The Nature and Origins of Mathematical Skills
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Chapter 2

RELATIONSHIPS CHILDREN CONSTRUCT AMONG ENGLISH NUMBER WORDS, MULTIUNIT BASE-TEN BLOCKS, AND WRITTEN MULTIDIGIT ADDITION

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Introduction and background

Arithmetic has arisen in many different cultures as a way to solve problems concerning quantitative aspects of real world situations. These quantitative aspects are described by words and, in many cultures, by marks that are written on some surface. In traditional cultures children learn arithmetic by observing and eventually using the quantitative words and written marks in their situations. In modern cultures, however, children are taught the arithmetic of single-digit whole numbers, multidigit whole numbers, integers (negative numbers), decimal fractions, and rational numbers. In much of this teaching, children do not learn to talk and write about quantitative aspects of real world situations, but rather stay within the arithmetic marks world and memorize sequences of written marks steps (routines) to accomplish each operation for each kind of number.[1] For too many children, this approach results in a verbal superstructure of hierarchical routines unrelated to anything. As a result, there is massive interference among the routines. Children have no way to reconstruct or verify forgotten routines and even no belief or expectation that one could do this, and they have poor ability to apply these routines to real world situations. Problem solving is often conceptualized only as the need to select and carry out the correct routine, and children have few ways to estimate answers to decide if an answer is sensible (e.g., Fuson, in press-a, in press-b). Before machine calculators were invented, there was a considerable need

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[1] The term written "marks" is used instead of the more usual term "symbol" in order to remind the reader continuously that for children, and any mathematical novices, the written mathematical squiggles are arbitrary and contain few cues to their referents in the real world. For adults and mathematical experts these meanings are so automatic that it is very difficult for us to remember and appreciate how arbitrary these written marks are.

for human calculators, so the school emphasis on producing human calculators, with its restricted calculator arithmetic focus, was understandable and probably even sensible. With the worldwide availability of inexpensive hand-held calculators, the need has shifted to humans who can apply and use these machines in real world situations and even in future situations not yet known. The need is now for meaningful arithmetic that can be related to real-world situations.

Meaningful arithmetic requires that quantitative operations on mathematical words and written marks be connected to real-world referents (objects or situations) in order for children to have an opportunity to understand the meanings of these operations (i.e., to see the attributes of their real-world referents). Various pedagogical objects have been invented and used in mathematics teaching for this purpose. For example, fraction pies, fraction bars (rectangles), or fraction strip divided into different numbers of units are used to consider various aspects of fractions. Studies have been carried out concerning the relative efficacy of learning with and without these pedagogical objects. However, the efficacy of a particular pedagogical object is limited by the extent to which it does actually present in its salient physical features the mathematical domain for which it is used. There has been little serious analytical research that has attempted to define the mathematical attributes of the written marks and spoken words used in a particular mathematical domain or to suggest the kinds of pedagogical objects that would present these needed attributes. Such an analysis is required for the necessary next step: The empirical investigation of the ease with which children can make links among written marks, spoken words, and pedagogical objects and use these links to construct full and correct conceptual structures for these marks and spoken words and for operations on them.

Such analytical research is complicated by the fact that the words and the written marks used to describe the mathematical entities may have different structural characteristics as well as some structural characteristics that are alike. Full understanding of the mathematical concepts being symbolized in verbal and written form may only be possible if both the system of words and the system of written marks are understood. The words and marks each fall along a positive to negative continuum, the positive side of which ranges from cueing many to cueing no important features of the symbolized concepts and the negative side of which ranges from containing no misleading features to containing many misleading features. For example, Chinese words for fractions convey more of the underlying fraction meaning than do English words: One says 3/5 as "out of five things three" in Chinese and "three fifths" in English. The word "fifths" conveys no sense of a fraction or of a ratio meaning and is even misleading because it is the same word used for the ordinal number meaning (fifth in the race, number five in an ordered

sequence). The Roman numeral VIII reflects the fiveness in a hand (the V-shape of the thumb and other fingers) plus three more fingers to make eight, whereas the Arabic numeral 8 contains no cues to its eightness. Multidigit numerals (e.g., 8625) are misleading because they look like adjacent single digits; no feature suggests that they tell how many tens, hundreds, and thousands or even that they involve such multiunits (larger units formed from multiple smaller units). Fractions similarly just look like two single digits separated by a line (3/5), thus seducing children into adding, subtracting, multiplying, and dividing these single digits as a way to carry out these operations on fractions (they are actually correct for two of these operations, thus giving partial reinforcement to this single-digit approach). Analyses of the structural characteristics of a mathematical domain and empirical investigations based on such analyses are needed. These can examine how the words and written marks used for a given mathematical domain appear on this continuum and explore how these supportive and misleading features may help or hinder children's learning.

This chapter presents such an empirical investigation in the domain of multidigit addition and subtraction. This investigation and the analyses presented here are based on an analysis of this domain presented in Fuson (1990a). This analysis is briefly summarized here in the next several paragraphs to provide a context for the chapter; the parts of the analysis concerning numbers larger than four digits and conceptual structures beyond those needed for multiunit addition are not summarized here because this investigation was limited to these areas.

Conceptual analysis of multidigit number marks and number words

English multiunit words and the usual multidigit marks have some features in common. Any system of words or written marks expresses large numbers of single units by combining several different larger multiunits (chunks of single units). English words and base-ten written marks both use the same multiunits based on powers of ten: Each multiunit consists of ten of the next smaller multiunit. Each of these systems also uses nine different symbols (words or marks) to denote the first nine numbers and then also uses these same nine symbols to tell how many of each multiunit there are. Words and marks also differ in important ways. Written marks require the perception of a visual layout of horizontal "slots" or positions into which any of the nine number marks can be placed, whereas English words require learning special multiunit words (thousand, hundred, ten) that are each prefaced by any one of the nine number words. English words immediately say the largest multiunit, but one has to look at the number of places in written marks to decide the value of the largest position.

Associations between these two systems enable translation between them. The first association is between the nine number words and the nine written marks (e.g., one for 1, two for 2, etc.), and the second association is between the English multiunit words and particular mark positions. To carry out a translation of marks to words using the second association (i.e., to say written marks as English words), one must count or subitize (immediately recognize visually) the number of positions in a multidigit number to find out the value of the leftmost position (e.g., "one, two, three, four -- oh, the fourth place is thousands"), or use the multiunit English word list in increasing order (ones, tens, hundreds, thousands) to find the name of the leftmost position. In both cases, these procedures are opposite to the order children are used to: they read and usually count (Fuson, 1988) from left to right, and they say multiunit English words in order from largest to smallest. To read this written multidigit number, one must then use the first association between the written mark and one of the nine English number words, say that written number, and follow it with the English multiunit word even though there is no cue in the multidigit marks to say this multiunit name. This process continues until the whole multidigit number is said. Translating in the opposite direction (writing spoken English words in marks) is much easier: one just uses the first association to write marks corresponding to the nine number words in the order they are said and ignores the special multiunit words. If one kind of multiunit does not appear in the number (e.g., 5096 has no hundreds multiunits), however, another difference between the two systems appears. In English words that multiunit just disappears and is not mentioned; in the written marks a special new mark, 0, must be used for that vanished multiunit so that all of the other marks will stay in their correct multiunit positions (they will move one position too far to the right if no mark is put into the position of the vanished multiunit). A final difference between the two systems is that one can easily say more than nine of a given multiunit and such constructions have a quantitative meaning even though they are not in standard form (e.g., twenty twelve or five thousand thirteen hundred fifty two), but one cannot write such a number because it pushes the larger multiunits into the wrong positions (e.g., 212 is not twenty twelve and 51352 is not five thousand thirteen hundred fifty two). The English words are concatenated in that they are independent and strung together successively. Young children and novices at learning the written marks (such as European adults used to concatenated Roman numerals first learning the new Arabic numerals in the Middle Ages) frequently learn written marks for each multiunit value and then concatenate rather than embed these multiunit values (e.g., one hundred sixty four is 100604 instead of 164 or seventy five is 705 instead of 75) (Bell & Burns, 1981; Ginsburg, 1977; Menninger, 1969). These errors result from using a cardinal notion of written marks (one hundred is three marks because it is written 100) instead of using the correct ordinal position meaning of the 100 mark: a 1 written in the *third* position from the right.

For either of these systems to mean anything, they must be linked to a conceptual multiunit structure. All of the above learning can take place without the learner having any idea what the English multiunit words or marks positions mean, and the translations can be carried out completely as rote procedures. This is in fact what seems to happen to many children in the United States under usual school instruction. They do not have any quantitative multiunit referents for either the English words or the written marks, and their conception of both of these systems, but especially the written marks system, is a concatenated single-digit conception: multidigit numbers are viewed as concatenated single-digit numbers (see the literature reviewed in Fuson, 1990a).

Two abstract conceptual structures (ways of thinking) seem sufficient for understanding multiunit addition and subtraction. These are the "multiunit quantities" conceptual structure and the "regular ten-for-one and one-for-ten trades" conceptual structure. The "multiunit quantities" conceptual structure supplies multiunit meanings for the English multiunit words and the marks positions. Its construction therefore requires experiences with multiunit collections of single units (collections of ten units, a hundred units, and a thousand units) that can be referents for the English multiunit words and the marks positions. In such multiunit situations, a viewer must focus on the cardinality of the units and conceptually collect these units to form the required multiunit (e.g., see the ten units as one ten formed from the ten units). Because the presence of the ten units cannot ensure that the viewer actually collects them into a multiunit of ten (thinks of them as one ten), a distinction is needed between the potentially *collectible* multiunits presented in a situation and the conceptual *collected* multiunits formed by an individual seeing the multiunits presented in that situation.

Base-ten blocks were invented by Dienes (1960) to support children's construction of multiunit conceptual structures. There are four kinds of blocks: single unit blocks (1 cc), ten-unit blocks (long blocks ten units long), hundred-unit blocks (flat blocks ten units by ten units by one unit), and thousand-unit blocks (large cubes ten units by ten units by ten units). Other physical referents and situations also can present such collectible multiunits, but base-ten blocks were used in the present study and so will be discussed as the exemplar collectible multiunits. The "regular ten-for-one and one-for-ten trades" conceptual structure is constructed from the multiunit conceptual structure, or from situations presenting collectible multiunits, by noticing that ten of one unit or multiunit makes one of the next larger multiunit and vice versa. This ten/one relationship can be learned as a rote rule; the ten/one conceptual structures are based on multiunits.

Much of the above discussion has ignored the fact that the English words, and most European number words, actually have many irregularities for the multiunit of ten while being totally regular for the multiunits of hundred and thousand. These irregularities create problems for English-speaking children learning single-digit sums over ten and learning place value and multiunit addition and subtraction (Fuson, 1990a, 1990b; Fuson, in press-b; Fuson & Kwon, 1991). In contrast, Asian languages based on Chinese are regular for the tens as well as for higher multiunits: 52 is said "five ten two." Naming in this regular way the multiunit of ten seems to facilitate children's construction and use of conceptual multiunits of ten (Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989) and their use of conceptual multiunits in multidigit addition and subtraction (Fuson & Kwon, in press-b).

Addition of multiunit numbers involves two major components that differ from addition of single-digit numbers. The first component is adding like multiunits, that is, adding tens to tens, ones to ones, hundreds to hundreds, etc. The need for this component arises because one cannot combine different multiunits to make only one of these multiunits (5 hundreds plus 4 tens does not equal 9 hundreds or 9 tens). The second component is recognizing and solving the problem of having too many ($\geq$ ten) of a given multiunit. This need only arises in the marks world, and not in the English words or with base-ten blocks, because it is only with marks that one cannot write more than nine of a given multiunit. Thus, when the sum of a given multiunit exceeds nine, one must trade ten of that multiunit for one of the next larger multiunit. The two multiunit conceptual structures, the "multiunit quantities" structure and the "regular one/ten trades" structure, can direct correct addition with respect to both of these components of multiunit addition and can help eliminate incorrect addition procedures. The physically salient different sizes in base-ten blocks suggest combining same-sized blocks to find a multiunit sum and also support trading when a multiunit sum exceeds nine. When teachers used base-ten blocks to model the standard United States algorithm (writing the traded 1 multiunit above that multiunit number in the top addend), second graders of all achievement levels and high-achieving first graders learned multidigit addition of four-digit numbers and gave conceptual multiunit explanations for their trading (Fuson, 1986; Fuson & Briars, 1990). These studies indicated that the use of base-ten blocks directly linked to written marks procedures and described in English words and block words could be a powerful instructional intervention. However, they provided little detailed information concerning the ways in which children formed multiunit conceptual structures, and they did not indicate the extent to which children could use the blocks in a similar linked way to construct their own multiunit blocks addition and written marks addition procedures.

Purposes of this study

This investigation had three main purposes. The first was to examine how easy it is for children to construct the relationships among English number words, written multidigit marks, and base-ten blocks and to maintain these relationships while exploring multidigit addition with the blocks and the marks. An important part of this first purpose was to identify trouble spots in this construction process for both practical and theoretical reasons. If teachers are to undertake this new approach, it would be quite helpful for them to have a roadmap marking typical roadblocks and error-prone routes as well as productive routes. Such knowledge can also contribute to theories about how to support such learning and about how this learning occurs. As part of this exploration, regular ten-based Asian word forms for the second position (e.g., "seven ten five" for 75) were taught to some groups.

The other purposes stem from the *Curriculum and Evaluation Standards for Teaching School Mathematics* (1989) and the emerging vision of new teaching methods described in *Professional Standards for Teaching Mathematics* (1991). These suggest supporting children's construction of their own arithmetic methods and recommend that children work in small groups. The second purpose of this study was to examine the kinds of procedures children would invent for four-digit numbers when given the support of base-ten blocks and to explore the amount of generalized place-value knowledge gained in making these constructions. The existing data on children's invented multidigit addition methods are limited almost entirely to addition of two-digit numbers (see the review in Labinowicz, 1985). Thus, we do not know how easy it is for children to use these two-digit procedures for three-digit and four-digit numbers (that is, to invent and see a general procedure across several digits) or to gain adequate generalized place-value understanding in such an approach. The third purpose was to explore the benefits and limitations of children working in small groups in this endeavor. We have almost no knowledge of the kinds of teacher support children need for multiunit addition approached in this way, and such knowledge would be extremely valuable for teachers attempting to support children's learning in the classroom. The analyses of these purposes are necessarily somewhat intertwined, but this report concentrates on the first purpose.

This initial study used children who were high-achieving in mathematics so that the mathematical work would not be limited by gaps in prerequisite mathematical knowledge or by huge motivational problems that interfered with their mathematical functioning. A second study using average and low-ability children has been done, but results are not yet analyzed.

Method

Subjects

The 26 children participating in this study were all the children in the highest achieving of the three second-grade math classes in a Chicago-area school that grouped children for reading and for math. Children in the school were from a wide range of SES backgrounds, from meeting the federal standards for receiving free school lunches to high parental income and education levels, and were racially heterogeneous. Children were given a written and interview pretest that assessed conceptual and procedural competence in place value and multidigit addition and subtraction. Children ranged on the pretest from solving no two-digit or four-digit addition problem correctly (6 children) to solving all three vertical and both horizontal problems correctly (4 children); they showed a similar range in place-value knowledge and conceptual explanations for 2-digit and 4-digit trading and alignment of uneven problems. There were many different patterns of place-value and addition knowledge, with some children showing strength in one but not the other.

Children were formed into three initial knowledge levels (high, medium, low) on the basis of pretest performance. Each initial knowledge level was split into two groups of 4 or 5 children balanced by gender. Because of the varied patterns of performance on the pretest the children in each group, especially in the middle and low initial knowledge levels, were quite heterogeneous with respect to the kind of domain knowledge they possessed.

It emerged during the study that the first-grade teacher of about half of the children had used a different kind of base-ten blocks to teach place value. She taught addition and subtraction but did not use the blocks for this teaching. Every group contained at least one child who had seen the blocks before. Thus, the data concerning children's initial exploration of the blocks must be considered as that possible when at least one group member has had some initial exposure to the blocks.

Procedure

The initial orientation to the study was done with the whole class during one class period. The study was described, and the session then focused on

establishing norms for the cooperative group work and describing the group roles of leader and checker that were used to facilitate group work and participation by everyone. The group approach was adapted from Cohen (1986) and Johnson and Johnson (1989). The principles of groupwork were to be brief, to listen to others and reflect on what they have said, and to make sure that everybody gets a turn. The leader's roles were to enforce these principles and to choose whose turn it was to speak. The checker's role was to be sure that group members said whether they understood or agreed with procedures being used or conclusions drawn by the group. These roles rotated around the group on a daily basis; children wore a special large button to identify their roles.

Each group had an adult experimenter who monitored the group learning. One experimenter was a Ph.D. candidate who had designed the study and groupwork approach and had considerable knowledge of the literature on children's mathematics learning. The other two experimenters were undergraduate honors students in psychology and education who had extensive experience with children. Each experimenter oversaw the videotaping of each group, took live notes of important mathematical discussions, and intervened when children's behavior became too rowdy or when the group became stuck on a mistaken procedure for too long. Math class was 40 minutes long, and about 35 minutes were effective working time as opposed to set up or clean up time.

An experimenter-intervention strategy was adopted that attempted to let children follow wrong paths until it did not seem likely that any child would bring the group back onto a productive path; the experimenter then intervened with hints to help the group but giving as little direction as necessary. This was done to provide maximal opportunities for the children to resolve conflicts and solve problems creatively. Because this necessarily always involved a judgment call, this loose description was replaced in the second session (see below) by a criterion of letting children follow a nonproductive path or engage in incorrect mathematical thinking for the length of one class session but then intervening. This criterion was intended to reflect the reality of a classroom where a teacher monitoring six or more groups might not get to a given group for a whole class session but would be able to give support by the end of that time.

Space and videotape equipment constraints resulted in a need for two successive data-gathering sessions. Each session used three groups--one high, one medium, and one low initial knowledge group. In each 3-1/2 week session the teacher worked in the classroom on a different topic with the half of the class not participating in that data collection session. Each experimenter supervised the two groups at the same initial knowledge level in the two sessions.

For the initial experience with the base-ten blocks the experimenters followed a script that asked children to do several successive tasks: 1) choose their own names for each kind of block, 2) find the (ten-for-one) relationship between adjacent-sized blocks, 3) find any similarity between these relationships, 4) establish the English words for each kind of block, and 5) establish the relationships among block arrays, English words, and the standard four-digit marks. The groups varied in the time they spent on these tasks, taking between one and three class periods to finish them. In the first session each group was given a file box of digit cards (small cards each with one numeral written on it) to show four-digit marks; these had been used successfully in the studies modelling the standard algorithm (Fuson, 1986; Fuson & Briars, 1990). When used for addition and subtraction by the children in the groups, the digit cards proved to be very time-consuming. It took children a long time to put all of the index cards away after a problem, and children frequently worked only in the blocks world or in the digit card world and did not link the two. Therefore, the digit cards were replaced in the second session (i.e., for the second three groups) by a "magic pad:" an 11" by 14" pad of paper which was magic because it had to show everything that was done with the blocks as soon as it was done but could not show anything that was not done with the blocks. Children were encouraged to "beep" whenever these constraints were violated. In both sessions children also wrote on individual papers after the first few days of addition. Because big cubes are expensive, each group set had two wooden big cubes and five big cubes made up of cardboard folded and taped into big cubes. Some groups also had for the addition phase a few hundreds blocks cut out of plain wood.

During the first session, the highest and lowest-achieving groups received the language intervention in which they were taught to use "Asian" number words for the tens place (68 was said "six ten eight"). We had intended to manipulate this variable across achievement and use it only in the middle-achieving group in the second session. But during the first session, only some of the children regularly used the Asian tens words. We did not want to interrupt the flow of children's work and the establishment of their autonomy by continually reminding them of this use. Therefore, we abandoned the manipulation of this variable and introduced this terminology to all three groups in the second session, intending to watch its survival and use with little support from the experimenters.

During the addition phase of the study, children were given an addition problem written horizontally on a long strip of paper and asked to use the blocks and the digit cards (or magic pads) to do that addition problem. After they had agreed upon a solution to one problem, they were given the next problem in a prepared list of problems. The first several problems in the list had four-digit numbers as

both addends (e.g., 1629 + 3747). No commas were used to write four-digit numbers. All problems required trades in one, two, or three columns. The number of thousands was generally one through four in each addend; this was smaller than the other numbers (which ranged up to addends of 9) because we did not have as many thousands blocks as other blocks. The issue of adding like multiunits was raised by giving the children some four-digit problems plus three- or two-digit problems later in the problem list. After several days of addition, children wrote marks problems on individual papers as well as doing them with blocks and/or digit cards on the magic pad. The goal was to move from doing coordinated block and individual paper solutions to doing just individual paper solutions connected to mental multiunit quantities.

We had hoped to have children work on multidigit addition until the group had agreed on one or more correct written procedures, and most children could add in written marks without using the blocks and could explain the addition in terms of multiunit quantities. Our agreement with the school regarding the dates for the study was based on the amount of time teachers in the earlier teacher-directed studies had spent with high-achieving second graders learning multidigit addition and subtraction with the blocks. Our original dates included at least 18 learning days, but math class was canceled on several days. Because we were obligated to teach both addition and subtraction, we moved on to subtraction before some of the children in the lower two groups displayed as much competence in addition as we desired (these subtraction results will be described elsewhere).

Because most groups did not get to subtraction problems with zeroes, the posttests were combined with a teacher-directed phase intended to be more like the original teacher-directed studies. The high initial knowledge children who had done such problems were split between the other two groups, and these groups worked for three days on such problems with considerable direction by the teacher for each group. During this time the experimenters interviewed children from their own group.

Analyses

All of the videotapes were transcribed by the experimenter for that group or by a work-study student. All mathematical conversations were transcribed verbatim and annotated with respect to actions with the blocks and marks (digit cards or magic pad), social-emotional interaction, and any other aspects of the group interaction not directly reflected in the verbal record. Off-topic digressions were to be summarized with an indication of their topic and length. All transcriptions were checked by a second transcriber.

To ascertain the relationships between actions on the blocks and actions on written marks, block and mark summaries were prepared for each day of addition. These showed in one column drawings of the successive block lay-outs and in another column the successive digit card lay-outs or magic pad writings; all of these entries were numbered with a line of the transcript and lettered to identify the child doing the action. These summaries made it easy to ascertain key features of the block or mark addition procedure and to determine how parallel the two procedures were. In spite of the emphasis to the transcribers on including complete accounts of the actions on the blocks, digit cards, and magic pads, the transcriptions proved to be variable in the extent to which these block and mark summaries could be prepared from the drawings already in the transcriptions; some tapes had to be viewed again in order to prepare adequate summaries.

A category scheme based on the analysis of multiunit knowledge in Fuson (1990a) was developed and used by one coder in a preliminary analysis of errors children made in the groups (Wallace, 1990). A major focus of these categories was associations between or among blocks, block words, English words, and written marks because we were interested in the extent to which children were constructing these associations. This category system was used over a three-month period by three coders to code every utterance. However, we were unable to achieve acceptably high inter-rater reliabilities. Coders would agree about the first level of association. For example, two coders would agree that a certain utterance was an English word/block association, (i.e., that it was an English word that referred to a block), but they would frequently disagree about further levels of association. For example, one coder would conclude that the child at that moment also had that English word and block associated with the written mark in the digit cards or on the original horizontal problem while the other coder would not agree with this further association. This was not a simple problem of different coders having more or less overall inclusive criteria; rather, they differed in their interpretation of what was in the mind of a given child at a given moment for a given utterance. We finally concluded that it is very difficult, and perhaps inherently impossible, to conclude for a given utterance at a given moment in time just which referents in the mathematical environment are intended by, or within the attention of, the child giving that utterance. The whole goal of this teaching/learning environment is to support the construction of a tightly linked web of interconnections among words, visuo-spatial (actual or mental) objects, and written mathematical marks. Therefore, a child who possesses or is in the process of building such a web can potentially be accessing all of these meanings or referents attentionally in the real world or mentally. However, a child does not necessarily do so at any given moment even though all of these meanings are

available in the actual environment or mentally. Therefore, we abandoned the attempt to code individual utterances and moved to descriptive methods in order to capture the complexity of the relationships children were (or were not) constructing among these different worlds; these methods permitted us to summarize the evolution of these webs within individual groups. We were concerned about the reliability of these more descriptive case-study methods, so the following criteria were established. For the descriptions that are relatively "objective" such as whether a given blocks procedure was accurate or not, the descriptive summaries in this chapter were written by the first author alone. For group interaction or social/emotional issues or other more complex issues, any summaries were prepared and agreed on by at least two authors.

Establishing relationships among blocks, block words, English words, and written marks: Results

The amount of time it took to establish relationships among blocks, block words, English words, and written marks varied by group from 1 2/3 to 3 40-minute class periods. The Session 1 high initial knowledge group (H1) took 2 days. The Session 2 high initial knowledge group (H2) took 1 2/3 days. The Session 1 middle initial knowledge group (M1) took 3 days (partly because of videotape failure that led to postponing the start of addition until the fourth day). The Session 2 middle initial knowledge group (M2) took 2 1/2 days. The Session 1 high initial knowledge group (L1) took 3 days. The Session 2 low initial knowledge group (L2) took 2 1/5 days.

Each group first chose names for the blocks. They then found the ten-for-one equivalencies between adjacent block sizes beginning with the little cubes and longs.[2] Children then ascertained the English words for the blocks by deciding how many little cubes were in each of the larger blocks. Some games were then played to practice the connections among the blocks, block words, and English words. Finally, each group worked on establishing relationships among blocks, English words, and marks. Results of each of these activities are described below. Some readers may wish to skip these detailed results and move straight to the discussion of this initial phase of establishing relationships. To facilitate this, and to provide an advanced organizer for readers of these results, a brief summary of the major results is provided in the next paragraph.

[2] Except where we are describing particular group block discussions, we will use the following names for the blocks: little cube, long, flat, big cube.

The names of the blocks chosen by various groups mostly depended on size and shape, and food related names were common. The naming process varied by group. Children easily found the ten-for-one equivalencies between the little cubes and longs, the longs and flats, and the flats and big cubes. They made few errors with respect to these equivalencies throughout this whole preaddition phase. Four of the 26 children did propose a four-for-one or six-for-one equivalency for the big cube and flat (i.e., they initially said that four or six flats made a big cube). Many verbal responses concerning equivalencies were not maximally helpful because they were so abbreviated. Children were very accurate in using English words and block words. They learned the Asian tens readily, though individuals varied in the extent to which they spontaneously used them in subsequent discussions. Children easily established relations among blocks, English words, and written marks. The need for zero arose in all groups and was successfully resolved. Some groups grappled with the issue of how to write block arrays that had more than ten of one kind of block (see Table 3).

Choosing names for the blocks

Experimenter direction. The experimenters gave the following directions to each group: "Choose a name for each kind of these blocks. Choose names that tell you something about the blocks so that you will be able to remember the names. You all need to agree on the names." Thereafter, the amount of experimenter involvement varied by group. Groups H1 and M1 (session 1 groups with high and middle initial knowledge) nominated and chose their names without any further comment from the experimenter. In Session 2 in all three groups some child nominated a block name that contained a number. Because we wanted the block names to be distinctive from the English words, which contain numbers in two different roles (as the multiunit name and as the number of multiunits), each experimenter said that the block names could not contain numbers. Two of the experimenters then suggested choosing a name that tells something about what the blocks look like. The group H2 and M2 experimenters both acted to ensure that the group agreed with the final choices. When a child in group H2 announced that they had named everything, the experimenter responded to the lack of clear voting procedure and choices by asking what names they had chosen and then asking if everyone agreed with those stated choices. She then asked the checker to verify that everyone agreed with the choices. The experimenter for group M2 also asked the checker to verify the final choices with everyone. The name-choosing process for the L1 and L2 groups involved considerable interaction with the experimenter. Early in the choosing process for group L1 the experimenter began trying to get

the leader to establish an orderly process, perhaps because the group was nominating names at random for various blocks; the experimenter gradually took over the role of leader. In group L2 the experimenter initially suggested a procedure (beginning with the smallest block) and then took over the leader role early in the process. The choosing process in these two groups did include all children because of the effective adult leadership. To the suggestion of "heavy" for the big cube by group L1, the experimenter pointed out that the cardboard big cubes were not heavy, though the wood cube was. The experimenter questioned the children's choice of "rectangular" in group L1 and "big ice cube" in group L2 because they took so long to say, but the children refused to change these names.

The block names. The number of block name nominations varied across groups. Groups H1, H2, M1, M2, L1, and L2 nominated 13, 50, 18, 14, 23, and 12 names, respectively (see Table 1). All six groups chose names for individual blocks rather than explicitly deciding upon a particular series or overall group category. However, four of the groups nominated a series of block names that did reflect some overall relationship, such as baby block, sister block, momma block, and daddy block. None of these series was the final choice. Group H2 nominated several different series names (see Table 1).

All six groups chose at least one block name that related to the shape of a block. Three of the groups (H1, M2, L2) used the shape of the block to designate names of food or food-related terms for all four blocks; the children in group L2 explicitly commented on the common theme of food in the names. The three remaining groups assigned two or three of the names based on the shape of the block, with only one of these being related to food ("pancake"). The other choices for these groups were based on size. The flat and long blocks were based on shape, and the small cubes were based on size in these three groups. The members of group L1 observed and discussed the fact that three types of blocks had square shapes and therefore this particular feature was rejected as a possible naming criterion.

Five of the six groups followed at their own initiative a consistent size order when choosing the names for three of the four block choices; the one block out of this order varied across groups. Two of the groups (H1 and M1) moved from the smallest to the largest block, and the other three groups (H2, M2, L1) moved from the largest to the smallest block. Group L2 moved from the smallest to the largest block at the direction of the experimenter.

The block naming process. Two different nominating and voting patterns emerged from the six groups. Three of the groups (H2, M1, M2) randomly nominated names for all the blocks, discussed the nominations, and postponed voting or choosing until the end of the discussion. Groups H1 and L2 discussed nominations for a particular block and agreed on a final name before nominating names for other blocks. Group L1 began with the first pattern but moved to the second. The procedure for L2 and the shift for L1 were initiated by the experimenter.

Table 1. Names Nominated and Ultimately Chosen By Each Group

Group	Thousands	Hundreds	Tens	Ones
H1	N: master ice cube N: <u>Big Mac</u> L: loaf of bread L: iceberg L: Mommy E: glacier E: meatloaf	L:flat L: <u>plate</u>	<u>L:carrot stick</u>	N&L: six-sided block L: baby C: <u>ice cube</u>
H2	Z: blockhead M: stegosaurus D: thousand block M: appletree Z: eight corner M: Christmas tree M: elm Z: redwood M: <u>fatty</u> M: facemask M: city Z: big square M: Papa square M: dunkey M: Susie M: Philip M: math four D: thousand math	D: hundred block D: birchtree M: house M: a hundred M: Mama square M: dinkey Q: Mama bear D: <u>flathead</u> M: math three D: hundred math	D: 10-block M: mapletree D: Pin Q: skinny guy M: child square M: donkey D: stick M: math two D: ten math Q: <u>skinny</u>	M: Pinnochio Z: one-square D: one-block M: Pinocchio(Pin) D: <u>small fry</u> Z: small square M: baby square M: dinkey Z: eight ball M: math one Q: baby bear
M1	U: thousands M: hundreds V: big guy O: <u>fatty</u>	U: hundreds O: kids U: kid guy U: baby U: <u>iceberg</u>	U: tens U: skinny guys U: <u>long legs</u> O: long guy	U&M: ones M: baby O: <u>little man</u> U: baby T U: <u>little guy</u>[a]
M2	N: one hundred T: square U: cube Dh: <u>ice cube</u>	T: square U: <u>pancake</u>	U: rod N: stick T: straw T: orange straw U: <u>licorice</u>	T: candy Dh: smallest cube in the world U: tooth
L1	N: <u>big block</u> N: Daddy block N: heavy block X: ten blocks	D: square blocks D: rectangular square D: flat block D: Momma block D: <u>pancake block</u> X: medium block	D: <u>rectangular block</u> N: rectangle blocks N: brother block N: sister block N: carrot stick X: skinny block X: medium blocks X: long block	D: little block D: small blocks N: baby blocks N: Daddy Junior X: <u>tiny block</u>
L2	T&K: thousands B: <u>big ice cube</u>	J: hundreds N: <u>bread</u> B: bread-bread	N: tens T&K: strange-strangey T: breadstick B: <u>pretzel</u>	B: ones B: <u>sugar cube</u> B: cheese

Note. The nominations are listed in chronological order and are prefaced by the nominating child's code.
[a] "Little man" is the chosen name, but "little guy" is the name actually used by the group. The change occurs without explicit group discussion.

The time devoted to the block naming task differed across the groups. Groups H1 and M1 completed the naming process in less than 3 minutes and 4 minutes, respectively. Groups M2 and L2 completed the task in 6 and 7 minutes, respectively. Both groups L1 and H2 spent 14 minutes nominating and assigning names to the blocks.

The block naming process varied considerably across groups. The personalities of the official leader and of the most dominant group member (these only sometimes coincided) strongly influenced this process. Some exercised their power with relatively little sharing, while others used their power to bring other group members into the choosing process.

The children in group H1 quickly grasped the task of assigning block names and worked well together to accomplish it. Each child seemed to enjoy the notion of assigning names to the blocks. The child who initiated the nominating activity, L, emerged as the dominant group member in the naming process. She nominated the most names (8), won the most final choices (2), and successfully vetoed nominations for blocks that she did not win. Ultimately, however, the group engaged in a voting process that included all group members for each decision.

Group H2 had difficulty in agreeing on what the block names should be. The process was meandering, with considerable disagreement between M and the official leader Q. M was a very dominant member in this process; she nominated 8 of the group's 50 nominations (many as a series of names) and gave 5 vetoes and 0 agreements to the other members. M monopolized the conversation and became frustrated when other members disagreed with her nominations without suggesting alternatives. Q was the least verbal member and nominated only 4 names. D, an involved member, won two of the final four block names. Although D was engaged for most of the activity, at times she would stray from the discussion and construct block buildings. Group member Z made seven nominations. He seemed frustrated with the prolonged inconclusive discussion and fairly early suggested that each person name one block. This was ignored for a long time, but was followed at the end when each person chose one block name and everyone agreed without voting.

Group M1 passively followed the strong leadership of U, their official leader for the day. U suggested 12 of the 18 nominations, ending with two of the four final names. U later changed the name for the unit block to her own original suggestion. The group expressed no direct vetoes and maintained a low level of engagement throughout the activity. One member, D, expressed his votes nonverbally by raising his hand and did not nominate any names.

Under the capable leadership of the official leader, Da, Group M2's voting process was expedient and smooth. Da initiated the voting process, objectively facilitated the discussions, and kept the group focused on the task at hand. Da's comments of "Does anyone have any suggestions for..." and "So who likes pancake for this?" propelled the voting process and invited involvement. Consequently, all group members were engaged and agreeable. The group offered many statements of agreements and few vetoes. Da suggested that the group choose names that would reflect the blocks' relative size. From his five nominations, U won 3 of the 4 final choices and openly displayed his pride.

Group L1 members contributed equally to their 23 nominations and maintained similar levels of involvement throughout the exercise. The experimenter acted early to elicit more leadership from the official leader and then began to function as the leader, eliciting these equal levels of participation and strongly guiding the name-choosing process. N, the official leader, offered tacit approval of D's and X's suggestions. D gained quiet dominance over the group by means of her parsimonious approval and frequent vetoes. X evaluated others' nominations with a balanced number of agreements and vetoes. Overall, the group was not strongly engaged and grew less interested over the 14-minute process. One group member, M, was absent during the naming process.

Group L2's voting process was structured from the beginning by the experimenter who suggested moving from the small to the large blocks. The official leader, T, relinquished her authority and strayed off task after her nomination of "breadstick" was not chosen by the group; the experimenter then took over the leader role. K, although nominating only two names, skillfully performed his job as checker. B suggested five of the overall 12 nominations and won three of the final four choices. The group at no time voiced vetoes of the few nominations but rather resuggested their preferred choice.

Finding the Ten-For-One equivalencies in the blocks

This activity was structured by a worksheet that presented three successive questions using block diagrams. The questions were all of the form: How many ___equal a ___ ? The first blank showed a diagram of the smaller block, and the second showed the block that was ten times larger. The questions moved from the small cube/long equivalency to the long/flat equivalency to the flat/big cube equivalency.

Table 2 presents the number of children giving verbal responses that stated "ten" as the equivalency or making block demonstrations showing that ten smaller blocks make one of the next larger blocks. Both kinds of responses are further classified into two types: a) simple block demonstrations or statements of a ten-for-one equivalency, for example, that ten longs make a flat or b) complex demonstrations or statements, which involved the use of an already established ten-for-one equivalency. Most of the complex cases were argumentations or demonstrations that used two adjacent equivalencies, for example, "Ten little guys in a long leg, ten long legs in each iceberg means one hundred little guys in an iceberg." The table includes not only children's responses to the worksheet but also spontaneous demonstrations of equivalence that occurred before this task (many children spontaneously put small cubes on top of a long or longs on top of a flat during the block name-choosing process) and after this task while children went through the rest of the preaddition activities (most of the complex uses occurred in the phases after the equivalency task). For each entry in Table 2, the number of responses a child made ranged from one to four.

Table 2. The Number of Children in Each Group Who Demonstrated Ten-For-One Equivalencies with Blocks, with Words, and with Blocks and Words Simultaneously

	Ten Hundreds = One Thousand		Ten Tens = One Hundred		Ten Ones = One Ten	
	Simple	Complex	Simple	Complex	Simple	Complex
Blocks and Verbal	12	8	10	3	13	1
Blocks only	-	1	8	-	8	-
Verbal only	10	7	10	9	14	9

Note. Over the total equivalency time, a child could produce equivalencies in each category (verbal only, blocks only, blocks and verbal at the same time). A child is entered in each category once regardless of how many equivalencies of that type s/he produced, but a child may appear in more than one category.

Every child in the study demonstrated understanding of at least one ten-for-one equivalency. All but two children demonstrated the ten/one equivalency verbally or with blocks or in both ways. These two children did identify both the thousand/hundred and the hundred/ten equivalencies. Twenty-three of the 26 children demonstrated the hundred/ten equivalency, and twenty children correctly showed or stated the thousand/hundred equivalency. Most of the demonstrations with blocks consisted of putting ten of the smaller blocks on top of the larger block (on top of the long or the flat) or beside the larger block (beside the long or the big cube). Children in three groups spontaneously put ten flats inside the open cardboard version of the big cubes as part of the discussion of how many small cubes make a big cube (see the next section). Verbal responses included simple answers of "ten" to the how-many question, counts of the blocks that ended in ten, equivalence statements that used indicating pronouns (e.g., "There's ten of these in this"), and full equivalence statements using the block words (e.g., "There's ten little guys in a long leg."). The majority of the responses were of the first two types. Most of the equivalence statements did not name the multiunits. The lack of demonstration of an equivalence by a given child may reflect only the structure of the group discussion as not requiring action of any kind from every child in the group. The "flavor" of the equivalency discussions in all groups was one of most children already knowing (from the place-value work with blocks in the previous year) or readily seeing the ten-for-one equivalencies. A few children said that they did not understand a particular equivalency, and other children in the group immediately demonstrated with blocks and verbally told the child the answer.

During the whole preaddition phase, there were very few errors concerning the ten-for-one equivalencies. There were two counting errors (final counts of nine and eleven instead of ten) that were immediately corrected by other children. In group L1, two children, N and D, answered "two" in response to questions like, "So, if this (single) is one, what's this (long)?" These responses of the ordinal number of the multiunit, the second multiunit, instead of the cardinal multiunit embodied by the blocks (ten) seemed to be misunderstandings of the question rather than lack of understanding of the multiunit value of the long block. Four children demonstrated confusion over what attribute of the big cube should be used to determine the equivalency. In group H1, E objected to determining the flat/big cube equivalency by stacking ten flats beside one cube, stating, "You can't do it by thickness." The other three group members finished their stacking and counted the flats to show ten. E then counted the blocks himself to verify the group's answer and agreed with it. There was a more prolonged confusion in group L1 that was initiated by N and X answering the flat/big cube equivalency question by focusing on the drawing of the big cube that was on the worksheet. The drawing directed them to the sides of the big cube, and they initially responded by saying four flats equal a big cube. The experimenter moved them from the drawing to the real blocks, where X immediately stacked ten flats beside the big cube and N put four flats on the four sides of the big cube. X reasserted that there were ten, but there was no discussion by N or the group to resolve this difference. During the task of finding the English word for the big cube, N answered "four hundred," demonstrating that he had not changed his view. D then counted the six sides of the big cube, so N changed his answer to "six hundred." X again made a stack of ten flats by the big cube. The experimenter then clarified that the question, "How many pancakes equal a big block?" means how many fill up the cardboard cube, not how many cover the sides. This seemed to resolve the issue because all three children spontaneously demonstrated this ten-for-one equivalency during the discussion of English names on the following day.

Within each of the six groups, at least one child identified the overall ten-for-one pattern that held across all three equivalencies. In all but one of the groups, this was a spontaneous observation that occurred during one of the particular equivalencies. In group L2 the observation that they were all ten was in response to the final worksheet question "Is there anything the same about what you found in (worksheet questions) 1, 2, and 3?"

Ascertaining the English words for the blocks

The English words for the first three multiunit values used in base-ten multiunit numbers are, like most words, arbitrary. But the quantity named by each of these English words ("ten" "hundred" and "thousand") can be ascertained by establishing how many little cubes are in each of the larger blocks. For the long, this is the same task as the ten-for-one equivalency task. For the flat and big cube, children can establish that one hundred little cubes make the flat and one thousand little cubes make the big cube. The hollow cardboard big cubes could be opened up to facilitate the task of ascertaining that one thousand little cubes fill the big cube.

Establishing the multiunit quantity of the flat and big cube in terms of the unit cubes, and providing the English names for these multiunits if children did not already know them, was the next task. Again, the process followed varied considerably by group. However, all groups established the hundred and thousand equivalencies fairly readily, but they did so with relatively little full verbalization.

Three groups had at least one child who began to explore the little cube/flat or little cube/big cube equivalencies during the ten-for-one equivalency task. Two of these groups (H1 and M2) continued on with the little cube/flat equivalency by covering a flat with longs and arguing that one hundred little cubes made a flat because ten little cubes were in a long. In both groups the experimenter then asked how many little cubes would fill the cardboard big cube. Both groups began to fill the big cube with little cubes. Both experimenters, perhaps prematurely, then suggested a more efficient approach (M2: "Is there an easier way?" H1: "How many little cubes if they were all level at the bottom?"). Group M2 then put ten flats into the big cube, and Da gave a full explanation based on all three adjacent ten-for-one equivalencies. In H1 E answered "one hundred" and suggested putting hundreds blocks in the big cube. The experimenter then asked how many carrots (longs) were in the big cube. L and E said "one hundred," and they all started filling the big cube with carrots. E then said he didn't think they had one hundred carrots and that they should start putting in some plates (flats). They filled a new cube with plates, saying that ten plates filled the big cube. The group then wandered into a discussion of the leader and checker roles, and an

explanation of why there were one thousand little cubes in the big cube was never elicited or given. The third such group (H2) only asserted verbally that there were one hundred little cubes in a flat; the experimenter never asked them to explain or demonstrate this with blocks. In this group the same pattern was followed for the little cube/big cube equivalence.

In the other three groups the experimenters specifically began a new phase in which they described the task as deciding what the English names were for each kind of block. Group M1 began like H2, but they asserted the answer verbally. When asked to demonstrate their assertions on the second day, U began putting little cubes on a flat while D put ten longs on a flat. M then counted the longs by tens (10, 20, ..., 90, 100). O reconciled these two approaches by putting little cubes on top of the longs on a flat to show the ten rows of ten little cubes each (only some of the rows of little cubes were made, but all ten rows were described). These children repeated these same roles with respect to the big cube except that O said they didn't have one thousand little guys so he repeated the approach of counting by hundreds the ten flats stacked by the big cube. When U said she did not understand, O began to show her. But she then said she understood, stacked ten flats, and said that it was one thousand because the nine flats are nine hundred and the last one makes one thousand.

Group L1 began by putting little cubes on the flat. N then put longs on a flat and counted them by tens to get one hundred. The group then moved spontaneously to the issue of the number of little cubes in a big cube and became involved in the ambiguity discussed above, with N and D asserting four hundred and then six hundred (based on four and then six flats to make the sides of a big cube) and X asserting that one thousand little cubes were in the big cube (because of a stack of ten flats beside a big cube). The experimenter clarified the meaning as filling up the big cube. On the next day two children verbalized several times the small cube/long and long/flat ten-for-one relationships to show that one hundred small cubes make one flat, but X said he did not understand. Most of these explanations used "these" rather than block words or English words. The experimenter elicited a counting by ten of the longs on a flat, which N had done spontaneously on the previous day. X then spontaneously moved from one hundred little cubes in a flat to saying there were one thousand little cubes in a big cube. M first stacked ten flats beside a big cube, and they then filled a cardboard big cube with the flats.

In group L2, B and J responded that they had learned last year that one hundred little cubes made a flat. B showed and verbalized that "these (sugar cubes) are ten (shows sugar cubes stacked next to a pretzel) and ten of these (pretzels) make this (bread)." J started putting lots of sugar cubes on top of a 'ead, and B pushed them into neat rows with a pretzel and covered the rest of .e flat with pretzels. To show how many sugar cubes in a thousand, B stacked n flats beside a big cube. To the task of filling the big cube, J recognized that .ey did not have one thousand little cubes to use and, when the experimenter said aybe they could fill it with other things, he excitedly said to fill it with flats. B anted to put in pretzels, which the other children began to do (after some sugar ıbes were in). B said that the pretzels had to be in rows (i.e., ten of them ·gether made a flat across the bottom of the big cube). J made a joke by using .e food block names to suggest that they make a sandwich with the sugar cubes heir little cubes) between two slices of bread (their flat). Time ran out before .ey finished filling the big cube.

Practicing labelling the blocks with english words and block words

Children played three kinds of games to practice the associations among the ock words, English multiunit words, and blocks. A child would choose a block ır say a block word or an English multiunit word), and the other children had to .y the English word and block word for it (or say the other kind of word and ıow the block). The experimenters were to continue the games until all children ere able to produce these English word/block/block word associations quite .pidly and accurately. These games were done because it was anticipated that rge numbers of inaccurate or very slow use of English multiunit words or block ords would interfere with children's ability to communicate during the multidigit ldition and subtraction phases.

All groups used the English words and the block words quite accurately during .e games, rarely making any errors. In some groups children also produced these ords very rapidly from the beginning of the games; these groups moved on ıickly to the next phase. No group spent more than half the period doing this ·actice.

To assess how accurately children used the block words and the English ultiunit words throughout the whole preaddition phase, all such utterances ɛtween the end of the block name-choosing phase and the beginning of multidigit ldition were identified. The English words included only those uses of an nglish word as a unit value (one) or as a multiunit value (ten, hundred, and ousand); not included were uses of these English words as the number of a given ıit or multiunit because such uses are unitary cardinal meanings that tell how

many of some kind of unit rather than telling what kind of unit.[3] So, for example, "a long legs is a *ten*" and "there are four *tens*" would be included as an English multiunit ten, but "there's *ten* of those in this" would not be.

Across all children, the words "thousand," "hundred," "ten, and "one" were used as multiunits 187, 161, 200, and 103 times, respectively, with the number of uses by each child ranging from 0 to 14, 1 to 16, 1 to 13, and 0 to 12 for these multiunits, respectively. There were no errors in the use of "thousand" or "one," and only four errors in the use of the words "hundred" and "ten" (all were in group L1). All but three children said each multiunit word at least once; the three exceptions were in group M1, where one child never said "thousand" and two children never said "one." These uses are pooled across examples of giving the English multiunit word for a block, for a block word, and for a written mark, so these children exhibited a very robust ability to give the correct English multiunit word for these various multiunit manifestations.

Children used the block words chosen by their group for the big cube, flat, long, and little cube 112, 92, 89, and 83 times, respectively, with the number of uses by each child ranging from 0 to 9, 0 to 7, 0 to 7, and 0 to 6 for these blocks, respectively. There were no errors in the block words for the small cube or long, one error in the use of the block word for the big cube, and four errors by three children in the word for the flat; all errors were in group L1. All but one or two children said each block word at least once (all but one of these exceptions were in group M1); two-thirds of the children said a given block word at least three times. It thus seemed to be quite easy for these children to use the block words they had chosen in their group.

Most of the English multiunit word and block errors in group L1 were confusions between the multiunits of ten and hundred. Three of the four errors in English multiunit words and block words for the hundred multiunit occurred at one point in a game where one child said pancake and three children said "tens" (this was quickly corrected to hundreds). One child used the block word "rectangular" instead of "pancake" for the flat and later grabbed a flat instead of a long block for the word "ten." The other errors were using a suggested but not chosen name for the big cube ("daddy" instead of "big") and giving the ordinal

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[3] During the discussion of how many little cubes make a flat and how many little cubes make a big cube, it was sometimes difficult to tell whether the words "hundred" and "thousand" were used as a single collected multiunit of small cubes or as the cardinal number of that many small cubes. Because the task in this case was to ascertain the latter in order to form the conception of the former, these meanings may be ambiguous or even simultaneously intended. Such uses were included in the analysis.

number of a multiunit (two, the second multiunit) instead of the multiunit value as ten.

Establishing relationships among blocks, English words, and written four-digit marks

The final two phases before multidigit addition focused on multiunit numbers composed of thousands, hundreds, tens, and ones. The first of these phases related block arrangements to English words, and the second established relationships among blocks, English words, and written 4-digit marks. A collection of blocks presents the same multiunit number no matter what order the blocks are arranged in because each block contains its multiunit value and thus carries this value to any new location. Although English words are ordinarily said in a standard order from the largest multiunit down to single units, the value of a multiunit number will be conserved if the multiunits are reordered or even split up and reordered: Three hundred two thousand five ten four hundred eight is obviously two thousand seven hundred five ten eight. However, written marks cannot be reordered because that will change their value. They do not carry their multiunit value within themselves in any feature except their relative left-to-right order--they are, after all, only ordinary single-digit numbers that tell how many there are of each multiunit and which multiunit is numbered depends only on the position of that number. It is therefore easier to say written marks if one uses the standard larger-to-smaller order of English words that matches the larger-to-smaller left-to-right order of written marks. It is also much easier for the multidigit written marks to take on the multiunit quantities presented by the blocks if the order of the blocks matches the order of the written marks. Therefore, in the first phase children were told by the experimenter that it was easier to say the English words for the blocks if the blocks were arranged left-to-right from largest to smallest. Children then practiced making several numbers by putting out several of each kind of block and saying each such block array in English words. In the final preaddition phase they did this while also making the marks for these block numbers by using digit cards (Session 1 groups) or writing on the magic pad (Session 2) or writing on individual papers (both sessions). Because the children during most of these phases made their own numbers by selecting some of each kind of block, not all of the issues discussed in this section arose equally in all groups.

Block arrays and English words. The initial phase of arranging block collections from the big cubes on the left through flats, longs, and little cubes on the right and saying the English words for such collections went smoothly in every group. The only error or difficulty occurred in group L2 when one child said that the ones go on the left.

Five of the six groups were told how to say Asian tens (the regular form of ten that parallels the English use of thousands and hundreds: 52 is said as "five ten two") after the first block arrangement.[4] They then practiced making several different block numbers and saying them in English words using the Asian tens. All groups learned the Asian tens readily, with no one making any errors for arrays having two or more tens. Teen numbers were not modeled by the experimenter, and most groups did not generate such words. One child in group L2 first said "one two" for twelve rather than "ten two." Groups L1 and L2 required some practice before the regular ten form replaced the usual English decade words reliably. Children in group M2 actually used the Asian ten form before they were told about it by the experimenter. They had just made a block arrangement and named it with block names (2 ice cubes 5 pancakes 4 licorice and 7 teeth), and they then produced the exact analogy with English words: 2 thousands 5 hundreds 4 tens and 7 ones.

In addition to the irregularities in how the multiunit of ten is said, English words for four-digit numbers have two other irregularities--omitting the multiunit word "ones" and omitting any mention of multiunits that do not exist in a given number rather than stating "zero tens." Although the multiunits thousand, hundred, and the various forms of ten are said, the word "ones" or "units" is not said. It is more consistent to say the ones (2 thousand 5 hundred 4 ten 7 ones) because each number is then followed by its unit. Children in all groups produced such forms spontaneously. In groups H1 and L2, the experimenter said that you don't have to say the ones. In group L2, N asked why, and K asked if they could say ones if they wanted to. M in group L1 ended a multiunit word with "and eight ones" and then asked if you are supposed to say ones or just the eight. So children are sensitive to this irregularity and seem predisposed to regularize the English word form.

Zero. Written marks explicitly signal when a multiunit is missing by putting a 0 in that position, but such cases are not said as "zero hundreds" or "zero tens." Instead, that unit is not said at all but must be skipped over, thus interfering with the regular production of the ordered multiunits. Again, it would be easier for novices to learn English words if each multiunit was named each time, and the relationship to written marks would also be simpler. Children in two groups, H1 and M2, actually said such a zero form. Also, M in group L1, after they had put

[4] According to the original design, group M1 was not given the Asian tens.

in a 0 digit card for the hundreds, showed explicit awareness that the zero is not said in English, "You couldn't say like 2 thousand and zero. You couldn't say something like that so it would be better just to say two thousand four ten four."

All of the groups spontaneously made at least one block array that omitted one kind of block, except for L1 in which each child was in charge of one kind of blocks so each kind was always used (the experimenter made a block array with no flats for this group). In groups H1, H2, and M2 a child correctly used a written zero for that block when making the marks for the block array with digit cards or writing them on the magic pad, and there was no discussion of whether or why a zero was needed. In groups M1, L1, and L2 the number was first written without a zero (e.g., 249 instead of 2049). In L1 and L2 one or more children then argued that there should be a zero. These arguments took two forms. One was the observation that there were none of a particular kind of block ("there are no ones"), so a zero needed to be written. The other was that the number without the zero is the wrong number (a block array of four big cubes five longs seven little cubes was written as 457, and K said, "It needs a zero because that'd be four hundred fifty seven."). The first argument addresses why you use a zero (to tell how many of that unit there are), and the second tells why you must use the zero: The marks show the wrong number if the zero is not there to push the single digits into their correct multiunit places, unlike the English words where one could say zero ones but it is not necessary, or even common, to do so. In group M1 the experimenter precipitated this kind of argument by asking the group what the blocks said ("two thousand forty nine") and what the digit cards said (249: "two hundred forty nine"); the group decided that they needed a zero to show the zero hundreds blocks in order to move the 2 into the fourth (thousands) place. The forward and backward thinking and counting of places that is required to understand this argument was nicely demonstrated by a discussion led by the experimenter in group L2. English words are written down just as they are said in order from left to right. But to read any given multidigit number, a child must do a reverse right-to-left process in order to decide the name of the farthest left place before beginning to say that number as an English word. This was described by a child answering the experimenter's question, "How do you know it is the thousands place?" as follows: "Cuz the four on the end-that's the one, and then the seven is the ten, and the three is the hundred, and the two is the thousand." The amplification of this response by another child beautifully captured the two reverse processes that must be gone through to read a number: "Because thousands is after--is before the hundreds." Thousands is after the hundreds in the initial right-to-left assignment of multiunits but is before the hundred when the number is said as an English word (the numerals are read from left to right).

One other issue concerning zero arose in two groups. This is the common attempt by children to make the written marks parallel the English words and explicitly name each multiunit by using zeros to show the multiunit, that is, to use cardinality (four positions show thousands) rather than ordinality (only the fourth position shows thousands)(see the discussion in the introduction). Because two thousand is written as 2000, or three hundred as 300, children want to write three zeros after a number to show that it is thousands or write two zeros to show that it is hundreds, yielding forms like 2000300405. A child in group M2 asked, "Why don't you put the zeros in for two thousand?" and a child in L2 similarly asked, "Why didn't I make zeros after my three hundred?" In the first case the child spontaneously then said, "I see now." and the issue was not pursued. In the second case the child asked again after the question was ignored. Another child responded that the zeroes were not necessary because "you can tell (it's a hundred) because you know how many numbers there are (pointing to the three places up to and including the hundred's place)."

Use of commas. In the United States, a comma is used to separate groups of three digits in a multidigit number. The comma is placed by counting each three places from the right so that each three digits will compose one of the larger multiunits based on a thousand that constitute the large English words. In the United States these base-thousand multiunits are called thousand, million (one thousand thousands), billion (one thousand millions), trillion (one thousand billions), etc.[5] The comma may make it easier to identify a 4-digit number as beginning with the thousands to someone who knows how commas should be interpreted. But it is an unnecessary feature of the written base-ten marks, arising instead from the base-thousand structure of the English words and from a desire to simplify perceptual processing of many numbers (other countries use a period or a space for the same purpose). Commas were not used in any numbers or problems presented to the children in this study. The issue of commas arose as an extended topic of discussion in two groups. In group M1 two children articulated a comma rule "Put a comma every three numbers," but they counted from the left and wrote a number as 204,9 rather than as 2,049. The third child Dh said that the comma should be on the other side, and they agreed. This difference arose later when they all wrote on paper very large numbers to show commas. The first two children wrote from left to right. They wrote three numbers, made a comma, wrote three more numbers, made a comma, etc. This

[5] In Great Britain a larger sub-base of a million is used instead of one thousand; the words are the same up to one million but then a billion is a million millions, a trillion is a million billions, etc.

process will only work if you end with three numbers. Dh wrote his whole long number, started from the right and made curves over each group of three numbers, and then wrote in all the commas. Group L2 had an argument over whether you have to write a comma. Children in each group sometimes wrote a comma between the thousand and hundred places in the addition phase and occasionally used them this way in the preaddition phase.

Ten or more of a given multiunit. The final issue confronted by each group in ascertaining the relationships among the blocks, English words, and written marks was how to write block arrays that had ten or more of a given multiunit. This is a crucial issue in multidigit addition, for it arises whenever the sum of a given multiunit is ten or more. This issue could arise in the preaddition phase if a group made a block array with ten or more of a given kind of block and wanted to write that block array in marks. In fact, all groups very early made such a block array. In all but group M2 the experimenter for that case and several others restricted the number of blocks by having children make that pile of blocks smaller. In groups H1, H2, and M1 the children later made such arrays, and then modified them in order to write the arrays in marks. Flats were stacked to make a big cube and the thousands were increased verbally by one, and other kinds of blocks were counted to make ten and the next larger multiunit was increased verbally, for example, finding the value of 21 longs as follows: "200 (there were two flats), 10, 20, ..., 90, 100 (counting the value of ten longs by counting each one as ten), 300 (incrementing the original 200), 1, 2, ..., 9, 10 (this time just counting ten longs as single units to get ten longs as another 100), 400 (incrementing for the second 100 made from longs)." Examples in group M1 included such large cases as these 21 longs and 30 small cubes. In group L1 this issue only arose in an array that had nine longs. M said, "If they (longs) were ten, they'd be like that (points to flat)." On the next block array, M then restricted the number of longs to less than ten, saying, "or else we'll be in the hundreds, and we don't want that to happen." Group L2 made an array with ten small cubes, and T wrote this as 3,3610. Three children said that you couldn't do a ten at the end because it has to go in the tens pile, you'd have to regroup. Regrouping was for them a procedure done with numerals; the discussion focused on "take the 1 and put the zero here." They did nothing with the blocks. The experimenter asked, "What are those (the small cubes) the same as?" B answered "one of those (a pretzel)," and N answered "a ten." B put down a pretzel, and they wrote 3,37 (and later added a 0 to make 3,370). B then stated a general rule about making block arrays and writing marks: "So you have to do any number *under* ten (her emphasis) cuz then you'd just put down one more of these (pretzel) and you wouldn't need the ten (ones)."

Group M2 embarked on a long exploration of this issue that extended over substantial parts of the second and third days. On the second day this group of five children made their second block array of five big cubes ten flats eight longs and twelve little cubes. The discussion went as follows: Several children: "Five thousand ten hundred." Da: "No, one thousand." U: "Wait, that makes six thousand." Several children: "Six thousand eight ten twelve." The next block array also had more than ten units that the children again read as teens ("seven thousand five hundred six ten fourteen"). Three block arrays requiring no trading followed, and then U made an array of three pancakes, four licorice, and twenty nine teeth. The group's pursuit of this problem over the rest of that day and part of the next is given in Table 3. Everyone in the group recognized that, although they could say twenty nine teeth, they could not write that many teeth in the usual written marks. They generated several different interesting solutions to the problem they posed to themselves in this situation: conveying in written form the number of blocks they had. They eventually needed the help of the experimenter to solve their original problem -- how to write that many blocks in standard marks -- because their reformulation of the problem (writing the blocks they had) did not solve the original problem of writing standard marks for those blocks. Their version of the problem actually cannot be solved: standard marks cannot write the blocks they had, standard marks can only write a value equivalent to those blocks. The blocks can be traded to find this equivalent value that can be expressed in standard marks. When on the next day they got into the same issue with a new number, the experimenter focused them on the task of changing the blocks to match their stated English word value for the blocks (see Table 3: they said "five hundred" but wrote the 5 in the fourth position). They had traded flats for a big cube on their first block array, so quickly saw its relevance here. They might have been able to think of trading with less direct support than the experimenter actually gave. This group went on in the addition phase to use this solution for writing too many blocks in a two-part "add and then fix" addition method they invented.

Establishing relationships among blocks, block words, English words, and written marks: Discussion

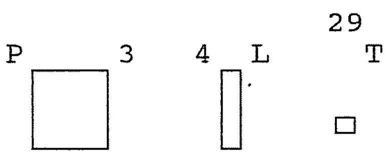
The success of the block nominating and naming procedures in groups H2, M1, and M2 in which the names for all blocks were nominated simultaneously indicates that teachers do not have to be concerned and intervene immediately in a messy choice process. In the groups in which the experimenter did not intervene much, children were able to select sensible block names. All of the block names chosen

Table 3. Identifying and Solving The Problem of Writing Ten or More of a Given Unit: Group M2

Day		Problem phases
2	All	Three hundred four ten twenty eight-twenty nine!
	T	So, what's the number?
	Dh	(writes 3429)
	T	How come when Dh writes it on the pad it looks like three thousand?
	Da	I think he should write a zero instead of three. Then they would know there's no ice cubes.
	U	But then it would say 0 3
	Dh	(writes 3429 more neatly) So, who cares?
	Da	Three thousand
	U	Three hundred - three hundred
	Da	To me it looks like three ten, four hundred.
	Da&N	Three thousand, four hundred, two ten, nine
	T	But that's not the number that we set up
	T	(to experimenter) But this is one number (points to 3) and this is one number (points to 4) and this is one number (points to 29)

Experimenter's question ("How would I know that?")
precipitates many marks solutions:

3 4 X 2 9   3 4 <u>2 9</u>   3 $\boxed{4}$ $\boxed{2\ 9}$



3 4 2 9 |3|4|29| <u>3</u> <u>4</u> <u>29</u>

Discussion of proposed solutions

U	That's what we did with the boxes before.
N	That's messy.
U	No, it isn't. It's the same thing though.
Da	OK. That's it, we're writing it down.

N	We're wasting paper
Da	T's idea is good. Write 3 4 29.
U	But that's the same thing as the boxes. We could have stopped a long time ago with boxes.
Da	OK, that's how we're doing it. How about we stop now?
Dh	(starts writing the number)
T	That's the wrong way.

More solutions

Da,T,N	Three hundred, four ten, twenty nine
T	So what's this number, guys?
U	But it still looks like three thousand.
Da	But see, this whole thing is underlined, so we can read it.
U	No we can't.
Da	(asks experimenter if she has any suggestions)
Exp	I want to see how you guys figure it out.
U	I have a different idea. I think I know what to do. (writes 300 4 ten 29 teeth)
N	I have an idea. (writes 3 4 29 with a bracket under the 3, the 4, and the 29)

No resolution

U	That's the same as boxes, again.
Da	All we're doing is wasting paper.
U	I know. We could have stopped a long time ago. This is the first day, too.

3   New similar problem

	The group made 5 flats, 3 longs, 17 small cubes which they wrote 5317 and said as "five hundred three ten seventeen ones."
U	That's the same problem we had yesterday.

	Experimenter questions support solution
Exp	Is there anything you can do with the blocks to make it look like five hundred? (shifting children's focus from trying to write the blocks in nonstandard marks to trying to change the blocks so they can be written in standard marks)
U	Yeah - um - should put like a thousand down, take these away and put five of those blocks (ice cubes) down
Da	What we could do with the magic pad is write hundreds and put an arrow up to the five and then write tens and put an arrow up to the three and then write ones and put an arrow up to the seventeen. That's what we could do and we can't do that with the blocks.
T	You put lines.
Exp	Let's concentrate on the blocks right now.
Da	What could you do to make the blocks look simpler?
T	You could take away some teeth to make it less than ten.
Exp	Is there anything you could do with the ones you take away?
Da	You can make another problem out of it.
Exp	Is there any exchanges you could make with the teeth and the licorice?
U	Yeah, you take one of these (a licorice) and that will make ten (teeth). Or if you take ten of these (teeth), it would be the same as that (licorice).
Da	You could take ten of these (picks up ten teeth) and then put them over here (in licorice pile) and then take one of these and just put them over here or you could just take ten away (teeth) and put another licorice there. Here, take ten of these away.
U	You need seven left, so count out seven.
Da	You take ten away and put another licorice there. And that makes our problem easier. five hundred four ten seven.
	(short discussion about writing a zero before the number)
Da	But yesterday we used about twenty pieces of paper for one problem.
U	That was my problem that I shouldn't have even thought up.
Da	That took forever.

Note. Group M2 chose the following names for the blocks, listed largest to smallest: big ice cubes, pancakes, licorice, and teeth.

by the groups proved to be easy for the children to remember and use. We wanted to observe this naming process and so allowed each group to choose block names. In a classroom it is probably advisable for all children to agree on the same names in order to facilitate communication among the children in that classroom.

The level of spontaneous verbalizations about most activities was disappointingly low, given the high verbal ability of many of these children. The discussions of the ten-for-one equivalencies and of the number of small cubes in the flat and big cube contained some good thinking and some complex arguments. But in general these children did not spontaneously produce verbal responses that would be maximally helpful to group members who did not understand. For the ten-for-one equivalencies, the many statements using "these" and "those" rather than the block names or English words required listeners to understand the referents for "these" and "those" in a sometimes complex physical and social environment. The even more frequent simple responses of "ten" or counts to ten required each listener to know the question being answered by these responses. The use of zero in written marks arose in all groups, and the issue of saying zero in English words arose in some groups.[6] Children successfully used zero in all groups. However, there was no spontaneous discussion or clear articulation of why zero is needed in marks but not really needed in words. Group L2, when asked by the experimenter, came close to such a discussion, so it seems likely that children at this level can clearly articulate these reasons if the teacher initiates and supports such conversations. Similarly, having children use multiunit words and block names to give full statements of any mathematical relationships would increase the ability of weaker or momentarily distracted group members to follow the mathematical discourse and might increase the accessibility of these relationships to the speaker.

The children's discussions of the equivalencies and their use of English words and block words with the marks underscore a limitation of English in this domain. The English language does not clearly differentiate between the use of the word "ten" as a unitary cardinal number telling how many there are of some unit and its use as a single multiunit of ten collected units that serve as a new higher unit. French, Spanish, Russian, and many other European languages do differentiate between these two meanings by providing a special ending for the single collected new multiunit meaning.[7] For example, "diez" is ten in Spanish, and "decena"

[6] Experimenters in fact were to make arrays that required a zero if the children did not.

[7] I am grateful for helpful conversations with Robert Streit concerning this issue in several different languages.

means a group of ten. Some sense of the collected meaning supported by these special endings is provided by the English word "dozen" which means a collected group of twelve (a dozen eggs); "dozen" in fact sounds as if it comes from the French special collection ending added to the French word for twelve: "douze" (12) plus "aine." However, the existence of this differentiation in the language does not necessarily mean that all users of that language comprehend the collective meaning. Recent conversations with some teachers from Puerto Rico indicate that the functional use even by teachers of the word "decena" may be, at least in some cases, limited to a label for the tens position in a written multidigit mark and may carry little or no cardinal meaning as "a group of ten." The common use of the collected-ten meanings may depend on the use in a culture of the metric system and the consequent frequent packaging of items into groups of ten or measures of ten units, as is common in the Soviet Union, for example. Without experiences of such actual collections of ten, or special experiences in the classroom, these special ending forms may not have multiunit meanings.

When there is more than one ten, hundred, or thousand, the "s" in the plural forms (e.g., five tens, eight hundreds) in English does provide a minimal cue that one is talking about collected multiunits. Children in this study did frequently use the English plural form just as they used the plural form for block words indicating multiunits (e.g., four thousands six hundreds five tens two ones" or "four ice cubes six breads five pretzels two sugar cubes"). But the fact that standard multidigit English words drop the "s" and say instead "four thousand six hundred fifty two" muddles even this possible difference, and it is not always easy to hear this plural form even when it is said. This lack of differentiation in English of ten as the number of multiunits and as a kind of multiunit, combined with the tendency noted in this preaddition phase for children not to use the multiunit words, caused communication difficulties and some addition errors in the addition phase.

Children's behavior in this preaddition phase indicated that they are predisposed to regularize the irregularities in English and generate full English forms that parallel the block words and name each unit. Children used the Asian regular tens words quite easily, though certain children and certain groups used them more than others. Sometimes a child added the units word "ones" so that every unit would be named and sometimes used "zero" to name a unit (zero tens) rather that just omitting that unit. It may be much easier for children, especially those not high achieving in mathematics, to see and use the multiunit structure of the English words and relate the words to multidigit numbers if they use full regular forms in the beginning of learning. Because none of the forms are "wrong" (just nonstandard or unnecessary), they could be dropped when children are older and understand the whole multiunit structure.

It may be particularly helpful to use the Asian regular ten forms because the English special words that hide the tens in two-digit numbers enable children to persevere erroneously in situations whose ten-structure would be much clearer with regular ten words. For example, the prolonged engagement of group M2 with writing the 29 teeth (see Table 3) seems much less likely to have occurred if their language said those teeth as "two ten nine" instead of as "twenty-nine." Saying the ten suggests trading the blocks to simplify the block display or adding in those two tens with the other tens. For saying words in the teens, the fact that one child in group L2 first said "one two" for twelve rather than "ten two" suggests that it might be better not to use the abbreviated form "ten two" (as Chinese do) but to use the full equivalent of the later decade form for the teens: Tell how many tens by saying "one ten two" or "one ten and two." Saying a full regular Asian form for numbers with zeroes (e.g., saying 100 as one hundred zero tens and zero ones) would also help to eliminate the cardinal/ordinal confusion that leads children to want to write forms such as 100406 instead of 146 for "one hundred forty six." It is the usual short-cut wording of 100 as one hundred, 10 as ten, and 40 as forty that suggests such errors.

The equivalency question for the flats and big cube actually is ambiguous because six flats (or four if you ignore the top and bottom) do "equal" a big cube in the sense that one can make one big cube out of six flats (dimensions are off by 0 to 2 cm depending on how you assemble the six flats). Because one cannot tell just by looking whether the wooden big cube is solid (made of ten flats) or empty inside (made of six flats), the former feature needs to be clarified from the beginning. The phrases "make as much wood as inside the big cube" or "fill up" (for the cardboard version that can be opened) might be better. The solid meaning also can be addressed by using weight, which some children did, for example, "See this (ten flats) is just as heavy as this (wooden big cube) except this (big cube) is much easier to carry around." One reason so few children had this possible confusion may be that, because the flat/big cube equivalency was last, children had an expectation that this relationship would be the same as the earlier ones (a ten-for-one relationship) and that one would show it in the same way--by stacking the smaller blocks next to or on top of the larger block. A new version of the blocks that is recently available does avoid this ambiguity. The blocks are clear plastic and fit together, so the big cube is seen to be filled with little cubes and can be made by sticking together ten flats. The disadvantage of this version is that they are somewhat difficult to put together and take apart, and the small extra parts that enable them to fit together may be distracting.

Our results concerning these preaddition experiences clearly are limited by the achievement level of these children and by our discovery that many children had place-value experiences with a different version of base-ten blocks in first grade. Establishing these relationships would presumably take longer and need more teacher support if all children were new to the blocks or were of varying achievement levels. However, the ease with which many children handled these ideas, and the high level of some spontaneous discussions, indicate that establishing relationships among blocks, English words, and written four-digit marks is well within the zone of proximal development of high-achieving children working in groups at the beginning of second grade. If teachers support discussion of these relationships, these children are probably able to articulate and explain clearly all of these relationships. Without such teacher initiation and support, even these high-achieving children do not spontaneously discuss all of the important issues in these relationships or articulate them clearly enough for weaker children to understand.

Results of the addition experiences

Adding like multiunits

Every group immediately added the like multiunit blocks. After making each addend with blocks, they either pushed the addend blocks of each kind together and counted all of the blocks of a given kind, or counted the blocks in place, or used extra blocks to make as many sum blocks for each kind of block as were in both addends. Evidently the visually salient collectible multiunits in the blocks supported the correct definition of multiunit addition as adding like multiunits. There was only one exception to this uniform definition of adding like blocks: One child in group M2 suggested that the answer should be obtained by counting all of the blocks of all kinds (he thus ignored the collectible multiunits in the blocks and considered each block as one countable unit item).

All groups also added two four-digit written marks addends by adding together the marks written in the same relative positions. In the groups that were clearly linking block addition and written marks addition, this carried the connotation of adding like multiunits. For some children, their written multidigit procedure already entailed the understanding that they were adding English multiunits (ones to ones, tens to tens, hundreds to hundreds, and thousands to thousands). For other children, multidigit addition was a procedure carried out on concatenated single digits, so these actions were based on a procedural rule and did not imply understanding of adding like multiunits. Evidence of these different bases for marks addition was not as clear as for multiunit understanding of the next component of multiunit addition, trading when one has too many of one kind of

multiunit, and it is linked to this trading knowledge. Therefore this issue will be discussed further in the section on trading.

The incorrect block arrays always involved making the first digit out of big cubes (for example, 287 would be made from 2 big cubes 8 flats and 7 longs or sometimes 7 units), and the incorrect alignments always aligned on the left. Although these both resulted in a failure to add like multiunits, such errors seemed to stem from the left-to-right manner in which block arrays were made and marks were written. Although block arrays can be made in any order from written four-digit marks, almost all block arrays for all addends in all the addition problems were made in the same order in which marks are written and English words are said: from the big cube to flats to longs to units. Only 5 out of 118 addends were made in any other order. Group H1 made two block arrays in the order longs, units, big cubes, flats and one array from units to big cubes, and group M1 made two arrays from units to big cubes. The initial several problems worked by all groups in which two four-digit numbers were added seemed to induce a "set" towards making a block array by using the big cubes first. Thus, when seeing a written multidigit number, children who had done several four-digit plus four-digit problems had a predisposition towards making the first number on the left out of big cubes. Similarly, there was a predisposition for groups who were writing the addends vertically to start writing the 287 under the 3458 on the left, putting the 2 under the 3.

Table 4 shows the relative correctness of making block arrays and aligning written marks problems for the problems in which the two addends had different numbers of digits. The performance reflected in Table 4 is group competence, not individual competence. Some children in some groups also verbally suggested making the three-digit number using the big cubes or writing the numbers aligned on the left, but they were ignored or corrected by other children. All but two block arrays (18 out of 20) were made correctly initially or were immediately corrected by some group member, while a lower proportion of marks problems (15 out of 24) were written in correct alignment. When incorrect marks problems were corrected, children justified or explained their correction by using multiunit words: by saying the English words for the marks ("That's two hundred not two thousand") or by saying the block words ("That's two pancakes, not two icebergs"). Thus, thinking of the multiunit values by saying the marks in English words or block words may be an effective way to reduce such alignment errors. Failing to think of the multiunit values seems to be more of a problem with the written marks than with the blocks, so teachers might suggest that children read problems in English words and/or block words.

Table 4. Correctness of Block Array and Marks Alignment for a Four-Digit and a Three- or Two-Digit Addend

	Block	Mark
Immediately made three-digit number correctly	13	10
Made three-digit number incorrectly but soon changed by group	5	5
Made three-digit number incorrectly but eventually changed by group	0	2
Made incorrectly and never changed	2	7

Note. The six groups worked a total of 24 problems in which one addend had four digits and the other addend had three or two digits. Children worked 20 of these problems with blocks and marks and worked 4 only in marks. The entries in the table reflect group, not individual, performance.

Trading or putting: Solving the problem of too many of a given multiunit

All of the groups recognized the problem of having too many of a given multiunit. This problem in fact arises only in the marks world: One can have as many blocks of a given kind as one wants, but one cannot write down a blocks display that has more than nine of a given kind of block. Thus, this problem really only arises when blocks addition is linked to written marks addition. The collectible multiunits in the blocks support the solution to the problem -- trading ten of the multiunit with too many for one of the next larger multiunit, but the problem presents itself in the marks world where writing down ten or more of a given multiunit pushes the other written marks too far to the left (see group M2 wrestling with this problem in Table 3). Group M2, because of their earlier extended experience, did indicate that they understood why they could not write two digits for a given multiunit. No other group clearly explained why having ten or more was a problem. Many children brought some awareness of this issue from

their knowledge of written multidigit addition, but for many of these the knowledge seemed to be formulated as an arbitrary rule such as "You can't write two numbers" or "You have to regroup (which meant writing little 1's in specified places)" or "You can't have more than nine." These rule statements were never accompanied by any hint of why this might be so or even that any justification of the rule was necessary.

These children clearly are capable of understanding and articulating this problem as did group M2; they do not have to memorize an arbitrary rule. Every group in fact did present to themselves this problem when they first made block arrays: Every group made at least one array that had ten or more of a given kind of block. But in all groups other than M2, the experimenter did not allow them to consider this problem and constrained block arrays to those that did not have this problem. Thus, this problem could come up for initial consideration when first making block arrays, and the recognition that writing two digits for a given kind of block makes the other digits in the wrong place would then be helpful in the addition context. When this issue arose in the addition context, children did not spontaneously try to understand why writing two digits for a given multiunit is a problem or seek to explain rules they had memorized. They needed outside support to raise this issue and focus them on trying to explain why writing two digits is a problem in the marks world. That group M2 easily saw this problem indicates that it is an easy one to solve if it does get raised.

Most of the addition phase was directed toward solving the recognized problem of what to do when there were too many of a given kind of multiunit. Each group had a different experience with this issue, and the nature of the experience was crucially affected by the extent to which the blocks addition procedures were linked to the marks addition procedures by that group. An overview of the evolution of co.:rect block trading and correct marks trading in the sequence of solution procedures for each group is provided in Table 5, and the addition experiences of each group are briefly summarized in the following sections. In Table 5, the nature of the trading in each successive block and written marks procedure is characterized, and each procedure is classified as linked or not. In linked block and mark procedures, children added or traded a given kind of block and marks position simultaneously (different children doing the blocks and marks step) or soon after each other before any other block or position was added or traded. In unlinked procedures the children worked in separate unconnected block and marks worlds. Either some children in a group worked on blocks while the others worked on marks -- and there was no communication or synchrony between these solutions -- or the whole group worked a problem in blocks and then in marks and there was no connection made between the solutions.

Table 5. Accuracy and the Linked Status of Blocks and Marks Addition Procedures by Group

Group	H1	H2	M1	M2	L1	L2
Block Addition						
Accurate trade						
Linked to marks		10,11	2,7, 7,9	1,2,3,4,5,6, 7,8,9,10,11, 12,13,14,15	2,8,9, 10,11	4,4,5,6,7, 8,9,10,11 12,13,14
Not linked to marks	2,3,4, 5,6		2,3,4, 4,5,5		3,4	
Correct sum but not show trade with blocks						
Mentally added the trade	1		2			
Copy traded answer from marks					9	
Added like multiunits and leave sums ≥ ten						
Linked to marks					1	
Not linked to marks		1,2,3,4,5 6,7,8,9	1		1,3	1,2,3
Trade inaccurately						
Linked to marks					5,7,8	
Not linked to marks					4,4,6	
Digit card/magic pad addition						
Accurate trade						
Linked to blocks	1[a]	10[a],11[a]	7,7,8, 8,9,9	11,12,13,14 15	1,2, 8,10	4,4,5,6,7, 9,10[a],11[a], 12,13,14,15
Not linked to blocks	4,4,7, 8,9, 10,11	1,2,3,4, 5,6,7,8, 9	3,4		1,4[a], 6[a]	1,3
Correct sum but not show trade with marks						
Copy answer from blocks	5,6		2	1,2,3,4,5,6, 7,8,9,10,11	2,4[a], 6[a]	
Copy answer from individual papers	2			13[b]		
Mentally added the trade	3		1,2,5		3,4[a],6[a], 9,11	
Trade inaccurately						
Linked to blocks					5,7,8	4,4,4, 4,8
Not linked to blocks	8	4	1,1,1,2,2 3,3,3,3,4, 4,4,4,5,5, 5,6,8,8,9		1,1,3, 10	1,1,1,2, 2,3,8

Note. Numerical table entries are the ordinal number of the problem. Many problems had multiple solutions proposed and used, and different parts of a problem may have been solved differently; each partial or whole solution is entered in the table. Groups H1, M1, and L1 used digit cards, and groups H2, M2, and L2 used a magic pad. Block counting errors and mark single-digit addition errors are ignored for the classification of accurate procedures. In linked block and mark procedures, children added or traded a given kind of block and a given marks position before moving to another kind of block or position, and the actions in the blocks word and the marks world were connected.

[a])Marks were on individual papers.

[b])These solutions were on individual papers and had some "unfixed" sums.

Patterns over all the groups and important issues that arose in any group are discussed in the main section following these group summaries (the Discussion of the Addition Experiences). Some readers might wish to move directly to that section. A brief overview of these group results is as follows. Children were much more accurate in their block than in their mark trading procedures. Across the six groups, only one group made any inaccurate block trades, while five groups made inaccurate mark trades. Three of these groups made a mean of 13 inaccurate marks trades. There is not space in this chapter to describe each of the solution procedures in detail; these are described in Fuson, Burghardt, and Fraivillig (1992). The focus here is on the relative accuracy and ease of addition with blocks and marks, and the nature of the relationships children established between these two worlds. The amount of supportive or misleading verbal descriptions and explanations, and the quality of such verbalizations elicited by the experimenter, are also briefly summarized. Each of the groups established a somewhat different relationship between working with the blocks and working with the marks. The groups also varied considerably in the accuracy of their marks procedures and in the number of different inaccurate marks procedures they used. These paths through addition were influenced by the extent to which the dominant and most socially skilled individuals in the group possessed and used conceptual multiunit knowledge versus rote marks rules.

Group H1. In group H1, L found the block sum for the first addition problem by looking at the two block addends and adding each kind of block, mentally trading over to the next larger multiunit when necessary, writing these sums on her paper under columns she made headed 1000, 100, 10, 1, and then making the block answer from her written marks answer. On the next and all successive problems on which block addition was done, whenever there were ten or more of a given kind of multiunit, children traded ten of these blocks for one block of the next larger size. There was relatively little discussion or justification of this block trading throughout all of these problems, and many trades were made without any verbalization. Each child at some time did make at least one comment articulating the ten-for-one nature of the trade. For example, E said for the very first trade, "There are twelve blocks (singles), so ten become one of the carrots (he makes this trade and puts the carrot with the carrots in the top addend)." Examples from the other children are: C: "Take one ten out of there;" L: "We took the ten pancakes and handed them in for a Big Mac;" and N: "Thirteen tens. And an extra hundred (adding a plate to the answer). And that's three tens." The children cooperated in physically making the block trades, indicating that they all understood these ten-for-one trades well. Most of the time the blocks were not related to the digit-card addition method. Frequently the girls did addition with

the blocks while the boys did addition with the digit cards (or vice versa), and there was relatively little discussion or linking of the two during addition (they usually were on different parts of the procedure) or afterwards.

These children did not devise a complete digit-card solution to the problem of trading until the sixth digit-card attempt (problem 7), when the experimenter said just to do the digit cards. In the earlier digit-card procedures children simply made digit-card addends and copied the sum from blocks or individual papers or added the digit cards mentally or devised only partial procedures (problem 4). For problems 7, 8, and 9, children showed the traditional algorithm by putting a card containing the digit 1 above any column that received a trade. There was little discussion or justification of the digit-card procedure or of written marks procedures done on individual papers. When the experimenter did ask for an explanation, these children did produce conceptual quantitative descriptions of their addition procedures that indicated that they had at least implicitly linked the block multiunit addition to written marks addition and that for them the traditional algorithm involved trades of ten of one kind of multiunit for one of the next larger multiunit, not just writing little 1's. Thus, they were capable of levels of discussion and explanation that were much higher than those generated spontaneously in the group. However, even these children would have benefitted from explicitly linking the blocks to their marks procedures, as when on problem 8 E changed the tens sum (which came from 7 + 3) from 0 to 1, saying, "I never can remember if you're supposed to put a 1 or 0." He soon changed it back, but this would have been a good opportunity for the experimenter to ask him to answer his question by thinking about the blocks.

Group H2. Group H2 immediately defined "adding blocks" as counting all the blocks for a given multiunit or pushing the blocks together, but, until the tenth problem, they did not do a full trade when they had ten or more of a given kind of block: The blocks answer for several days had at least one multiunit with ten or more blocks. On the magic pad, they used the traditional vertical algorithm writing the 1's above the next left column. They all knew this algorithm before the addition phase began. They sporadically linked the blocks and magic pad addition for one or two columns, but they did not link or really even compare a whole blocks addition and answer to their magic pad written addition until the tenth problem, worked during the fifth day of addition. On that problem M, who had from the beginning focused more on the blocks and tried to link the blocks and magic pad more than any other group member, related her whole individual written marks problem to the blocks problem by describing blocks addition using blocks words and showing what marks she had written for each kind of block. The experimenter asked about the difference between the blocks answer and their individual magic pad answers (these all showed traded answers for tens and hundreds while the blocks were not traded), and the group immediately traded ten skinnies for one flathead and ten flatheads for one fatty. The group then spontaneously traded blocks on the next problem. No one in this group ever spontaneously gave a full conceptual explanation of trading, and such explanations were not elicited by the experimenter. Their later subtraction work indicated that they were capable of high level thinking and did have ten-for-one conceptions of trading.

Group M1. Group M1 quickly constructed accurate blocks addition, trading with the blocks on the second and all subsequent problems. They began with the blocks linked to the digit cards, and almost constructed a correct, fully linked, addition procedure on the second problem, where discussion of the blocks hundred/thousands trade led them to make a correct digit-card trade instead of writing the hundreds sum as two digits. They did not spontaneously make a similar blocks and digit-card link for the ones/tens trade, and so ended with a blocks answer of 5376 and a digit-card answer of 53616. On the next day they did a complete correct blocks addition with trading, and then moved to the digit-card world, where they never connected the digit cards to the blocks. This separate pattern continued for four more problems over three more days. During this time they generated multiple incorrect digit-card marks procedures, suggesting or carrying out as many as four different incorrect procedures on one problem. On three of these days they also wrote marks problems on their own individual papers. Much of the time these individual solutions were different from the digit-card solution showing at that time, and the incorrect procedures circulated among the group members like a virus, popping out on the digit cards or on individual papers with little predictability. For most of this part of the addition phase, these children were operating in three different unrelated worlds: the blocks world in which they carried out correct multiunit addition and seemed to understand their trading procedures, a digit-card marks world, and an individual paper marks world. Occasionally they even operated in a fourth world, the problem card on which the problem was written horizontally, because they would discuss a procedure by pointing to that card, and the discussion or proposed solution might differ from the solutions in the digit cards or on individual papers.

On problems 5 and 6 the experimenter tried to have them connect the blocks to the marks procedures by having them talk about the marks procedures using blocks words; on problem 5 they actually used blocks, and on problem 6 they just used block words. But they did not carry out the linking consistently and continued to use wrong marks trading procedures. Finally on problem 7 the experimenter enforced block and digit card links, making the children relate their procedures column by column (e.g., as soon as they added and traded the unit blocks, they had to show that in the digit cards). They carried out correct linked methods in both worlds. For some parts of the problem, they even constructed two different correct linked methods (increasing the bottom addend by one and the standard algorithm of putting the traded new multiunit block above its multiunit column). On the next day the experimenter asked them to do the digit-card addition while saying block words. As long as they said block words, they traded the digit cards correctly, beginning by using the traditional algorithm. But they stopped using block words, and then reverted to their most frequent wrong digit-card procedure (in which the ten from any two-digit sum was written above the tens column because it was a ten). The experimenter again forced them to use block words, and they solved the problem correctly in two ways, using the traditional algorithm and increasing the top number by one to show a trade. Each of these procedures was described by two different group members using block words.

On the next day, the experimenter continually had them describe the digit-card actions with block words (the child doing the digit cards was a "digit-card robot" that could only move after the blocks had moved) and they did a digit-card procedure tightly linked to their blocks solution. They traded correctly in blocks and digit cards and described their digit-card trades in block words except for one long digression in which they used English words, leading two members to argue for the procedure in which the "ten" from any two-digit sum (e.g., the ten from 8 + 7 in the hundreds column) is written above the tens column. They then all worked the problem on their individual papers and all got the same answer; until they were forced by the experimenter to link the blocks to the digit cards, at least two children on every problem did incorrect marks procedures on their individual papers. One child spontaneously used Asian tens to read his answer aloud: "four thousand two hundred one ten nine" (this was the one group that were not taught the Asian tens.) The children then did another problem individually on paper, getting the same correct answer. They all used the traditional algorithm, but there was not time to discuss it using block words.

Group M2. The five members of group M2 cooperated very well throughout the addition phase, kept the blocks and magic pad procedures completely linked throughout this phase, and invented the only correct nonstandard marks procedure used for any length of time by any group. From the very first problem, they added one kind of multiunit block by pushing all of them together, wrote that sum on the magic pad, pushed together another kind of blocks, wrote that sum on the magic pad, etc. This resulted in at least one two-digit answer for every problem. The group then "fixed" this answer, trading ten of any block for one of the next larger block wherever possible. For ten problems solved over four days, they copied the fixed answer from the blocks onto the magic pad. Initially they did all the block trading and then copied the whole traded answer. But these group members were very conscious of keeping the blocks linked to the magic pad, and someone beeped whenever some action with the blocks was not immediately recorded on the magic pad (this was supposed to be the modus operandi for all the session 2 groups). This linking soon led them to write each intermediate answer after each trade. For example, for 1947 + 4185, they wrote each fixed answer as they changed them; they changed 5 10 12 12 to 5 11 2 12 to 6 1 2 12 to 6 1 3 2. All members clearly understood their two-phase addition method. They called the second phase "fixing" or "changing" the answer and realized that this phase was necessary in order for the marks not to show an answer that was too large (recall that this was the group that spent a long time in the preaddition phase figuring out how to write a block array with 29 units).

However, they did not really reflect on the marks procedure itself, and the successive traded answers were often written unaligned below the problem, or scattered across the page, or even on a different page from earlier answers and from the problem so that such reflection was not easy to do. On problem 9 the answer was written after all the trades were made because there was no official writer. One child did not understand this answer, and other children explained the answer using English words to explain the fixing that had been done while pointing to the magic pad problem. On problem 10 and 11 they did the problems with blocks and recorded on individual magic pads (pieces of paper). The child making the blocks for problem 11 put out one too many flats and one too few longs. N added the columns on her individual paper rather than writing the announced block sum, as had been done on earlier problems and was done by other children for this problem. She noticed and stated that her sums were different from the block sums. The group decided that the block sums were wrong and corrected them, and then fixed the magic pad problem. Thus, at this point they could carry out the first phase of their addition procedure just in the marks world by adding each column of marks.

To support reflection on trading in the marks procedure, the experimenter began the sixth addition day by showing the group their unfixed and fixed marks magic pad answer from the day before and asked them how they could fix such an answer without actually doing the block trading. The subsequent conversation included many descriptions of imagined individual block trades that related to particular marks, and children could write the marks trades if they thought about the blocks for that particular trade. In response to the experimenter's repeated request for a marks procedure that did not involve thinking through the individual

block trades, two group members evolved a method in which the 1 to be traded was written above the sum to the left and a small x was put below the 1 in the sum. The fixed answer could then be written in one step by increasing each sum number that had a 1 above it (each 1 reflected a ten in the column to the right). This written procedure was described in block words, so it was clear that these children understood the traded 1 as ten coming from the right. For example, an explanation of the new procedure of writing the 1's above the sum to the left was: "I took ten of these (teeth) and put one licorice up. Then I took ten licorice and put one pancake up. Then I took ten pancakes and put one ice cube up."

Over the next two days the experimenter supported a fading procedure in which children increasingly worked in the marks world while still relating the marks to the blocks by describing whatever they did with the marks in blocks words. On problem 12 they described the blocks trades before they fixed their marks answer. For each of the final three problems they first solved the problem on their individual magic pads and then did the problem with blocks and discussed the problem. On the first such individual problem, solutions ranged from completely correctly fixed problems to partially correct fixes to sums not fixed at all (e.g., 2 15 10 8). The last two problems had almost completely correct fixed solutions by the three children (the weakest three) present on this last addition day; there were two errors (one sum with one too many and one sum with one too few) in the 18 fixing steps these three children did on these two problems. Again there were full explanations in block words of the marks trading, and both group magic pad solutions integrated the group's fixing procedure with the traditional algorithm: the two-digit sums were written below the problem, 1's were written above the problem in the columns to the left where necessary as in the traditional algorithm, and the fixed answer reflecting the sum of these 1's and the unfixed sum in that column was written below the unfixed sum. In the final solution N also wrote little x's below the 1's in the unfixed sum because "I just wanted to show what I put up there when I carried."

Group L1. Group L1 immediately added like blocks, but they did not trade blocks on the first problem. For a long time, the whole group was driven by procedural rules and usually used concatenated single-digit words in describing their marks procedures (e.g., "Put the two from the twelve there and put the one at the top."). On the first problem they suggested or did with the digit cards two incorrect and two correct trading procedures, a correct one of which was linked to the blocks (looking at the twelve tinies led D to take away the 8 and 4 digit cards in the ones column, replace them with a digit card 2, and replace the digit card in the tens column of the top addend by a number one larger: change a 3 to a 4). On the second problem the two girls traded blocks from the ones to the tens, used the traditional algorithm with the digit cards, and clearly described the trade both in English words and block words. They also traded the blocks accurately for the hundreds/thousand trade. D was the leader in the mathematical aspects of these activities, while M was more socially dominant but was the mathematical follower. Their physical trading procedure involved counting as many blocks as the number of ones in the sum and moving them aside to keep them for the block answer, removing the remaining ten blocks, and adding one of the next larger block. For example, for 7 + 6, they counted three pancakes and moved them aside, removed the remaining ten pancakes, and added in one big block. The two boys did not understand the hundreds/thousand trade, and the girls only described it in single-digit terms or described their counting actions literally. This led the boys, and the whole group thereafter, to describe trades as "take away three" instead of saying anything about the ten traded blocks. In the third problem one block trade was made and the other was described but not done because the boys were being silly. The digit-card answer was obtained by mentally trading and adding in the extra multiunit to that sum.

Then began a six-problem sequence over five days in which the most dominant member of the group, M, imposed an incorrect procedure on the group. M reacted to the correct digit-card solution of problem 3 by carrying out her new procedure instead. This procedure stemmed from the "take away" language used in the early block trading and a rule repeatedly stated by M: "You can't have ten in any column." In M's procedure, when the sum of any column was more than nine, the addend digit cards were taken away and replaced by a 9 digit card, nine blocks were left in that column, and the rest of the blocks were thrown away. This resulted in answers with many nines in them (e.g., 4995 and 6999). Everyone agreed with M's rule (you can't have ten in any column), but there were repeated rebellions over the five days as various group members objected to the nines procedure and tried to discuss alternative procedures. M was very domineering in her responses to this resistance, and usually won by stating her rule. But several times she expressed frustration and said that she did not understand what she was doing, and also proposed substantive objections to her procedure (e.g., when someone else did her nines procedure with the blocks, she said, "You can't just throw some blocks away. You have to use them.") Almost all of the discussions of proposed correct procedures and of the nines procedure used single-digit procedural words, and the multiunits in the blocks were not used by the children in this discussion.

On problem 8, the experimenter focused them on the blocks, asked why they threw away one rectangle (they were making 7 + 3 = 9 with the rectangles), and asked if they could do something with ten of the rectangles instead of just throwing one away. M immediately responded that that was what she had been trying to do earlier when "I was trying to explain that we should put the one in the other column." (i.e., do a traditional marks trade); she had proposed this earlier when they were trading. Over this and the next problem the experimenter supported the children's block trades, sometimes perhaps giving suggestions before it was necessary. On the final two problems the children did the block trading independently and spontaneously, but they never evolved a digit-card trading procedure to show the trades. They added in the trade mentally without showing it with a marks 1. This may have partly been because their digit-card procedure was to take away the two addend cards and replace them with the sum card. Thus, in both the blocks and the digit cards, only the answer showed at the end of addition. Three of the four children did the same marks procedure on their paper as with the digit cards -- they added in the trade mentally and did not show it with the traditional 1 mark written above the top addend. The children never spontaneously gave a full explanation of the marks procedure in English words or blocks words, and such explanations were not elicited by the experimenter.

Group L2. From the first problem the five children in group L2 added like multiunit blocks, though they continued throughout the addition phase to argue about whether they should show the sum with extra blocks or just push the addend blocks together. For the first three problems over two days the added blocks were not traded or linked to the magic pad written marks procedure. Various incorrect trading procedures as well as the correct traditional procedure (called regrouping by the children) were done on the magic pad. "Regrouping" for all the children involved writing a little 1 somewhere. They always referred to the regrouped number as a one (never as a ten or hundred or thousand) and never explained what they were doing or why. A typical such interchange is the following: N: "Well, then what's the one there for?" T: "It's just because you regrouped, and you keep the one there for a little mark."

On the third addition day (problem 4) the experimenter emphasized that they had to write on the magic pad each time they did something with the blocks and enforced those links throughout the problem solution. The experimenter also asked children for explanations several times during the problem solution; explanations of particular actions with the blocks or marks written on the magic pad were seldom given spontaneously. There were three trades required by this fourth problem, and each was discussed at length by the group. For each place in which a trade was needed, some children suggested or physically did a ten-for-one block trade or put ten of one block together in the column to the left. Discussion of these trades often used ten to describe the 1 written on the magic pad. However, for each trade at least one marks magic pad or blocks procedure stemming from a regrouping notion as writing a little 1 somewhere was also suggested or done; these were always described by using procedural descriptions of writing single digits somewhere. The children were solving the problem from left to right, so some wanted to write the 1 above the column to the right (a correct procedural analog of the usual right-to-left solution: write the 1 above the next column to be added). The children who focused on the block values and block trades and those who focused on single-digit marks regrouping varied across the problem solution. For each place, children were eventually convinced by the blocks and by the explanations of the blocks trades, and they agreed to trade the blocks and agreed on the correct marks writing of these trades.

Except for marks errors on problem 8, the children solved ten more problems over five more days in which the blocks were traded accurately and a correct written marks procedure linked to the blocks trade recorded these trades. The group solved problems left-to-right and right-to-left and, at the suggestion of the experimenter, did some problems both ways in order to decide which way they thought was best. This issue for most of the addition phase was a boys against girls issue, with the boys wanting to add left to right. In this group the two boys happened to be the weakest mathematically; whether this was related to their preference is not clear. They decided by the end that right-to-left was easier because they did not have to cross out sums they had written and write new ones.

This part of the addition phase was not as smooth as indicated by the uniformity in Table 5. Although the group worked hard and well over the whole period, there were continuing controversies about who got to write what, who got to do each kind of block (five children and four blocks meant one person was without blocks), whether to begin on the left or on the right, and whether to use extra blocks or just push the blocks together. Some of these controversies were carried on simultaneously, and the discussion became quite confused. The experimenter needed to intervene at times to facilitate their resolution of these issues. As in group L1, these children also rapidly moved toward a procedural take-away description of the block trading in which the trading was not described and ten was not mentioned. Instead, only the number of blocks remaining was stated: for twelve pretzels, "Leave two out, take away the rest." This was more efficient than counting ten and taking them away and was based on conceptual knowledge: twelve consists of ten and two, so if we count two and take the rest, we will be taking ten. But for the two weakest group members, the failure of the conceptually strong group members to give conceptual explanations, or even descriptions that included the word "ten," for the block trades as they were carried out meant that these members still retained their procedural single-digit marks regrouping orientation along with the new blocks trading procedure and that they could be quite fuzzy about the ten involved in the trade. Once the experimenter asked the group how many were left after they had taken away the ones of the two-digit sum, and the answers "nine" and "eleven" were given. On the last day of the addition phase, the experimenter asked each group member to give block word and English word descriptions of the block trading when they were just doing marks problems. Three of the members gave several good explanations that indicated that by now the marks procedure was no longer just marks single-digit regrouping but was firmly grounded in conceptual multiunit quantities. The two weakest members could not consistently give such explanations. Their linking of the blocks to the marks would have been facilitated by conceptual multiunit descriptions from the stronger group members rather than the short-cut procedural take-away description of block trades.

Other aspects of multiunit addition

The other component of multiunit addition, single-digit addition of each multiunit, and the technical aspects such as copying the problem, writing digits, and counting the blocks did not present much difficulty to these children. These children either knew single-digit facts or had fairly rapid solution procedures for finding them (such as sequence counting on). Thus, adding like multiunits in the marks world was fairly easy when they began to fade into just doing the marks problems without the blocks. One problem was miscopied from the problem card. Blocks were miscounted in making the block addends several times. These sometimes were caught quickly and other times resulted in long derailments of a problem solution because the source of the difference between the blocks and the marks was not seen immediately. Such errors helped us to ascertain which children were working from the blocks to the marks and which were only in the marks world (the marks of the former reflected the incorrect number of blocks), but these derailments were frustrating to the groups. The block addends and sums also sometimes became incorrect during the solving of a problem because children played with blocks in the problem (and removed them in so doing) or because the block problem became quite messy and the blocks in the problem merged into the nonproblem blocks in the block bank reserve.

Total time of the addition phase

Groups H1 and H2 took five days for the addition phase, and all of the other groups took eight days. During this time Groups H1, H2, and L1 solved 11 problems, Group M1 solved 9, and Groups M2 and L2 solved 15 problems. Thus, the high-achieving groups averaged about 2 problems per day, while all the others averaged from 1 to 2 problems a day. The two high groups did have fairly good conceptual understanding of marks addition at the end of that time, though the experimenter did not force full linked procedures or full conceptual explanations by everyone in both groups, partly because she felt that these children had good understanding. All of the other groups were moved into the subtraction phase earlier than was ideal because of our contract with the school that required us to do subtraction as well as addition within the allotted time.

Discussion of the addition experiences

Linking the blocks and marks

The purpose of base-ten blocks is to enable children to construct conceptual multiunit structures as meanings for multidigit written marks and English number words. Their function in addition of multidigit numbers is to enable children to use their conceptual multiunit structures to understand how to add multiunit numbers and, eventually, how to carry out meaningful written marks addition without needing to use the blocks. To facilitate both of these goals, each experimenter was supposed to establish and enforce linked procedures in which blocks addition was tightly linked for each kind of multiunit to marks addition with the digit cards (first session) or on the magic pad (second session). At the beginning of the addition phase each experimenter gave such linking directions -- that as soon as children did something with the blocks, they were to do that same thing with the digit cards or on the magic pad. Because children in session 1 frequently violated this directive, the session 2 children had the further linking directive that each child was to beep whenever something was written on the magic pad that had not been done with the blocks or something was done with the blocks that was not written on the magic pad.

In spite of these directions all groups, except group M2, did not link marks addition to blocks addition. For some groups this separation continued for days. When children were functioning in a marks world separated from the blocks worlds, this unlinked marks world proved to be a fertile ground for generating many different incorrect addition procedures (the incorrect entries in Table 5 are described in Fuson, Burghardt, and Fraivillig, 1992). The group M1 children on some days were even functioning, at least briefly, in four different unlinked worlds: blocks addition, digit card addition, marks addition on their individual papers, and marks addition on the horizontal problem card. In all groups, when the experimenter imposed these links after children had devised and persisted in

incorrect marks addition, one simple directive was not enough. Children might record the marks for the block addition of one or two kinds of multiunits, but the blocks and marks would then become separated. The experimenter had to continue to monitor and support linking in order for children to execute a fully linked procedure in which each addition with blocks was immediately recorded. Such linking support was necessary for at least one day, and in some groups, for two days. Children then seemed to be able to carry out a linked addition solution without any support.

There were several identifiable sources of this difficulty in linking. First, spontaneous comments by children in some groups indicated that children constructed different interpretations of the chunks involved in the experimenter's linking directive. Group M2 wrote down each digit as a block array was made; the other groups usually wrote the whole multidigit number after the blocks were made. One child in another group argued that they should not write the first addend number after making it with blocks because they "had to make the whole problem (i.e., both addends)" before writing it. Thus, the crucial linking of writing the result of adding and trading (if necessary) one kind of block needs to be clearly articulated, emphasized, and monitored by the teacher.

Second, the practical division of work sometimes contributed to this lack of linking. Who got to do what, and when, was an extremely important and emotionally charged issue in every group (except perhaps group M2, which, under an effective and fair initial leader, quickly evolved an atmosphere of equal participation). There were long arguments in many groups about turns and fairness. In most groups the leader chose who got a turn at something. In some cases these decisions were based on who had not yet received a turn, but in many cases the choices seemed to reflect friendship or criteria other than equal turns. A time-consuming counting rhyme was chosen as a fair procedure in one group. Sometimes a whole problem would be worked by one child chosen by the leader; linking then depended upon that child, and to a lesser extent, on the rest of the group. Sometimes some children would do the blocks while the others did the digit cards (e.g., girls the blocks and boys the digit cards); this set-up proved to be quite difficult to link, with each subgroup having its own momentum. To facilitate participation by everyone, the experimenter for groups L1 and L2 instituted the agreement that each child had one kind of blocks. This meant for group L2, with five members, that one child did not get to do anything or wrote on the magic pad (a potential source of unlinking as this child might move ahead of the blocks or lag behind the block solution). Distributing the blocks in this way did involve everyone on every problem and worked fairly well, but children sometimes went out of turn or were so involved in the problem solution that they did another child's blocks (with consequent protests).

Third, at least initially, the direction of the link from the blocks to the written marks--and the consequent status of the digit cards or magic pad as the written record of the blocks procedure--was not emphasized enough. The fact that many of these children already had a procedure for adding written marks also interfered with establishing the link in this direction because there was some tendency to use the written marks procedure they already knew and make the backwards link from the marks to the blocks. Nor was the purpose of the blocks underscored sufficiently, which was to enable children to construct written marks procedures they could understand, explain, and defend conceptually in terms of attributes of the multiunit numbers they were adding. The learning task should have been presented from the beginning as one of using the blocks to help explain in terms of the blocks, and in English words, why one or more written marks procedures worked. Without these needed emphases, the children assimilated this task into their usual school mathematics set: Learn how to do something -- add with the blocks and add with the marks -- and these procedures do not need to be connected or explained except by rote rules. The emphasis on explaining why a marks procedure works might also have elicited much more discussion and explanation than these children generated spontaneously.

Fourth, the relatively small space shared by groups H1 and L1 meant that the blocks problem and the digit-card problem were crowded together and sometimes children did not have room to lay out the digit cards exactly as they had laid out the blocks. Thus, sufficient space must be provided to support linking.

In all groups in which the experimenter imposed links between the blocks and the written marks, these links did prove to be sufficient to direct a correct marks procedure and to eliminate incorrect marks procedures that were done before the links were made. The collectible multiunit quantities in the blocks were salient enough to direct and constrain correct block trading, and any block trading was easily recorded as a written marks procedure. Block trading always involved a ten-for-one trade, but the one next-larger block could be placed in various places in the block problem. No group ever made any incorrect block trades when they approached the trading problem within the block world. Some children needed their attention directed to the collectible multiunits as a potential source of the solution to their problem of having too many of a given multiunit, but once attention was focused on this feature of the blocks, all children saw the sense of block trading immediately. No one ever objected to a block trade, unlike objections or reservations that were voiced about the incorrect marks procedures. Even the weakest members could think fruitfully about the block trades, as with one of the weakest members of L2, who said after the first block trade was made (this happened in a case in which the sum of the hundreds was exactly ten flats, so all ten flats were traded for a big cube), "I don't like this idea if we go put 'em all on." In this case the trade had been ambiguous and could be interpreted as trading ten or trading all of the blocks; this child was checking to be sure that they were doing the former and not the latter.

The behavior of group M2, which most clearly exemplified the desired blocks-to-marks link approach, reveals another function of a teacher that might be necessary during the fading procedure to the marks. For several days, these children did not reflect on the marks procedure at all and often did not even record their blocks addition in such a way that they could really reflect on it; they wrote the successive fixing answers on different pages or disorganized all over the problem page. Therefore a teacher should to monitor the recording process to ensure that it is eventually done in such a way that children can reflect on what they are doing in the marks world. Children may also need to be helped to do this reflection in the marks world -- still strongly connected to the blocks world by blocks words -- to facilitate the fading process from the blocks to just the marks but with multiunit meanings attached to the marks.

The final step in the fading process is to think about the blocks while doing the marks procedure. For children who had instruction in which the blocks modelled the standard algorithm, this step proved to be very powerful in helping those who later started to make errors in the marks procedure self-correct these errors (Fuson, 1986). The collectible multiunits in children's mental images of the blocks were sufficient to direct them to a correct trading method (when they made an error, they were asked to think about the blocks), and they verbalized these corrections with block words or English words or mixtures of the two. Although in no group was there a great deal of spontaneous verbal description of the block trades in block words or English words, most children who were asked by their experimenter were able to make such descriptions while looking only at the marks procedure. Weaker children were not always able to do so. This suggests that it would be very helpful if the task of using blocks included describing and explaining what one is doing with the blocks. This would mean that initially the abler children in a group would give full block multiunit descriptions of their block trades, enabling the weaker children to follow the block trade and link it to the written marks recording. With such modelling, the weaker children could become able to verbalize their block actions. Describing the marks addition in block words would help ensure that children were constructing and using multiunit conceptual quantities for those multidigit numbers instead of the inadequate single-digit conception of those numbers. Verbalization would facilitate all phases of the linking process and support children's use of the blocks to monitor and self-correct errors that might otherwise creep into their marks procedures.

Aspects of helpful verbalization

The previous section described several aspects of verbalization that can help establish the initial links between blocks and marks addition and then support the reverse marks-to-blocks link mentally when the child is no longer using blocks. The importance of initially emphasizing explaining and justifying block and mark addition--not just saying what one did, but saying why one did it or could do it--was also discussed. This section focuses on other results concerning verbalization.

Children may initially need the teacher's support to say the multiunit word as well as the number of multiunits. As in the preaddition phase, many spontaneous descriptions of the block trades named how many one was trading but did not name the multiunits involved in the trade: "I'm taking ten and putting one here." Perhaps because of the lack of differentiation in English between these two uses (as in diez and decena in Spanish), the failure to say the multiunit word after ten (ten whats) led children to confuse the function of these two meanings of ten (the unitary ten telling how many of a multiunit and the multiunit ten in the second position). Such confusions led to the prolonged use in group M1 of the incorrect marks procedure of putting all trades in the tens column. Each 1 was written above the tens column because it was one ten coming from the sum of the two numbers in a column. Thus, the ten (the number of a particular multiunit) went to the tens column (a kind of multiunit). As soon as these children consistently focused on the kind of block involved or on the multiunit word (ten whats?), they saw that ten hundreds or ten tens would not go to the tens column (would not be one multiunit of ten). The confusions from not saying the multiunit word were briefer in other groups, but children did other combinations of these two functions of the word ten. For example, "OK, six (carrots) plus ten (actually one traded carrot, a ten multiunit) is sixteen." Using block words is one powerful way to reduce these confusions, because the block words say the multiunit quantities and the ten will tell how many of a block there are.

For all mathematical quantities children learn about after small whole numbers, the small whole numbers in fact are always used in special new ways to tell how many of some particular new kind of quantity. In multidigit numbers it is how many of larger and larger multiunits. In fractions it is how many of a particular unit divided into how many parts. Multidigit numbers present a good opportunity for children to prepare for all of these new mathematical ideas by recognizing that

these small numbers are "how many" numbers that tell how many of something there is. Thus, they can begin the very useful practice of asking "how many *whats*?" for any number they see in these new uses.

The dysfunctional nature of the take-away descriptions groups L1 and L2 used to describe trading, with this language supporting and perhaps even suggesting the erroneous nines procedure, indicates that it would be helpful if teachers monitor the language used to describe trading. For this take-away description, and any other procedural short-cut definitions that suggest wrong marks trading, teachers need to ensure that children, instead, give a full description using multiunit words and the numbers of these multiunits ("I'm trading ten of these pancakes for one ice cube." or "I'm putting these ten hundreds here together to make one thousand.").

These differing positive and negative effects of language indicate that future research needs to examine the effects of the kinds of verbal descriptions children produce. Resnick and Omanson (1987) reported that the amount a child verbalized when using base-ten blocks was positively related to their correction of written marks errors. In the present study some kinds of verbalization seem instead to have had negative effects. Furthermore, even though questions by the experimenter at the end of the addition phase indicated that most children could produce good verbal descriptions, these highly verbal children did not spontaneously produce large amounts of such descriptions. Therefore, both teachers in the classroom and researchers studying multiunit learning need to support children's production of the positive kinds of language (full multiunit descriptions of trading).

We are unable to make any strong conclusions about the efficacy of using the regular Asian ten-structured words compared to the irregular English words because children used relatively few full multiunit verbalizations (English irregular or Asian regular). Children did learn the Asian words readily, and some children seemed to like their regularity. Other occasions on which they seemed advantageous have been discussed elsewhere. A preliminary report of a teaching experiment in which Asian tens words are used in a first and third grade class is in Fuson and Fraivillig (in press).

Time to build multiunit thinking

The four groups that spent eight days on addition all had agreed upon an accurate blocks procedure that was understood and could be carried out by all children. Each of these groups also had centered on some marks procedure that, for some children, was linked closely to the blocks procedure and was conceptual, but that for other children in the group, was less closely linked and not yet fully conceptual. None of these four groups had enough time in the reverse marks-to-blocks linked direction. They would have benefitted from two more days doing fully linked blocks-to-marks procedures with full multiunit descriptions to help the weakest group members, and two to four days doing faded reverse marks-to-blocks links where children did marks procedures and described them in block words and English words and used the blocks where necessary to clarify problematic points. If blocks-to-marks links and full multiunit descriptions had been supported earlier in the addition phase than occurred spontaneously in this study, these children might have been where they were at the end of eight days three or four days earlier. They also would have avoided adding the several incorrect marks procedures they invented to their repertoire of solution procedures, with those procedures needing to be suppressed by thinking about the collectible multiunits in the blocks.

These results indicate that it takes a long time--days and even weeks--for high-achieving second graders to construct multiunit quantities and ten/one trade conceptual structures and use these structures in devising and being able to explain and justify an accurate method to add multidigit marks. For children with weaker backgrounds it might take two or three times longer for this construction, and children might need even more support from the teacher or other expert to maintain links and produce full verbalizations in order to enable the blocks to function most effectively. In studies in which the standard algorithm was modelled with blocks (Fuson, 1986; Fuson & Briars, 1990), the amount of time spent on addition varied with the achievement level of the second-grade class from about a week to three weeks. When children are inventing their own addition method, the required time would seem to be greater, even with teacher support to curtail the long unlinked incorrect marks sidetrips taken by some groups here. Therefore the appearance in textbooks of base-ten blocks for three or four pages, as is becoming typical now (they do appear but only for a short time: Fuson, in press-b), is not nearly long enough even for the most able children. Of course, the extent to which children need to move the blocks themselves as opposed to seeing the collectible multiunits in pictures is also unknown at this time, so this may be another limitation of the blocks in book pictures.

Some interventions with blocks (e.g., Resnick & Omanson,1987), and with other physical materials used to support conceptual understanding (e.g., Byrnes & Wasik, 1991), consist of a single instructional session. When this single session fails to lead to full conceptual understanding or to accurate written computation, that physical material, or the whole approach of using physical materials to provide quantitative referents for mathematical symbols and operations, is judged to be a

failure (e.g., Byrnes & Wasik, 1991; Siegler, 1991). Instead, the real question should be why anyone would think that a single session would be sufficient for all but the brightest child to construct all of the necessary new conceptual structures in the mathematical domain in question and clearly link the quantitative features in the situation to the new operations on the mathematical marks. If a child already had all of the requisite knowledge, a single session might be sufficient to make new connections in this knowledge that would lead to an insight kind of new learning. But in most cases, children must build the requisite knowledge as well as make the connections. Such a single session is even more problematic when the subjects are not novices for whom the target written procedure is a new discovery but are instead children who have for months and even years carried out an incorrect procedure (e.g., Resnick & Omanson, 1987; Byrnes & Wasik, 1991).

Finally, as we discussed above with respect to our children in their small groups, children in such a single session will bring to this session their usual interpretation of the goals and purposes of mathematical activities--learning correct written procedures. Until these social norms can be renegotiated, and a new focus established on conceptual understanding and explanation, use of materials is likely to be assimilated into this expectation concerning mathematical activity. Children then are likely to see the session as having two separate components--learning to add with the blocks and learning to add with the marks--rather than trying to use the perceptual support of the materials to understand the marks procedure.

Children also took a long time to work through a problem, averaging only one to two problems per 35-minute working time. This is in contrast to the range of 7 to 12 three-digit subtraction problems worked in the single tutoring session in Resnick and Omanson (1987). Our groups' time was spent discussing various group issues and some off-task topics as well as proposing and carrying out and arguing about various block or mark solution procedures. But this group pace is much more typical of the slower pace, relatively fewer problems solved per class, and more discussion found in Japanese classrooms compared to classrooms in the United States (Stigler, 1988). Blocks can be quite effective when they are used over a longer period both in the standard algorithm studies (Fuson, 1986; Fuson & Briars, 1990) and in the invention approach used here, but time is needed to work through a single example and extended time over days and weeks is needed to build and connect all of the new conceptual structures.

Supporting multiunit thinking

Children in all groups were capable of much higher levels of thinking than they produced spontaneously. Group L2, with the support of the experimenter, profitably compared the relative advantages of adding from the left and from the right. All groups could have discussed this issue. Most groups had at least one member who wanted to add the blocks from left to right, and several groups carried out full left-to-right blocks procedures. The exploration of left-to-right procedures was shortcut in several groups by assertions of a rule (e.g., in reading we go from left to right and in math we go from right to left), so such an exploration might be especially natural for children who have not yet learned such rules.

We had hoped that all groups might construct at least one marks procedure that differed from the standard algorithm many of them already knew. However some groups, especially H1 and H2, were so focused on the standard algorithm that they did not really do this. In such a case it would seem worthwhile to demonstrate one or more nonstandard procedures described here or in Fuson, Burghardt, and Fraivillig (1992) and ask the children to decide if they are correct or not and what might be their relative advantages or disadvantages. Children also might pursue extensions of their four-digit experience such as deciding what would be the size of the next three multiunit blocks (blocks for the fifth, sixth, and seventh positions) and how would they add two seven-digit numbers and why. Role-playing activities, such as explaining an addition method to a new student who has just come into the class or helping that student figure out a method for herself, might also be interesting and help children try to articulate every step they were doing. Some children in this study showed remarkable ability to scaffold other children's learning, and even first graders can do so. During the first year of the block work reported in Fuson and Briars (1990), a new student came into the high-achieving first-grade class just as the children were completing the block work (they had been working in individual groups each helped by a fourth grader). The teacher asked one of the strongest students to show this child how the blocks and marks procedure worked, and in one day the new child added correctly and could explain the marks addition in terms of the blocks.

If children are working in the same group for a prolonged period of time, it also may be helpful from time to time to have each group make a report to the whole class on their addition methods, discoveries, and current difficulties. The class then might discuss these alternative approaches and discoveries and suggest solutions to the difficulties. Having to make such a report may help to focus the work of the group. If any group member can be chosen at random to make the report, all group members may feel more need to support the understanding of all members of their group. Having occasional outside input might help groups whose socially dominant members have weaker conceptual understanding. These comparative discussions would require children to understand the thinking of other groups and would seem likely to extend everyone's thinking. Of course, as with the groups here, such discussions may need to be supported by the teacher, at least initially.

Generalizing to other achievement levels

How well these results extend to second-graders who are low- or average-achieving in mathematics is an open question. We collected similar data on a class of such children, but have not yet completed the data analysis. This class, especially the low-achieving children, clearly needed more support from the experimenters than did the children in this study. In the studies modelling the standard algorithm (Fuson & Briars, 1990) even low-achieving second graders did learn to add four-digit numbers and explain their trading using ten/one multiunit concepts (they did this late in the year rather than at the beginning of the year, as in this study); in the lower-achieving classrooms the teachers initially modelled the blocks and marks procedures and the children participated in justifying what was being done with the blocks. When classrooms or groups are more heterogeneous, it seems likely that the higher-achieving children might do more of the initial discovery of an addition method and then play the role of the teacher modelling this method for the lower-achieving children. The quality of the explanations given by the children in the present study, at least those elicited by the experimenters, seems sufficient to facilitate the learning of their low-achieving peers.

Lower-achieving children may also have more trouble with the technical aspects such as copying the problem, counting the blocks, and doing single-digit addition. For the first two, It would be helpful if teachers emphasized that disagreements can sometimes be resolved by checking that the problem was copied correctly or checking that the number of blocks is correct or by starting the problem again with carefully checked blocks. To keep the problem blocks accurate and eliminate the frustrating digressions that occasionally occurred when blocks were played with or merged with nonproblem blocks, horizontal block trays for each addend and for the sum would be helpful. These would also provide perceptual support for the horizontal multiunit number versus the vertical columns in the written marks; this was a special problem in subtraction (Fuson & Burghardt, 1991). Finally, though the children in this study did not need the blocks to find single-digit sums, some lower-achieving children might need them at least initially for the larger sums. Because the trading in addition requires a given multiunit sum to be in the form of a ten and the left-over amount, and the block collectible multiunits display this tenness, multiunit addition with blocks can help children learn the efficient ten-structured up-over-ten method in which the second addend is split into a) the part that makes ten with the first addend and b) the left-over amount. For example, 8 + 5 is done as 8 + 2 (to make ten) + 3 (the rest of 5) = ten plus three. With the blocks 8 longs plus 2 of the 5 longs makes one flat plus 3 longs of the 5 original longs left. This is the addition method taught to Chinese, Japanese, and Korean children (Fuson & Kwon, in pressa; in press-b; Fuson, Stigler, & Bartsch, 1988), and it seems to be readily learned by these children whose language supports these methods. Thus, work with base-ten blocks might support this more advanced single-digit addition procedure.

Conclusions

The base-ten blocks present key quantitative features of multidigit English words and marks. Empirical questions about these blocks are (a) whether and when and how do children use these features of the blocks to carry out correct blocks addition and (b) whether and when and how do children use these features of the blocks and of blocks addition to carry out correct addition with multidigit marks. Our results indicated that (a) was fairly straightforward. The blocks strongly directed children toward correct block addition procedures. For (b), we found that second graders could easily link the quantitative features of the blocks to the marks and English words. Such linking did enable them to invent addition methods for the written marks that were accurate, based on the quantitative features; methods twhich the children were explain and justify. Such linking also enabled them to self-correct incorrect marks addition procedures and justify these corrections based on quantitative features.

However, we identified two crucial roles of an adult in accomplishing (b). First, most children did not spontaneously make such links but instead worked within separate blocks and marks worlds. In the separate marks world, they primarily used concatenated single-digit meanings of the multidigit marks and made many different kinds of errors. When an adult supported links between blocks and marks over at least a class session, children did find it relatively easy to carry out linked procedures. It was these linked procedures that enabled them to correct incorrect marks procedures and carry out correct ones. Second, these high-achieving children spontaneously produced relatively few explanations or even full descriptions of blocks addition that could support the linking of the multiunit quantities to the marks procedure. This lack of explanations was sometimes detrimental to the mathematically less advanced children in a group. Again, when an adult elicited such verbalizations, most children could give them.

The ease with which children functioned within the linked setting once the adult had helped them create this setting, and the fact that many could give when asked adequate verbal descriptions and justifications, suggests that these inadequacies may have at least partly stemmed from an inadequate communication of the goals of the block activity as including describing and justifying steps in an addition procedure. It may take some time for a teacher to define the goals of a classroom as using physical materials and situations to enable children to carry out marks activities that are comprehensible to and justifiable by them and to help children learn how to use materials to approach mathematics in this way. The power of the blocks to direct correct addition and constrain incorrect marks addition indicates that this can be a powerful approach to meaningful mathematics learning and will be well worth the extra class time and extra initial teacher preparation it takes.

These results illuminate several views of children's learning that have been or are currently rather widespread. These might be summarized in the following three different views or models of children's learning: (a) the Monkey See-Monkey Do imitation view of learning, (b) the Computer programmed view of learning, and (c) the Instamatic Camera view of learning (especially with pedagogical objects such as base-ten blocks). In a classroom using the Monkey See-Monkey Do view of learning, an expert (usually the teacher) models a mathematical activity or procedure, and children imitate that model. With the Computer view of learning, children are programmed to carry out the mathematical activity or procedure by being told the rule or procedure to carry out. With the Instamatic Camera view of learning, children are briefly shown pedagogical objects that present mathematical features; children are viewed as cameras that can instantly picture these objects and use them internally to direct their mathmatical thinking. Learning views (a) and (b) have dominated traditional school mathematical instruction. Learning view (c) has directed some uses of manipulative materials (pedagogical objects) for a long time; it recently appears in some textbooks and has marred some instructional research. None of these views of learning results in successful mathematics learning for most students, though all of these views can be effective with some students (usually the most advanced, who can construct conceptual understanding with little support). There is considerable evidence that children do not imitate correctly (the Monkey See-Monkey Do view is not effective), they do not stay told (the computer view is not effective), and they do not stay shown--or showing does not even work initially (the Instamatic Camera view does not work) (e.g., see Grouws, in press, for results concerning the first two views, and Byrnes & Wasik, 1992, and Resnick & Omanson, 1987, concerning inadequacies of the third view).

The view of children's learning that accounts for much of the current evidence (for example, concerning the many different computational errors children make) is that children are goal-directed active Meaning Makers and Rule Derivers. Children in any mathematical classroom are individually constructing meanings for mathematical symbols and deriving rules for mathematical procedures. They do this within the particular mathematical situations presented in that classroom, and their individual constructions are affected by the meanings and rules derived by the other children and the teacher. A major reason that traditional mathematics calculation instruction fails for so many children is that much of this calculation (with multidigit numbers, decimal fractions, ordinary fractions) uses a meaning of written marks as single digits. Multidigit numbers and decimal fractions are concatenated single digits, and ordinary fractions are two single digits separated by a bar. This meaning is sufficient for the rules in standard calculation procedures. That it is sufficient is part of the power and simplicity of these systems of written marks. But the single-digit meanings of these systems are insufficient to constrain incorrect rules or direct correct ones because there are so many differences between calculation with single digits and calculation in these more complex systems.

Various mathematical pedagogical objects have been invented in order to address this issue of helping children learn adequate meanings for various mathematical systems. However, the research literature on their use has been quite mixed concerning the success of various pedagogical objects. Many of the failures, we believe, are due to two sources: (1) their use has been governed by inappropriate or inadequate learning theories and (2) an inadequate understanding of the mathematical system or procedure they are intended to support which results in an inherently inadequate pedagogical object.

When the efficacy of particular pedagogical objects has been tested using one of the three learning theories (a) through (c), children's understanding does not improve considerably, and they may not learn the targeted mathematical procedure as well as children not using the pedagogical objects. These failures occur because the teaching effort does not recognize children as active Meaning Makers and Rule Derivers and thus it ignores where given children are at the beginning of the teaching effort. Use of pedagogical objects has to begin where children are. Such use has to recognize that these pedagogical objects will be viewed by a given child with the conceptual structures that child has at that moment. For most children, there will be some distance between the conceptual structures they possess for the targeted domain at the beginning of teaching and the desired conceptual structures that the pedagogical objects are designed to support. Our results here, and the results of successful use of pedagogical objects (e.g., Wearne & Hiebert, 1989), all indicate that the successful use of pedagogical objects requires a process of interiorization of the features of and actions on the pedagogical objects. This

process of interiorization is neither rapid nor veridical. It depends on the conceptual structures the child already has, and it will for most children take days and, for some children, weeks or even months. These conceptual structures, and the amount of sensitively adapted conceptual support in the environment in the form of adults and other children, determine for a given child, the rapidity of interiorization and the nature of the interiorized conceptions. The brief group summaries of addition given in this paper give some indication of individual differences in this process of interiorization; more detailed analyses of individual learning paths as affected by the conceptions of others in the group are given in Burghardt (1992).

Successful uses of pedagogical objects also depend, as we have seen in the study reported here, on children's making constant close links between the pedagogical objects and the mathematical symbols (here, multidigit marks). If these two worlds remain separate, the meanings potentially supportable by the pedagogical objects cannot and will not become linked to the mathematical symbols. Further, our results indicate that children do not naturally link these two worlds. If anything, strong social forces may continually seek to separate these worlds. Thus, successful use of pedagogical objects may depend upon a teacher's support of such linking.

Unfortunately, other evidence indicates that teachers do not recognize the need for this linking (Hart, 1987). Their typical pattern is to use the pedagogical objects (base-ten blocks, in Hart, 1987) for some period alone, and then to move to marks with little (one day) or no linking of the pedagogical objects to the marks. Our results indicate that this is just the opposite of what children need. The children had little difficulty in adding with the blocks. The blocks were successful pedagogical objects in that their features did direct children to correct multiunit addition and correct trading. But children did have considerable difficulty with addition with marks if this addition was not connected to the blocks. Therefore, children need much of the learning time spent on experiences in which the blocks world and marks world are tightly connected in order for the marks to take on the quantitative meanings supported by the blocks.

Furthermore, our results indicate that, after strong connections between the worlds are made, children may need time working just in the marks world while using interiorized blocks meanings for these marks. This can permit children to reflect on and connect various marks procedures (e.g., group M2 connecting their invented fixing method to the standard algorithm) and give them practice in using these interiorized meanings to direct and correct marks operations. The results of Fuson (1986) indicate that second graders of all achievement levels and even high-achieving first graders do interiorize base-ten blocks, and most of them can use these interiorized blocks to self-correct errors that may arise over time in their marks procedures. A few children who began making marks errors needed to use the actual blocks to self-correct their errors, another indication of the individual variation in the process of interiorization.

The combination of the (usually) slow process of interiorization and the need for prolonged linking of the pedagogical objects with their verbal and written symbols results in a prolonged and complex learning experience with different phases. Initially there is a period of close linking of actions on the pedagogical objects and actions on spoken or written mathematical symbols. Then there may be a phase in which the marks are used without the objects but verbal descriptions of pedagogical object actions are given to keep the meanings linked to the marks. Finally, there may be a phase of use of the marks in a particular solution in which the meanings are not explicitly accessed during the solution (e.g., 5286 and 2749 are added without accessing multiunit meanings). However, the goal of the use of pedagogical objects is that the interiorized meanings are available at any time for the solution of nonroutine problems or for the justification of a particular solution. Unfortunately, many oversimplistic interpretations of the use of pedagogical objects assume the Instamatic Camera theory of learning with pedagogical objects or have a simpler view than indicated in this study (e.g., Byrnes and Wasik, 1991, in their reduction of pedagogical objects to their described "simultaneous activation" view).

The second limitation of pedagogical objects is not in the learning theory employed in their use, but rests in the pedagogical objects themselves. An inadequate understanding of the mathematical domain can lead to the design of pedagogical objects that have features that are irrelevant or even misleading with respect to the targeted mathematical domain. For example, colored chips are widely used as place-value activities. However, they present neither the quantitative features of the English words and multidigit marks (red does not show tenness, green does not show hundredness, etc.) nor the use of position to show multiunit quantities as in the written marks (one needs to use chips all of one color to do this, e.g., Bell, Fuson, & Lesh, 1976). Thus, these pedagogical objects do not help children learn multiunit meanings, and they support incorrect responses (Labinowicz, 1985). Many pedagogical objects are designed and marketed without any accompanying analysis of the mathematical features of the English words and written symbols that indicate how the pedagogical objects can support the desired learning. Nor is there empirical research indicating what conceptual structures children need to have already in order to use particular pedagogical objects successfully, i.e., there is no sense of the zone of proximal

development for particular pedagogical objects (Vygotsky, 1962). Both of these kinds of analyses are needed.

An inadequate mathematical analysis can also result in measures of conceptual understanding that are not the mathematical prerequisites for learning a particular mathematical operation. For example, Byrnes and Wasik (1991) used measures of conceptual knowledge in the domain of fractions that were much too simple for the targeted operations: addition and multiplication of fractions. Only the conceptual measure of order was even close to the difficulty of these operations; understanding order of fractions (i.e., is $1/3 > 1/5$?) can inhibit the almost universal incorrect addition procedure used by Byrnes and Wasik's subjects (add the numerators and add the denominators). However, there was no measure of understanding of equivalence classes of fractions (e.g., $1/2 = 2/4 = 3/6 = 4/8 = \ldots$) or of the ability to change one fraction into a targeted related member of the equivalence class. These are the conceptual prerequisites for addition of fractions. One can only add fractions that have the same fractional unit (e.g., eighths of a whole unit) just as one can only add like multiunits in multidigit addition. Therefore, one must change fractions that have different fractional units into those with the same fractional units in order to add them. Byrnes and Wasik's use of pedagogical objects to support correct fraction addition also seemed (from the brief description available) to be limited by an inadequate linking of actions on the pedagogical objects (plastic wedges in which different fractional units were a different color) to operations on the written fraction symbols (using a least common denominator procedure).

In this paper we did not focus on the social aspects of learning. Many of our group summaries did convey some of these social results, however, because these aspects are inherently inseparable from the cognitive aspects whenever learning occurs in a social setting. Ways in which the personalities, initial knowledge, and knowledge-under-construction of individual children interacted with the personalities, initial knowledge, and knowledge-under-construction of other group members is described in Burghardt (1992). A recent theoretical discussion of social/cultural aspects of the Meaning Maker theory of children's learning is given in Cobb, Yackel, and Wood (1992). That paper articulates very sensitively and well these social/cultural aspects of a Meaning Maker theory of learning. But, in our view, the paper confounds a "representational view of the mind" with the theory of learning used by the person espousing such a view of the mind, especially when the research used pedagogical objects. The paper therefore mislabels the target of its analyses and loses effectiveness as a result. The real targets of the arguments mounted against the "representational view of the mind" are the Monkey See-Monkey Do, Computer, and Instamatic Camera theories of learning.

The paper reads quite well if one simply substitutes these targets whenever the "representational view of the mind" is mentioned. A major problem with the mislabelling is that some representational view of the mind, i.e., some view of children's conceptual structures, is required by a constructivist Meaning Maker theory of learning. Such a theory requires, more than any other, considerable understanding of children's conceptual structures in a given mathematical domain. For researchers who have experience in watching children's learning and are keenly aware of the power of the conceptual structures possessed by a given child to affect what that child sees and hears in a given learning situation, it is absurd to imagine that most children could learn a complex mathematical procedure or system of concepts in one session of use of pedagogical objects, no matter how powerful (the Instamatic Camera view), or by simple imitation (the Monkey See-Monkey Do) or by following rules (the Computer view). Their discussion of these theories is insightful. However, it would have been very helpful if these theories of learning had not been lumped together and mislabelled the "representational view of the mind."

Ohlsson and Rees (1991) raise the question of whether one can learn "why" and "how" at the same time. Their analysis of the function of conceptual understanding in the learning of arithmetic procedures emphasizes using knowledge about "why" to correct errors that arise in an already learned "how" procedure. In their analysis, conceptual understanding can support the self-correction of errors by constraining problem states so that errors can be detected and corrected. Our study provided support for this position. When blocks were linked to marks, the multiunits in the blocks led children to see the errors in their marks procedures. However, our study also indicates that the "why" can precede and direct the "how": Conceptual understanding has an equally important role in directing the construction of initially correct problem states (i.e., here, correct multiunit addition).

Our results also emphasize the critical role of the teacher in the classroom in ensuring that the pedagogical objects are linked to the written marks (i.e., that the available conceptual understanding is related to the marks) and in directing children's attention to critical features of these objects to facilitate their use. Conceptual understanding can enable children to construct correct arithmetic procedures and to find and eliminate errors in incorrect procedures. However, children must understand and accept this conceptual approach to mathematics learning in order to carry it out, and they may need to be helped along the way in seeing and using critical quantitative aspects of the domain. Likewise, researchers and teachers using pedagogical objects need to use a theory of learning that is consistent with the focus of pedagogical objects on meaning (i.e., they need to use a Meaning Maker and Rule Deriver theory of children's learning).

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