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Issues in Place-Value and Multidigit Addition and Subtraction Learning and Teaching

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Research on multidigit addition and subtraction is now sufficient to question some present textbook practices and suggest alternatives. These practices revolve around the organization and placement of topics within the curriculum and around teaching/learning methods. These questions are being raised because the evidence indicates that U.S. children do not learn place-value concepts or multidigit addition and subtraction adequately and even many children who calculate correctly show little understanding of the procedures they are using (e.g., Cauley, 1988; Kamii & Joseph, 1988; Kouba et al., 1988; Labinowicz, 1985; Lindquist, 1989; Resnick, 1983; Resnick & Omanson, 1987; Ross, 1989; Stigler, Lee, & Stevenson, in press; Tougher, 1981).

Initially, children's conceptual structures for number words are only *unitary conceptual structures* in which the meanings or referents of the number words are single objects (as in counting objects) or a collection of single objects (as in the cardinal reference to a collection of seven objects). These unitary conceptual structures become more integrated, abbreviated, and abstract and progress through a developmental sequence that enables children to carry out increasingly abstract and efficient addition and subtraction solution procedures (Carpenter & Moser, 1984; Fuson & Hall, 1983; Fuson, 1988, in press b; Kamii, 1985; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). However, even the most sophisticated of the procedures based on these unitary conceptual structures is time-consuming and error-prone for adding and subtracting two-digit numbers greater than 20 or so. For such two-digit and larger multidigit numbers, children need to construct *multiunit conceptual structures* in which the meanings or referents of the number words are a collection of entities (such as counting "one hundred, two hundred, three hundred," in which the referent for each "hundred" is a collection of 100 entities of some kind) or a collection of collections of objects (as in the cardinal reference of "seven hundred" to a collection of seven collections of 100 entities). Both English number words and written number marks for multidigit numbers are built up of increasingly larger multiunits related to ten (ten, hundred, thousand, etc.). Understanding multidigit numbers requires being able to think about these various sizes of multiunits, and understanding operations on multidigit numbers requires understanding how to compose and decompose multidigit numbers into these multiunits in order to carry out the various operations.

With present school instruction many children in the United States do not build any multiunit conceptual structures for multidigit numbers but instead treat them as concatenated single digits (Fuson, in press a). Multidigit addition and subtraction are learned as procedures carried out on columns of single digits, and meanings other than single-digit meanings are not constructed or are not accessed. In this article I will argue that certain characteristics of present textbook presentations of place value and multidigit addition and subtraction contribute to the failure of U.S. children to build adequate multiunit conceptual structures, and I will propose new characteristics to replace some present textbook characteristics. Both the present textbook and the proposed characteristics are listed in Table 1. Present textbook characteristics depend primarily on a skills analysis approach in which children learn and practice skills, slowly adding more difficult skills. How children think about numbers and the differing attributes of multidigit number words and written marks are not considered. The proposed new characteristics stem from considerations of children's thinking and of analyses of attributes of place value and multidigit addition and subtraction.

Table 1
Present Textbook Characteristics and Proposed New Characteristics of Teaching and Learning About Place Value and Multidigit Addition and Subtraction

Present Textbook Characteristics	Proposed New Characteristics
1. Reading and writing two-digit numerals is initially related to multiunit conceptual structures	1. Reading and writing two-digit numerals is initially related to unitary conceptual structures and later is related to multiunit conceptual structures
2. Addition/subtraction of large two-digit numbers with no trading precedes addition/subtraction of the most difficult single-digit sums to 18 (e.g., 41 + 57 precedes 8 + 8)	2. Addition/subtraction of all single-digit numbers (use of unitary conceptual structures) precedes all multidigit addition/subtraction (use of multiunit conceptual structures)
3. Understanding of place value must precede understanding of multidigit addition and subtraction	3. Understanding of place value is multifaceted and prolonged and accompanies and follows understanding of multidigit addition and subtraction
4. Multidigit addition and subtraction work is piecemeal and extends over years:[a] Problems with no trades (G1) precede those with trades (G2) Two-digit (G2) precede three-digit (G2, 3 mo. later) precede four-digit (G3) precede five-digit (G4) problems Ones trades (G2) precede tens trades (G2 or G3) precede ones and tens trades (G3) precede hundreds trades (G3)	4. Multidigit addition and subtraction work can be integrated and done in the second grade or whenever children are ready to construct multiunit conceptual structures Problems with and without trades are presented at the same time Multidigit addition and subtraction begins with (or moves rapidly to) four-digit problems All possible combinations of trades are done from the beginning
5. Rote rules are given initially for multidigit addition and subtraction, and inadequate support is provided for constructing multiunit conceptual structures	5. Multidigit addition and subtraction procedures arise from multiunit conceptual structures, and adequate support is provided for constructing multiunit conceptual structures

[a] Grade placements (G1 is Grade 1) are those of at least six out of eight recent reviewed textbook series (Fuson, in press b).

INITIAL FOCUS ON UNITARY CONCEPTUAL STRUCTURES

The first two textbook characteristics in Table 1 are related. Both ignore the considerable evidence that reading and writing two-digit numerals up to 20 or 30 and addition and subtraction of all single-digit sums and differences (i.e., all sums through 9 + 9 and their subtraction inverses) can be based on unitary conceptual structures; this period of learning extends for a long time as children progress through the extended developmental sequence of unitary conceptual structures. Multiunit conceptual structures are needed for work with multidigit numbers but are not needed for addition and subtraction of numbers less than 20. Thus, this work can come quite early and can precede any work fostering children's construction of multiunit conceptual structures. This reallocation of tasks according to children's thinking rather than to a skills analysis leads to the first two proposed new characteristics to replace the first two present textbook characteristics in Table 1.

UNDERSTANDING PLACE VALUE

The third textbook characteristic is that understanding of place value is assumed to be required for understanding multidigit addition and subtraction, and thus fairly extensive work on place value precedes work on multidigit addition and subtraction in most U.S. textbooks. The assumption underlying this characteristic is undermined if one undertakes an analysis of the actual understandings required for place value and for multidigit addition and subtraction. Understanding of place value requires that one understand both the English system of number words and the system of written multidigit marks. These two systems have different features (Fuson, in press a). English number words are an irregular named-value system in which larger numbers are made by pairing a number word between one and nine with a word that names a higher value to be given to these small number words. *Five* thousand *six* hundred *for*ty-*eight* is similar to *five* miles *six* yards *four* feet *eight* inches in that the value/measure is explicitly stated for each number. The written multidigit marks are a positional base-ten system in which larger numbers are made by placing a written mark (0 or 1 or ... or 9) in a different relative position to the left of the units mark. Each position implicitly takes a value ten times larger than the value of the position on its right.

Although the third textbook characteristic in Table 1 has considerable face validity, many of the features of the named-value English words and the positional base-ten marks—and the differences between them—are required by and can be motivated by multidigit addition and subtraction, and the same multiunit conceptual structures are required to understand multidigit addition and subtraction and many aspects of place value. The spoken number words (or a physical multiunit embodiment of the words, like base-ten blocks) can direct the basic insight underlying multidigit addition and subtraction: that like quantities are added to or subtracted from each other. These number words (or embodiments of them) also can direct and constrain correct trade rules when one has too many or not enough of a given value. But it is the written marks that require trading (borrowing, carrying, regrouping) in multidigit addition and subtraction because only they limit the number of each value to nine or less. Such English phrases as "sixteen tens" and "fifteen hundred" are meaningful and unambiguous but are difficult to write using written place-value notation. Thus, addition and subtraction of multidigit numbers can help clarify the following positional base-ten features: (a) the regular ten-for-one trade rules that make each larger value in each position to the left, (b) the need for zero to keep each mark in its correct relative position, (c) the lack of conservation of values if the positions of marks are changed, and (d) the continued generation of larger numbers by trading to the left again and again.

Understanding place value requires understanding the differences between the English named-value words and the written positional marks, and understanding both of these systems requires the construction of multiunit conceptual structures. As a consequence, understanding place value requires a considerable amount of time and extended activities supporting such constructions. Because so many aspects of place value, and of the differences between these systems, can be understood by meaningful engagement in multidigit addition and subtraction, the third proposed new characteristic is that much place-value work be done within the multidigit addition and subtraction setting rather than being done separately before any multidigit addition and subtraction work is begun. This characteristic also suggests that there may be aspects of place-value understanding that go beyond those that are necessary for understanding multidigit addition and subtraction. It may be that some features of a mature conceptual structure for the written positional base-ten marks cannot really be appreciated until multidigit multiplication is understood (e.g., that multiplying by 10 shifts any number one place to the left).

INTEGRATED MULTIDIGIT ADDITION AND SUBTRACTION

A major strength of both the system of number words and of written marks is that they have consistent one/ten trade rules between adjacent values/positions. This feature enables systematic repetitive procedures to be devised to carry out multidigit addition and subtraction. The textbook attributes that constitute Characteristic 4 in Table 1 may prevent children from seeing these regularities because multidigit addition and subtraction is dismembered into tiny separate problem exemplars and exposure to these is spread out over a period of years. Furthermore, the early textbook emphasis on two-digit problems with no trades may support typical errors like writing both numbers in a given column and subtracting the smaller from the larger number in a column. The lack of trading emphasizes the view of multidigit addition and subtraction as unrelated vertical column procedures; only trading requires relating adjacent columns to each other. Such a single-digit view may be further exacerbated by Characteristic 2 in Table 1 (large two-digit numbers with no trades preceding the much smaller single-digit sums to 18) because children who have experience only with sums that do not exceed ten seem particularly likely to view each column just as single-digit sums.

The prolonged use only of two-digit numbers in U.S. textbooks may also extend the period of time during which U.S. children use unitary conceptual structures for multidigit numbers. English number words are regular and name the values "hundred" and "thousand" and thus support children's construction of multiunit conceptual structures for hundreds and thousands. However, English number words do not explicitly name the tens and are irregular in several ways; even many adults have not realized that "teen" and "ty" in the decade words mean "ten," if the reactions of many of my Northwestern undergraduates are typical. Delaying U.S. children's opportunities for interacting with numbers in the hundreds and thousands may only delay opportunities to construct multiunit conceptual structures for multidigit numbers.

MEANING FOR MULTIDIGIT ADDITION AND SUBTRACTION PROCEDURES

The final textbook characteristic focuses on the method of presentation of multidigit addition and subtraction. In almost every case in the textbook series analyzed for Fuson (in press b), multidigit addition and subtraction began with a statement of a rote rule and inadequate support was provided for children's construction of multiunit conceptual structures. First, the rule preceded any attempt to develop meaning for the procedure. Second, even though most texts used drawings of multiunit base-ten blocks on one or a few pages, these were abandoned in too short a time for children to build new conceptual structures. Furthermore, the textbook pictures showing base-ten blocks and written marks were in many cases so confusing that it was difficult to see the relationship between the blocks and the written marks. Because the whole point of using blocks with the spoken words and the written marks is to build multiunit meanings for these words and marks, the blocks must be linked very tightly and clearly to the words and the base-ten marks.

An alternative to this rule-based approach is the fifth proposed new characteristic: provide a teaching/learning setting in which children can build multiunit conceptual structures and allow the multidigit addition and subtraction procedures to arise out of the aspects of these multiunit conceptual structures. This approach was taken in a series of studies in which first- and second-grade teachers used base-ten blocks to support children's construction of named-value collection multiunit conceptual structures and digit cards to support positional base-ten multiunit conceptual structures (Fuson, 1986; Fuson & Briars, 1990). This teaching/learning setting possessed all of the proposed new characteristics listed in Table 1: 1 and 2) Children learned advanced unitary methods for adding and subtracting sums to 18 (counting on for addition and counting up for subtraction) before doing any multidigit addition and subtraction; 3) a short introduction to place value linked the English words, block words, and written marks, and then children moved immediately to extended work on multidigit addition and subtraction; 4) problems with all combinations of trades were presented immediately and multidigit addition and subtraction began with or moved immediately to four-digit problems; and 5) no rules were given for addition or subtraction, but the traditional procedures arose out of the multiunit values in the blocks and the positional characteristics of the digit cards. Second graders of all achievement levels and high-achieving first graders showed considerably more understanding of place value and of multidigit addition and subtraction than do third graders and even older children receiving usual instruction as reported in the research literature. This approach enabled children to learn much more than U.S. children are ordinarily given the opportunity to learn, and it put their accomplishments much more in line with those of children in countries mathematically more advanced (Fuson, Stigler, & Bartsch, 1988). These studies are existence proofs that the proposed new characteristics are possible for classroom teachers to carry out and that they yield considerably more competence than does usual instruction.

U.S. CHILDREN'S LINGUISTIC AND CULTURAL DISADVANTAGES IN CONSTRUCTING MULTIUNITS OF TEN

English-speaking U.S. children may need considerable and extended support in the classroom for constructing multiunit meanings based on ten, because the English language and the U.S. culture provide relatively little support for such meanings. English does not explicitly name the tens in 2-digit numbers, in contrast to several Asian languages (Chinese, Japanese, Korean) that do name the tens (12 is said "ten two" and 58 is said "five ten eight"). Children in these countries seem to construct multiunit conceptual structures for two-digit marks while their U.S. agemates construct only unitary conceptual structures (Fuson & Kwon, in preparation; Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989). These cultures provide other supports for multiunit structures of ten. Asian children learn in first grade addition and subtraction methods structured around ten for single-digit numbers with sums between ten and 18 (see Fuson & Kwon, in press a, in press b; Fuson, Stigler, & Bartsch, 1988); the named ten for 11 through 18 ("ten one" through "ten eight") make these methods considerably easier than when they are carried out in English. The metric system, with all its examples of ten/one trades, is used in these countries but not in the U.S. The abacus has been widespread in these countries, reflecting a long cultural focus on operations on numbers structured around ten. There are even in some countries methods of showing numbers on fingers that allow easy use of addition and subtraction structured around ten, while our methods of showing addition and subtraction on fingers instead support unitary conceptual structures (Fuson & Kwon, in press b).

Because this research on single-digit sums and differences structured around ten is just emerging, how easily these solution procedures can be transferred to this culture and the nature of the support required (e.g., linguistic support of regular named tens such as "ten three" for 13, physical embodiments that show tens and ones) is not yet clear. This is one possible path to the construction of multiunits that might be explored by research—following children's construction of unitary solution procedures for single-digit sums and differences by classroom experiences that support multiunit solution procedures structured around ten. These experiences would then be followed by place-value and multidigit experiences possessing proposed Characteristics 3, 4, and 5 of Table 1. Thus, two alternative paths through the new characteristics in Table 1 might be examined in future research: a) a path moving from unitary conceptual structures to single-digit sums and differences structured around ten and then (directly or rapidly) to four-digit addition and subtraction, and b) a path supporting children through unitary conceptual structures for single-digit addition and subtraction and then (directly or rapidly) to four-digit addition and subtraction. The base-ten block studies indicate that teachers can support children along the second path. How well and how rapidly they can do so along the first path is an important question. Deciding which of these paths through the new characteristics is ultimately more sensible for children in our culture must await empirical resolution.

IN CONCLUSION

The argument here is not that the base-ten block approach as applied in our studies is the best possible approach for helping children understand place value and multidigit addition and subtraction. An open-ended problem solving use of a base-ten embodiment or multidigit addition and subtraction explored in real world contexts such as bank accounts or the use of Asian regular named-value tens may prove to be more effective for some or all children. Future research should certainly focus on helping children learn even more about multidigit addition and subtraction and place value. But the arguments advanced above concerning the reasons for replacing the present textbook characteristics with the proposed new characteristics and the evidence that these new characteristics can be implemented and do lead to comparatively higher levels of learning suggest that we rethink many attributes of present place-value and multidigit addition and subtraction instruction and begin to provide children with an opportunity to learn the mathematics that they are capable of learning.

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