Instruction Supporting Children's Counting on for Addition and Counting up for Subtraction

BRIEF REPORT

Instruction Supporting Children's Counting On for Addition and Counting Up for Subtraction

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Children in the United States ordinarily invent a series of increasingly abbreviated and abstract strategies to solve addition and subtraction problems during their first 4 years in school (Carpenter & Moser, 1984; Fuson, 1988, in press–a, in press–b; Steffe & Cobb, 1988). Several studies have shown that instruction can help children learn specific strategies in this developmental sequence. Fuson (1986), Fuson and Secada (1986), and Fuson and Willis (1988) demonstrated that by the end of first grade children of all achievement levels could add and subtract single-digit sums and differences (sums to 18) by sequence counting on and sequence counting up. Sequence counting on and counting up are abbreviated counting strategies in which the number words present the addends and the sum. In both strategies the counting begins by saying the number word of the first addend. For example, to count on to add 8 + 6, a child would say, "8 (pause), 9, 10, 11, 12, 13, 14." The same sequence of number words is used to find 14 – 8 by counting up, but the answer is the number of words said after the first addend word rather than the last word in the sequence. When the second addend is larger than 2 or 3, some method of keeping track of the words said for the second addend is required. In the studies above this method was one-handed finger patterns that showed quantities 1 through 9 (the thumb is 5) so that children could hold their pencil in their writing hand all of the time. The counting-on and counting-up instruction related the counting words to objects showing the addends and the sum, thus focusing on conceptual prerequisites for these abbreviated counting procedures and enabling children to relate counting and cardinal meanings of number words (Secada, Fuson, & Hall, 1983). The counting-up instruction provided interpretations of subtraction and the "–" symbol as adding on, as well as the usual take-away interpretation that leads children to count down for subtraction.

PURPOSE AND PROCEDURES

The purposes of this brief report are (a) to provide new data concerning children's accuracy from the earlier studies (Fuson, 1986; Fuson & Secada, 1986; Fuson & Willis, 1988) and (b) to report September pretest data and May posttest data from a new year of instruction to clarify the progress of first graders during the entire year on difficult single-digit addition and subtraction combinations. We then discuss advantages of an adding on interpretation of subtraction and emphasize the importance of conceptual rather than rote learning of counting on and counting up.

The measure reported in the earlier studies provided an incomplete picture of student learning. This measure, the mean number of problems correct on a 2-minute timed test of 20 of the most difficult single-digit problems (sums and minuends from 11 through 18 excluding doubles), confounds speed of solution with accuracy of solution. The means, which ranged from 6 through 16 problems correct, could have resulted from accurate but somewhat slow solutions or from rapid but inaccurate solutions (e.g., completing all 20 problems but getting many of them wrong). Using the measure of percent accuracy (the percent of problems attempted that were correct) in conjunction with the mean correct measure provides a more complete picture of children's ability. This measure is reported here for the earlier studies as well as for the new sample.

Data from the new sample adds several pieces to our overall picture of counting-on and counting-up learning experiences. This third year of work in the same two schools that participated in the earlier studies focused (a) on giving first and second graders opportunities to solve a wide range of addition and subtraction word problems with the support of schematic drawings (Fuson & Willis, 1989; Willis & Fuson, 1988) and (b) on giving all second graders and the higher-achieving first graders experience with four-digit addition and subtraction with the support of base-ten blocks (Fuson & Briars, 1990). Children in both schools were assigned to a low-, middle-, or high-achieving math class by recommendation of the kindergarten teacher and informal tests given at the beginning of the year; children were transferred during the year as their performance warranted. Counting on and counting up were integrated into the work on word problems as individual teachers desired, except that all first graders solved simple canonical addition and subtraction word problems and number facts (numerals only) with small numbers before they discussed counting on and counting up. Children completed work on counting on and counting up before doing any multidigit work, for these counting strategies enabled children to find any single-digit sums or differences they did not know. The multidigit work was successful in helping children understand and carry out multidigit addition and subtraction considerably better than is usual for children in this country (Fuson & Briars, 1990). During this year staff efforts were directed at training in-service teachers with respect to the word problem and multidigit teaching. Two of the 10 teachers were new, so they had to learn counting on and counting up from colleagues in their school. Finally, in the third year all pretests were given in the first week of school before any work on addition and subtraction rather than immediately before the instructional unit, as had been done in the studies during the first two years of the project. The tests were 2-minute, 20-item tests of nondouble (not $a + a$) combinations with both addends ≥ 6 on half the items; on the other half, one addend was ≥ 6 and the other addend was 3 or 4 or 5. The early pretest data provide a picture of the growth of the first graders over the year rather than just over the counting-on or counting-up units. For more complete details of the methods and instructions for the third year, see Fuson and Briars (1990) and Fuson and Willis (1989).

ACCURACY OF CHILDREN'S SEQUENCE COUNTING ON AND SEQUENCE COUNTING UP

Year 1 and Year 2 Results

Percent accuracy measures were calculated for all classes in the earlier studies. These measures are reported for the posttests in which addition and subtraction number combinations were given in vertical form; the accuracy measures for the posttests in which number combinations were given in horizontal form were similar.

The five classes in the first year studies (reported in Fuson, 1986, and in Study 2 of Fuson & Secada, 1986) had a mean percent accuracy rate of 88% on the addition posttest and 85% on the 1-month–delayed posttest. Class means ranged from 82% to 97% and from 63% to 93%, respectively, and only one class showed any sizable decrease. The five classes had a mean percent accuracy rate of 93% on the subtraction posttest and 89% on the 1-month–delayed posttest. The class means ranged from 91% to 95% and from 82% to 94% on these posttests, respectively.

Two low-, average-, and high-achieving first grades and two low- and two average-achieving second grades participated in the second-year studies (Fuson & Secada, Study 3; Fuson & Willis, 1988). Children in both grades were quite accurate in counting on and counting up. The mean percent accuracy rate for the addition posttest, the 1-month–delayed posttest administered before any review, and the 1-month–delayed posttest administered after a few minutes of review was 92% (class mean ranges 87% to 95%), 84% (class mean ranges 59% to 96%), and 90% (class mean ranges 68% to 97%), respectively. The measures for the subtraction posttests were 91% (class mean ranges 79% to 96%), 89% (class mean ranges 81% to 94%), and 91% (class mean ranges 80% to 96%), respectively. On the final end-of-the-year posttest, administered from 1 to 6 months after instruction (the time at which instruction took place varied by grade and achievement), most class means for the accuracy measure were above 95%, and all but one were 87% correct or higher.

Year 3 Results

First grade mean correct and mean accuracy pretest and posttest scores are given in Table 1. The pretest addition scores reveal initial achievement differences: high-achieving first graders already possessed fairly accurate methods for finding sums between 11 and 18, though these methods were somewhat slow; other children did not have reliable methods for finding these sums. Children of all achievement levels had only inaccurate methods for finding differences. On the posttests children at all achievement levels were very accurate (except for one class that had only a 61% accuracy rate on the addition posttest) and completed more problems than they had on the pretest. Speed was directly related to achievement level; the higher-achieving classes completed more problems that the lower-achieving classes. Informal observations indicated that the high-achieving classes were more rapid at all aspects of addition and subtraction including writing the answers. The four classes of second graders ($n = 82$) solved a mean of 16 and 15 correct on the addition and subtraction posttests, respectively, and had 93% and 91% percent accuracy on these tests.

Table 1
Mean Number Correct and Mean Accuracy Rate on Addition and Subtraction Pretests and Posttests for First Graders in the Third Year Study

		Addition				Subtraction			
		Pretest		Posttest		Pretest		Posttest	
Achievement level	*n*	Mean number correct	Percent accuracy[a]	Mean number correct	Percent accuracy[a]	Mean number correct	Percent accuracy[a]	Mean number correct	Percent accuracy[a]
Low	31	0	5	7	73	0	9	10	90
Average	44	2	35	13	89	0	6	10	87
High	48	4	78	16	94	1	31	14	91

Note. In cases in which students attempted very few problems, the mean number correct rounded to 0 although the mean percent accuracy rate was greater than 0.
[a]The percent of problems attempted that were correct.

DISCUSSION

Accuracy

In all three years of the studies, children learned to count on and count up quite accurately. The pretests given at the beginning of the third year indicated that the low- and average-achieving children moved, over the course of the first grade, from not being able to solve addition and subtraction combinations involving single-digit sums between 11 and 18 to being quite accurate with these combinations and being fairly fast. High-achieving first graders moved from using fairly accurate but slow methods for finding such sums and having no accurate method for finding differences to using highly accurate and much more rapid methods for both sums and differences.

Advantages of an Adding-On Interpretation for Subtraction

In all three years of our studies, children were as accurate and fast at counting up for subtraction as at counting on for addition. This contrasts with the usual finding that subtraction is much more difficult than addition over the whole range of development of addition and subtraction solution strategies. For example, Siegler (1987) reported that children used primitive subtraction methods for small numbers considerably later for subtraction than for addition, Steinberg (1984, 1985) reported that second graders had much more difficulty with subtraction-derived facts than with addition-derived facts, and the control data reported by Thornton and Smith (1988) indicated the superiority of addition over subtraction on a range of problems grouped by different strategies.

Two factors may contribute to this similarity between addition and subtraction performance when counting on and counting up. First, subtraction carried out by counting up uses ordinary forward counting and thus avoids the much more difficult backward counting down that is required by the backward strategies invented by children who learn "take away" as the only meaning of subtraction. Second, with the forward sequence counting strategies, the monitoring process for stopping the second addend count is actually easier for subtraction than for addition. The words said must be monitored in subtraction counting up in order to stop when the minuend word is said, and the unknown addend is recorded using finger patterns or by some other keeping-track procedure. For addition by counting on, the finger patterns or some other keeping-track procedure must be monitored in order to stop when the second addend has been made with the fingers, and the last word in the counting sequence is the sum. Addition thus requires knowing the keeping-track procedure well enough to monitor it while counting on, but subtraction requires only recognizing a number word said while counting up.

Thornton's (1990) study provides new evidence that children who are given an opportunity to learn a counting-up meaning for subtraction as well as a counting-down meaning prefer the counting-up meaning. These children counted up to solve the difficult subtraction combinations with sums over 10 more than they solved them by counting down, even though their instruction focused more on take-away backward interpretations of subtraction than on forward interpretations. Children did have a forward interpretation because they were taught counting up for small differences of 1, 2, and 3 (they kept track of these small differences auditorily). Children who only learn a take-away meaning of subtraction (and who therefore count down for subtraction) ordinarily cannot choose to count up until they have advanced conceptual understanding of relationships among the addends and the sum so that they can transform a backward take-away conception to a forward count-up conception. Carpenter and Moser (1984) reported that children in these more advanced stages did choose to solve all subtraction word problems by counting up. Children in our counting-up studies were introduced to subtraction and the minus sign as having several different meanings. They solved subtraction number fact problems by counting up, but they also had available adequate take-away meanings and solution strategies for word problems describing take-away action (Fuson & Willis, 1988). The results from the studies reported here and Thornton's study underscore how important it is to provide alternative interpretations of subtraction so that children can choose as early as possible to use the easier forward subtraction solution procedures, including counting up. In contrast, most textbooks provide only a take-away meaning for subtraction and do not include situations involving comparison or adding on, or they provide them much later or much less frequently (Fuson, in press–a; Stigler et al., 1986).

Counting On and Counting Up Based in Conceptual Knowledge

The initial focus of our counting-on research was to ascertain what understandings are necessary and sufficient for children to count on. Three such abilities were identified in Secada et al. (1983): the ability to begin counting from an arbitrary word in the number-word sequence, a cardinal-to-count transition in word meanings, and a counting extension to the second addend embedded within the sum (see also Fuson, 1988, and Steffe & Cobb, 1988, for discussions of the counting and cardinal relationships involved in these solution procedures). The latter two abilities require children to relate counting words to objects organized into addends and their related sum. It proved to be easy for children to learn these abilities in individual interviews, a result that prompted the original classroom teaching study. The teaching studies examined whether children could learn object counting on, sequence counting on, and sequence counting up in a meaningful way that was connected to addition and subtraction situations. Some early evidence indicated that some children may show interference between counting on and counting up if experiences are limited to problems given as written numerals (e.g., 8 + 7 and 16 – 9), but that this interference can be eliminated by using addition and subtraction word problems and discussing differences between addition and subtraction. Thus, the best conceptual opportunities would seem to be provided by the related use of dot array problems, as in Secada et al. (1983), various addition and subtraction word problems, and numeral problems, an approach used in the third-year study reported here.

We underscore these conceptual recommendations because the conceptual bases of counting on and counting up can be ignored, and these can be taught as rote procedures (e.g., Frank, 1989). Counting on also appears in some textbooks with no recognition of the relationship between cardinal and counting concepts that children must construct in order to count on with objects meaningfully.

Some children may have difficulty in devising an efficient and comprehensible method of keeping track of the second addend when they move from counting on and counting up with objects to sequence counting on and counting up. For example, Thornton (1990) reported that all children who received her instructional treatment could count up to solve the difficult subtraction facts but they were considerably less accurate on the larger differences (about 50% and 75% in the two studies reported) than on the count-up problems with differences of 1, 2, or 3 that were the focus of instruction (around 87% across the studies). The method of keeping track taught in these studies, auditory patterns, seems to work well for the small differences of 1, 2, or 3 but not as well for larger differences. Ascertaining why the methods spontaneously used by these children for the larger differences were less accurate than the one-handed finger patterns used in the counting-up studies, and determining the relative merit of various keeping-track methods, would be of interest.

For most children, counting on and counting up are only stops on the developmental path. Many children in our studies went on to use derived and known facts. But for some children the counting strategies persisted for a long time, providing them with reliable solution strategies that enabled them to solve these large single-digit combinations accurately. Siegler (1988) indicated how important it is for children to have at least one such accurate method. Fuson and Briars (1990) also concluded that counting methods that use fingers are not necessarily crutches that later interfere with more complex tasks. They found that counting on and counting up with one-handed finger patterns were accurate and fast enough for use in four-digit addition and subtraction with regrouping. Thus, helping children learn sequence counting on and counting up in the third-year study enabled these children to tackle single-digit and multidigit addition and subtraction topics at the grade levels seen in China, Japan, the Soviet Union, and Taiwan rather than at the usual delayed pace of the United States (Fuson et al., 1988) and permitted children of all achievement levels to engage in a wide range of addition and subtraction activities.

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