18 Chinese-based regular and European irregular systems of number words: The disadvantages for English-speaking children*

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Chinese and some other Asian languages (e.g. Burmese, Japanese, Korean and Thai) have regular named-value systems of number words in which a number word is said and then the value of that number word is named (five *thousand* seven *hundred* two *ten* six). Many European languages have regular named-value number-word systems for the values of 100 and 1000, but they are irregular in different ways below 100 (see Table 18.1). These irregularities have serious consequences which affect children's numerical learning adversely in several different ways. Sufficient research is available to describe these consequences for English-speaking children (mostly in the USA), and therefore the focus in this chapter is on the English language. However, many of the consequences discussed here would seem to apply to the other irregular European number-word systems, though some details might vary. This chapter will briefly review English-speaking children's relative difficulties compared to Asian children *in learning the number-word sequence, in adding and subtracting numbers with a sum between 10 and 18, in constructing adequate mental representations for multidigit numbers*, and *in adding and subtracting multidigit numbers accurately and/or meaningfully*. Making linguistic comparisons is complex because most such comparisons also involve many non-linguistic cultural factors that might affect learning. This problem is reduced in this case because the linguistic effects are supported by data concerning different kinds of errors made or different solution procedures used in the regular and irregular languages, rather than a simpler accelerated learning in one language that might be due to more general cultural factors.

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Table 18.1 Number words

	French	*Spanish*	*Italian*	*German*
1	un, une	uno, una	uno, una	eins
2	deux	dos	due	zwei
3	trois	tres	tre	drei
4	quatre	cuatro	quattro	vier
5	cinq	cinco	cinque	fünf
6	six	seis	sei	sechs
7	sept	siete	sette	sieben
8	huit	ocho	otto	acht
9	neuf	neuve	nove	neun
10	dix	diez	dieci	zehn
11	onze	once	undici	elf
12	douze	doce	dodici	zwölf
13	treize	trece	tredici	dreizehn
14	quatorze	catorce	quattordici	vierzehn
15	quinze	quince	quindici	fünfzehn
16	seize	dieciséis	sedici	sechzehn
17	dixsept	diecisiete	diciassette	siebzehn
18	dixhuit	dieciocho	diciotto	achtzehn
19	dixneuf	diecinueve	diciannove	neunzehn
20	vingt	veinte	venti	zwanzig
21	vingt et un	veintiuno	ventuno	einundzwanzig
22	vingt-deux	veintidós	ventidue	zweiundzwanzig
23	vingt-trois	veintitrés	ventitre	dreiundzwanzig
24	vingt-quatre	veinticuatro	ventiquattro	vierundzwanzig
25	vingt-cinq	veinticinco	venticinque	fünfundzwanzig
26	vingt-six	veintiséis	ventisei	sechsundzwanzig
27	vingt-sept	veintisiete	ventisette	siebenundzwanzig
28	vingt-huit	veintiocho	ventotto	achtundzwanzig
29	vingt-neuf	veintinueve	ventonove	neunundzwanzig
30	trente	treinta	trenta	drei-ssig
31	trente et un	treinta y uno	trentuno	einunddreissig
39	trente neuf	treinta y nueve	trentnove	neununddreissig
40	quarante	cuarenta	quaranta	vierzig
50	cinquante	cincuenta	cinquanta	fünfzig
60	soixante	sesenta	sessanta	sechzig
70	soixante-dix	setenta	settanta	siebzig
80	quatre-vingt	ochenta	ottanta	achtzig
90	quatre-vingt-dix	noventa	novanta	neunzig
99	quatre-vingt-dix-neuf	noventa y nueve	novantanove	neunundneunzig
100	cent	cien	cento	hundert
101	cent et un	ciento uno	centouno	hunderteins
125	cent vingt cinq	ciento veinticinco	centoventicinque	hundertfünfundzwan
4313	quatremilletrois-centstreize	cuatromiltres-cientostrece	quattromille-trecentotredici	viertausenddreihund-dreizehn

Notes: The words for 4313 are one word in the first four languages; they are hyphenated here for lack of space. The Chinese words are given in English to show their structure. Burmese, Japanese, Korean, Thai, and Vietnamese number words have the same regular structure as the

English	*Chinese*	*Positional base-ten*
one	one	one
two	two	two
three	three	three
four	four	four
five	five	five
six	six	six
seven	seven	seven
eight	eight	eight
nine	nine	nine
ten	ten	one zero
eleven	ten one	one one
twelve	ten two	one two
thirteen	ten three	one three
fourteen	ten four	one four
fifteen	ten five	one five
sixteen	ten six	one six
seventeen	ten seven	one seven
eighteen	ten eight	one eight
nineteen	ten nine	one nine
twenty	two ten	two zero
twenty-one	two ten one	two one
twenty-two	two ten two	two two
twenty-three	two ten three	two three
twenty-four	two ten four	two four
twenty-five	two ten five	two five
twenty-six	two ten six	two six
twenty-seven	two ten seven	two seven
twenty-eight	two ten eight	two eight
twenty-nine	two ten nine	two nine
thirty	three ten	three zero
thirty-one	three ten one	three one
thirty-nine	three ten nine	three nine
forty	four ten	four zero
fifty	five ten	five zero
sixty	six ten	six zero
seventy	seven ten	seven zero
eighty	eight ten	eight zero
ninety	nine ten	nine zero
ninety-nine	nine ten nine	nine nine
one hundred	one hundred	one zero zero
one hundred one	one hundred one	one zero one
one hundred twenty five	one hundred two ten five	one two five
four thousand three hundred thirteen	four thousand three hundred ten three	four three one three

Chinese words, and Bahasa (the Indonesian formal language) and Tagalog (the language used in primary schools in the Philippines) are like Chinese words except they have a teen word and a few irregularities (Vietnamese also has a few tonal and consonant irregularities).

Relative difficulties in learning the English number-word sequence

The English system of number words does not directly name the ten and one values in two-digit numbers. Several features of English even make it difficult to see this underlying tens and one structure and to see how the first nine numbers are re-used to make the decade words:

1 The existence of the arbitrary number words 'eleven' and 'twelve' that do not indicate their composition as 'ten and one' and 'ten and two'.
2 The irregular pronunciation of 'three' in 'thirteen' and 'five' in 'fifteen' that obfuscate the re-use of the words 'three, four, . . ., eight, nine' with 'teen' to make the 'ten three' to 'ten nine' words.
3 The tens/ones reversal only in the teen words so that the 'four' is said first in the teen word ('fourteen' instead of 'teenfour' or 'ten four') but is said second in all of the other decade words ('twenty four').
4 The irregular pronunciation of the decade words 'twenty', 'thirty' and 'fifty' that mask for many children the relationship of the decade names to the first nine number words.
5 The use of two different modifications of 'ten' (i.e. 'teen' and 'ty') neither of which clearly says 'ten'.

Table 18.1 indicates that other European languages have many of these irregularities, and French, Italian and Spanish also have a reversal and change of form in the middle of the teens (see Menninger, 1969, for more about European number words).

These irregularities in the words between ten and one hundred require children to memorize major parts of the English number-word sequence without seeing patterns other than the one through nine repetition within the decades (Fuson *et al.*, 1982; Siegler and Robinson, 1982) and without seeing units of tens within this sequence. Consequently, children make more errors and more kinds of errors in saying the English sequence than do their peers who are learning the Chinese regular named-value sequence in which tens are explicitly named; errors in English are made in many different places, whereas those in Chinese are concentrated at decade or hundred changes, indicating that Chinese children do see the pattern in their number-word sequence (Agnoli and Zhu, 1989; Miller and Stigler, 1987). Chinese children say their regular words 'ten' through 'ten five' faster than English children say their irregular 'ten' through 'fifteen' words (Agnoli and Zhu, 1989). Other details of difficulties children have learning the English number-word sequence are given by Fuson (this volume).

In general, and not surprisingly, the kinds of errors made in learning a number-word sequence seem to depend upon the structure of the sequence. Deaf preschoolers learning American Sign Language (ASL) from their deaf parents may skip signs that are difficult for them to make with their fingers or may confuse the production rules that generate the number words (Secada, 1985). Italian-speaking children show particular difficulties with the reversal in the upper teens (see Table 18.1); Agnoli and Zhu, 1989).

Relative difficulties in adding and subtracting numbers with a sum between 10 and 18: Construction by English-speaking children of a unitary representation of number words

The lack of an obvious tens and ones structure in English number words between ten and one hundred results in the construction by English-speaking children of a unitary mental representation of number words in which each number word is a single unit. This unitary representation goes through several developmental levels of increasingly efficient and abstract solution procedures for addition and subtraction situations (see Fuson, this volume, for a summary of these levels and Fuson, 1988, for a more detailed discussion of these successive unitary representations). With these unitary representations children do not group objects into tens or count by tens; each number is composed of that number of single units, whether the units are presented by objects or by number words.

This developmental sequence of US children's addition and subtraction solution procedures is almost entirely a secret underground movement within school classrooms. Children invent most of these procedures for themselves without support from their textbooks or teachers. Textbooks merely move from addition and subtraction problems where objects are given with the numbers to problems given in numbers with no available objects (Fuson *et al.*, 1988). Through practice, children are supposed to remember all of the addition and subtraction facts. Because most are unable to remember all of these facts, they move through the above developmental sequence of solution procedures for facts not yet memorized. However, they do so slowly, with a considerable number of second-graders still not at the highest level (Carpenter and Moser, 1983, 1984; Steinberg, 1984). The solution of subtraction problems with sums up to 18 is particularly delayed. Many US textbook series do not even present the most difficult single-digit addition and subtraction problems in the first grade (Fuson *et al.*, 1988).

Asian children learn single-digit sums and differences more rapidly than US children (e.g. Song and Ginsburg, 1987). All such sums and differences are presented in the first grade in China, Japan, Korea and Taiwan (Fuson *et al.*, 1988; Fuson and Kwon, in press). Children in these countries are taught particular methods of adding and subtracting numbers with sums between 11 and 18. These methods all depend on the clear tens and ones composition of these numbers 'ten one' through 'ten eight'. The up-over-ten method is taught for addition: one addend is partitioned into the number that makes ten with the other addend and the left-over number. *Eight* plus *five* is thought of as '*eight* plus two from the *five* is ten plus the three left over from the five is *ten three*'. Some US children also invent this over-ten method (e.g. Carpenter and Moser, 1983, 1984). However, this method is more difficult in English because the 'ten plus x' sums (e.g. ten plus three is thirteen) have to be learned rather than being given automatically by the counting sequence as 'ten three'; many US first- and even second-graders do not know these sums and have to count up from ten to find out how many 'ten plus two' or 'ten plus four' is (e.g. Steinberg, 1984). US children also commonly lack another prerequisite for the over-tens method: there is much less emphasis in the USA than in China or Japan concerning the number that makes ten with a given number (e.g. for eight plus five, one needs to know that eight plus two is ten) and thus many first-graders have to count to find out how many to put with a given number to make ten.

Two different methods are taught for subtraction in Japan, Korea and Taiwan. The 'down-over-ten' method is the reverse of the up-over-ten method: the number being subtracted is split into the number that is over ten, and the rest is then subtracted from ten (ten three – 8: the 8 is split into 3 and 5 – because the three taken from 'ten three' leaves ten – and the 5 is then subtracted from this ten = 5). The 'subtract-all-from-ten' method essentially turns subtraction into an additive procedure: the number being subtracted is taken from ten and the resulting difference is added to the amount over ten (ten three – 8: ten – 8 = 2 plus the three in the 'ten three' = 5).

The use of a unitary representation that has no larger units of ten does not mean that such users cannot learn to read and write two-digit numerals. These written numerals are related to the patterns in the English number-word sequence: the first digit suggests (but does not necessarily specifically name, as in 'twen') the decade name and the second digit tells the word said after the decade name, except for the reversed irregular teen words. Ross (1986) found that all second-graders sampled from a wide range of classrooms were able to count collections of as many as 52 objects and could write the two-digit numeral that told how many objects there were. However, more than half of the second-graders and 15% of the fourth-graders showed no knowledge of tens and ones even as labels for the two digits, indicating that they had only unitary representations for these numbers. The teens reversal in English (and other European languages) does create special problems for writing the numerals for the words between ten and twenty. The words for the ones digits are said first but written second: one says 'fourteen' but writes 14. Thus, many children write 41 for fourteen, following the pattern of the words (some children say that the 1 with the 4 'teens' it, i.e. makes it a teen word). This strategy of writing the ones word wherever it is said works for all the words between twenty and one hundred: one says 'twenty four' and writes 24. So the reversal in the English teen words is particularly troublesome. In Chinese, there is of course no problem with writing any two-digit number or with linking the digits to tens and ones labels: one says 'two ten four' and writes 24 and one says 'ten four' and writes 14.

Relative difficulties in learning adequate representations for multidigit numbers: Interference of the unitary representation when constructing the necessary named-value and positional base-ten representations

The English and Chinese systems of number words are measure named-value systems, and the system of written multidigit number marks used in most countries is a positional base-ten system (see Fuson, in press, for a fuller discussion of the features of these systems and of the different mental representations children construct for these systems, and see Table 18.1 for positional base-ten number words that illustrate some of these features). The numbers in both of these systems are composed of different kinds of multi-units – larger and larger units – and not just of single-unit items as in the unitary number representations. Measure named-value systems have the following features:

1 Each named value (e.g. tens, hundreds, thousands) is a collection of units.
2 New larger values are made by regular ten-for-one trades.
3 Because each new value must be named, large numbers are limited by known names.
4 The value is conserved if named value/digit pairs are out of order.
5 Zero is not needed.
6 Quantities of each value direct multidigit addition and subtraction by adding or subtracting like values.
7 Trading for too many (+) and for not enough (−) in multidigit addition and subtraction are directed by the values of the quantities.
8 Values can have ten or more of that value (e.g. the expression 'fifteen tens' is meaningful).

Positional base-ten systems have the following features:

1 Values are not named but are implicit in the position of a digit relative to the ones position.
2 New larger values are made by regular ten-for-one trades.
3 Because these new larger values just take positions to the left, large numbers are not limited – one can make as large a number as places one can write.
4 The value is not conserved if digits are out of order (e.g. 14 is different from 41).
5 Zero is needed for missing values to keep digits in their correct relative positions.
6 The meaning of positions built up by ten-for-one trades (their values as products of ten) directs multidigit addition and subtraction as adding or subtracting like products of ten (i.e. like positions).
7 Trading for too many (+) and for not enough (−) in multidigit addition and subtraction are directed by the values of the positions.
8 Positions cannot have ten or more in that position.

In order to understand these features of named-value English words and positional base-ten written number marks for large multidigit numbers, children must construct named-value and positional base-ten mental representations for the words and the marks, and relate these representations to each other and to the words and the marks.

English-speaking children show considerable difficulty in constructing multi-unit named-value and positional base-ten representations in contrast to children who speak regular named-value Asian languages. These languages evidently help children construct named-value and base-ten representations for two-digit numbers. Miura (1987) found that Japanese-speaking first-graders living in the

San Francisco area used the tens and units in base-ten blocks to make named-value representations of five numbers between 11 and 42 considerably more than did English-speaking first-graders. The latter instead made unitary count/cardinal representations from single units: 28 little cubes instead of two longs (each long is ten units long) and eight units. Similar results were reported for Chinese, Japanese and Korean children compared with US children (Miura *et al.*, 1988) and even for Japanese first-graders before any work on tens compared to US first-graders after instruction on tens (Miura and Okamoto, 1989). The persistence of this unitary view of number even in US adults is indicated by the difficulty adults had accessing a base-ten rather than a unitary meaning for a two-digit number, even when it would have helped them in a task (Heinrichs *et al.*, 1981).

Because most English-speaking children find sums and differences to 18 using a unitary representation (even the result of a memorized number fact is likely to be in the form of a unitary representation), English-speaking children must shift back and forth between a unitary representation and a multi-unit named-value or positional base-ten representation to add or subtract multidigit numbers meaningfully. This representation shifting, and the difficulty English-speaking children initially have in thinking in tens and ones, is illustrated by the extra step for multidigit addition that was invented by first-graders using base-ten blocks in Fuson (1986). At the beginning of the addition learning, some children would find the sum of a given column, e.g. that seven plus five is twelve, but they would not know how many tens and ones were in twelve because it had only a unitary sequence or cardinal meaning for them (twelve as coming after eleven or as a pile of objects they would get if they counted out twelve things). These children would write down the twelve as 12 using an available rote word-numeral association. They would then look at the 12 and think of it as 'one two' or 'one ten two ones' in order to find that twelve has one ten and two ones; then they would trade the 1 ten and record the 2 in the ones column. Many second-graders in Fuson and Briars (1990) used this extra step on some problems in the post-test. All of these children evidently found the support of the written two-digit marks helpful in shifting from the unitary to the named-value representation that was required to decide what to trade.

It may be that working with four-digit numbers, as the children who invented this step did, facilitates this named-value view of two-digit marks. In contrast, Madell (1985) found that many 6-year-olds using base-ten blocks to invent procedures for adding two-digit numbers held to their unitary mental representation rather than constructing a named-value representation for the blocks: to do 48 – 14, they would trade a long for ten units in order to get eighteen units from which they could take fourteen units. They did not use the multi-unit solution supportable by the blocks: think of the 14 as 1 long and 4 unit and take the 1 long from 4 longs (i.e. one ten from four tens) and the 4 units from the 8 units to get 3 longs and 4 units (i.e. 34). Even when English-speaking children get better at knowing the tens and ones in single-digit sums between eleven and eighteen and do not need the support of the written two-digit sum to decide the tens and ones, in most cases they must still switch back and forth conceptually between the unitary representation with which they find the sum of the single-digit numbers in a column and the multi-unit representation that directs their trading when they have too many in a given column.

Asian children speaking a Chinese-based language have linguistic support for the multi-unit representation and do not necessarily even need to shift representations. Their regular named-value words give all sums over ten in a linguistic multi-unit form: seven plus five is 'ten two'. Even if this 'ten two' is initially thought of with a unitary counting/sequence representation with which the sum of seven and five was found, the words themselves suggest what to do with these 'too many' ones. They suggest that the ten in 'ten two' be put with the tens in the other multidigit number and that the 'two' ones be recorded in the ones column. Furthermore, these words are likely to have quantitative 'tens' and 'ones' meaning (and not just be verbal tens and ones labels) because the over-ten method taught in mainland China, Japan, Korea and Taiwan to first-graders for adding sums to 18 supports this quantitative interpretation by using the value of ten in the addition or subtraction procedure (this ten is built up by adding one addend to part of the other addend).

Chinese-based languages clearly are better than English for dealing with situations in which there are more than nine ones because the named-value of ten suggests what to do with the extra ones (i.e. with the ten). When dealing with sums over ten in other columns – situations in which there are more than nine tens, hundreds, thousands, etc. – there are two different approaches that might be taken within a multi-unit named-value representation and within a positional base-ten representation. Again the different language forms support different thinking. A unitary representation within a named-value representation was used by English-speaking children who had used base-ten blocks (Fuson and Briars, 1990) in explaining tens sums that exceeded one hundred: 'That's 8 tens and 8 tens is sixteen tens and ten of those tens makes one hundred and six tens left, so trade the hundred to the hundreds place and write the 6 tens here. It's one hundred and six tens.' With this unitary representation of tens, ten of the tens must be traded for one hundred, i.e. a value trade must occur. In contrast, many Korean children are not permitted to say such 'illegal' forms as 'ten six ten' (the Korean named-value equivalent to 'sixteen tens'), but must say 'one hundred six ten', keeping to a pure named-value representation. Such sums can be found by a generalization of the over-ten method to the larger value: '8 ten plus 8 ten is one hundred (putting 2 ten from the second 8 ten with the first 8 ten) and 6 ten (left over from the original second 8 ten)'. Sums of a given value that exceed nine can also be found by using a base-ten positional representation that ignores (at least momentarily) the value of given digits and says (using an English unitary two-digit representation) '8 plus 8 is sixteen – that's one six: one to be traded and six to be recorded in this column' or (using a Korean two-digit named-value representation) '8 plus 8 is ten six: move the ten over to the next left column and record the six in this column'. Each of these Korean named-value approaches was used by some Korean second- and third-graders in explaining their multidigit addition (Fuson and Kwon, under review).

Relative difficulties in adding and subtracting multidigit numbers accurately and/or meaningfully

The lack of verbal support in the English language for multi-unit named-value/base-ten representations of tens and ones makes it particularly important that support for constructing such representations be provided in other ways to English-speaking children and other children with number words that do not clearly state the underlying tens structure of the words and the base-ten written marks. Unfortunately, in the USA, such support is rarely given. Children are taught multidigit addition and subtraction as step-by-step procedures of adding and subtracting single-digit numbers and of writing digits in certain locations. These experiences result in many US children constructing a mental concatenated single-digit representation of multidigit numbers in which multidigit numbers are viewed as composed of single-digit numbers placed next to each other (M. Kamii, 1981, called this 'glued together' digits). This representation is inadequate in many ways and results in many errors in place-value tasks and in multidigit addition and subtraction.

Children indicate use of the concatenated single-digit representation in several different place-value tasks. When shown, for example, the numeral 16 and sixteen objects and asked successively to show the objects made by each part of the numeral (the 6 and then the 1), many elementary school children indicate six objects for the 6 but indicate only one object for the 1 instead of the ten objects to which the 1 really refers (C. Kamii, 1985; M. Kamii, 1981). When asked to read a three-digit number and then write the number that is one more than the given number, half the third-graders increased by one the digit in one or more places other than the ones place: giving for 342 the answers 1342, 453, 442, 452, 352 (Labinowicz, 1985). Children also sometimes seem to use a concatenated single-digit representation to decide which of two multidigit numbers is larger: half of the third-graders sometimes ignored the position of digits and focused on a single digit in one number as being larger than a single digit in another number, choosing 198 as being larger than 231 (Labinowicz, 1985). Ginsburg's (1977) Stage 1 for children's understanding of written number – no verbalizable meaning for the digits – fits this concatenated single-digit representation: in the example protocol, the child says about the 123 just written for the words 'one hundred twenty three' that the '1 is just 1, the 2 is just 2, and the 3 is just 3'. Ross (1986, 1988) found a level of place-value knowledge in which children seem to possess quantitative meaning for the tens and ones (they relate object subgroupings of tens and of ones to two-digit numbers) but in which they actually are just relating digit values or digit positions to presented groupings without regard for the size of the group: they will say that the 6 in 26 refers to six groups of four and the 2 refers to the two left-over objects if presented with this non-ten grouping of 26 objects. Bednarz and Janvier (1982) reported for many Canadian third- and fourth-graders 'digit by digit' strategies that ignored the tens and hundreds values of the digits.

Use of the concatenated single-digit representation may not be evident when children are given multidigit addition or subtraction problems written correctly vertically (with like relative positions aligned) if the sum of the addends in each column does not exceed ten. Children add (or subtract) the digits in each column and write each sum (or difference) in the space below the column. Inadequacies in calculation performance appear if the problem is written horizontally, if the columns are aligned incorrectly, if the multidigit numbers have different numbers of digits, or if the sum in a column is ten or greater. If asked to add two multidigit numbers written horizontally, children may not even keep the digits in each given number together (Fuson and Briars, 1990) or they align the numbers on the left (Ginsburg, 1977; Labinowicz, 1985; Tougher, 1981). Friend (1979, discussed in Davis, 1984) identified several different kinds of errors Spanish-speaking children make in addition and subtraction problems in which the numbers have different numbers of digits (see Table 18.2). Further inadequacies of the concatenated single-digit representation in directing multidigit addition and subtraction are revealed when the sum in any given column exceeds nine; examples of several kinds of errors made in such problems are given in Table 18.2. For some of these errors children at least maintain the values of the single digits: if they subtract from one digit they add the same amount to some other digit. But in other errors, children do not even conserve the values of the single digits that make up the multidigit numbers.

Many US children who carry out multidigit addition and subtraction correctly do not understand multidigit numbers and do not have adequate meanings for the multidigit procedures. Many third-graders who correctly add two-digit numbers nevertheless identify the 1 written above the tens column as a 'one' and not as ten ones or as one ten (Resnick, 1983; Resnick and Omanson, 1987), and in a three-digit problem correctly added similarly identified the traded 1 as a one rather than as a hundred (for ten traded tens) despite probes such as 'What does this 1 stand for?' and 'What do the 3 and the 2 [the hundreds digits] stand for?' (Labinowicz, 1985). Only 24% of the second- and third-graders who subtracted correctly identified their trade from the hundreds place as borrowing a hundred (Cauley, 1988); the others said they had borrowed a 'one'.

Thus, there seem to be levels within the concatenated single-digit representation, both with repect to place-value performance and multidigit addition and subtraction performance. With respect to place-value performance, children may initially not even be able to label the digits as 'tens' and 'ones'. They then may begin to label digits reliably, but these digits are based on ordinal position (the name of the first column is 'ones', the name of the second column to the left is 'tens', the . . . third . . . is 'hundreds') and not on any quantitative meanings of these names. Later, some children may select grouping referents for these digits, but these grouping referents are general aspects of any given grouping rather

Table 18.2 Multidigit addition and subtraction errors that reflect a concatenated single-digit representation

Addition errors	*Subtraction errors*
Carry-to-the-leftmost[a] ^{11}1 6 8 1 5 6 ——— 4 1 4	Always-borrow-left[e] 2 ~~3~~ 6 ^{1}5 1 0 9 ——— 1 6 6
Wrong-align-long-algorithm[b] 8 7 3 9 —— 1 6 —— 2 7	Borrow-unit-difference[e] 4 ~~8~~ ~~5~~ 9 1 9 —— 3 0
Write-sum-for-each-column[c] 5 6 8 7 7 8 ——— 12 13 16	Borrow-across-zero[f] 5 ~~6~~ ~~7~~ 0 ^{1}0 ^{1}2 2 5 ———— 5 0 8 7
Vanish-the-one[c] 5 6 8 7 7 8 ——— 2 3 6	Stops-borrow-at-zero[e] 6 ~~7~~ ^{1}0 ^{1}0 ^{1}2 3 2 5 ———— 6 7 8 7
Reuse-digit-if-uneven[d] 6 3 2 —— 8 5	Top-smaller-write-zero[g] 2 5 2 1 1 8 ——— 1 4 0
Add-extra-digit-into-column[d] 6 3 2 —— 1 1	Smaller-from-larger[h] 2 5 2 1 1 8 ——— 1 4 6
Ignore-extra-digits[d] 6 3 2 —— 5	Reuse-digit-in-uneven[d] 7 8 6 —— 1 2

Note: The following are source notes: [a] Baroody (1987); [b] Ginsburg (1977); [c] Fuson and Briars (in press) and Fuson (1986); [d] Friend (1979, in Davis, 1984); [e] Van Lehn (1986); [f] Van Lehn (1986) and Davis (1984); [g] Fuson and Briars (in press) and Van Lehn (1986); [h] Davis *et al.* (1979, in Davis, 1984), Fuson and Briars (in press), Fuson (1986), Labinowicz (1985) and Van Lehn (1986).

than specific tens and hundreds groups. Within multidigit addition and subtraction performance, the concatenated single-digit representation for some children seems to serve as the basis for a spatial pattern analysis of multidigit procedures (see Van Lehn, 1986) in which the quantitative values of the single digits are not accessed, whereas for other children these values are accessed and used to constrain the trading of units from digit to digit. However, because these digits do not have tens or hundred values, these values do not constrain the columns between which trading occurs and so the errors shown in Table 18.2 occur.

Practical implications

Support for English-speaking children in constructing mental multi-unit named-value and positional base-ten representations might be provided in at least three ways. First, size embodiments such as base-ten blocks that perceptually display the relative sizes of different named values and a positional base-ten embodiment such as digit cards (cards on which a single digit is written) can be used to help children understand the features of the named-value and positional base-ten systems. Figures 18.1 and 18.2 illustrate the use of these embodiments in multidigit addition and subtraction. Some children may need to use such embodiments for a long time in order to construct the multi-unit mental representations required for multidigit numbers. During this time it is important that the named-value blocks and number words be closely linked to the positional base-ten digit cards and written marks problems so that the named-value and positional base-ten meanings can be related to each other (see Fuson, in press, for a discussion of other features of the effective use of these embodiments).

Secondly, multidigit learning/teaching might begin with four-digit numbers because the regular named-value English words for hundreds and thousands support the construction of a named-value representation and can provide a strong context into which the irregular tens can be pulled. This was done successfully for second-graders with base-ten blocks used as in Figs. 18.1 and 18.2 (Fuson, 1986; Fuson and Briars, 1990), and anecdotal evidence indicates that these larger numbers may have a similar effect for children using a multi-unit sequence representation in which they count on by tens and hundreds.

A third alternative is having children learn a 'Chinese' version of English number words, that is named-value for tens. Thus, 8653 would be said '8 thousand 6 hundred 5 ten 3' and 12 would be read as 'ten two' or 'one ten two'. These 'Chinese' number words could be introduced in a cross-cultural context and might help to focus English-speaking children on constructing a named-value rather than just a unitary representation of two-digit numbers.

Because so much multidigit school instruction occurs without sufficient perceptual or linguistic support of these kinds, it is no wonder that so much two-digit place-value and addition and subtraction instruction goes awry in the USA

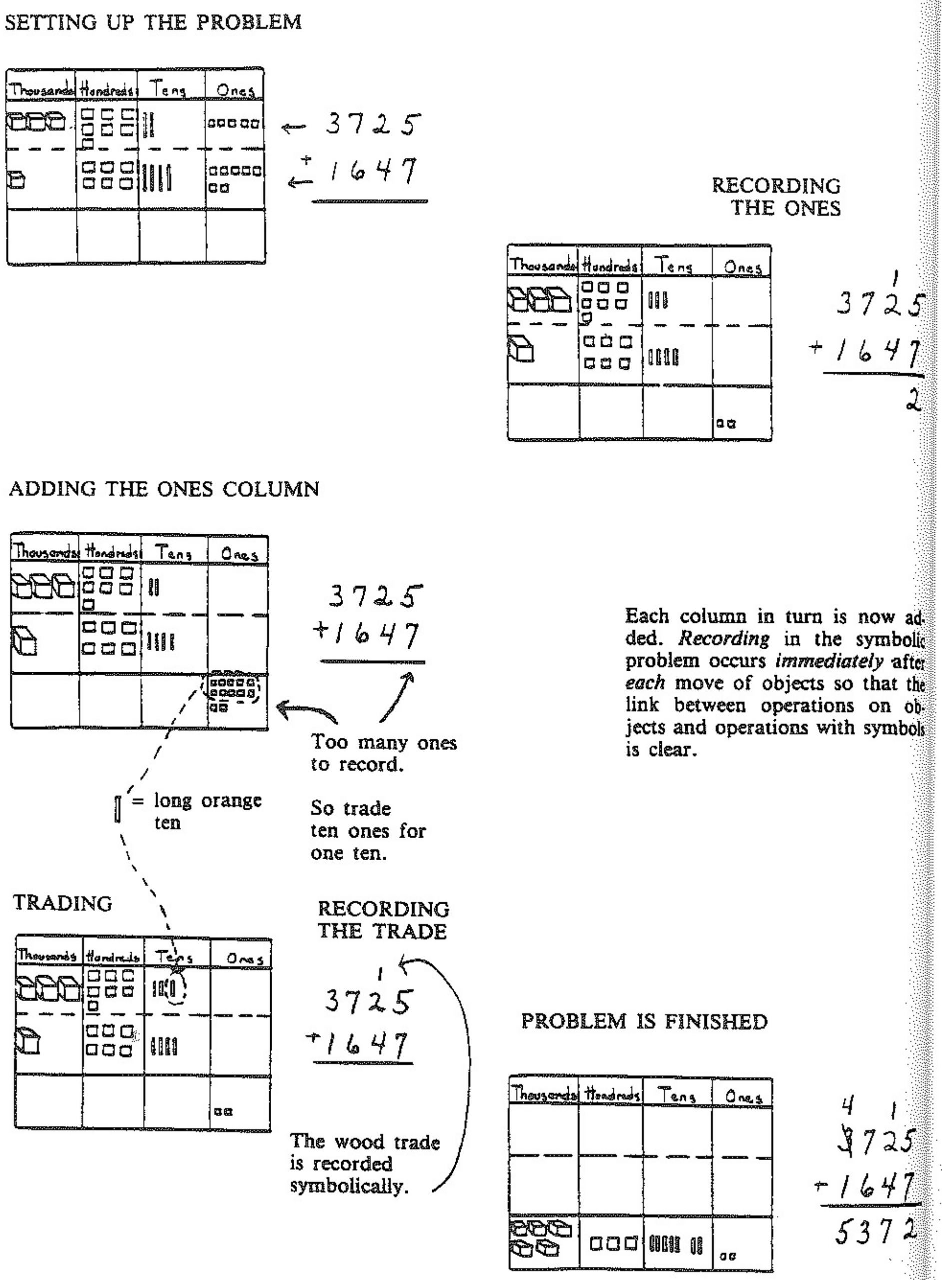

Fig. 18.1. Multidigit adding and recording with base-ten word embodiment.

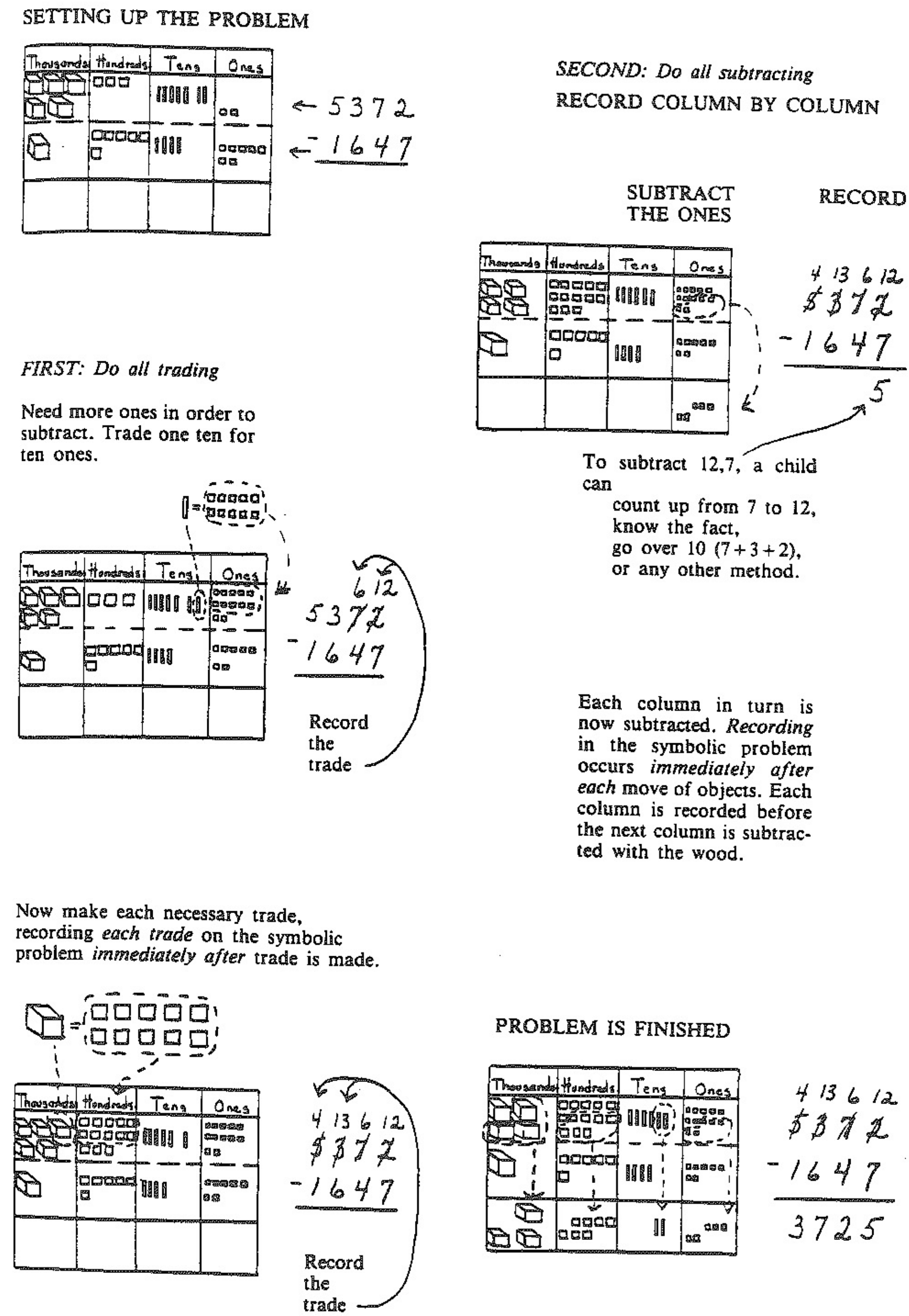

Fig. 18.2. Multidigit subtracting and recording with base-ten word embodiment.

(and possibly in many European countries): teachers talk about tens and ones but children see the written two-digit marks as unitary sequence/counting numbers that are counted collections of single objects (or words), or see these marks as concatenated single digits, each with only a unitary meaning.

Conclusions

Asian languages that have a regular named-value system of number words that name the ten values in a regular way as well as naming the hundred and thousand values, help Asian children construct multi-unit mental representations for multidigit numbers. These mental representations allow Asian children to add and subtract numbers with sums between 10 and 18 and to add and subtract multidigit numbers earlier, more easily and more accurately and render multidigit addition and subtraction more meaningful. English-speaking children construct and use for a long time unitary representations of number instead of multi-unit representations and are much more likely to construct an inadequate concatenated single-digit representation of multidigit numbers that allow them to make errors in place-value tasks and in multidigit addition and subtraction. Because of the lack of support for understanding tens and ones in English, perceptual or linguistic support for constructing adequate multi-unit representations needs to be provided in the classroom.

Further reading

Fuson, K.C. (submitted). *Children's representations of multidigit numbers: Implications for addition and subtraction and place-value and teaching.*

Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking.* Menlo Park, Calif.: Addison-Wesley.

Menninger, K. (1969). *Number words and number symbols: A cultural history of numbers* (translated by P. Broneer). Cambridge, Mass.: MIT Press. (Translated from original publication, *Zahlwort und ziffer*, 1958, Vandenhoeck & Ruprecht.)