PATHWAYS TO NUMBER

Children's Developing Numerical Abilities

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15 Learning Addition and Subtraction: Effects of Number Words and Other Cultural Tools

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There are thousands of different systems of number words; on the island of Papua New Guinea alone there are over 700 languages (Lancy, 1983; see also Ifrah 1981/1985; Menninger, 1958/1969; Zaslavsky, 1973). Many cultures, especially those surrounding and within the Pacific Ocean, even have several different number-word systems that are used for different purposes or for counting different kinds of objects. The features of a system of number words affect how easily it can be learned and used to add and subtract numbers less than 10, to add and subtract numbers between 10 and 100, and to add and subtract numbers larger than 100. This chapter will focus on some of the features that affect these different aspects of numerical learning. The concentration will be on a comparison between European systems of number words, which are irregular up to 100 (with English used as the main example), and the Asian systems that are based on Chinese, which are totally regular. Most of the points made also generalize to a wide variety of other systems of number words, and examples will be given where available. Nonlinguistic cultural supports for learning addition and subtraction, especially the different uses of fingers as countables, will also be discussed.

Asian number-word systems that are based on Chinese and most European number-word systems are named-value systems in which the values are successive powers of 10: there are words for the numbers 1 through 9, and larger numbers are made by saying one of these number words followed by a power-of-ten-value word that tells the value of the 1 through 9 word. One says 5353 in English as five *thousand* three *hundred* fif*ty* three, in French as cinq*mille*trois-*cent*cinq*uante*trois, and in Chinese as five *thousand* three *hundred* five *ten* three (using English words to show the values of the Chinese words; the actual Chinese words are wu *chien* san *bai* wu *shi* san). Most European languages are irregular (in many different ways) up to 100 but are regular named-value systems after 100, whereas Asian systems based on Chinese are regular named-value systems, explicitly naming the *ten* beginning with 11 (*ten one*) and continuing to 100 (e.g., 16 is *ten six,* 24 is *two ten four*). Mandarin Chinese, Japanese, Korean, and Burmese are totally regular named-value systems, and many other Asian languages have only minor irregularities in the second decade (words for 11 through 19) and in some decade words (e.g., Thai, Vietnamese, Bahasa used in Indonesia, Tagalog used in the Phillippines, at least some versions of Maori used in New Zealand, and Austronesian languages used on the coast and islands of Papua New Guinea). Some African languages also have regular named-value systems based on successive powers of 10 (e.g., Dioula).

LEARNING THE SEQUENCE OF NUMBER WORDS

How difficult it is to learn a sequence of number words depends on the features of the number-word sequence; the nature of errors made in saying a sequence depends on these features. Deaf children learning the number-word sequence of American Sign Language gestures make errors on the signs that are difficult for their fingers to form and show confusions about the rules used to make the separate related parts of the sequence for 1 to 5, 6 to 10, 11 to 15, and 15 to 20 (Secada, 1985). Most English-speaking children in the United States learn the English number words to twenty largely as a rote sequence in which the words between *ten* and *twenty* are not related to the words below *ten* (although some children do show awareness that these words are teen words and may overgeneralize and say "eight, nine, ten, eleventeen, twelveteen, thirteen"). The errors children in the United States make are largely omissions of words rather than reversals, and the portions of the sequence from which words are omitted may be stable for a long time (Fuson et al., 1982). English-speaking children do show awareness of the decade structure (the pattern of x-ty, x-ty one, x-ty two, . . . , x-ty nine), but they take a long time (as much as a year and a half) to learn the decade words in their correct order. Children learning Italian show particular difficulties with the reversal from 16 to 17 (the *ten* is said second for 11 through 16—*undici, . . . , sedici*—but is said first for 17 through 19—*diciassette, diciotto, diciannove*) (Agnoli & Zhu, 1989). Korean children, whose language has a formal and and informal system of number words, show more errors in decade words when counting in their informal system, in which all decade words are new different words, then when counting in their formal system based on Chinese, in which decade words are regular named tens (Song & Ginsburg, 1988).

The Asian systems based on Chinese (see Table 15.1) are very easy for children to learn. They only need to learn the first nine words, the words for the

TABLE 15.1
French, English, and Chinese Systems of Number Words

	French	English	Chinese: English words	Chinese: Chinese words
1	un, une	one	one	yi
2	deux	two	two	er
3	trois	three	three	san
4	quatre	four	four	si
5	cinq	five	five	wu
6	six	six	six	liu
7	sept	seven	seven	qi
8	huit	eight	eight	ba
9	neuf	nine	nine	jiu
10	dix	ten	ten	shi
11	onze	eleven	ten one	shi yi
12	douze	twelve	ten two	shi er
13	treize	thirteen	ten three	shi san
14	quatorze	fourteen	ten four	shi si
15	quinze	fifteen	ten five	shi wu
16	seize	sixteen	ten six	shi liu
17	dix-sept	seventeen	ten seven	shi qi
18	dix-huit	eighteen	ten eight	shi ba
19	dix-neuf	nineteen	ten nine	shi jiu
20	vingt	twenty	two ten	er shi
21	vingt et un	twenty-one	two ten one	er shi yi
22	vingt-deux	twenty-two	two ten two	er shi er
23	vingt-trois	twenty-three	two ten three	er shi san
24	vingt-quatre	twenty-four	two ten four	er shi si
25	vingt-cinq	twenty-five	two ten five	er shi wu
26	vingt-six	twenty-six	two ten six	er shi liu
27	vingt-sept	twenty-seven	two ten seven	er shi qi
28	vingt-huit	twenty-eight	two ten eight	er shi ba
29	vingt-neuf	twenty-nine	two ten nine	er shi jiu
30	trente	thirty	three ten	san shi
31	trente et un	thirty-one	three ten one	san shi yi
39	trente neuf	thirty-nine	three ten nine	san shi jiu
40	quarante	forty	four ten	si shi
50	cinquante	fifty	five ten	wu shi
60	soixante	sixty	six ten	liu shi
70	soixante-dix	seventy	seven ten	qi shi
80	quatre-vingt	eighty	eight ten	ba shi
90	quartre-vingt-dix	ninety	nine ten	jiu shi
99	quartre-vingt-dix-neuf	ninety-nine	nine ten nine	jiu shi jiu
100	cent	one hundred	one hundred	yi bai
101	cent-un	one hundred one	one hundred one	yi bai ling yi
125	cent vingt cinq	one hundred twenty-five	one hundred two ten five	yi bai er shi wu
4,313	quatre mille trois cent treize	four thousand three hundred thirteen	four thousand three hundred ten three	si qian san bai shi san

powers of 10 (shi, bai, qian, etc.), and the order in which words are said (from the largest value to the smallest). Chinese children make many fewer errors in saying the words to 19 than do English-speaking children in the United States (Miller & Stigler, 1987), and Chinese children show earlier learning of the sequence between 109 and 200 than do English-speaking U.S. or Italian children (Agnoli & Zhu, 1989). Errors reflecting imperfect knowledge or use of the decade structure, which the Chinese, English, and Italian number words all possess, were made by children speaking all three languages: They jumped to the wrong decade, forgot the current decade they were saying, and had trouble at transition points in counting backward (e.g., erroneously saying 72, 71, 70, 60, 69, 68 . . .). Fuson et al. (1982) and Siegler and Robinson (1982) also reported such difficulties for English-speaking children in the United States.

Most European languages clearly say neither the *ten one, ten two, . . . , ten nine* structure for 11 through 19 nor the *two ten, three ten, four ten, . . . , nine ten* pattern for the decade names, but most of them show some traces of both of these structures. In many languages, some words have lost their original meaning. For example, the English twelve for 12 comes from the Anglo-Saxon *twa-lif* meaning "two remain" (presumably two remain over ten; Greenberg, 1978), and *eleven* probably has a similar derivation (e-lif-un: "remain one" or even "ten left one"). The multisyllabic nature of European languages (compared to the single syllables used in Chinese) has, over time, led to the omission of parts of words, to changes in consonants, and to the addition of short syllables in order to facilitate pronunciation of the underlying words for 11 through 19 and for the decade words from 20 to 90. These phonetic changes then make it difficult for children to see the underlying structure of many European words as composed of x tens and y ones. Examples in English are the use of *thir* in *thirteen* (13) and in *thirty* (30) instead of *three,* the use of *fif* in *fifteen* (15) and *fifty* (50) instead of *five,* the use of *-teen* for 13 through 19, and *-ty* for 20 through 90 instead of *ten.* These phonetic substitutions and the quantitative meaning of these substitutions may not be understood even by adults; many of the first author's university undergraduates have never realized that -teen and -ty sound like *ten* and mean "ten"; they just used these syllables in a counting pattern without ever reflecting on their meaning. French and most other European languages have several such examples of phonetic changes: for example, *quatre* (4) becomes *quator* in 14 and *quar* in 40. Some Asian languages that have a regular structure except for a few irregularities also exhibit such phonetic changes: *isa* means "one," but *sam, san,* and *sang* are all used to mean "one" for larger numbers in Tagalog (Philippines); *satu* means "one" in Bahasa (Indonesia), but *se* is used when forming larger numbers that require "one"; and *tit* means "one," but *ta* is used with larger numbers in Burmese. All of these phonetic changes make it more difficult for children learning the language to understand and use the underlying tens and ones structure.

The preceding discussion has only focused on the nature of the patterns involved in producing (and, thus, in learning) a number-word sequence and has ignored any quantitative aspects of the actual words used in a sequence. Any number-word sequence can, of course, be learned as a totally arbitrary sequence of sounds like saying the alphabet (*A, B, C, D, E, F, G, . . .*). Because the native sequence of number words is so overlaid with quantitative meaning for adults, a useful technique for understanding what children must learn for a given language is to generate a particular number-word sequence using the alphabet. The patterns revealed by such alphabetic sequences are given in Table 15.2 for three European languages, Chinese, a Papua New Guinea language using some body parts, and an African language using a base of 20 and subtraction. The base-ten pattern is the pattern of the written base-ten positional numerals. Note the similarities and differences between the base-ten and Chinese patterns: the tenth symbol (*j*) is "zero" in base ten and is "ten" in Chinese (10 reuses the first symbol 1 and needs a new symbol 0 while shi in Chinese is just another new word) and the values are not named in base ten (e.g., 55 is *ee* [*five five*] in base ten rather than *eje* [*five ten five*] as in Chinese). It is clear that some patterns would be easier than others to learn, and the patterns lead to predictions about where errors might occur. Some languages use a word that has a quantitative meaning—like *hand* for five or *man* for twenty—for a particular number. Such meanings may be ignored in the original learning of the sequence, but they may facilitate the linking of quantitative meaning to related words when the sequence is used for cardinal purposes. Thus, languages may vary in how easily individual words and patterns can be related to cardinal meanings. These differences have important implications for addition and subtraction.

RELATING SPOKEN NUMBER WORDS TO WRITTEN NUMERALS

Written Numerals Having Sequence/Count Meanings

Children learn associations between written numerals and spoken number words, and these written numerals take on the meanings of the spoken number word. For small number words, the meanings may be cardinal (3 may mean three cookies) or sequence meanings (3 may mean what is said after two and before four), but for most larger words, children have few cardinal meanings. Thus, the meanings of the number words and of the numerals are initially only sequence meanings (8 means the word coming after seven and before nine). The pattern in the sequence of written numerals used in most languages is a simple one: It is just the regular Chinese pattern with the value words omitted and a 0 numeral used for any missing value, so that all the values stay in their correct relative position. This numeral pattern is given in Table 15.2 as the base-ten pattern. Children can learn the sequence of written numerals by its pattern, but in order to say a given written

TABLE 15.2
Patterns in Different Number-Word Systems

Number	*Base Ten*	*Chinese*	*English*	*French*	*German*	*Kilenge*[a]	*Yoruba*[b]
1	a	a	a	a	a	a	a
2	b	b	b	b	b	b	b
3	c	c	c	c	c	c	c
4	d	d	d	d	d	d	d
5	e	e	e	e	e	hand	e
6	f	f	f	f	f	hand a	f
7	g	g	g	g	g	hand b	g
8	h	h	h	h	h	hand c	h
9	i	i	i	i	i	hand d	i
10	aj	j	j	j	j	b hands	j
11	aa	ja	k	km	k	b hands a only	ak
12	ab	jb	l	lm	l	b hands b	bk
13	ac	jc	mn	nm	cj	b hands c	ck
14	ad	jd	dn	om	dj	b hands d	dk
15	ae	je	on	pm	ej	c hands	l
16	af	jf	fn	qm	fj	c hands a	d reduces m
17	ag	jg	gn	jg	gj	c hands b	c reduces m
18	ah	jh	hn	jh	hj	c hands c	b reduces m
19	ai	ji	in	ji	ij	c hands d	a reduces m
20	bj	bj	pq	r	mn	one man	m
21	ba	bja	pqa	rsa	aomn	one man a only	a on m
29	bi	bji	pqi	ri	iomn	one man hand d	i on m
30	cj	cj	rq	t	cn	one man a b hands	o
31	ca	cja	rqa	tsa	aocn	one man a b hands a only	a on o
39	ci	cji	rqi	ti	iocn	one man a b hands hand d	i on o
40	dj	dj	dq	uv	dn	one man b over	mb
50	ej	ej	oq	ev	en	—	j reduces mc
55	ee	eje	oqe	eve	eoen	—	e reduces mc
60	fj	fj	fq	wv	fn	one man c over	mc
70	gj	gj	gq	wvj	gn	—	—
80	hj	hj	hq	dr	hn	one man d over	—
90	ij	ij	iq	drj	in	—	—
91	ia	ija	iqa	drkm	aoin	—	—
99	ii	iji	iqi	drji	ioin	—	—
100	ajj	ak	ar	x	p	—	me
101	aja	akla	ara	xsa	pa	—	—
125	abe	akbje	arpqe	xre	peomn	—	—

[a]Kilenge is a Type III Papua New Guinea language typical of 40% of the languages (Lancy, 1978); not enough information was available to fill in all the numbers.

[b]Yoruba is a West African language based on 20 that uses subtraction frequently (Zaslavsky, 1973); not enough information was available to fill in all the numbers; k is an abbreviation for "on ten" or "in addition to ten," l is an abbreviation of "e reduces m," and several other higher forms are slightly abbreviated in actual use but are given in the table as if they were not.

numeral, they must relate the pattern in the numerals to the pattern in their own number-word sequence (or learn a very large number of numeral to number-word associations by rote).

Clearly, the ease with which children can relate the numeral and number-word patterns depends on their number-word sequence. Chinese (and Japanese and Korean) children have a very simple relationship to learn, because the patterns share many features and have no special irregularities. For European languages, this relationship is much more complex. The English words do not even signal a pattern break at 10 because the first 12 words are rote, arbitrary words. For many languages (e.g., French, Spanish, Italian, German, English, and Swedish), all or part of the words for the numerals between 11 and 19 have a number-word order opposite to the numeral order: One says the *four* first (*quatorz*e or *catorc*e or *quattor*dici or *vier*zehn or *four*teen or *fjor*ton), but writes the *four* second (14). Some languages switch the order of the *ten* and the *ones* words at 15 or 16, but the written numerals keep the single ten-then-one order. Many European languages have the decade word before the ones word (*vingt et un, twenty-one, ventuno*) as in the written numerals, but in German all words between 20 and 100 are ordered opposite to the written numerals, with the ones words before the tens word (e.g., *einundzwanzig* is *one and twenty*).

The difficulties European children sometimes have in learning the sequence of number words and in relating this sequence to the pattern of written numerals is illustrated by a report by Neuman (1987) about an 11-year-old Swedish boy in a remedial math class who, after some work on structuring by tens, made a drawing of rows of numerals so that 1 through 10 were lined up, with 11 through 20 lined up just below, and 21 through 30 lined up just below that. This boy shouted out excitedly,

> You see . . . I sat the other day and thought about numbers . . . and so . . . so I wrote on a bit of paper like that . . . and then I *saw* . . . you see? . . . Look!! . . . Have you ever noticed? . . . That one comes under one the whole time, and two comes under two . . . and three under three . . . Then it's much *easier* to count!! (pp. 318–319)

This boy had been in school for 4 years and undoubtedly had had the tens and ones structure of the numerals "explained" to him many times, but he still had not seen the numeral pattern or related this pattern to the Swedish number words. For some European children, the easier regular pattern of the numerals may provide the structure necessary to understand the pattern of an irregular system of number words.

Written Numerals Meaning Tens and Ones

Children need to understand the quantitative meaning of written numerals as tens and ones and not just learn the nonquantitative alphabetic pattern of the numerals. Evidently, children speaking regular, named-value Asian languages, which name the "ten," learn these tens and ones meanings much more easily than do English-speaking children in the United States. Miura (Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989) has reported, in a number of studies, that Chinese, Korean, and Japanese kindergarten and first-grade children chose to show two-digit numerals as combinations of ten-unit blocks and unit blocks, whereas their English-speaking age-mates in the United States showed the same numerals only with unit blocks (e.g., they counted out 42 unit blocks by ones instead of choosing 4 ten-unit blocks and 2 unit blocks). This was true even though the Japanese first-grade children had had no instruction on tens and ones in school and the U.S. first-graders had (Miura & Okamoto, 1989). Children in the United States have considerable difficulty in replacing their unitary sequence meaning of numerals by a meaning in which the first digit means "ten." Many first and second graders, and even substantial proportions of fifth graders, show the meaning of the *1* in *16* as one object and not as ten objects (C. K. Kamii, 1985, 1986). The *1* may be said by children to "teen" the *6*, that is, to make the *6* be *sixteen* instead of *six*, but there is no comprehension of 16 being composed of a ten and a six. M. Kamii (1981) argued that this "glued-together" pattern meaning of numerals is similar to spelling: *16* and *61* are reversals, just as are *dog* and *god*, and each of the glued-together composites in the pair has a different meaning, but the *1* and the *6* within *16* and *61* do not have quantitative meaning aside from their single-digit meanings as "one" and "six." The path to quantitative named-value multiunit meanings (e.g., the hundreds-digit telling the number of hundred-units) is a difficult one for English-speaking children, and many of them do not negotiate this path successfully. They may, at best, be able to use verbal named-value labels, that is, to tell which numeral in a four-digit numeral is called the hundreds digit. The very strong unitary meanings of number words and written numerals continue to create difficulties for English-speaking children in carrying out single-digit and multidigit addition and subtraction (see Fuson, 1990, in press-a, in press-b).

Even though the pattern of regular named-value number words relates fairly easily to the regular pattern of positional base-ten numerals, these systems do have several differences that can cause difficulties (see Fuson, 1991, for a discussion of the difference between named-value number words and the positional base-ten written numerals). It is quite common to write incorrect named-value numerals that mirror the named-value words (e.g., writing 300408 for *three hundred forty-eight*). English-speaking children make such errors (Bell & Burns, 1981). Europeans first changing from Roman to Arabic numerals also made such named-value errors (Menninger, 1958/1969), and Dioula and Baoule African children also do so (Ginsburg, Posner, & Russell, 1981b). We know of no evidence concerning how frequently Asian children may make such named-value number-word intrusion errors when first learning to write numerals.

SINGLE-DIGIT ADDITION AND SUBTRACTION

Cardinal Meanings for Number Words

In order for number words to be used for addition and subtraction, they must take on cardinal meanings; that is, they must tell how many there are. The structure of the system of number words, and the number words themselves, affects which cardinal meanings are easily understood. In Table 15.2, the Kilenge system supports cardinal meanings for 5, 10, 15, and 20 because the *hand* and *man* words can have quantitative meanings. In Chinese, once a cardinal meaning for *shi* (10) is understood, the cardinal meanings for 11 (*ten one*) through 99 (*nine ten nine*) follow quite readily. In English there is little similar support for these meanings for two-digit numbers. The words through *twenty* are just a linear sequence of piles of entities that get one larger, and the words between *twenty* and *one hundred* are just a similar sequence of very large piles that suggest, at most, a composition of a large plus a small pile of things (57 is 50 and 7 of the same single units, not 5 tens and 7 ones). Features of number-word (or number-gesture) systems can even interfere with the construction of these cardinal meanings. Papua New Guinea Oksapmin children, who learn a body-parts number sequence in which a succession of body locations constitute the numerical sequence, show cardinal confusions between similar body parts (e.g., left elbow and right elbow) even though these are quite separated in the sequence (Saxe, 1981). Some number-gesture systems have many clear cardinal references, whereas others do not (Zaslavsky, 1973).

Most number-word systems have a considerable number of words with no cardinal meaning: These words take on cardinal meaning through counting objects. The last counted word tells how many there are in (i.e., has a cardinal reference to) the pile of counted objects. How children first make this connection between counting and cardinal meanings of number words is discussed by Fuson (1988; this volume, chap. 6), including the developmental sequence of continuing relationships children construct to relate sequence, counting, and cardinal meanings. Once children can move from a count meaning to a cardinal meaning and vice versa, they can add by "counting all" and subtract by taking away or separating. In *counting all,* a child counts out objects for the first addend, counts out objects for the second addend, and then counts all of the objects. In taking away or separating, the child does the reverse: Counts out objects for the known sum, counts some of those sum objects up to the known addend and moves them away, and then counts the remaining objects to find the unknown addend objects.

Developmental Paths to More Advanced Addition Procedures

These original object counting procedures become increasingly abbreviated and abstract. Fingers are frequently chosen as the objects to be counted, and children eventually begin to learn finger patterns that make certain numbers. At this point there are at least three developmental paths children can take through addition and subtraction of single-digit numbers. Different cultures seem to support certain paths, although there is also individual variation within a culture. Fingers are used in conceptually different ways in these different paths. These differences seem to be related to the way a particular culture shows the numbers 1 through 10 on fingers, although other factors may also be involved. On all of these paths, children construct relationships among sequence, count, and cardinal meanings of numbers words, but the meanings that predominate differ. These paths describe children in the United States (the pertinent research is summarized by Fuson, 1988; 1990, in press-a; in press-b; and this volume, chap. 6), in Sweden, (Neuman, 1987), and in Korea (Fuson & Kwon, in press-a, in press-b). Some examples of these paths that occur in other cultures will also be given. These paths clearly depend on fingers and not the structure of the number words because English and Swedish number words have identical structures. The regular Korean words do confer some advantages even beyond the Korean finger methods.

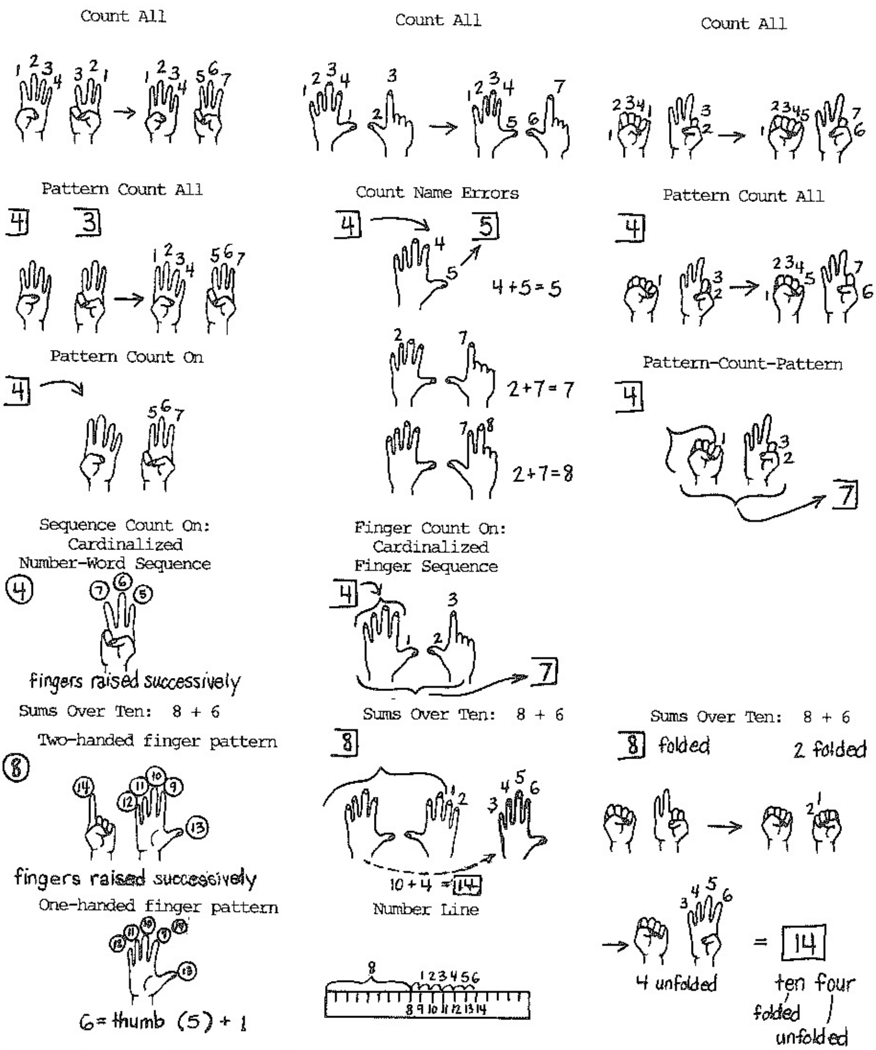
Sequence Counting. In one path, taken by many children in the United States, the number words themselves eventually become the objects that present the addends and the sum within addition and subtraction situations (see Table 15.3 for steps along this path), and the fingers are only used to keep track of the second addend. One begins this path by using fingers on one hand to count out one addend and fingers on the other hand to count out the other addend; all of the fingers are then counted to find the sum. When counting, fingers are typically raised beginning with the finger closest to the thumb and moving across the other fingers to the smallest finger; the thumb is raised last.[1] The child holds both hands up in the air, usually with the palm toward the face. Children eventually learn patterns for each number from 1 through 5 on either hand; they can then just raise finger patterns for each addend and count all of the fingers (Baroody, 1987c; Siegler & Robinson, 1982). Children eventually learn that they do not have to count all of the fingers in the sum count, but can begin the counting from the first addend word, that is, they can count on from the first addend. This is not a rote procedure but requires them to shift from the cardinal meaning of the first addend word to a counting meaning of that word (see Fuson, this volume, for a more detailed discussion). Finally, children do not need the perceptual support of the fingers to see the addends and the sum; they simply say the number words in sequence, and these sequence words themselves present the addends and the sum

[1]No claim is made in this chapter that the finger patterns shown in Table 15.3 are those used by all children or adults in the United States, Sweden, or Korea. Data concerning the range of finger patterns that may be used in different geographic and subcultural areas of these countries are not now available. The patterns shown are those reported in the references cited.

TABLE 15.3

Three Developmental Paths Through Single-Digit Addition

Fingers Keep Track of Sequence Counting	*Fingers as Count Names*	*Cardinal Finger Counting*



Note. [4] means a cardinal meaning for four, 4 means a count meaning for four, ④ means a sequence meaning for four, [14] means a ten and four meaning for 14. Only the folding down Korean methods are shown.

to the child. If the second addend is very large, some method of keeping track of how many sequence words have been said is required. Fingers are the most usual means of keeping track. Here, the fingers function as a cardinal finger pattern that is matched to each sequence word as it is said: Fingers are raised in succession with each word (rather than being put out before and then counted as in object or pattern counting on), and the sequence counting stops when the desired finger pattern has been made (see Table 15.3). New Guinea Oksapmin children use such sequence counting on for addition problems that exceed their native body-parts sequence, which only goes to 27: They count on in English and use their body-parts sequence to keep track of the second addend (Saxe, 1985).

The Fingers as Count Names. In a different path, taken by many Swedish children (Neuman, 1987), each finger takes on a particular count name from *one* through *ten;* see Table 15.3 for steps along this path. Swedish school entrants were interviewed by Neuman; in Sweden children begin first grade at age 7. These children counted on their fingers by placing both hands on the table in front of them with the palms down and the thumbs in the middle and counted from left to right (some counted similarly with their hands raised in the air). When adding two small numbers, they did not put the second addend on the second hand, but counted it continuing across the fingers, beginning with the finger to the right of the last finger used for the first addend. All fingers were then counted to find the sum by beginning from the left and counting to the right. With this method each finger always receives a standard word during the counting of the first addend and the sum: The left little finger is always *one,* the left thumb is always *five,* the right thumb is always *six,* the right little finger is always *ten,* and the middle fingers take on the words between these words. The word received by a given finger always varies during the second addend count because those words depend on the size of the first addend. Through repeated standard counting, each finger takes on its own count name from *one* through *ten.*

Many children stay in this count-name stage for a considerable period of time (a substantial proportion of these school entrants displayed this level) and make errors in adding and subtracting that result from their failure to connect these count names to a cardinal meaning for these names: Thus, for example, the word *four* is the count name for the index finger on the left hand, but *four* does not also have a cardinal meaning as referring to all of the first four counted fingers. Children therefore make the three kinds of mistakes shown in Table 15.3.[2] For example, 4 + 5 is found to be 5 by using the count meaning of 5: 4 fingers (or possibly, the finger named four) plus the finger named 5 (the thumb on the left hand) is 5 fingers; 2 + 7 = 7 by using count meanings for both *two* and *seven:*

[2]Children at this level were frequently not very articulate about the meanings they were using, so increased understanding of the conceptual bases for these mistakes awaits further research.

The finger *two* and the finger *seven* go along the fingers to finger seven. Also, 2 + 7 = 8 because each count finger is just one finger: "the *seven* finger plus one more finger (which happens to be named *two*, but this name does not matter) equals finger *eight*." Children showed a strong predisposition at this level to add by beginning with the larger number regardless of which number was given first in the problem.

Eventually, the fingers became a cardinalized finger sequence in which the second addend word has a cardinal value as the number of fingers counted past the first addend. Children may first pass through a period in which they estimate the second addend by counting an approximate number of words past the first addend but do not actually keep accurate track of the second addend. Neuman (1987) does not provide much information about how Swedish children come to keep track of large second addends, because almost all of her problems had a small addend that could just be subitized when counting on. The obvious way to keep track of the second addend accurately is to use number words to count the second addend fingers; the fingers then show the sum. This is opposite to the use of fingers and spoken words in the first path, where the spoken words present the sum and the fingers present (or keep track of) the second addend.

New Guinea Oksapmin children follow this second path in solving addition problems within the range of their 27-unit body-parts number sequence. They count on from a given body location while calling each counted-on location a body part from the second addend body-part list (Saxe, 1985). The number line used in some textbooks is structured like this second path: the written numerals present both the first addend and the sum, and the spoken number words present the second addend as children go up the number line. Chisenbop, the Korean method of finger calculation that attracted national attention in the United States in the 1970s, is also structured as in the second path: Finger patterns on one hand present 1 through 9 (the thumb pressed to a surface is 5, so the thumb plus the index finger—the *one* finger—is 6, etc., through the thumb and all four fingers, which equals 9), the first addend is put onto the right-hand fingers, number words are spoken aloud to present the second addend as each successive finger number pattern is made, and the fingers then present the sum (if the sum is over ten, tens are made on the left hand). These one-handed finger patterns can also be used successfully by first graders learning to add by counting on in the first path: Words say the sum, and the one-handed finger patterns show the second addend and match the sum count to stop it at the correct sum (see Table 15.3; see also Fuson & Secada, 1986). In that study we used the one-handed finger patterns rather than the two-handed finger patterns usually used by children in the United States for second addends over 5 because children frequently put down their pencil in order to use two-handed finger patterns, slowing their addition considerably.

The first path is easier than the second path to carry out for sums over 10 because the fingers can easily show any single-digit number through 9 as the second addend (either with one-handed or two-handed finger patterns, see Table 15.3), whereas the second path requires that fingers be reused in some way to show any sums over 10 because the fingers show the sum. Table 15.3 shows one possibility for doing this: moving the fingers for 1 through 5 (the left hand) over to the right of the right hand and reusing them. This has the advantage of clearly showing the tens and ones structure: Eleven is two hands (ten) and the named *one* finger, 12 is two hands (ten) and the named *two* finger, and so on. This, however, requires a move of the second hand for sums over 15, which might get too complex for some children. Neuman (1987) did not report how Swedish children use their named fingers to solve sums over 10, so it is not clear how children solve (or how the culture solves) this reusing problem. In her experimental teaching, Neuman used Cuisenaire ten-rods and one-rods for teaching sums over 10 rather than using fingers. It would also be possible for children to shift to the first path and say the sum words while keeping track of the second addend with the named fingers. How difficult this shift would be is not clear.

Cardinal Fingers Reused Over Ten. A third path through addition is a cardinal approach, in which fingers are counted or patterned to make finger patterns for 1 through 10 and fingers are reused to make numbers between 11 and 19. In this approach, the 10 fingers support the construction of addition methods based on structuring numbers by 10. This path was evidenced by first-grade Korean children interviewed to ascertain how they solved single-digit addition and subtraction problems (Fuson & Kwon, in press-a), and it is the path supported by Japanese teaching tiles structured around 10 (Hatano, 1982).

When a Korean child is counting all, the hands are held up facing the person with the thumbs out, as in the first path. However, the counting starts with the thumb and moves linearly across the fingers to the little finger, then continues onto the other thumb, and moves across to the little finger on that hand (see Table 15.3), or counting may be done in the reverse fashion beginning with the little finger and moving to the thumb. Some children begin on the left hand and move to the right, and some begin on the right hand. Children may count all by folding down fingers as each count is made, or they may begin with folded fingers and unfold the fingers while counting.[3] The first step in finding sums by counting all is like the second Swedish path: The first addend is counted as the fingers are folded, the second addend is counted as the next fingers are folded (the second addend does not begin separately on the second hand), and then all of the fingers

[3]Koreans typically show small cardinal finger patterns for age or small numbers of objects (e.g., three apples) by raising their fingers. *One* may be shown by the thumb or by the index finger, 2 may be shown by the thumb and index finger or by the index and second finger, 3 may be the shown by the thumb and next two fingers or by the three fingers other than the little finger, 4 is shown by the four fingers, and 5 is the thumb and four fingers. In the pattern-count-pattern procedure shown in Table 15.3, 4 might be made with the four fingers rather than with the thumb and three fingers, as in the earlier unfolded finger counting all.

are counted to find the sum. Children then may learn finger patterns of folded fingers so that they can fold fingers for the first addend without counting, count and fold fingers for the second addend, and then recognize the folded fingers for the sum. Counting all by unfolding is done in the same way (the unfolding second addend fingers are next to the unfolded first addend fingers), and unfolded finger patterns are learned for pattern adding.

To find sums over 10 by counting all in the folding-down method, all 10 fingers are folded (i.e., counted to 10), and then the fingers are unfolded beginning with the little finger of the second hand and moving across the fingers toward the thumb (i.e., the last fingers folded are the first fingers to be unfolded). A child using the method of unfolding fingers would count numbers over 10 by folding fingers again beginning with the last finger unfolded. With either method, the sum over 10 is easily said in the ten-structured Korean words as *ten* (all the fingers were used) *the-number-of-fingers-reused* (e.g., ten four in Table 15.3).

Korean children learn in first grade the over-ten method for adding sums over 10. In this method, addition (usually) begins with the larger addend and the smaller addend is broken up into (a) the number that will make ten with the first addend and (b) the remaining number. So eight plus six equals eight plus two (which makes ten) plus four (the rest of the six) = *ten four*. This method is easy to do in Asian languages in which 11 to 19 are said *ten one, ten two, . . . , ten nine,* because the sum is said as just ten and the rest of the second addend over ten. In English one has the extra step of finding this ten sum as a teen word (e.g., ten plus four is fourteen). Many first and second graders in the United States do not know these ten sums and have to count on from ten to find the sum (e.g., they find ten plus four by saying, "ten, eleven, twelve, thirteen, fourteen"). The Korean finger methods of folding and unfolding fingers support the learning of the over-ten method because (a) they make it easy to learn all of the complements to ten (i.e., the pairs of numbers that equal ten) just by looking at or thinking of the folded versus unfolded fingers, and (b) the counting of the second addend by folding and then unfolding fingers gives visual pattern support for breaking the second addend into the part that makes ten and the rest over ten. Most of the Korean first graders interviewed in Fuson and Kwon (in press-a) used the over-ten method even before they had been taught this method in school, and most of them knew, without calculating, which number made ten with a given number.

The second and third paths, thus, begin in the same way: counting all by showing the second addend on fingers following the first addend fingers. Korean first graders interviewed midway through the school year did not show any of the count-name errors shown by Swedish children just entering first grade: The Korean children were considerably more advanced in their addition methods, so it is possible that younger Korean children might show such count-name errors. Alternatively, the visual salience of the folded (or unfolded) fingers and the common reusing of fingers in the folding and unfolding methods may help

Korean children keep the cardinal view of fingers paramount and avoid count-name errors. This reuse of fingers was demonstrated in two other ways, both of which fall along the first developmental path. A couple of Korean children counted all by unfolding fingers for the first addend while counting them and then folding some of those same fingers to make the second addend while counting them; this unfolding and folding were then repeated while counting all. A couple of children also made one addend on one hand by unfolding and then folding fingers (e.g., made 7 by unfolding 5 fingers and folding 2), and the other addend on the other hand by unfolding 5 and folding 3 fingers, and then stated the sum as *ten* (the 5 fingers unfolded on each hand) *five* (the sum of the 2 and 3 folded fingers).

Some Korean children did show methods from the first path. The examples of reusing fingers just described both use addends in this nonsuccessive way. A few Korean children also counted on by the first method, saying the sum words for the second addend. Their method of keeping track of the second addend was usually not evident, but either folded or unfolded finger patterns could have been used. Ascertaining whether some children follow the whole first developmental path or only adopt some steps within it (e.g., count on before learning the over-ten method) requires more research.

In Japan many first-grade teachers use tiles that show the numbers 6 through 9 as 5 + 1, 5 + 2, 5 + 3, and 5 + 4 (Hatano, 1982), just as the one-handed finger patterns do and as the two-handed finger patterns can do if one thinks of the first hand as 5 (Neuman, 1987, reported that Swedish children do think of 6 as the *5* finger plus 1 more finger, 7 as the *5* finger plus 2 more fingers, and the *9* finger as 1 finger less than the *ten* finger). These tiles support the over-ten method, and the over-ten method is taught to all Japanese children (or at least appears in all Japanese textbooks; Fuson, Stigler, & Bartsch, 1988).

Stucturing Sums Around Ten

These paths have different advantages and disadvantages for supporting methods of finding sums that are structured around ten (such methods are advantageous for multidigit addition and subtraction as well as being effective general methods). The first path entails no difficulty in one's finding sums over ten, but its usual application by children in the United States is in a unitary sequence counting on in which ten plays no special role: The sum count moves across ten without marking ten with the fingers or with the words, because the English language moves across ten without showing any strong difference in the words below and above it. Sequence counting on could support an over-ten method if the finger pattern for the second addend was separated into two parts: the words counted up to ten and the words over ten. Thus in 8 + 6, for example, fingers on one hand could be raised for the words nine and *ten,* and fingers on the other hand could be raised for the words *eleven, twelve, thirteen, fourteen,* showing a total of 6 fingers, but separated into the two that made ten and the four that make

14 (i.e., ten four). Using an English version of Chinese words would also structure sequence counting on around ten: As the six fingers go up in 8 + 6, the words would say *eight, nine, ten, ten one, ten two, ten three, ten four* (14), thus showing the two fingers to make ten and saying the four fingers over ten. In the second path, the need to reuse fingers on both hands for sums between 10 and 19 has already been discussed. Such a reuse can support an over-ten method because the fingers for the first 10 count names allow a child to learn the complements to ten visually and kinesthetically, and the second addend is broken into the two needed parts visually (see Table 15.3). The number line, as used in schools, is always taught as a unitary procedure, in which the second addend jumps to make ten and the jumps over ten are not differentiated, but the number line could potentially be used to support an over-ten method. Both children and teachers show considerable difficulty with a number line, using it as a count model (and, thus, sometimes ending up with answers off by 1) rather than as the measuring model it actually is: Each number is shown by the length from 0, not by the number itself. A better support for the second path might be a number list of the written numerals (a count model like the Swedish row of finger numbers), in which the numerals over 10 were written in a different color to make the shift more salient (the numerals do mark the different structure before and after ten better than do English words). Some children might temporarily show the same use of count meanings of the numerals unconnected to the cardinal meanings seen in Swedish children (i.e., the numeral 4 would be the only meaning for *four* rather than *four* also having the cardinal meaning of the first four numerals), but they would presumably move on to connect the count and cardinal meanings.

An alternative method of structuring addition by ten could be carried out by using the first method of putting each addend on a separate hand if each addend over 5 is presented on one hand as a pattern involving 5 (e.g., 7 = 5 + 2). For two such numbers, their sum is easily found by combining the two fives to make ten and combining the parts of each over 5 to make the x in *ten* x (in Asian words). The use of this method by two Korean children was described earlier. Two first graders in the United States who learned one-handed finger patterns in one of the first author's instructional studies also invented the same approach by putting one number on each hand: 7 + 8 would be the thumb plus two fingers on one hand touching the desk (7) and the thumb plus three fingers on the other hand touching the desk (8), so the two thumbs made ten (5 + 5 = 10) and the 2 + 3 fingers down made 5, for a sum of 15. Roman numerals also support this 5-based approach, with 7 + 8 = VII + VIII = XV. The Japanese tiles have this five sub-base and therefore can also support this method.

Subtraction

Subtraction can be carried out in all three paths by both forward (adding up or counting up) and backward (taking away or counting down) methods. The drawings in Table 15.3 can be interpreted as showing the forward subtraction procedures for each path: In each drawing the first addend and the sum are known, and the second addend is the unknown number to be found. For the top drawings this means that the order of the counts is reversed; for the lower drawings, the feedback loop governing the stopping of the counting depends on the sum word or the sum finger pattern being reached, and the unknown addend is then read from the second addend words or fingers. There is insufficient space here to present and discuss backward procedures along each path. These backward procedures for the first and second paths are more difficult than are the forward procedures, because they require counting backward (which is much more difficult than is counting forward; Fuson et al., 1982) and because two alternative counting-down procedures can be carried out and children sometimes confuse them (e.g., Steinberg, 1984). The backward procedure for the third path is not so difficult because it is supported just by looking at the fingers; the most difficult part of the procedure (the separation of the second addend) involves forward rather than backward thinking. For example, to carry out the down-over-ten method that is the reverse of the addition over-ten method for 14 − 6, one just makes "ten four" on the hands, looks at the folded four fingers, thinks of how much more makes 6 (an additive procedure), and then takes that away from ten by folding down two fingers from ten unfolded fingers or by thinking of the complement of 2. Korean children also use a take-from-ten method in which the known addend is subtracted from ten and the difference is added to the amount over ten: 13 − 6 (ten three minus six) is thought of as ten minus six is four plus the three is seven (Fuson & Kwon, in press-a).

ADDITION AND SUBTRACTION OF MULTIDIGIT NUMBERS

Addition and subtraction of multidigit numbers requires that the same values be added to each other or subtracted from each other. Thus, to add 2,489 and 3,765, the thousands are added to each other, the hundreds are added to each other, the tens are added to each other, and the ones are added to each other. When there are too many of a given value, 10 of them must be given over (or carried, or traded, or regrouped) to the next larger value; when there are not enough of a given value to subtract from, 10 of them must be given over from the next larger value (or borrowed, or traded, or regrouped) to make enough to subtract. There is nothing explicit in the written multidigit numerals either to show the values, i.e., to show what should be added to or subtracted from each other, or to show that 10 must be given over sometimes. Therefore, understanding how to add and subtract multidigit numbers must be supported in some way.

Systems of number words vary in the extent to which they support these two different understandings crucial to understanding multidigit addition and subtraction. Regular named-value systems support these understandings, while irregular systems do not. For example, the Dioula language used in West Africa is a regular named-value language that names tens, and adult unschooled Dioula merchants add two-digit numbers mentally by adding the tens, then adding the ones (often by an over-ten method), and then adding the ones sum to the tens sum (Ginsburg, Posner, & Russell, 1981a). Unschooled Oksapmin adult store owners adapted their unitary body parts counting system to early Austalian currency that used 20 shillings to the pound by counting up to 20 (the inner elbow on the second side of the body) and then beginning the count again at 1; with this base-20 system, pound and shilling amounts could be added or subtracted by adding the shillings within this 20-value and then adding the pounds (Saxe, 1982).

Asian named-value systems, which are regular, make it very easy to see that one adds and subtracts like values. They also support transferring 10 when there are too many or not enough because sums over 10 are actually said as tens and ones. Thus, when adding 2,489 + 3,765, 9 + 5 is *ten four,* which indicates that there are 4 ones left and 1 ten to be added to the other tens. We found, in interviewing Korean second and third graders (Fuson & Kwon, in press-b), that some of them used named-value conceptual structures for the tens (and also for the hundreds) so that, in 489 + 765, a child would say, "eight tens plus six tens is hundred four ten"[4] (found perhaps by using an over-ten analogue within the tens: eight tens plus two more tens from the six tens make one hundred and the four tens left from the six tens make one hundred four tens). Other Korean children used conceptual structures in which the numbers in each column were said without values, but the values were kept in mind and used for correct giving of tens when a sum was greater than 10 or a value of a minuend needed to be larger in order to subtract from it: In such a case a child said, "eight plus six is ten four" and wrote down the four and traded the ten over to the next column to the left. Some children used mixed conceptual structures within the same sum ("eight plus six is hundred four ten"), using or not using the named-values very freely. For both addition and subtraction, the Korean children showed remarkable accuracy and could explain and justify their procedures. For example, in sharp contrast to children in the United States, many of whom say that 1 written at the top of an addition problem to show a carried ten or hundred is a one (rather than saying it is a ten or a hundred), every Korean child said that the 1 written in the tens column was a ten, and only one Korean child said that the 1 written in the hundreds column was a one. Thus, these Korean children were aware of and used the different values of the numerals when they added and subtracted multidigit numbers, whereas many children in the United States do not show such awareness.

Children in the United States have great difficulties in learning place value

[4]There are no singulars or plurals in the Korean language, and Koreans say only *hundred,* and not *one hundred.*

(i.e., learning the values of the written numerals and how these values relate to the English words) and in carrying out multidigit addition and subtraction accurately (much of this evidence is reviewed in Fuson, 1990). Many build neither named-value conceptual structures for the English words nor positional base-ten conceptual structures for the written numerals. Either of these are conceptual structures adequate to support understanding of multidigit addition and subtraction. Differences between named-value systems of words, including most European and Asian languages, and the positional base-ten system of written numerals used in most countries are discussed in Fuson (1990). Many children in the United States instead construct inadequate conceptual structures for written multidigit numerals in which numerals are interpreted as concatenated single digits: numbers from 1 through 9 placed adjacent to each other. These children make many errors in multidigit addition and subtraction that reflect this inadequate single-digit conceptual structure (see Fuson, 1990). French-speaking Canadian children show many similar difficulties with place value and multidigit addition and subtraction (Bednarz & Dufour-Janvier, 1986; Bednarz & Janvier, 1982).

The irregularities in the English language and characteristics of the mathematics curriculum in the United States seem to contribute to the failure to construct adequate conceptual structures and adequate multidigit procedures. First, most children in the United States find sums over 10 by using unitary conceptual structures that do not involve tens and ones. Thus, when they get too many in a given column, neither the English words for the sum nor their conceptual structures for this sum suggest giving over the ten to the tens column: for 489 + 875, a child would find or know 9 + 5 to be *fourteen* but this would not be thought of as a ten and four ones. Such children have to switch between using unitary conceptual structures for finding sums over 10 to named-value tens and ones structures to trade when there are too many. This switch is exemplified by the step invented by first graders who had used base-ten blocks (see further on) to understand multidigit addition (this step was also used by many second graders who were shown it in a study the following year, Fuson & Briars, 1990). When these children attempted a problem without the blocks, they would add a column (by using a known fact or one-handed finger patterns to count on) to find the sum (e.g., "eight plus six is fourteen"), and then they would write this sum (14) out to the side by using their pattern relationship between 2-digit numerals and counting words, look at the written 14 as a tens and ones structure because they knew they had to break the 14 down into a ten and the left-over ones in order to trade the ten, and then write the 4 in the ones column and write the 1 ten in the tens column. They could explain what they were doing and used named-value conceptual structures for multidigit addition and subtraction, but they needed the written support of the numerals to switch from their unitary meaning of the sum word *fourteen* to a tens and ones structure for this word (they did not automatically know that fourteen meant 1 ten and 4 ones). In contrast, regular Asian languages automatically provide sums over 10 in a tens and ones structure, so children speaking these languages either do not have to switch meanings as they add or the switch is a very easy one.

Second, the curriculum in the United States gives children only 2-digit addition problems with no trades for a long time. The English language cannot support named-value meanings for these problems because the words are irregular for the tens. Furthermore, because no trades are required, the tens meaning of the digits on the left is not evident: they look and act like single digits that are added to and subtracted from each other rather than looking or acting like tens and ones values. These children may not add or subtract 3-digit numbers until the third grade and 4-digit numbers until the fourth grade (Fuson, Stigler, & Bartsch, 1988), so the support of the regular named-value English words for hundreds and thousands is not available until quite late. Finally, the support provided in textbooks is insufficient for children to construct named-value conceptual structures: Multidigit addition and subtraction is approached as a rule-based manipulation of written digits in most textbooks (Fuson, in press-a).

English-speaking children can construct adequate conceptual structures that enable them to understand multidigit addition and subtraction and carry out these operations accurately and meaningfully. Base-Ten blocks[5] (Dienes, 1960) show the values of the English words and of the positions of the written numerals. Second graders of all ability levels who linked procedures with the blocks tightly to procedures with the written numerals learned to add and subtract 4-digit numbers, could justify their procedures (e.g., they never said that a traded-over 1 was a one but said its actual value), and many generalized these procedures to addition and subtraction of 10-digit numbers (Fuson, 1986a; Fuson & Briars, 1990).

The unitary sequence methods used to find single-digit sums and differences can be extended to find 2-digit sums and differences: One can count up or count down by ones to find 26 + 37, and many children initially do this even though it is time-consuming. One can also structure such counting by constructing tens units within the counting, and can then count on, count up, and count down by tens and ones (e.g., 47 + 35 is 47, 57, 67, 77, 78, 79, 80, 81, 82 or, separating the values, 40, 50, 60, 70, 77, 78, 79, 80, 81, 82). Many children invent such counting procedures to solve two-digit sums or differences not presented in vertical form. Almost all invented procedures reported in the literature involve such sequence procedures (Labinowicz, 1985). However, such procedures are difficult for many children, and their counterparts for three- and four-digit numbers are difficult, because these sequence procedures do not generalize well to general multidigit addition and subtraction. Therefore, educators in the United States and Europe, who use languages that are irregular for two-digit numbers, face a choice: (a) spend time allowing children to construct ten-unit items (count-

[5]A ones block is a cubic centimeter, a tens block is 1 cm × 1 cm × 10 cm, a hundreds block is 1 cm × 10 cm × 10 cm, and a thousands block is 10 cm × 10 cm × 10 cm.

ing up and down by tens) within their sequence conceptual structures and use these for solving two-digit addition and subtraction; later, shift to some other support for named-value conceptual structures to add three-digit and four-digit and larger numbers in a meaningful way; or (b) move directly to supporting children's construction of named-value conceptual structures for three-digit and/or four-digit numbers by using perceptual multiunit supports such as base-ten blocks. The former seems to be done in the Netherlands: Second graders use only mental methods of adding and subtracting two-digit numbers, and written multidigit addition and subtraction and three-digit numbers are delayed until third grade (at least as reported by Beihuizen, 1985). Some U.S. educators advocate supporting counting approaches for two-digit numbers, even if this means delaying two-digit subtraction until third grade, because such subtraction procedures are difficult for many second graders (e.g., C. K. Kamii, 1989). The classroom base-ten block research by Fuson (1986a; Fuson & Briars, 1990) suggests that alternative (b) leads to accelerated performance compared to alternative (a), similar to that shown by children in Asian countries (i.e., strong multidigit competence for four-digit numbers in second graders). These alternative educational paths will probably continue to be debated, and it will be interesting to compare the paths for children in different countries.

Different mental and written algorithms (or repetitive procedures) are used to add and subtract multidigit numbers in different countries. Some of these support conceptual understanding better than others, and some are more efficient or easier to carry out. For example, in the base-ten block studies (Fuson, 1986a; Fuson & Briars, 1990), a subtraction algorithm was used in which any necessary trading was done for all columns first, followed by single-digit subtraction for each column; this eliminated the need to shift back and forth between a named-value conceptual structure for trading and a unitary conceptual structure for the single-digit subtraction (done by counting up from the known addend to the known sum; Fuson, 1986b; Fuson & Willis, 1988). Korean children learn a written subtraction algorithm in which they write a *10* above any column that requires a ten traded over to subtract; this written *10* supports both types of single-digit subtraction methods structured around 10 (Fuson & Kwon, in press-b). However, little research exists in English that compares children's understanding using different algorithms and even less that relates aspects of an algorithm to aspects of English number words that may make a given algorithm easier or harder for children to understand.

OTHER CULTURAL SUPPORTS FOR ADDITION AND SUBTRACTION

It is clear that systems of number words that are regular and that name the values used in the system support, in many different ways, the construction of concepts of number and facilitate the learning of addition and subtraction. Cultures also provide other experiences that can support or interfere with the construction of number concepts structured around 10 and around multiples of 10. Most countries in the world use the metric system, which provides many examples of 1-for-10 exchanges of value within and between different kinds of measures. Some countries have systems of money that have regular 1-for-10 exchanges. A traditional calculator based on 10—the abacus—has had widespread use in many Asian countries and in the Soviet Union. The United States has none of these supports: the English system of measure, with its many irregular, non-10 trades is still used, and the system of money has irregular intrusions of nickels and quarters (5¢ & 25¢) and of $20 and $50, which interfere with the tens and ones structure within the monetary system. Therefore, children in the United States need considerable support from materials and special activities within the classroom in order to construct multiunit named-value or base-10 positional conceptual structures, and they often do not get that necessary assistance. Children in other countries in North, Central, and South America and in Europe who speak one of the irregular European systems of number words do at least have the support of the metric system, but the systems of money in these countries vary in how much they are structured only by tens. Whether and how various countries choose to attempt to redress the linguistic disadvantage of their irregular number words is an interesting question for future comparative research.

CONCLUSION

This chapter has focused heavily on various consequences of the difference between the regular named-value Asian number words and the irregular European number words. Of course, the superiority in mathematical performance of Japanese and Chinese children over children in the United States (e.g., Stigler, Lee & Stevenson, 1990; Stigler & Perry, 1988) is due to many factors other than the systems of number words: more time spent in mathematics learning; higher teacher status; more activities in the classroom using particular concrete materials; more focus on understanding, explanation, and alternative solution procedures; and more emphasis on the role of effort in childrens', and parents', and teachers' views of mathematics learning (see Fischer, 1991, for a review). Although some of these cultural differences may be relatively difficult to change, providing support to compensate for the irregularities in English number words may be easier, and may result in considerable increases in children's understanding of numbers and of addition and subtraction. Our work with counting on and counting up with one-handed finger patterns and with base-ten blocks indicates that children in the United States can perform considerably more like Asian children, at least in single-digit and multidigit addition and subtraction situations.

Three different developmental paths through single-digit addition and subtraction have been identified here. These paths seem to be supported by different uses of fingers to show addition and subtraction. These different uses of fingers may then lead to different addition procedures even when the number-word sequences are similar (as English and Swedish are). Therefore, understanding how children in a given culture construct concepts of, and procedures for, addition and subtraction may require knowing how that culture uses fingers to show numbers, as well as knowing the structure of its number-word sequence. Exploring how teaching children finger or number-word practices that support their construction of useful concepts and procedures also seems likely to prove fruitful.