Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case of Mathematics

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We argue that reform in curriculum and instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems. In mathematics, this principle fits under the umbrella of problem solving, but our interpretation is different from many problem-solving approaches. We first note that the history of problem solving in the curriculum has been infused with a distinction between acquiring knowledge and applying it. We then propose our alternative principle by building on John Dewey's idea of "reflective inquiry," argue that such an approach would facilitate students' understanding, and compare our proposal with other views on the role of problem solving in the curriculum. We close by considering several common dichotomies that take on a different meaning from this perspective.

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We are in the midst of educational reform. National and state standards for curricula, teaching, and assessment are emerging at a rapid rate. Decisions are being made at national, state, and local levels that aim to change classroom practice. So many different perspectives and criteria are being used to make these decisions that, as David Cohen (1995) observed, we are experiencing the "gathering babel of reform ideas and practices" (p. 13). Although the widespread attention to reform holds promise for change, the confusing array of ideas on how classrooms should look leaves teachers with the difficult task of sorting out what is really essential.

The purpose of this article is to propose one principle for reform in curriculum and instruction. The principle is this: students should be allowed to make the subject problematic. We argue that this single principle captures what is essential for instructional practice. It enables us to make sense of the chaos, to sort out what is indispensable from what is optional. By itself, the principle does not specify curriculum nor prescribe instruction. But it does provide a compass that points classroom practice in a particular direction and that checks the alignment of its basic elements.

Allowing the subject to be problematic means allowing students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas, and questions for students. We do not use "problematic" to mean that students should become frustrated and find the subject overly difficult. Rather, we use "problematic" in the sense that students should be allowed and encouraged to problematize what they study, to define problems that elicit their curiosities and sense-making skills.

To develop our proposal, we focus on mathematics. Although we recognize that the subject matters (Stodolsky, 1988) and mathematics possess some unique features, we believe that the principle is relevant for all school subjects and that the issues we raise will be familiar to educators working in other disciplines.

To illustrate what the principle means for practice and to provide a point of reference, we first present an example from a second-grade classroom in which the development of arithmetic is treated as problem solving. We chose this example because the distinctions we will try to articulate are revealed most clearly when considering a topic that is usually treated as a routine skill. After presenting the example, we begin our analysis by reviewing briefly the history of problem solving in the curriculum. We note that the classroom episode we have just presented is at odds with most historic views of problem solving. We then re-establish the principle of problematizing the subject by building on John Dewey's notion of reflective inquiry. We argue that the benefit of this approach is that it yields deep understandings of the kinds that we value. We close by contrasting our view of problem solving with other current

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views on how problems should be used in teaching mathematics and by pointing to several common dichotomies that seem to collapse from this perspective.

A Classroom Example

Ms. Hudson's second-grade class is located in an urban school with a large Latino population. Prior to this lesson, in January, the students had been working on addition and subtraction problems, developing their own methods for solution. Some students had been using base-10 materials, such as sticks with 10 dots on them. This day's problem was to find the difference in the height of two children, Jorge and Paulo, who were 62 inches tall and 37 inches tall, respectively. After most of the students had worked out a solution, Ms. Hudson asked for volunteers to share their methods.

Gabriela, the first student to share, had solved the problem by counting up from 37 to 62. In the process, she counted by 1s and 10s, keeping track of her counts by drawing single dots to represent 1s and drawing sticks to represent 10s. She counted from 37 to 40, making 3 dots as she counted. Then she counted from 40 to 60 by 10s, drawing 2 sticks to show the 2 10s. Finally, she counted up to 62, making 2 more dots. She described her solution as follows:

> Gabriela: "I said, 'How much does Paulo have to grow?' so 37 plus 3 more [pointing to the 3 dots] is 38, 39, 40, and 50 [pointing to a 10 stick], 60 [pointing to another 10 stick], 61, 62 [pointing to 2 more dots]. So this is 23, 24, 25 more he has to grow."
>
> Ms. Hudson: "OK. Roberto?"

Roberto had first drawn a picture of Jorge and Paulo and extended a horizontal line from the top of Paulo's head across to Jorge. He had then drawn 6 10 sticks and 2 dots to represent 62. He took away 37 by first crossing out 3 sticks to take away 30. Then he put a mark on the 4th stick about three tenths of the way down and wrote a small 7 below the mark to show the 7 taken away and a 3 above the mark to show that 3 were left over. He then combined the 3 above the mark, the 2 dots, and the remaining 2 sticks to get 25. Here is what Roberto said:

> Roberto: "I shrunk the big guy down by taking away the little guy from him [pointing to his drawing of Paulo and Jorge]. I took 3 10s from the 6 10s and 7 from this 10 [pointing to the 4th stick]. That leaves 3 and these 2 are 5 and 2 10s left is 25."
>
> Ms. Hudson: "OK. Now I am going to ask Jose how he did it."

Jose had added up from 37 to 62 using a combination of numerals and drawings to help. He had written, in a single line, "37," then "3" (to make 40), then 2 sticks (to make 60), then "2" (to make 62). He then looked back and combined the 3, the 2 sticks, and the 2 to get 25.

> Jose: "I did it like Gabriela, but I wrote 3 and then my 10 sticks and 2 and then added them to get 25 more the little guy needs."

Maria was the next student to share a strategy. She had written 62 minus 37 in vertical form and used a more conventional subtraction procedure except that she first subtracted 7 from 10 (rather than 12), combined the 3 left over with the 2 to get 5, then subtracted 3 10s from the 5 remaining 10s to leave 2 10s and 5.

> Maria: "I subtracted Paulo from Jorge like Roberto did, but I used numbers. I took one of the 10s to get enough to take away the 7, so that was 3 and 2 more was 5 1s, and there were 2 10s left, so 25."
>
> Ms. Hudson: "Can someone tell how Roberto's and Maria's methods are alike?"
>
> Carlos: "They both took away the little guy."
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> Ms. Hudson: "Anything else?"
>
> Jazmin: "They both had to open a 10 because there weren't 7 1s to take away. So Roberto took his 7 from that 10 stick. He took 7 and left 3. And Maria took a 10 from the 6 10s and wrote it with the 1s and then took the 7 to leave 3."
>
> Ms. Hudson: "So they were both thinking kind of alike but wrote it in different ways?"
>
> Students: "Yes."

For the students in this class, the generation of procedures for computing with multidigit numbers was a problem-solving activity. Ms. Hudson had not demonstrated any of the methods that the students shared. The students constructed their own methods, either individually or collectively through peer interactions, using their knowledge of the base-10 number system.

What is striking about this example is not that students solved an exceptionally difficult problem nor that they displayed brilliant insights but rather that students became engaged in genuine problem solving in what is potentially the most routine of activities. These second graders worked diligently on the problem for over 10 minutes and then spent another 10 minutes presenting and discussing alternative solution methods. The students did not perceive the task as routine, and they were motivated to find and explain their alternative methods. This all happened because they were allowed to problematize what is usually taught through demonstration and repeated practice.

A Brief History of Problem Solving in the Curriculum

Most historic accounts of problem solving in school mathematics would not characterize finding the difference between 62 and 37 as a genuine problem nor would they identify the activity of the students in Ms. Hudson's classroom as an instance of problem solving. The reason is that conceptions of problem solving have been colored by a distinction between acquiring knowledge and applying it. The distinction suggests that computation procedures should be acquired first and then applied to solve problems. But the distinction is more pervasive than this.

Acquisition and Application

Since the early 20th century, the mathematics curriculum has been shaped periodically by concerns about preparation for the workplace and for life outside of school (Stanic & Kilpatrick, 1988). These concerns have framed the debates about curricula around applications. The salient distinction has been between acquiring knowledge in school and applying it outside of school. Problems have been used as vehicles for practicing applications.

The 1930s and 1940s witnessed a movement to design the mathematics curriculum around real-life situations. Students were to learn the mathematics they needed in the context of solving problems. This would ensure that the knowledge they acquired would be useful. Opposing views argued that this "incidental" learning was haphazard and insufficient. Important mathematics would get lost. The curriculum should rather be designed around the important ideas and skills that should be acquired (see Brownell, 1935; Reeve, 1936).

The tension between acquiring knowledge and applying it is not special to mathematics. It is at work in most disciplines that comprise the school curriculum. How much emphasis should be placed on acquiring the concepts and skills of the subject and how much on applying them in realistic situations?

The Focus on Application

Recent concerns about school learning have noted the wide gulf between acquiring and applying knowledge. One response has taken the form of "problem-based learning." A number of faculty in professional schools around the country have noted that the knowledge acquired in the classroom does not transfer well to the profession, whether it be medicine, engineering, social work, or education (Boud & Feletti, 1991a). In order to increase the usefulness of students' knowledge, some schools have adopted the model of problem-based learning or case-based instruction (Shulman, 1992). As described by Boud and Feletti (1991b), problem-based learning "is not simply the addition of problem-solving activities to otherwise discipline-centered curricula, but a way of conceiving of the curriculum which is centered around key problems in professional practice. Problem-based courses start with problems rather than with exposition of disciplinary knowledge" (p. 14). As was the case 50 years ago, critics claim that this approach focuses only on applications and worry that important information will get lost in the less predictable curriculum.

Mathematics education has witnessed a similar resurgence of interest in developing curricula that encourages the study of mathematics in the context of real-life problems. In contrast to the basic-skills curricula of the 1970s, with its emphasis on acquiring the mechanics of mathematics, the recent reform recommendations place a heavier emphasis on applications and connections of mathematics to the real-world (National Council of Teachers of Mathematics, 1989, 1991). Large-scale, real-life problems are proposed as appropriate contexts for learning and assessment (Burkhardt, 1981; Cognition and Technology Group at Vanderbilt, 1990; Lesh & Lamon, 1992; Romberg, 1992). Mathematics acquired in these realistic situations, proponents argue, will be perceived by students as being useful. Rather than acquiring knowledge that is isolated from real situations, students will acquire knowledge that is connected to such situations, and they will be able to apply this knowledge to a range of real-life problems. Although these approaches have been widely endorsed, we believe they do not resolve the difficulties that are inherent in the distinction between acquiring knowledge and applying it. The distinction may be somewhat blurred, but it still exists.

An Alternative View of Problems and Problem Solving

We believe that the distinction between acquiring knowledge and applying it is inappropriate for education.[1] By making the distinction, educators have separated mathematical activity into two artificial categories and then have created equally artificial methods to bring them back together. To understand some of the philosophical roots of the distinction and to develop the alternative principle of problematizing the subject, we reconsider John Dewey's analysis and his central notion of reflective inquiry.

Revisiting the Distinction Between Acquisition and Application

The separation between acquiring knowledge and applying it builds directly from the distinction in philosophy between knowing and doing. Dewey (1929) argued that its ancient roots can be found in humans' desire for certainty. We have a long-standing belief, said Dewey, that knowing produced by reason and thought is potentially certain. Ideas can be abstracted from the particulars of experience and thereby become stable and reliable. Doing, on the other hand, is unreliable and uncertain. The outcomes are not always predictable. Doing involves interacting with the real world, and such interactions are filled with changing circumstances that we cannot control.

What is important here is that the distinction between knowing and doing has become so pervasive and so subtle that it permeates our thinking. "We are so accustomed to the separation of knowledge from doing and making that we fail to recognize how it controls our conceptions of mind, of consciousness and of reflective inquiry" (Dewey, 1929, p. 22). Among other effects, the distinction has spawned a number of familiar dichotomies such as theory versus practice, reason versus experience, and acquiring knowledge versus applying knowledge.

Dewey's View of Problem Solving

Dewey moved beyond the pervasive, almost inescapable distinction between knowing and doing by a strikingly simple approach: he considered the methods people ordinarily use to deal with everyday problems to turn doubtful and uncertain situations into ones that are more predictable and certain. Dewey (1910, 1929, 1938) observed that thoughtful but ordinary methods of solving problems share fundamental features with the more refined methods of scientists, and the differences are in degree, not in kind.[2] Dewey placed great faith in scientific (and ordinary) methods of solving problems. He referred to the methods by several names including the "experimental practice of knowing" (1929) and "reflective inquiry" (1933). He believed reflective inquiry was the key to moving beyond the distinction between knowing and doing, thereby providing a new way of viewing human behavior. More than that, he believed that the method provided a target for intelligent human behavior. To the extent that we could use the method of reflective inquiry, we would be acting intelligently. "The value of any cognitive conclusion depends upon the *method* by which it is reached, so that the perfecting of method, the perfecting of intelligence, is the thing of supreme value" (Dewey, 1929, p. 200, emphasis in original).

The fundamental features of reflective inquiry can be stated simply: (1) problems are identified; (2) problems are studied through active engagement; (3) conclusions are reached as problems are (at least partially) resolved. It is worth elaborating briefly on each feature.

Identifying problems. The process begins with the recognition or definition of a problem. Problems are identified as such if the participant sees a quandary or feels a difficulty or doubt that needs to be resolved. "The origin of thinking is some perplexity, confusion, or doubt. Thinking is not a case of spontaneous combustion" (Dewey, 1910, p. 12). Stated in slightly different terms, "All reflective inquiry starts from a problematic situation" (Dewey, 1929, p. 189). The importance of this claim for Dewey lay not only in the fact that problems trigger reflective inquiry but also in the proposition that those who engage in reflective inquiry look for problems. They problematize their experiences in order to understand them more fully. This results in a radical reorientation. Familiar objects, including subject matters in school, are treated as "challenges to thought. . . . They are *to be* known, rather than objects of knowledge. . . . [t]hey are things *to be* understood" (Dewey, 1929, p. 103, emphasis in original). "The subject-matter which had been taken as satisfying the demands of knowledge, as the material with which to frame solutions [becomes] something which sets *problems*" (Dewey, 1929, p. 99, emphasis in original).

When we treat an object as a problem to be solved and examine it carefully, said Dewey (1929), we begin to understand it, to gain more control over it, and to use it more effectively for our advantage. The objects can be school topics, including things as ordinary as arithmetic computation procedures. As in Ms. Hudson's class, treating procedures as problems and examining them carefully affords students the chance to understand them, gain more control over them, and use them more effectively.

Searching for resolutions. Once a problem has been identified, the participant actively pursues a solution by calling up and searching out related information, formulating hypotheses, interacting with the problem, and observing the results. Several characteristics define this activity. It involves action, overt doing, that changes something about the problem and/or the situation in which the problem is embedded. Activity is central to the process. This is why Dewey (1929) claimed that, "The experimental procedure is one that installs doing as the heart of knowing" (p. 36).

A second defining characteristic of searching for solutions "involves willingness to endure a condition of mental unrest and disturbance" (Dewey, 1910, p. 13). It is always tempting to establish certainty too quickly by jumping to conclusions. But this undermines the process. Reflective inquiry, which Dewey (1929) equated with scientific forms of investigation, takes a much different view: "A disciplined mind takes delight in the problematic. . . . The scientific attitude may almost be defined as that which is capable of enjoying the doubtful" (p. 228).

School instruction, said Dewey, is plagued by a push for quick answers. This short-circuits the necessary feeling of uncertainty and inhibits the search for alternative methods of solution. The result is a single, mechanically executed procedure that may yield the correct answer but shifts the attention away from the quality of methods. "Probably the chief cause of devotion to rigidity of method is, however, that it seems to promise speedy, accurately measurable, correct results. . . . Were all instructors to realize that the quality of mental process, not the production of correct answers, is the measure of educative growth something hardly less than a revolution in teaching would be worked" (Dewey, 1926, pp. 206–207). As the excerpt from the second-grade class demonstrates, attention to method and process can be realized even in areas of the curriculum where rigidity of method and speedy answers have been the norm.

Reaching conclusions. Eventually some conclusion is reached, some resolution is achieved, some hypotheses are refined. The outcome of the process is a new situation, and perhaps a new problem, showing new relationships that are now understood. "The outcome of the directed activity is the construction of a new empirical situation in which objects are differently related to one another, and such that the *consequences* of directed operations form the objects that have the property of being *known*" (Dewey, 1929, pp. 86–87, emphasis in original). The benefits of reflective inquiry lie not in the solutions to problems but in the new relationships that are uncovered, the new aspects of the situation that are understood more deeply. When the second graders were finding the differences in heights, they were exploring relationships within the number system, not just finding an answer. The relationships constructed are the things of primary value.

According to Dewey (1929), these relationships and understandings are what is left after the problem has been resolved. They constitute knowledge for the participant. "Fruits remain and these fruits are the abiding advance of knowledge. . . . Knowledge is the fruit of the undertakings that transform a problematic situation into a resolved one" (pp. 192, 242–243). This does not mean, of course, that every participant will be left with the same knowledge. The nature of the knowledge will depend on the prior knowledge available to the participant when engaged in inquiry and the kind of operations that were used during investigative activity. But new understandings of some kind are the expected outcome of the process.

Moving Beyond Dewey

Dewey's notion of reflective inquiry provides a useful starting point for elaborating the principle of problematizing the subject. His concern with both the mental and social processes of learning also foreshadows the kinds of classrooms that take shape when teachers and students treat the subject as problematic.

However, as we continue our argument, we depart from Dewey, in practice if not in spirit, in two respects. We extend the range of tasks that can become problematic beyond those he cited as exemplary problem situations, and we extend the arguments that link reflective inquiry with understanding. Although Dewey's voice will still be heard in the discussion and we will show that our proposition is largely consistent with Dewey's position, we build the details of our argument from our own work and that of others.

Problematizing Mathematics and Developing Understanding

We work from an assumption that understanding is the goal of mathematics instruction. In fact, we justify the practice of problematizing the subject by claiming that it is this activity that most likely leads to the construction of understanding. To support this claim, we look at how problematizing fits within two very different views of mathematical understanding: a functional view and a structural view. These views can be seen as competing and even incompatible. The reason we include both is to show that the principle of allowing students to treat the subject problematically can be interpreted meaningfully from both perspectives. This allows us to consider students' construction of understanding from different perspectives and to uncover aspects of this process that might otherwise remain hidden.

Functional Understanding

From a functional perspective, understanding means participating in a community of people who practice mathematics (Brown, Collins, & Duguid, 1989; Derry, 1992; Lave, Smith, & Butler, 1988; Lave & Wenger, 1991; Schoenfeld, 1988). Understanding is participating. "Knowing is not the act of an outside spectator but of a participator" (Dewey, 1929, p. 196).

The functional view focuses on the activity of the classroom. Understanding is defined in terms of the ways in which students contribute to and share in the collective activity of the here and now. We argue that the key to shaping classroom activity that invites participation is to allow the subject to be problematic. We can be more specific.

The role of the teacher. The teacher bears the responsibility for developing a social community of students that problematizes mathematics and shares in searching for solutions. A critical feature of such communities is that the focus of examination and discussion be on the methods used to achieve solutions. Analyzing the adequacy of methods and searching for better ones are the activities around which teachers build the social and intellectual community of the classroom. In our example, Ms. Hudson can be seen guiding the discussion so the focus was placed on eliciting methods and analyzing their features. Other examples are contained in a growing number of case-study descriptions of classrooms in which the teacher emphasizes the open and constructive examination of methods of inquiry and solution (Ball, 1993; Cobb, Wood, Yackel, & McNeal, 1992; Fawcett, 1938; Fennema, Franke, Carpenter, & Carey, 1993; Fraivillig, Murphy, & Fuson, 1996; Murray, Olivier, & Human, 1993; Lampert, 1989; Resnick, Bill, & Lesgold, 1992; Schoenfeld, 1985).

We touch on two specifics of the teacher's role: providing information and setting tasks. Dewey (1910) recognized the importance of the first issue: "No educational question is of greater importance than how to get the most logical good out of learning through transmission from others" (p. 198). Clearly students can benefit from having access to relevant information; they would make very slow progress if they were asked to rediscover all of the information available to the teacher. On the other hand, too much information imposed with a heavy hand undermines students' inquiries. Our position is that the teacher is free, and obligated, to share relevant information with students as long as it does not prevent students from problematizing the subject.

In Dewey's time, as in ours, teachers more often erred on the side of providing too much information with too prescriptive a tone. Recall Dewey's concern with a single rigidly prescribed method. However, in an instance of history repeating itself, Dewey (1933) noted later that some teachers who were applying his ideas had the mistaken impression that they were supposed to withhold information and ideas from students and simply let them explore. There are some today who advocate such an approach. We agree with Dewey's (1933) observation: "Provided the student is genuinely engaged upon a topic, and provided the teacher is willing to give the student a good deal of leeway as to what he assimilates and retains (not requiring rigidly that everything be grasped or reproduced), there is comparatively little danger that one who is himself enthusiastic will communicate too much concerning a topic" (p. 270).

The teacher will need to take an active role in selecting and presenting tasks. Tasks do not just appear, and it is unlikely that students spontaneously will create tasks that sustain reflective inquiry in mathematics. To select appropriate tasks, the teacher must draw on two resources: knowledge of the subject to select tasks that encourage students to wrestle with key ideas and knowledge of students' thinking to select tasks that link with students' experience and for which students can see the relevance of the ideas and skills they already possess.

It is at this point that we part company with Dewey. Although Dewey (1926) identified the same two sources of knowledge that are essential for selecting tasks and although he did not explicitly preclude the range of tasks that we endorse, he usually pointed to tasks that were drawn from students' outside-of-school experiences (Cremin, 1964; Dewey, 1915). Dewey (1933) later criticized the practice in some "progressive" classrooms of simply importing out-of-school activities and assuming that learning would occur incidentally (see Prawat, 1995), but he continued to underscore the benefits of relatively large-scale real-life problems.

We propose that reflective inquiry and problematizing depends more on the student and the culture of the classroom than on the task. Although the content of tasks is important, the culture of the classroom will determine how tasks are treated by students. Tasks such as 62 minus 37 can trigger reflective inquiry because of the shared expectations of the teacher and students although they may look routine and are contained entirely within the domain of mathematics. Given a different culture, even large-scale real-life situations can be drained of their problematic possibilities. Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them. This means that tasks of much greater variety than described by Dewey can be used by teachers to help students problematize mathematics. This is a central point of our proposal, and we will take it up again later.

The role of the students. Students share the responsibility for developing a community of learners in which they participate. We highlight two aspects of the students' role in reflective inquiry classrooms. First, students must take responsibility for sharing the results of their inquiries and for explaining and justifying their methods. This creates the openness that is essential for examining and improving the methods and for becoming full participants in the community. "One of the most important factors in preventing an aimless and discursive recitation consists in making it necessary for every student to follow up and justify the suggestions he offers. . . . Unless the pupil is made responsible for developing on his own account the *reasonableness* of the guess he puts forth, the recitation counts for practically nothing" (Dewey, 1933, p. 271, emphasis in original).

A second responsibility for students is to recognize that learning means learning from others, taking advantage of others' ideas and the results of their investigations. This requires students to listen. We have in mind more than listening out of politeness or respect, but also listening because of a genuine interest in what the speaker has to say (Paley, 1986). In this sense, listening serves both a social and intellectual function. To become full participants in a community of peers doing mathematics, students must become good listeners.

Structural Understanding

From a structural view, understanding means representing and organizing knowledge internally in ways that highlight relationships between pieces of information (Hiebert & Carpenter, 1992). Whereas the functional view focuses on the activity of the classroom, the structural view focuses on what the students take with them from the classroom.

To deal with what knowledge is retained after classroom lessons end, we build on (1) Dewey's (1929) idea that knowledge is the fruit of activity that resolves problematic situations, (2) Brownell's (1946) observation that understanding is better viewed as a by-product of activity than as a direct target of instruction, and (3) Davis' (1992) more recent formulation of this idea as the residue that gets left behind when students solve problems. Residue provides a way of talking about the understandings that remain after an activity is over. We noted earlier that the nature of the residue will depend, in part, on the prior knowledge with which the student enters the activity. It will also depend on the nature of the problem that is being solved. We highlight three kinds of residues.

Insights into structure. Insights into the structure of the subject matter are left behind when problems involve analyzing patterns and relationships within the subject. In Ms. Hudson's class, the second graders analyzed the ways in which procedures worked and how procedures were the same and different. To do this they needed to use what they knew about the base-10 number system and relate it to using their 10 sticks, counting by 10s and 1s, regrouping 10 1s as 1 10, and so on. This kind of reflective activity is likely to yield new relationships, new insights into how the number system works. In fact, the evidence suggests that young students who are presented with just these kinds of problems and engage in just these kinds of discussions do develop deeper structural understandings of the number system than their peers who move through a more traditional skills-based curriculum (Cobb et al., 1991; Hiebert & Wearne, 1993, in press).

Strategies for solving problems. Two kinds of strategies are produced by working through problematic situations. One is the particular procedures that can be used for solving particular problems. The second is the general approaches or ways of thought that are needed to construct the procedures.

When second graders solve whole-number addition or subtraction problems, when sixth graders solve fraction multiplication problems, or when ninth graders solve algebra equations, they acquire specific procedures and techniques for solving specific problems. The procedures that get left behind depend on the kinds of problems that are solved. These procedures make up the kinds of skills that ordinarily are taught in school mathematics. The evidence suggests that students who are allowed to problematize arithmetic procedures perform just as well on routine tasks as their more traditionally taught peers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., in press; Cobb et al., 1991; Hiebert & Wearne, 1992, 1993, in press; Kamii & Joseph, 1989). In other words, specific procedures for specific tasks constitute one kind of residue.

A second, and perhaps more important, kind of strategic residue could be called meta-strategic. By working through problematic situations, students learn how to construct strategies and how to adjust strategies to solve new kinds of problems. What gets left behind are the conceptual underpinnings and methods for actually working out new procedures when they are needed. The best evidence for this residue is the fact that students who have been encouraged to treat situations problematically and develop their own strategies can adapt them later, or invent new ones, to solve new problems (Fennema, Franke, Carpenter, & Carey, 1993; Fuson & Briars, 1990; Hiebert & Wearne, 1993, in press; Kamii & Joseph, 1989; Wearne & Hiebert, 1989).

It is worth noting that when students develop methods for constructing new procedures they are integrating their conceptual knowledge with their procedural skill. This is significant because one of the most common findings in research on students' mathematics learning is that they often show a separation between conceptual and procedural knowledge (Hiebert, 1986). Given traditional instruction, students possess understandings that they do not use to inform their procedures, and they memorize and execute procedures that they do not understand. When students experience curricula that treat mathematics as problematic, this separation is infrequent (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fuson & Briars, 1990; Fuson, Smith, & Lo Cicero, 1996; Hiebert & Wearne, 1992, 1993, in press; Kamii & Joseph, 1989; Murray, Olivier, & Human, 1992). This is not surprising. Students who treat the development of procedures as problematic must rely on their conceptual understandings to drive their procedural advances. The two necessarily are linked.

Dispositions toward mathematics. What students take away from any instructional activity is only partly accounted for by cognitive descriptions (Doyle, 1988; Schoenfeld, 1985). Students also form attitudes and beliefs about the subject which, in turn, influence their orientation toward future activities. These dispositions are constructed from the way in which the subject is treated by the curriculum and the teacher, the kinds of tasks students complete, and the everyday rituals of the classroom. We believe that problematizing mathematics provides an opportunity for students to "recognize the inventiveness of their own practice" (Lave, Smith, & Butler, 1988, p. 69) and to see mathematics as an intellectual activity in which they can participate. There is evidence that students who engage in reflective inquiry, who are allowed to treat mathematics as problematic, develop these and other positive dispositions toward mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef 1989; Cobb et al., 1991).

Summary

Approaches to instruction and curriculum design that are based on treating mathematics as problematic allow room to view classroom activity from both functional and structural perspectives. The notion of reflective inquiry captures the activity of communities of students and teachers engaged in practicing mathematics. The notion of residue provides a way of thinking about the understandings and skills that individual students take with them from classroom experiences. Either perspective can be used to link the problematizing of the subject with the development of understanding.

Other Views of Problem Solving in the Curriculum

Because of its multiple connections to understanding, we believe the principle of treating mathematics as problematic is the most powerful and practical way to think about problem solving. It is different than many historic views on the role of problem solving in the curriculum. It is also different than many current views. By comparing it with several popular views, we can clarify further some of the principle's distinguishing features.

Problem Solving Makes Mathematics Useful

As noted earlier, the belief that mathematics should be useful, outside of school, has a long history. The current version of this approach emphasizes the presentation of real-world problems as a major part of the curriculum (Boud & Feletti, 1991a; Cognition and Technology Group at Vanderbilt, 1990; Streefland, 1991). The logic of this approach usually runs as follows. Mathematics is useful if it helps to solve professional or everyday tasks. Students will be more likely to see appropriate applications if they spend considerable time working in applied situations and, in fact, will acquire domain-specific knowledge while doing so. Problems, then, become valued to the extent that they embed mathematics in outside-of-school contexts.

Prawat (1991) expressed concern that the emphasis on solving problems can easily become too utilitarian. When useful mathematics becomes synonymous with learning strategies for solving problems, attention shifts to procedures and away from ideas. Practical skills become overvalued and important ideas are neglected.

Our critique is somewhat different than Prawat's (1991). We believe that real-life problems provide a legitimate context for problematizing mathematics. If students are engaged in solving as reflective activity, then the concern about an overemphasis on skills disappears. Our concern rests with the narrowness of this approach. Real-life or everyday problems are one context, but only one context, for reflective inquiry.

The value of a problem depends on two things: whether students problematize the situation and whether it offers the chance of leaving behind important residue. The first depends not so much on the task as on the culture of the classroom. This issue will be revisited in the next section. The second does depend on the task. Tasks with different content are likely to leave behind different residues. But the residues identified earlier depend as much on the mathematical ideas embedded in the task as on the way it is packaged. Of course, important mathematical residues can be left by grappling with real-life problems. We argue only that the mathematical content be considered seriously when selecting tasks and that the definition of usefulness be expanded to a variety of problem situations, including those contextualized entirely within mathematics. The students in Ms. Hudson's class were gaining useful insights into the number system and developing general methods for modifying and inventing procedures, and the task appeared to be rather routine—find the difference between 62 and 37. Useful tasks come in many different packages.

Problem Solving Engages Students

A common argument for problem solving is that good problems are motivational. Intriguing or relevant problems will pique the interests of students and engage them in mathematics. There is an overlap between the advocates of this view and the previous one because it is often proposed that the problems with which students will become most easily engaged are those which are taken from their everyday lives.

Our concern with this view is that it can easily lead to the belief that the source of interest and motivation is the task. We believe that the basis for engaging a task is not the task itself but the prior knowledge of the student and the conditions under which the task is completed (Hatano, 1988). Whether students perceive a task as a problematic situation and whether they become actively involved in searching for solutions depends on the knowledge they bring to the task, the opportunities that are provided for solving it, and the values and expectations that have been established in the classroom (Ball, 1993; Cobb, Wood, Yackel, & McNeal, 1992; Fennema, Franke, Carpenter, & Carey, 1993; Fuson, Fraivillig, & Burghardt, 1992; Lampert, 1991; Murray, Olivier, & Human, 1992, 1993; Resnick, Bill, & Lesgold, 1992). If presented at an appropriate time, tasks such as the difference between 62 and 37, tasks that some teachers might see as boring and routine, can be engaged by students as genuine problems. The students in Ms. Hudson's class were intensively engaged in the task, not because they had a burning interest in how much taller Jorge was, but because the class had established a culture in which the students knew they had the freedom and responsibility to develop their own methods of solution.

Earlier we noted that this view represents a departure from Dewey. In particular, it represents a departure from his belief that outside-of-school tasks have a higher interest value for students and are more likely to be treated problematically. Much of his essay *The Child and the Curriculum* (1956) is devoted to critiquing the curriculum in the schools as lifeless, predigested by adults, and unconnected to the lives of children. The tasks, Dewey said, are dull and do not allow children to experience the subject. Later, when pointing to real-life tasks as more naturally engaging, he said, "Probably the most frequent cause of failure in school to secure genuine thinking from students is the failure to insure the existence of an experienced situation of such a nature as to call out thinking in the way in which these out-of-school situations do" (1933, p. 99).

We agree that genuine thinking is too often absent from classrooms, but we believe that the source of the problem is not so much the tasks themselves as the way in which students are expected and allowed to treat them. Too often students are shown a procedure and asked to apply it in a straightforward way. They have few opportunities to treat situations of any kind problematically. Outside-of-school problems can provide contexts for important mathematical work, but the packaging of the task is not the primary determinant for engagement.

Problem Solving Is What Mathematicians Do

Some advocate problem solving in school mathematics because such activity is like the practice of mathematicians (Collins, Brown, & Newman, 1989; Lave, Smith, & Butler, 1988; Schoenfeld, 1985, 1988). Learning is treated as enculturation into a community of practice. The goal is that "children might learn, by becoming apprentice mathematicians, to do what master mathematicians and scientists do in their everyday practice" (Lave, Smith, & Butler, 1988, p. 62).

Our perspective has much in common with cognitive apprenticeship (Brown, Collins, & Duguid, 1989; Collins, Brown, & Newman, 1989; Lave, Smith, & Butler, 1988), and many examples of instruction that are used to characterize cognitive apprenticeship represent good examples of the kind of instruction we envision as well. From both perspectives, learning is embedded in activity, students engage a variety of problem situations, and artificial distinctions between acquiring knowledge and applying it are eliminated.

The two perspectives seem to be complementary rather than competing. Differences in descriptions emerge from differences in focus and emphasis. The master or expert plays a more prominent role in cognitive apprenticeship. The teaching activities of the expert—such as modeling, coaching, scaffolding, and fading—are central features of the model. In contrast, our perspective highlights the inquiry processes of students as they problematize the subject and search for solutions.

The difference in emphasis may stem from the fact that problem solving is not a central feature of classical trade apprenticeship from which the model of cognitive apprenticeship is drawn. Novice apprentices often learn the trade by observing and imitating the expert master. This requires that the techniques needed for handling each part of the task be made visible and demonstrated clearly. The master's role includes modeling, scaffolding, and so on. Apprentices monitor their progress by checking their work against the master's. The goal is usually a visible, identifiable product.

Reflective inquiry emphasizes the process of resolving problems and searching for solutions rather than manufacturing a product. Tasks are seen as problems and quandaries to be resolved rather than as skills to be mastered. Methods of solution are as much dependent on inventiveness as imitation. Feedback on the appropriateness of methods and solutions comes from the logic of the subject rather than from the master/teacher.

Focusing on the inquiry processes of students also suggests that the metaphor of children as small mathematicians can be pushed too far. Children are different than mathematicians in their experiences, immediate ambitions, cognitive processing power, representational tools, and so on. If these differences are minimized or ignored, children can be thought of as small adults and education can become a matter of training children to think and behave like older adults. Dewey (1956) cautioned against such programs because they can easily overconstrain the activities in which children engage.

From our perspective, children need not be asked to think like mathematicians but rather to think like children about problems and ideas that are mathematically fertile. Finding the difference between 62 and 37 does not contain the complexities of the problems on which mathematicians work, and the procedures developed by the second graders are not even those they are likely to use as adults. The similarities between mathematicians and children lie in the fact that they are both working on situations that they can problematize with the goal of understanding the situations and developing solution methods that make sense for them.

Implications for Classroom Practice

The Locus for Change

Treating mathematics as problematic requires changing the entire system of instruction. It is not achieved by injecting interesting problems into a curriculum that retains a distinction between acquisition and application. It is not achieved by adding problem solving into the mix of ongoing classroom activities. Rather, it is achieved by viewing the goal of instruction and the subject from a very different perspective.

Because the conditions that determine whether students will treat the subject as problematic reside in the classroom, the locus for change resides here as well. The culture of classrooms will need to change, and this kind of change begins with teachers. Many teachers, having experienced more traditional classroom cultures and more conventional approaches to problem solving during their education, will need to change their conceptions of the subject in fundamental ways. Working out new orientations to a subject and changing classroom practice are not easy things to do. But Ms. Hudson and a growing number of teachers have shown that it is possible to move toward such practice (Heaton & Lampert, 1993; Fennema, Franke, Carpenter, & Carey, 1992; Fennema, et al., in press; Fraivillig, Murphy, & Fuson, 1996; Resnick, Bill, & Lesgold, 1992; Schifter & Fosnot, 1993).

The Collapse of Old Dichotomies

We began this article by noting that there is some value in formulating basic principles for reform in curriculum and instruction because such principles may help to sort out what is essential from what is optional. We have elaborated the essential features for treating mathematics as problematic. We close by reviewing briefly some classroom practices that are optional and some dichotomies that lose their significance.

First, as we mentioned earlier, the choice between telling students and letting them discover is redefined. Allowing students to treat tasks as genuine problems may involve various configurations of sharing information and discovery. Teachers do not need to do only one or the other. Of course, some invention is likely to be part of the kind of problematizing and reflective inquiry that leads to understanding. As Piaget (1971) remarked, "The essential functions of intelligence consist in understanding and in inventing. . . . It increasingly appears, in fact, that these two functions are inseparable" (pp. 27–28). But, as Dewey (1933, 1956) noted, information and direction from the teacher still play an important role.

A second issue that can be seen in a new way is the distinction between "real-life" problems and "school" problems. The question of which are better turns out to be irrelevant. The important questions are (1) has the student made the problem his or her own, and (2) what kind of residue is likely to remain. These are the criteria that address, respectively, whether problems are appropriate and whether they are important. Because both school and real-life problems can fit these criteria, the question of which is better is not useful. Many configurations are possible.

A third common dichotomy is between cognition and affect. Should instruction focus on the development of intellectual competence or positive attitudes? The easy answer, of course, is both. But once these are separated, there is a tendency to consider what must be done to foster each. Activities are weighed in terms of their likely contribution to one or the other. In contrast, the inquiry process that attempts to resolve problems is necessarily driven by both affect and cognition. "This conception of the mental [treating situations as problematic] brings to unity various modes of response; emotional, volitional, and intellectual" (Dewey, 1929, p. 225). Students do not separate them. There is no need for the teacher to choose between the two.

A final dichotomy concerns the control of the curriculum. Should the curriculum begin with the child and move from the bottom up (Gravemeijer, in press), or should it begin with the structure of the discipline and move from the top down (Davydov, 1990)? At the local classroom level, should the problems be generated by the students or presented by the teacher? Once again the notions of reflective inquiry and mathematical residue provide the relevant criteria. Reflective inquiry can occur and important residue can be left whether the problems come from the child's everyday world or from the world of mathematics, whether they are generated by the child or presented by the teacher. It is not necessary to choose one or the other.

Treating mathematics as problematic is a principle that provides a different vantage point from which to look. It resolves some old problems, creates some new challenges, and helps us see things in a different way. Having a different place to stand is a wonderful thing.

Notes

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[1]We are not the first to point out the dangers of the distinction nor to propose an alternative perspective. Educational approaches drawn from models of cognitive apprenticeship, for example, have implicitly or explicitly eliminated the distinction. Nevertheless, the distinction is alive and well in many arenas of education and warrants additional analysis. We contrast our perspective with others, including cognitive apprenticeship, later in the paper.

[2]More recently, Brown, Collins, & Duguid (1989) and Lave (1988) have drawn a similar parallel between the practice of "just plain folks" and expert practitioners.

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