problem solving. At the moment, we have little data on children's problem-solving methods in classrooms. Ultimately, models of word problem solving should also deal with the social contexts of learning to solve such problems. For example, classrooms in which counting up to or adding on is discussed might lead to substantially more solving of Match conceptions by these methods instead of by subtraction.

Although there were systematic effects of additive or subtractive and of Compare versus Equalize wordings, there were also variations in the errors on and approaches to given problems. Clearly there are strong individual differences and, perhaps, class learning history differences in conceptualizing and solving matching problems. More research is necessary to understand more thoroughly these interactions of presentational structure with individual solution approaches. Studies in which children are asked to draw the situation and then describe their solution method would seem helpful as well as those in which languaging and symbolizing are observed and discussed after problem solution. Instructional studies in which children are helped to understand Compare More or Compare Less language or to reverse relational sentences would also seem fruitful. We hope that the analyses given here of matching situations, of Compare and Equalize language issues, and of varying solution approaches will clarify future research and instructional interventions concerning the solving of such situations.

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