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Conceptual Structures for Multiunit Numbers: Implications for Learning and Teaching Multidigit Addition, Subtraction, and Place Value

Karen C. Fuson
Northwestern University

Multiunit numbers are whole numbers composed of one or more kinds of multiunits (collections of single units) and possibly some single units. Multiunit numbers are expressible by number words and by written number marks. This article identifies conceptual structures necessary for understanding the named-multiunit-value English system of number words, the unnamed-value positional system of written marks for multiunit numbers, and several different kinds of conceptual multiunits that give meaning to these two different systems. Conceptual components of multiunit addition and subtraction are described. The special difficulties caused by irregularities in the English words for multiunit numbers composed of tens and ones are discussed and compared with the much simpler learning task for children speaking a system of number words that names the tens and ones in a regular fashion (Asian systems based on Chinese). Literature concerning the poor performance of U.S. children on place-value tasks and on multidigit addition and subtraction is reviewed, and two special conceptual structures (sequence multiunit and concatenated single-digit structures) used by U.S. children are identified. The latter conceptual structure is associated with use of many different partially correct procedural rules for multidigit addition and subtraction that lead to characteristic errors made by U.S. children and that violate particular conceptual components of multiunit addition or subtraction. Classroom experiences that support children's construction of the requisite conceptual structures are discussed, with particular attention to the role of objects that display collectible multiunits. Limitations of current U.S. textbook treatments and curricular placement of place value and multidigit addition and subtraction are described, and possible alternative paths to children's multiunit addition and subtraction are summarized.

Requests for reprints should be sent to Karen C. Fuson, School of Education and Social Policy, Northwestern University, 2003 Sheridan Road, Evanston, IL 60208-2610.

Large whole numbers are expressed by systems of spoken number words or systems of written number marks that use multiunits to build up the large whole number. Different cultures have used different combinations of multiunits. Five, ten, and twenty have been multiunits used by many different cultures (see Bell, Fuson, & Lesh, 1976, chapter B7, for a more general analysis of features of number-word and number-mark systems; for descriptions of many different systems, see Ifrah 1981/1985; Menninger, 1958/1969; Zaslavsky, 1973). Children in a given culture have to learn the multiunits used by their own system of number words and by their system of written marks, and they have to learn how these systems use these multiunits to express large numbers. To learn how to add and subtract large numbers, they have to learn how the multiunits function in such addition and subtraction. This article focuses on the nature of these tasks for English-speaking children in the United States. Features of the English system of number words and of our system of written marks are identified, and difficulties in relating these two systems are discussed. Because understanding these systems requires understanding these features, each feature can be considered a conceptual structure that any given individual may or may not have.

It is important for the reader to remember that the meanings the reader has for multiunit English words and for the standard multiunit written marks may not be shared by others, especially by young children. The meanings do not lie in the English words or in the written multidigit marks; they lie in children's minds and are constructed and linked to the words or the marks by individual mental activity in given individual situations. There are many different meanings that can be linked to the words and to the marks. This article analyzes these different meanings, identifying each different kind of meaning as a different conceptual structure that can be linked to the words and/or to the marks. Meanings supported by the words or by the marks are identified. Conceptual understandings required for addition and subtraction of multiunit addition and subtraction are discussed.

The analysis of the English system of number words and the teaching and learning of this system are complicated by various irregularities in English words between ten and one hundred. These irregularities can obfuscate the underlying features of the system and the relationships between the system of words and the system of written marks. To clarify the analysis, the first part of this article considers a system of number words that is structurally like English but does not have the English irregularities—namely, the Chinese system. Because most readers already possess the appropriate conceptual structures for English words and written marks, it may be difficult to separate these and understand what children must learn. Therefore, this discussion will sometimes use Chinese word examples to permit the reader to see the child's task in constructing the necessary conceptual structures.

This article focuses next on the irregularities in English and on the many difficulties these irregularities cause English-speaking children. Finally, implications of the conceptual analysis and of these irregularities for learning and teaching multidigit addition, subtraction, and place value are discussed. The conceptual structures required in these domains are complex, especially given the special difficulties caused by the English irregularities. These complexities place considerable demands on the kinds of learning opportunities that need to be provided to children within mathematics classrooms.

This effort is undertaken because present instructional methods in the United States result in unacceptably low levels of competence with place-value and multidigit addition and subtraction. With respect to place value, less than 50% of third graders in the National Assessment of Educational Progress (NAEP) could do items identifying the hundreds digit, and only 65% identified the tens digit correctly (Kouba et al., 1988). Many children in a heterogeneous sample from 33 second- through fifth-grade classrooms showed wrong or inadequate understanding of place value for two-digit numbers, with more than half of the fifth graders failing to demonstrate understanding of the ten-for-one trading that underlies the standard addition algorithm (Ross, 1986). Less than half of the third graders interviewed identified tens and hundreds in three-digit numbers (Labinowicz, 1985). Other inadequacies in place value are detailed in Ginsburg (1977), C. Kamii (1985, 1986), M. Kamii (1981), and Labinowicz (1985). Performance on multidigit addition and subtraction is likewise very inadequate. A third of third graders in the NAEP survey gave incorrect answers on a two-digit subtraction problem requiring trading, and half did so for three-digit problems (Kouba et al., 1988). Half of the middle-class third graders interviewed by Labinowicz (1985) solved a two-digit subtraction problem requiring trading incorrectly, and most of them were confident about their incorrect procedure. Davis and McKnight (1980) interviewed third and fourth graders from several schools with above-average students and teaching and found not a single child who solved 7002 – 25 correctly. Many different kinds of errors have been documented, especially for multidigit subtraction (e.g., Ashlock, 1982; Brown & VanLehn, 1982; VanLehn, 1986). Furthermore, many or most children who carry out multidigit addition and subtraction correctly do so only as a rote procedure. They do not understand crucial features of this procedure and cannot explain them or relate them to features of the English number words or the written marks (Cauley, 1988; Cobb & Wheatley, 1988; Davis & McKnight, 1980; Ginsburg, 1977; Labinowicz, 1985; Resnick, 1982, 1983; Resnick & Omanson, 1987; Silvern, 1989).

These results are not due to inevitable age limitations on children's ability to understand these concepts and procedures. On items measuring place value and multidigit addition and subtraction, Japanese and Taiwanese first and fifth graders showed much higher scores than did U.S. first and fifth graders on curriculum-fair tests (Stigler, Lee, & Stevenson, 1990). Korean children ages 6, 7, and 8 carried out multidigit addition and subtraction much more accurately than their U.S. counterparts (Song & Ginsburg, 1987), and Korean second and third graders showed better understanding of multidigit addition and subtraction than did U.S. children (Fuson & Kwon, 1990b). Korean, Japanese, and Chinese children showed better understanding of place value than did U.S. children (Miura, Kim, Chang, & Okamoto, 1988). Mainland China, Japan, the Soviet Union, and Taiwan all begin place-value and multidigit addition and subtraction topics earlier than does the United States, and they complete this instruction earlier (Fuson, Stigler, & Bartsch, 1988). Clearly, considerable improvement needs to occur in the United States in the teaching and learning of these topic areas. The goal of this article is to provide mathematical, cognitive, and instructional analyses to facilitate efforts at such improvement.

NAMED-VALUE MULTIUNIT WORDS AND UNNAMED POSITION-VALUE MULTIUNIT WRITTEN MARKS

In a named-value system of number words, the multiunits are explicitly stated. A number word tells how many of a given multiunit there are, and that number word is followed by the multiunit value word. An example of an English version of the regular named-value Chinese system is two *ten thousand* nine *thousand* five *hundred* eight *ten* three (or, to use the Chinese value words, two *wan* nine *qian* five *bai* eight *shi* three). In the unnamed position-value multiunit system of written marks now used in most of the world, the multiunit values are not indicated by any written marks. They instead are understood and implicit; the only marks written are those for the numbers that tell how many of each multiunit there are. These named-value or unnamed position-value features are not inherent in words or in marks. One can have a system of named-value written marks and a system of unnamed position-value words (see Table 1).

The system of Chinese named-value words and the system of unnamed position marks have two attributes in common. First, they use the same multiunits—successive multiples of ten—to make larger numbers. The multiunits can be thought of in several different ways: as collections of single

TABLE 1
Named-Value and Unnamed Position-Value Words and Written Marks

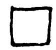
Numbers Expressed As	*Named-Value System*	*Unnamed Position-Value System*
Spoken words	*Two ten-thousand nine thousand five hundred eight ten three* or *two wan nine qian five bai eight shi three*	Two nine five eight three
Written marks	*2 TTh 9 Th 5 H 8 T 3*	2 9 5 8 3

units, as generated by a ten-for-one trade from the next smaller multiunit, as values of cumulative ten-for-one trades, as cumulative multiples of ten, and as exponential word or mark expressions of the multiples of ten (see the multiunit structures in Table 2). These different conceptions of the multiunits range from simple multiunit quantities to abstract expressions of repeated multiples. To understand either the words or the marks, one must understand at least one of these notions of multiunits. Second, both systems use nine different words/marks for the first nine numbers and reuse these words/marks with the multiunits based on ten to tell how many multiunits there are. Therefore, to relate these systems to each other, one must learn associations between the first nine words (one, two, three, . . . , nine) and the first nine marks (1, 2, 3, . . . , 9).

These systems also differ in fundamental ways (see Table 2). The system of written marks requires (a) the perception of a visual layout of horizontal "slots" or positions into which the nine number marks, 1 through 9, can be written to show the number of each kind of multiunit and (b) learning that these positions are ordered in *increasing* value from the *rightmost* position. The system of words requires (a) learning the value words *shi, bai, qian,* and *wan,* (b) learning that these value words are said in order of *decreasing* value from *wan* to *shi,* and (c) learning that each number word one through nine (except the number word that tells how many ones there are) is followed by a multiunit value word.

Being able to move between the words and the marks requires two sets of associations: the association between the nine small-number words and the nine small-number marks (as mentioned earlier) and the association of multiunit value words to particular positions relative to the rightmost position (see Table 2). This latter association involves the conceptual structure listed in the second column of Table 2: being able to find a particular position from the rightmost position. This requires being able to start on the right and then counting or subitizing (recognizing visually) or naming places by moving to the left. Thus, one might count "one, two, three, four" and know the fourth place is the *qian* place, or subitize the fourth place and know it is the *qian* place, or say the value list in increasing value order "*yi, shi, bai, qian*" pointing from right to left to find the *qian* place. In all cases children must move from right to left, the direction opposite to that used in reading and commonly used in counting objects. One must also say the value list in *increasing* value order, the opposite to the value order in saying a multiunit English word. To say in words a multiunit number written in marks, one must go through one of the right-to-left processes to ascertain what value name to give to the first (leftmost) written digit. Once one knows this value name, one moves from left to right in saying the rest of the multiunit word from the written marks and uses the word order of decreasing values. One also moves from left to right when writing multiunit marks from the word, ignoring the value words and using the association between

TABLE 2
Conceptual Structures for Multiunit Numbers

Name of the Conceptual Structure	*Nature of the Conceptual Structure*				
Features of the marks					
Visual layout	____	____	____	____	____
Positions ordered in increasing value from the right	Fifth	Fourth	Third	Second	First
Features of the words					
Multiunit names	*Wan*	*Qian*	*Bai*	*Shi*	*Yi*
Words ordered in decreasing value as they are said	Ten-thousand	Thousand	Hundred	Ten	Ones
Multiunit structures					
Multiunit quantities					
Regular ten-for-one and one-for-ten trades	Ten thousands one	Ten hundreds ten one	Ten tens ten one	Ten ones ten one	ten
Positions/values as cumulative trades	Four trades	Three trades	Two trades	One trade	No trades
Positions/values as cumulative multiples of ten	Four multiples of ten ($t \times t \times t \times t$)	Three multiples of ten ($t \times t \times t$)	Two multiples of ten ($t \times t$)	One multiple of ten (t)	No multiples of ten
Positions/values as exponential words for multiples of ten	Ten to the fourth power	Ten to the third power	Ten to the second power	Ten to the first power	Ten to the zero power
Positions/values as exponential marks for multiples of ten	10^4	10^3	10^2	10^1	10^0

the small-number words and the small-number marks. For young children, many of whom are prone to reversal problems, it may be quite problematic to sort out these directional complications and decide between moving from left to right and from right to left and between saying words in increasing order and in decreasing order.

Once these two sets of associations have been learned, there are still three problem areas that involve additional distinctions and additional learning. First, if a value word is missing from a given multiunit number (e.g., four *qian* two *bai yi),* writing the digits in order (4 2 1) does not give the correct written marks. Some indication must be given for the missing *shi* value, or the larger values (*qian* and *bai*) end up written in the wrong places (in the third and second instead of in the fourth and third places). Any mark could be used to show the missing value; the mark that is used is 0. This word-mark association would be easier to learn if missing values were stated with a word counterpart to 0 (e.g., "no *shi*" or "zero *shi*"); instead children must learn the consequences of omitting a written mark if a value is missing (i.e., the larger values will be in the wrong place) and so must learn to write a zero for the missing value. Second, if there are more than nine of a given value, one has the reverse problem: Larger values get pushed one place too far to the left. This problem arises in multiunit addition when the sum of the multiunits of a given value may exceed nine. Thus, children may add 38 + 24 and get 512 (especially if the 24 is written below the 38) and may even call that answer "fifty twelve" (as some U.S. second graders did in Behr, 1976). Words such as fifty twelve are well defined, even though unusual; the problem in written marks is that using more than one position for the ones (i.e., using two positions to write 12 for the twelve ones) pushes the digit for the tens from the tens to the third (hundreds) place. Thus, one cannot write more than nine of a given value, even though one can say more than nine of a given value. This constraint comes from the written marks and not from the words. Third, multiunit numbers are made with words by concatenating the small-number/value pairs, but the effect of leaving out the value marks in the standard written marks is that written multiunit numbers are embedded rather than concatenated. Writing the marks by concatenating rather than embedding the marks is a frequent and widespread error. In this error, the written marks for a given value word (e.g., 100 for *bai* and 10 for *shi*) are concatenated just as the words are concatenated: "two *bai* four *shi* six" becomes 200406 or "five *shi* eight" becomes 508. This type of error is frequently made by U.S. children (Behr, 1976; Bell & Burns, 1981) and by Dioula African children speaking a named-value system of words (Ginsburg, Posner, & Russell, 1981), and it was made by European adults when our written marks were beginning to replace concatenated Roman numerals (Menninger, 1958/1969). Such errors might occur less frequently if named-value systems used a zero word rather than not saying zero values because 100 would not be just "one *bai*" but rather "one *bai* no *shi* no *yi*."

One could learn to make the associations between the written marks and the spoken words using only the knowledge discussed so far. This would allow one to say words for any written marks and write any spoken number words. To have any numerical understanding of the words or marks, however, the words and marks must have multiunit quantities associated with the multiunit names and the multiunit positions. The "multiunit-quantities" conceptual structure shown in the fifth row of Table 2 must be constructed, and these multiunit quantities have to be associated with the conceptual structures for the marks and words. This construction requires experiences with situations that present multiunit collections of single units. Three examples that present the first four named values in visually perceptual ways are:

1. *Bundled sticks:* single sticks, bundles of ten sticks, bundles of one hundred sticks, bundles of one thousand sticks.
2. *Base-ten blocks:* single small cubes; longs, one cube × one cube × ten cubes; flats, ten cubes × ten cubes × one cube; big, ten × ten × ten cubes (these are shown in Table 2).
3. *String lengths:* one-cm strings, ten-cm strings, one-m strings, ten-m strings.

For objects from any of these embodiments to be considered as a multiunit (e.g., the long ten-unit block), the viewer must focus on the cardinality of the units in a given bundle/block/string (e.g., must see that there are ten small cubes in the long block) and must conceptually collect or unite those separate units into a single multiunit (e.g., must see the ten ones as one ten). To emphasize this conceptual collecting activity, a conceptual quantity so formed is called a *collected multiunit*. The physical collections of objects from which a viewer can construct a conceptual collected multiunit are called a *physical collectible*[1] *multiunit* to emphasize the fact that such a physical item is only potentially, and not necessarily, a multiunit in the mind of a particular viewer (or at a given particular moment).

The "regular ten-for-one and one-for-ten trades" conceptual structures (the second multiunit structures in Table 2) can be used in any situation in which one has too many or not enough of a given multiunit. These conceptual structures guide the trades that can be made without changing the quantity of the overall multiunit number. The ten-for-one trades can arise from the multiunit-quantities conceptual structure by looking at contiguous

[1]The word *collectible* is taken from Cobb and Wheatley's (1988) term *abstract collectible units*, which they used to refer to the conceptual multiunit a child constructed from a physical collection of single units.

multiunits and noticing that ten of one always makes one of the next larger multiunit. They can also be learned as a rote feature of the multiunit names (ten tens make a hundred, ten hundreds make a thousand) or of the visual layout of written marks (one can trade ten in any position for one in the position to the left). The regular ten-for-one and one-for-ten trades conceptual structures are required for addition and subtraction of any multiunit numbers that have one or more multiunits exceeding ten in the sum.

The last four conceptual structures shown in Table 2 require increasing reflection on the whole multiunit structure (at least through several positions) and movement from additive to multiplicative notions. Each of the four structures builds on the structure above it. The conceptual structures in the first six rows can be constructed in isolated bits focusing on any given position or named quantity. For example, one could remember that the name for the fourth position is *qian* without remembering that *bai* is the name for the third place. Or one could have experiences that lead one to see that ten *bai* equal one *qian* without having first had experiences that help one notice that ten *shi* equal one *bai*. In contrast, each of the last four conceptual structures in Table 2 requires reflection on the pattern in the whole conceptual structure immediately above it in the table. First, each value or position can be seen as the cumulative result of regular ten-for-one trades. Second, the act of trading can be seen as the act of creating a multiple, a single multiunit, out of smaller units, so values or positions can be seen as cumulative multiples of ten. Third and fourth, denoting the number of successive multiples of a given base number (here, multiples of ten) is exactly the function of spoken words and written marks for exponential expressions.

The discussion of conceptual structures has thus far been limited to numbers with only four multiunits. The system of named-value multiunit words and the system of unnamed position-value written marks begin to have different structures after several multiunits (the number varies with the system of number words). One can continue to make larger and larger positions to the left (i.e., larger multiunits for the written marks) by using regular ten-for-one trades. Thus, the multiunit value of any given position to the left, no matter how large, can be ascertained by using the regular ten-for-one trades structure or one of the conceptual structures below it in Table 2. To get larger and larger multiunits in a named-value system of words, however, one needs a new name for each new larger multiunit. To avoid the necessity of memorizing a huge list of multiunit names, most systems of number words meet this problem by creating certain large multiunits within which a small list of multiunit names is reused. Thus, in English very large numbers are chunked into large multiunits of a thousand, and the smaller multiunits of hundred and ten are used within these thousand-unit chunks. Thus, the fourth through sixth positions have the names (*one*) *thousand, ten thousand,* and *hundred thousand.* In the United States these thousand-unit chunks show a base-thousand structure like the base-ten structure present in the first four places (the trades between contiguous chunks are thousand-for-one trades), and each of the new thousand chunks has a new name: *thousand, million* (a thousand thousands, the word *million* actually taken from the Latin *mille* meaning "thousand"), *billion* (a thousand millions), and *trillion* (a thousand billions). Each three multiunits are read as if they were in the three rightmost positions (e.g., *three hundred forty seven*), and then the base-thousand multiunit name is added (e.g., *three hundred forty seven million or three hundred forty seven thousand*). These base-thousand chunks are shown by separating them by commas (4,735,735). In Great Britain, *million* is taken as a new base. Words in the 7th through 12th positions are read as if they were in the 1st through 6th positions, and then million is added as the large multiunit value name. A British billion is a million millions (10^{12}) rather than a thousand millions (10^9) as in the United States. In Chinese, five rather than four positions have new multiunit names before the names are reused. *Wan* is then a new overarching multiunit, and the smaller multiunits are repeated: *shi wan, bai wan, qian wan. Wan wan* is then a new large multiunit *yi* that is used with the small multiunit names: *shi yi* (10^9), *bai yi* (10^{10}), *qian yi* (10^{11}), *wan yi* (10^{12}).[2]

ADDITION AND SUBTRACTION OF MULTIUNIT NUMBERS

The components of multiunit addition and subtraction are given in Table 3. Addition of two multiunit numbers requires addition of the like multiunits that make up each number. Addition of the same kind of multiunits is just like adding small numbers, except that the result is some number of that multiunit. For example, four hundred plus five hundred is nine hundred, just as four plus five is nine. Two different kinds of multiunits cannot be added together like small numbers: Four hundred plus five ten is not nine hundred or nine ten, it is only four hundred five ten. The conceptual structures in Table 2 give different amounts of support to this fundamental understanding of the nature of multiunit addition as adding like multiunits. The visual layout and relative position structures give little such support. There is nothing in them to suggest that multiunits even exist and nothing to direct which of the small numbers should be added together. Many second graders who had not yet studied multiunit addition (except perhaps two-digit addition with no trades) demonstrated this lack of support by responding to a question like "Rewrite 4273 + 56 so that it easy to add but

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[2]*Zhao yi* is also used instead of *wan yi*. These Chinese large number words are courtesy of K. Miller and J. Zao (personal communication, March 18, 1990).

TABLE 3
Components of Multiunit Addition and Subtraction

Operation	*Components*
Addition	Add like multiunits.
	Carry out the addition of a given kind of multiunit: single-digit addition of the numbers of a given kind of multiunit.
	Recognize and solve the problem of having too many (≥ ten) of a given multiunit:
	1. Recognize this as a problem (two digits cannot be written for a multiunit sum).
	2. Trade ten for one to the immediate left (and know trading conserves the quantity of the traded multiunit and of the whole multiunit number).
Subtraction	Subtract like multiunits.
	Recognize and solve the problem of having too few of a given multiunit (the number being subtracted has more of that multiunit than the number being subtracted from):
	1. Recognize this as a problem (and know the correct order of subtraction and that subtraction is not commutative).
	2. Trade one for ten to the immediate right (and know trading conserves the quantity of the traded multiunit and of the whole multiunit number).
	Carry out the subtraction of a given kind of multiunit: single-digit subtraction of the numbers of a given kind of multiunit.

gives the same answer" by writing vertically 427 + 356 or 42 + 73 + 56 (Fuson & Briars, 1990). Multiunit names do give some sort of cue. If asked to add 3 *bai* 2 *shi* 7 and 3 *bai* 5 *shi* 6, a person might well hazard a sum of 6 *bai* 7 *shi* 13, even if the meaning of *bai* or *shi* was not known. Their different names give them different identities, so it seems sensible to combine the things with the same name. Multiunit quantities give even more support to the strategy of adding like multiunits. Presented with an array of 3 flat (hundred) blocks 2 long (ten) blocks 7 unit blocks and 3 flat blocks 5 long blocks 6 unit blocks, it is clear that pushing them all together (i.e., adding them) makes 6 flat blocks 7 long blocks 13 unit blocks as a sum. None of the other conceptual structures in Table 2 gives such clear support for the combining of like multiunits as does the multiunit quantities structure. Physical collectible multiunits seem particularly to support understanding this aspect of multiunit addition.

When the sum of a given kind of multiunit exceeds nine, that multiunit cannot be written with the standard multiunit marks, because doing so would push marks for larger multiunits one position too far to the left. Thus, in the previous example, writing the sum of 6 flat blocks 7 long blocks 13 unit blocks as 6713 puts the 67 in the third and fourth positions instead of in the second and third positions where those multiunits belong. Either the multiunit quantities or the regular ten-for-one trade suggests the solution to this difficulty of having too many of a particular multiunit:

Trade ten of that multiunit for one of the next larger multiunit. This will give one more of the next larger multiunit in the sum and leave only the excess over ten of the original "too-many" kind of multiunit. In this case, 6713 trades into 683: 7 + 1 traded = 8, and the 3 is left from the 13. This requirement for trading when you have too many (i.e., ≥ ten) of a given multiunit arises from the written marks, not from named value words. Neglecting to trade creates written marks that are incorrect, whereas neglecting to trade with words creates nonstandard but comprehensible word forms (e.g., *thirteen hundred* or *thirteen tens* in English or *shi three bai* in Chinese). Whenever one must trade a given multiunit, the sum for that multiunit will be smaller than either of the addends. This is counterintuitive (addition usually makes larger) unless one understands that the sum is actually larger than either addend but ten of that sum is written with the next larger multiunit; thus, the sum for that multiunit appears to be smaller than either addend if one just looks at the written-marks problem. The sum is smaller than either addend for a traded multiunit because, if one thinks of adding the smaller addend to the larger and breaks it into a part that will make ten with the larger number, the sum actually written for that multiunit will be the rest of that smaller addend, which must be smaller than the whole smaller addend and thus also smaller than the larger addend.

Subtraction of multiunit numbers has the same three components as addition: (a) One operates on (subtracts) like multiunits, (b) this subtraction can be carried out as single-digit subtraction of the numbers of each kind of multiunit, and (c) trading is required for problems where the sum of a multiunit is ten or more. With addition, one can carry out the addition of like multiunits and only confront component (c), the problem of trading, if the sum exceeds nine. For subtraction, if a trade is necessary, one cannot even begin the subtraction of like multiunits until one has traded. Addition and subtraction are inverse (opposite) operations, and each multidigit addition problem is inversely related to two subtraction problems (those made by subtracting each addend from the sum). One will need to trade in a subtraction problem for any multiunit that was traded in the related inverse addition problem, because the number of that multiunit in the minuend (sum) will be less than the number of that multiunit in the subtrahend (addend being subtracted). Thus, trading in subtraction is just undoing the original trading that was required in addition, because one could not write the whole two-digit sum for that multiunit. Therefore, trading in subtraction is just one-for-ten trading to the right, the opposite of the ten-for-one trading to the left that occurs for addition. So if one makes a subtraction problem from the addition example discussed earlier, 683 – 327 requires the subtractions 6 *bai* – 3 *bai*, 8 *shi* – 2 *shi*, and 3 – 7. There are not enough units in the minuend to carry out the subtraction of the units (3 – 7), because the sum of the units in the original addition problem exceeded nine (6 + 7 = 13), and ten of the 13 were traded to the *shi* position, leaving only

3 units. Now to subtract 7, one needs the original sum of the units, 13; so one needs to carry out the reverse trade of one *shi* for ten units, resulting in 13 units. This reduces the number of *shi* to 7, leaving the problems 6 *bai* – 3 *bai*, 7 *shi* – 2 *shi*, 13 – 7 for an answer of 3 *bai* 5 *shi* 6, or 356. Thus, the traded minuend (6 7 13) is just the sum before trading (6 7 13). Seeing this inverse relationship between trading in addition and trading in subtraction may be quite a late understanding, because it requires knowing trading in addition and in subtraction fairly well before the reflection on the relationship between these different kinds of trading can be carried out. Table 3 shows in parentheses the knowledge that can lead to subtraction trading before this inverse addition–subtraction relationship is understood. One needs to know the correct order of subtraction (in a vertical problem, that 3 on top and 8 on bottom is 3 – 8, which is 3 minus 8 or 3 take away 8) and that subtraction is not commutative (without this knowledge, one can solve this problem by commuting 3 – 8 to 8 – 3 and saying 5, an extremely common mistake to be discussed later).

In summary, addition and subtraction of multiunit numbers require three components: (a) understanding that one operates on (adds or subtracts) like multiunits, (b) making a ten-for-one trade to the left when one has too many of a given multiunit in addition and making a one-for-ten trade to the right when one has too few of a given multiunit in subtraction, and (c) being able to carry out addition and subtraction of single-digit numbers to find the sum or difference of any given kind of multiunit. Initially, the multiunit-names and multiunit-quantities conceptual structures may support the understanding of these three components for the first three or four values or positions separately (e.g., one sees that one must add hundreds to hundreds and that if one has too many, one can trade ten hundreds for one thousand). Noticing that the structure of multiunit addition (or multiunit subtraction) is the same across several multiunits, that is, noticing the similar ten-for-one (or one-for-ten) trades and the similar addition (or subtraction) of multiunits, is a more advanced understanding. This abstraction may be facilitated by written-marks problems and by the ten-for-one and one-for-ten trades conceptual structures more than by words and the multiunit-quantities conceptual structure, because in the former the multiunits look alike, whereas in the latter they sound and look different. Emphasizing the differences across multiunits is important initially in deciding what can be added (or subtracted), but later on the generalization of multiunit addition (and subtraction) to very large problems with many positions may occur more readily by emphasizing the similarities across the multiunits (and thus looking at written-marks problems). Conversely, even raising the issue of how to add and subtract larger and larger numbers (i.e., those with more and more positions) may force (or enable) students to step back from a focus on separate individual multiunits and facilitate the construction of the regular ten-for-one and one-for-ten trades conceptual structures and a general understanding of the three components underlying multiunit addition and subtraction. Finally, reflection on inverse multidigit addition and subtraction problems may enable students to understand these as inverse operations.

THE SPECIAL DIFFICULTIES POSED BY THE IRREGULARITIES IN ENGLISH NUMBER WORDS

There are several irregularities in the English number words for two-digit numbers that obfuscate the tens and ones structure of two-digit numbers. These irregularities include:

1. The existence of the arbitrary number words *eleven* and *twelve* that do not indicate their composition as *ten* and *one,* and *ten* and *two.*
2. The irregular pronunciation of *three* in *thirteen* and *five* in *fifteen* that interferes with the "digit-teen" pattern for number words between thirteen and nineteen.
3. A reversal in the teen words that makes them opposite to the order of saying all other decades (one says *fourteen* but *twenty four*) and of writing the digits (one says *fourteen* but writes *one four:* 14).
4. Words for the one-ten decade that are different in structure from the words in the two-ten through nine-ten decades.
5. The use of two different modifications of *ten* (*-teen* in the first decade and *-ty* in successive decades), neither of which clearly says ten.
6. An irregular pronunciation of the decade words *twenty, thirty,* and *fifty* that interferes with seeing the words *two, three,* and so on being reused in the decades and, thus, masks for many children the relationship of the decade names to the first nine number words (Fuson, Richards, & Briars, 1982).

Difficulties in Learning the Number-Word Sequence and the Written Marks

These irregularities make it more difficult for English-speaking children than for children speaking a regular named-value system without these irregularities to learn the sequence of counting words, to differentiate teen and decade words, to make links between words and two-digit written marks, and to see multiunits of ten within the sequence either in the teens or in the decades. The last and the first two irregularities result in many children memorizing the English number–word sequence without seeing patterns other than the *x-ty one, x-ty two, . . . , x-ty nine* repetition within the decades; for most children there is a long period of months or even years during which they learn the teen words and then another one during which they learn the order of the decade words (Fuson et al., 1982; Siegler

& Robinson, 1982). During this extended learning period, children make more errors and more kinds of errors in saying the English sequence than do children learning the Chinese regular named-value sequence in which tens are explicitly named (Miller & Stigler, 1987).

The reversal of the ten (teen) and unit words in the teen words makes the teen words and decade words sound very much alike (*thirteen* and *thirty* or *eighteen* and *eighty*). This becomes particularly problematic when children are learning the links between the English words and two-digit written marks, especially for children who may be able to process phonetically the beginning but not the end of a given word (Kirtley, Bryant, MacLean, & Bradley, 1989). Thus, some children experience considerable interference and frequently write, for example, 80 for *eighteen* and 18 for *eighty* (Behr, 1976). This reversal in the order of the ones word and the tens word (teen) in the teens also leads children to considerable difficulty in writing two digits for the teens words, because they want to write them in the order in which they hear the words: *fourteen* as 41 or *eighteen* as 81. This tendency is reinforced by the fact that such writing in the order in which the words are said works for all decades other than the teens (e.g., *eighty-one* is written with the eight before the one: 81).

This obfuscation of the underlying tens structure in English number words results in the construction by English-speaking children of unitary conceptual structures for numbers between ten and one hundred. In these conceptual structures, numbers consist of single units. Young preschool children have separate sequence, count, and cardinal meanings for number words. The sequence meaning of twenty is as the number coming just after *nineteen* and just before *twenty-one.* The count meaning or count reference of "twenty" is to the object to which the word *twenty* is attached when counting. The cardinal meaning is "twenty" as the cardinality (numerosity) of a pile of twenty entities. Between the ages of 2 and 8, children construct increasingly complex relationships among these three kinds of meanings (see Fuson, 1988, for a more detailed treatment of these separate meanings and of the increasing integration of these meanings). Almost all first and second graders have related count and cardinal meanings and, thus, can count a group of objects to tell how many there are and can count well into the decades, so they have unitary sequence, count, and cardinal meanings for numbers well toward one hundred (Bell & Burns, 1981; Fuson, 1988; Fuson et al., 1982; Gelman & Gallistel, 1978; Ginsburg & Russell, 1981; Resnick, 1983; Siegler & Robinson, 1982; Starkey & Gelman, 1982; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Children having only these unitary structures can learn to read and write two-digit numbers (i.e., to link English words and written marks). They relate the patterns in the written digits to the patterns in the English number-word sequence below one hundred. The first digit suggests the decade name, and the second tells the number following the decade word, except for the irregular teen words as discussed earlier. Ross (1986) found that all second graders sampled from a wide range of classrooms were able to count collections of as many as 52 objects and could write the two-digit numeral that corresponded to the count, evidencing a link between the count or cardinal unitary meaning of that pile of objects and the written marks. Children at her Level 1 interpretation of two-digit numbers, however, showed no knowledge of tens and ones even as labels for the digits. More than half of the second graders and 15% of the fourth graders were at this level, indicating that they had only unitary structures. C. Kamii (1985) discussed how first graders can generate written numbers to 99 by repeating the cyclical counting order of the counting words.

Difficulties in Constructing Multiunits of Ten

Named-value Asian words support the construction of multiunit conceptual structures of tens and ones more than do English words. The named Asian ten makes it easier than in English to learn the name for the second marks position, because *shi* is used in every word above nine (i.e., it appears in 90 different number words below one hundred). This omnipresent *shi* is a constant reminder of the presence of tens within numbers between ten and one hundred. In contrast, the English word *ten* is used only once in the English words for those same 90 numbers. Furthermore, named-ten Asian words make it easier to link the written marks to any word because the pattern is the same for all words between nine and one hundred. For *one shi eight* or *six shi eight,* one just writes the two number words said in the order in which they are said and ignores the word *shi* (18 or 68), and to say any two-digit written marks, one just says the first mark, says *shi,* and says the second mark.

English words have all of the problems and special patterns already discussed with respect to the teens and to confusions between teens and decade words. The English words have three further problems with respect to decade words. First, the change in pronunciation from *two, three,* and *five* to *twen-, thir-,* and *fif-* masks the pattern *four-ty, six-ty, seven-ty, eight-ty, nine-ty* that is partially present in the English decade words. As a consequence, many children memorize a list of decade words (*twenty, thirty, forty, fifty,* etc.) to learn to count to one hundred. They then may use this list to decide what mark to write for a given English word. Behr (1976) reported examples of second graders who count through such a decade word list on their fingers and then know that they write a 6 for *sixty eight* because *sixty* is said with their sixth finger. This 6–six correspondence is given directly in the Asian words (*six shi eight*); no mapping to a decade word list is required. Second, the unitary conceptual structure elicited by the English words leads many children to write 608 for *sixty eight:* They know 60 is *sixty,* and sixty-eight is *sixty* followed by *eight* (or 60 plus 8), so 60 followed by 8 (or 60 and then 8) seems a sensible way to write sixty-eight. Finally, in named-ten regular Asian words, just counting to one hundred shows the incrementing of tens as well as the incrementing of ones. The use of *nine ten* for the ten words before one hundred (*nine ten, nine ten one, . . . , nine ten nine*) shows the composition of one hundred as ten tens (as nine tens and another ten), whereas the lack of named tens in English results instead in a unitary conception of one hundred.

Children who speak regular named-value Asian languages based on Chinese construct conceptual multiunits of tens, demonstrating both multiunit names for ten (in their named-ten number words) and conceptual multiunit quantities of ten. English-speaking children find it much more difficult to construct conceptual multiunits of ten, and instead primarily construct unitary structures for two-digit numbers. Miura (1987) found that Japanese-speaking first graders living in the San Francisco area used the tens and units in base-ten blocks to show five written numerals between 11 and 42 considerably more than did English-speaking first graders. The former showed 11 as one long and one little cube and 28 as two longs and eight little cubes; the latter made unitary count/cardinal collections of single units of eleven or twenty eight little cubes. Similar results were reported for Chinese, Japanese, and Korean children compared with U.S. children (Miura et al., 1988) and even for Japanese first graders before any work on tens compared with U.S. first graders after instruction on tens and ones (Miura & Okamoto, 1989). The failure of English-speaking children to show conceptual multiunits of ten is also reported by M. Kamii (1982), Richards and Carter (1982), and C. Kamii (1985, 1986). In the Kamii task, for example, children count out a pile of sixteen chips when shown the written marks, 16; when asked to show with the chips what this part (the 1 is circled) means, many children show only one chip rather than ten chips. For them the 1 is a mark that "teens" the 6 (children I have interviewed have said exactly this), but it does not mean one ten. The strength and persistence of this unitary view of number even in U.S. adults are indicated by the difficulty adults had focusing on just the tens digit in a two-digit number, even when doing so would have helped them in a task (Heinrichs, Yurko, & Hu, 1981).

Difficulties in Using Multiunits of Ten in Single-Digit Addition and Subtraction

This difference in English and Asian conceptual structures for numbers between 10 and 20 leads to different methods of addition and subtraction of single-digit numbers with sums between 10 and 18. Children in the United States invent a whole developmental sequence of unitary conceptual structures used to add and subtract single-digit numbers (see Fuson, in press-b, for a review), whereas Asian children learn addition and subtraction methods structured around ten. The use of the unitary conceptual structures becomes highly automatized in U.S. first and second graders and interferes with their construction and use of multiunits of ten. The unitary solution procedures begin with adding two numbers by counting all: counting out objects for each number and then counting all the objects to find the sum. This is sufficient to carry out addition and subtraction up through two-digit combinations, first by counting objects and later by counting forward and backward by ones within the sequence (e.g., Baroody & Ginsburg, 1986; Fuson, 1988; Steffe et al., 1983). Many first and second graders then go on to relate count/cardinal and sequence number-word meanings, so that the sequence words themselves become the objects that are counted, and the first addend becomes embedded within the sum, and its counting can be abbreviated (Baroody, 1987; Baroody & Ginsburg, 1986; Carpenter & Moser, 1983, 1984; Fuson, 1988; Fuson et al., 1982; Steffe & Cobb, 1988; Steffe et al., 1983). Addition situations now can be solved by counting on—beginning the final sum count at one of the addends. Thus, 5 + 7 would now be solved more efficiently by counting on 7 words past 5 (counting on from first) or by counting on 5 words past 7 (counting on from larger) rather than counting all twelve words. Some method of keeping track of the number of counted on words has to be used in this unitary sequence solution procedure. For example, 9 of 14 second graders interviewed early in the school year by Cobb and Wheatley (1988) counted on by ones, but several had difficulty in keeping track of the counting on of the second addend.

Subtraction is initially carried out by U.S. children using a unitary count/cardinal structure. Children separate from: They count out 12 objects, pull away 7 of these, and count the remaining objects (Carpenter & Moser, 1983, 1984). With the embedded unitary sequence-count-cardinal conceptual structure that comes later, children can count down from (count down 7 words from 12 to find out how many words are left), count down to (count down from 12 to 7 to find out how many words are in between), or count up to (count up from 7 to 12 to find out how many words are in between) (e.g., Carpenter & Moser, 1983, 1984; Fuson, 1988; Resnick, 1983; Steffe & Cobb, 1988; Steffe et al., 1983).

Children in the United States then go on to derived-fact addition and subtraction procedures in which a needed sum or difference is derived from a known sum or difference by relating the addends and the sums (e.g., Carpenter & Moser, 1983, 1984). These procedures involve chunking the addends and sums in different ways. "Doubles + 1" is a frequent derived-fact procedure: 7 + 6 is related to the well-known double 6 + 6 = 12 and is seen to be one more than 12 (6 + 7 = 13), because 7 is one more than 6. Only one of the several derived-fact procedures involves chunks of ten. This is the over-tens method in which one number is broken into the part that makes ten with the other number and the part that is over ten. The combination seven plus five is thought of as "seven plus three from the five is ten, plus the two left over from the five is ten plus two, which is twelve."

Asian children learn addition and especially subtraction through 18 much earlier than do children in the United States (Fuson et al., 1988; Song & Ginsburg, 1987; Stigler et al., 1990). The over-ten method for addition is taught in mainland China, Japan, Korea, and Taiwan to first graders. This method is easier in these countries than in English because the numbers over ten are said as *ten* and any leftover ones. The sum of seven plus five is *ten two,* so Asian children have only to find the part that is leftover after making the ten and then say *ten left-over-part.* In English the ten plus x sums have to be learned rather than being given in the counting sequence; many U.S. first and even second graders do not know these sums and count up from ten to find out how many are "ten plus two" or "ten plus five" (e.g., Steinberg, 1984). U.S. children also commonly lack another prerequisite for the over-tens method: There is much less emphasis in the United States on number pairs that make ten; therefore, many first graders have to count to find out how many to put with a given number to make ten. In Korea, children also learn methods of folding and unfolding fingers that support this over-ten method, whereas U.S. children learn finger patterns that support the unitary structures and make it relatively difficult to add sums over ten on the fingers (Fuson & Kwon, in press). Korean children unfold fingers to show the first number to be added, unfold successive fingers to show the second addend, and fold fingers down again if the sum is over ten.[3] So for 7 + 5 they would unfold 7 fingers, unfold the remaining 3 fingers and fold 2 fingers to make the 5 more fingers; these fingers then clearly show both how many more to make ten (7 plus how many fingers to make all ten fingers unfolded) and show the answer in Korean number words: ten two (the ten fingers unfolded and the two fingers folded). Children in the United States do not have a culturally supported way to reuse fingers, so they have difficulties when the sum exceeds ten (e.g., Steinberg, 1984).

Asian children are taught two different subtraction methods that are structured around ten (Fuson & Kwon, 1990a, 1990b). One is the reverse of the over-ten addition method. In this down-over-ten procedure, one splits the known addend into the part over ten in the sum and a leftover part, which is then subtracted from ten to give the unknown addend: For 12 – 5 (ten two minus five), the 5 is split into two (the part over ten) and the leftover part, three; the three is then subtracted from ten to give seven. In the subtract-from-ten method, the known addend is subtracted from the ten part of the sum, and the difference is added to the part over ten to make the unknown addend: For 12 – 5 (ten two minus five), the five is subtracted from ten (ten – five, or five + how much is ten) to leave five, which is then

[3]Some children begin with open fingers, fold them to show the addends, and then unfold fingers over ten.

added to the two to make seven. This method essentially turns subtraction into addition. Both subtraction methods require knowing complements to ten (the number that will make ten with any given number) and are supported by the Asian named ten: The ten is named and provides a partial sum around which the addition or subtraction can be oriented, and the part over ten is named and does not have to be counted to find the teen name as it does for many children in English.

Difficulties in Adding and Subtracting Two-Digit Numbers

When Asian children first face addition of two-digit numbers, the named ten in their number words supports all three major components of multiunit addition: (a) knowing that one adds like multiunits, (b) trading when one has too many of a given multiunit, and (c) expressing single-digit sums as a ten and ones to facilitate trading. Reading the problem 38 + 45 as "three ten eight plus four ten five" suggests what one needs to do to add these numbers: One needs to add the tens together and add the units together. This strategy would be suggested just by a multiunit-names conceptual structure but is directed even more strongly by a multiunit-quantity conceptual structure of tens and ones, which Miura's research suggests many Asian-language children have. Trading when one has too many is also suggested because the sum itself contains a ten: In this example, 8 + 5 is ten three (*shi* three). This very name suggests what to do—put that ten with the other tens—giving eight tens altogether for a sum of eight ten three (83). The notion of trading is further supported by the method of adding sums over ten: The over-ten method actually partitions the units so that they form a ten and some ones during the addition procedure. So the "ten-ness" (the multiunit quantity of ten) is supported by the addition procedure and is not just limited to the name of the sum.

With the named-ten words, adding multiunits of ten is just a straightforward extension of adding single units. In the earlier example (38 + 45), the sum of the tens before the trade is easily seen to be three plus four is seven tens. When the sum exceeds ten tens, children have two options. They can operate within a multiunit-name or multiunit-quantity conceptual structure and use an analog of the over-ten method to find the sum of the tens. So, for example, for 82 + 65, a child might think "eight *shi* plus six *shi* is eight *shi* plus two of the *shi* from the six *shi* gives one *bai* and the rest of the six *shi*—the four *shi*—gives one *bai* four *shi* (one hundred four ten) so the sum would be one *bai* four *shi* seven. The other alternative is to do the tens sum ignoring the particular multiunit involved until the very end. This would give thinking such as "eight plus six is eight plus two from the six is *shi* plus the four leftover is *shi* four and *shi* four *shi* are one *bai* four *shi.*" It is also possible to carry out the addition of the tens numbers with no use of the fact that these numbers refer to multiunits of ten. This may be particularly likely if one just looks at the written marks—which do not name or otherwise show the multiunit of ten but instead look just like the single units in the ones position—and does not think of the words for the marks. In this case, one would just use the over-ten method for the numbers written in the tens position; this would sound like the second example without the last nine words—The sum would just be *shi* four. The named ten in this sum, however, suggests what to do to avoid writing two digits in a single place. If one is thinking of the tens position as a units position, then the position to the left would be the tens position, so one would put the ten as one ten in the next position to the left. Thus, even this single-digit approach could yield the correct procedure. At present, it is not clear how much each of these alternatives is supported in schools or used by children in Asian countries. In interviews with Korean second and third graders after the second graders had learned two-digit sums with trades from the ones to the tens but had not learned to trade from the tens to the hundreds, some second graders discussed the sums using the beginning and the end of the first and third methods to explain addition in three-digit problems involving a trade from tens to the hundreds (Fuson & Kwon, 1990b). The children's actual solution procedures for finding sums over one hundred were not evident from their verbalizations, and they were not asked about these procedures. But some children did use multiunit quantities and some used regular ten-for-one trades conceptual structures in discussing their solutions. Most of the second graders solved such problems correctly, indicating a robust support from their available conceptual structures for solving such problems.

The situation for English-speaking children trying to add two-digit numbers is much more complex. A considerable proportion of U.S. second graders do not even have a multiunit-names conceptual structure for tens (e.g., they cannot say that the second position is the tens place) (Ross, 1986), and few have a multiunit-quantity conceptual structure for ten (C. Kamii, 1986, 1989; C. Kamii & Joseph, 1988; M. Kamii, 1982; see also Labinowicz, 1985). They all do have available unitary structures, so they can add two-digit numbers by making a pile of objects for each addend and then counting all of the objects or by counting on by one from one of the numbers (e.g., for 37 + 25 counting on 25 words from 37: 37, 38, 39, . . . , 62). The power of these unitary structures may be so strong that they can even interfere with structural materials intended to help children construct and use multiunits of ten in addition (i.e., materials showing collectible tens). For example, Madell (1985) reported that 5- and 6-year-olds using base-ten blocks (longs and units) to solve addition word problems requiring two-digit sums were reluctant to combine the tens. They counted by tens (used a decade word list: *ten, twenty, thirty, forty,* etc.) to find the name of a collection of blocks (e.g., counting "ten, twenty, thirty, forty, fifty, fifty one, fifty two, . . . , fifty nine" to find the name for 5 longs and 9 little cubes). To add 35 and 24, however, they counted by ones from 35: They counted each of the ones on the 2 tens in 24 and then counted the 4 ones. Numbers in the teens were particularly difficult to deal with as tens and ones, and children frequently were 8 or even 9 years old before they did so. Before this time, they used unitary procedures. For example, they showed 48 – 14 as 4 longs and 8 units but did not subtract 14 as 1 long and 4 units. Instead they traded one long for ten units so that they had a unitary presentation of 14 as fourteen units and then subtracted the fourteen units.

This fixation on unitary conceptual structures interferes with the construction of multiunits of ten and, thus, has three negative consequences:

1. Children do not see that they need to combine like multiunits (because they do not have conceptual multiunits of ten).
2. They do not see what they need to do when they get too many ones.
3. Even if they do come to understand these aspects at some level, they must switch back and forth between a unitary conceptual structure for finding the single-digit sum of a given multiunit and a multiunit conceptual structure for understanding the combination of like multiunits and of trading for too many.

Trading requires that a unitary sum be changed into a multiunit sum so that the multiunit can be traded (e.g., fourteen must be conceptualized as one ten four ones. Many first and second graders do not automatically know these ten-structured multiunit conceptions for the teen words and have to count on their fingers to find how many tens and ones are made from fourteen (Madell, 1985). Even when English-speaking children get better at knowing the tens and ones in single-digit sums between eleven and eighteen, in most cases they must still switch back and forth between the unitary procedure with which they find the sum of the single-digit numbers for a given multiunit and the multiunit conception that directs their trading.

Sequence ten-units. Some English-speaking children eventually begin to construct sequence/counting multiunit items of ten within the unitary sequence structure (Cobb & Wheatley, 1988; C. Kamii, 1985, 1986; Resnick, 1983; Richards & Carter, 1982; Steffe & Cobb, 1988; Steffe & von Glasersfeld, 1983; Thompson, 1982). These sequence ten-unit items permit children to count all, count on, count down from, count down to, and count up to by tens and ones. The earliest procedures use the decade word list to count by tens; all tens are counted on this decade word list, and then all the ones are counted. For example, 35 + 47 could now be solved by counting all by tens and ones: "10, 20, 30, 40, 50, 60, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82" or by counting on from 30 by tens and ones: "30, 40, 50, 60, 70, 75, 76, 77, 78, 79, 80, 81, 82." It is more difficult to count by tens from words not in the decade list, but eventually children may count on by tens and ones from one of the addends: "35, 45, 55, 65, 75, 76, 77, 78, 79, 80, 81, 82." Under usual classroom conditions, these ten-unit items take considerable time to construct and seem initially to require perceptual support of physical collectible multiunits of ten (objects grouped into tens). The previous examples were taken from Labinowicz (1985) and were carried out by 23 of 29 third graders using base-ten blocks to show both 35 and 47. The remaining 6 third graders were not even able to count all by tens and ones when using the blocks; they counted by ones. When blocks were available only for one number (46 was given in digits and 57 was given in blocks), only 13 children were able to count on by tens and ones from 46. The others counted on by ones, evidently requiring the support of the tens blocks for both numbers to count on by tens and ones.

Considerable other evidence describes the many difficulties English-speaking children have in constructing and using sequence ten-units for addition of two-digit numbers. Steffe and Cobb (1988) outlined six kinds of increasingly complex conceptual ten-units that second-grade children construct within the number–word sequence over a period of many months and even years. They carefully described children's use of these ten-units in various counting tasks. These successively more abstract ten-units culminate in counting by tens without the perceptual support of materials (they call this use of iterable ten-unit items); this level was not reached by some of the second graders in their study, even at the end of the 2-year teaching experiment facilitating such constructions. Cobb and Wheatley (1988) found that only 3 of 14 second graders interviewed early in the year could use iterable ten-unit items on any tasks. Three others could use sequence ten-unit items in counting tasks with the support of collectible ten-unit items; the remaining 8 showed only a labeling use of ten similar to that identified as Level 2 by Ross (1986). C. Kamii (1986) reported that no Genevan first through third graders spontaneously counted a large number of objects by ten, and only 14% and 5% of fourth and fifth graders, respectively, did so. Even imitating an experimenter solving two-digit addition problems by counting on by tens was beyond the capacity of most first graders who counted on by ones (unpublished data from the counting-on subjects in Secada, Fuson, & Hall, 1983). Case studies in Behr (1976) document the slowness and complexity of second graders' construction of sequence ten-units. Children in various groups used different materials to support this construction (base-ten longs, bundled sticks, bead abacus). Many activities required them to write a number in three forms: (a) multiunit name (3 tens and 7 ones), (b) sequence decade form (30 + 7 ones), and (c) written marks (37). Children showed many confusions between the first two forms.

There is little evidence about U.S. children's spontaneous solution procedures for two-digit subtraction problems. Madell (1985) described procedures requiring known decade facts (e.g., knowing that 50 – 30 is 20) and combining such known decade facts with counting down by ones or subtracting ones from the known decade fact. C. Kamii (1989) reported that two-digit subtraction requiring trades was difficult for children in a Piagetian second-grade classroom focused on supporting children's mathematical thinking; she suggested that it be postponed and multiplication (an additive procedure) be focused on instead. Because subtraction in U.S. classrooms is usually interpreted as take-away, and take-away leads to counting-down procedures, the expected subtraction sequence multiunit solution procedures would involve counting down by tens and by ones. Because counting down is considerably more difficult for most children than is counting up, such sequence ten-unit procedures would be slower to appear even than the addition ten-unit sequence procedures. Thus, the evidence at present indicates that the use of sequence ten-unit solution procedures is somewhat difficult for two-digit addition but is really formidable for subtraction.

Difficulties in Adding and Subtracting Three-Digit and Four-Digit Numbers

English words describe the values for the third and fourth positions in a totally regular way. Thus, these named hundred and thousand values can provide the same kind of support to the three components of multiunit addition and subtraction that are provided by the named tens in Asian words. Four hundred plus three hundred can be easily seen to be seven hundred; two thousand plus six thousand can be easily seen to be eight thousand. As with Asian named tens, trading for multiunit sums above ten can be supported at least minimally by the multiunit-names conceptual structure and supported considerably more by the multiunit-quantities and regular ten-for-one and one-for-ten trades conceptual structures. Thus, providing English-speaking children with physical collectible hundred and thousand multiunits to use in addition and subtraction situations could make their task of understanding multiunit addition and subtraction for these values as simple as are addition and subtraction of tens for Asian children.

Unfortunately, the irregular and unnamed English tens create such barriers to understanding two-digit addition and subtraction that U.S. children are often not even given an opportunity to see problems with thousands until 2 or 3 years after they first see two-digit problems without trading (Fuson, in press-a; Fuson et al., 1988). Thus, they cannot take advantage of the potential support of the English regular named hundreds and thousands. Furthermore, the extensive experience with two-digit problems with their unnamed and irregular tens frequently leads to the construction of two different conceptualizations of multidigit numbers that interfere with the construction of collected multiunit quantities that are just added or subtracted and traded if necessary. These conceptualizations are sequence multiunit conceptual structures and concatenated single-digit conceptual structures. Children may have both of these, using the former for horizontal problems and the latter for problems written with positions aligned vertically (Cobb & Wheatley, 1988). The sequence-multiunit conceptual structures involve counting forward or backward by chunks (sequence multiunits) within the sequence, as in the counting by tens and ones procedures discussed earlier for two-digit problems. The concatenated single-digit conceptual structure uses only the visual layout conceptual structure in Table 2; the phrase "concatenated single digit" specifies that no other conceptual structure is being used with the visual layout structure, so that each written mark is taken just as it appears—as a single digit written next to other single digits.

Sequence-multiunit conceptual structures. Quantitative sequence-multiunit conceptual structures for three- and four-digit problems require sequence pattern skills of skip-counting by tens, hundreds, and thousands, and they require that these skills be connected to cardinal collected multiunits of tens, hundreds, and thousands so that counting by tens actually means to the counter that the quantity is being increased by ten with each count. Sequence pattern skills can be learned by rote just based on the sound patterns in the sequence. Thus, a child can learn the decade word list (*ten, twenty, thirty, forty,* etc.) as a memorized list without any understanding that the words refer to one ten, two tens, three tens, and so on, or that each word is a quantity ten more than the previous quantity. Similarly, children can learn to say patterns such as "fourteen, twenty four, thirty four, forty four," and so on, without seeing the tens and ones in this pattern (e.g., Thompson, 1982). It is not until such auditory counting patterns are related to cardinal quantities of tens and ones that they become multiunit conceptual structures that give quantitative meaning to multiunit words and written marks. In this way, sequence-multiunit conceptual structures are analogous to the unitary sequence solution procedures invented by most U.S. children. The sequence pattern skills of counting up or down beginning with an arbitrary number word, or of counting up or down a given number of words, may appear much earlier than quantitative sequence counting on and counting back that is connected to cardinal situations and can be used to solve cardinal addition and subtraction situations (Fuson, 1988; Fuson et al., 1982; Steffe et al., 1983).

In regular named-value Asian number words, it is difficult to separate sequence counting (counting by chunks of ten) and counting of collected multiunits (counting of tens) because the words collapse these: An Asian child looking at the physical collectible multiunits in base-ten longs and saying "one ten, two ten, three ten, four ten, five ten, six ten" may be saying the sequence via chunks of ten (ignoring the *one ten one, one ten two,* and so on, sequence words between the tens) or may be counting collected multiunits of cardinal tens. In English, these methods are differentiated by using the decade word list for sequence counting (*ten, twenty, thirty, forty, fifty, sixty*) and by counting the multiunits of ten for collected multiunits (*one ten, two ten, three ten, four ten, five ten, six ten or one, two, three, four, five, six tens*). In actual practice, Asian children who are adding two-digit numbers probably rarely use either procedure because by the time they face such problems they know most sums to ten. Thus, a problem such as 35 + 32 is just seen to be 3 *shi* plus 3 *shi* is 6 *shi,* using their knowledge of 3 + 3. English-speaking children could use collected multiunits of ten to solve the problem in the same way (3 tens plus 3 tens is 6 tens). However, their extensive use of unitary sequence solution procedures leads many of them to the sequence ten-unit procedures of counting by tens (*ten, twenty, thirty, forty, fifty, sixty,* or *thirty, forty, fifty, sixty*) rather than seeing thirty plus thirty as 3 tens plus 3 tens. Cobb and Wheatley (1988) reported that some second graders used each of these methods (sequence multiunits of ten and collected multiunits of ten) to solve two-digit problems written horizontally. Thompson (1982) found that, when children were provided with physical collectible multiunits of ten (base-ten block longs), they could do tasks using collected multiunits before they did similar tasks using sequence multiunits.

Children's use of sequence ten-units to solve problems with two-digit numbers leads many researchers to focus on the extension of sequence multiunit procedures to the hundreds and thousands as the meaningful way for children to understand three-digit and four-digit addition and subtraction. In some treatments, such counting skills are assumed to be required for comprehension of three-digit and four-digit addition and subtraction and of place value (e.g., Labinowicz, 1985; Resnick, 1983). Conceptual sequence hundred units enable children to solve three-digit sums and differences by counting on, counting down from, counting down to, and counting up to by hundreds, tens, and ones. Thus, 527 + 435 could be solved by counting on the hundreds and then the tens and then the ones (500, 600, 700, 800, 900, 920, 930, 940, 950, 957, 958, 959, 960, 961, 962) or by counting on by hundreds, tens, and ones from 527 (527, 627, 727, 827, 927, 937, 947, 957, 958, 959, 960, 961, 962). However, such counting is evidently quite difficult for many second and third graders (Labinowicz, 1985; Resnick, 1983; Thompson, 1982). They experience considerable difficulty changing from the hundred-unit items to the ten-unit items to the unit items. More than half the third graders interviewed in one study had difficulty making such unit changes even when they were counting these units only for a single multidigit number and had physical collectible multiunits in base-ten blocks to support such counting (Resnick, 1983). They made mistakes such as counting the ten units as hundreds (counting 6 hundreds and 5 tens as "one hundred, two hundred, . . . , eleven hundred") or as ones (counting 2 hundreds 7 tens 4 ones as "one hundred, two hundred, two hundred one, two hundred two, . . . , two hundred eleven"). There are few data on subtracting three-digit numbers by counting with sequence hundred-unit, ten-unit, and unit items, but such counting down from (or down to or up to) would be at least as difficult as counting on and possibly considerably more difficult for the procedures involving counting backward. One could also construct sequence thousand-unit items and use them in counting procedures to solve four-digit addition and subtraction problems. These, of course, would be even more difficult than the three-digit problems, for they would require yet another unit change and keeping track of four rather than three different kinds of multiunits.

With these conceptual sequence multiunits, addition or subtraction situations in which the sum for a given multiunit exceeds nine require that children be able to count over the transition point to the next higher sequence multiunit. Thus, a child must count by ones over a decade to solve a sum with more than nine ones, count by tens over a hundred to solve a sum with more than nine tens, and count by hundreds over a thousand to solve a number with more than nine hundreds. Such counting seems to be more difficult than counting by ones, tens, or hundreds without such changes, because children's counting is less automatized at these transition points (Fuson et al., 1982; Miller & Stigler, 1987; Siegler & Robinson, 1982) and because counting over a hundred or a thousand introduces a new unit that is especially salient (and thus may be distracting) because it is said first each time in the counting (Labinowicz, 1985; Miller & Stigler, 1987). Thus, for such problems, children may make errors at transition points such as "one hundred nine, two hundred, three hundred, . . ." or ". . . , one hundred nine, ten hundred." Bell and Burns (1981) reported that 9 of 30 beginning third graders could not count by tens from 180 to 210. Labinowicz (1985) found that 20 of 29 beginning third graders made errors in counting between 94 and 124 when counting by tens. Combining all three units (ones, tens, hundreds) may be particularly difficult for speakers of irregular English number words: 8 of 13 U.S. 6-year-olds who counted above one hundred counted by hundreds after 109 ("one hundred nine, two hundred"), whereas no Chinese 6-year-old did (Miller & Stigler, 1987). Negotiating these transition points is more difficult in English than in the Asian languages, because the latter have clearer transition points because the composition of numbers in the multiunits of hundreds, tens, and ones is always named and one cannot say values over ten. In English, children can easily say "ten hundred, eleven hundred, twelve hundred," and so on, and even be correct in such counting. The extensive use by adults as well as by children of unitary conceptions for two-digit numbers creates many nonstandard usages such as "nineteen eighteen" for 1918 Orrington Avenue, a street address, or "twenty-five hundred" for 2500.

Use of a named-value size embodiment that shows physical collectible multiunits of ones, tens, and hundreds (e.g., base-ten blocks, bundled sticks) seems to facilitate children's counting by sequence ten-unit and hundred-unit items. Children were more successful in counting by ones, tens, and hundreds with base-ten blocks than without them, but second and third graders still made many unit change errors (Labinowicz, 1985; Thompson, 1982). The different sizes seem to help children keep track of the multiunit item with which the counting is being done, and the embodiment pieces keep track of how many multiunits have been counted on, up, or down as the pieces are counted. Labinowicz (1989) described many difficulties third graders had in finding the number of tens in larger numbers. Many of these difficulties involve confusion between a sequence multiunit conceptual structure and a collected multiunit conceptual structure; children combined sequence and collected multiunit structures for different parts of the multidigit number, finding the number of tens in one multiunit value but not in the other. For example, in answering how many tens are in 132, children said that there were forty tens (adding the collected "ten tens" in one hundred to the sequence word *thirty*) or that there were one hundred three tens (adding the sequence word *one hundred* to the collected "three tens" in thirty two). Labinowicz found to be helpful a 15-min exercise with base-ten blocks on hundreds grids in which children coordinated their counting by tens (use of a multiunit sequence structure) with their counting of tens (use of collected multiunits).

Most of the literature concerning child-invented algorithms for solving multidigit addition and subtraction problems involves a sequence multiunit conceptual structure (see Labinowicz, 1985, chapter 14, for a review of this literature; see also Cobb & Wheatley, 1988). Some of these methods involve the separation of the multidigit numbers into their decade parts and ones parts so that 54 + 28, for example, is solved as "50, 60, 70, 78, 79, 80, 81, 82." The counting may be done from the larger tens and the larger ones, which may involve commuting the sequence decade and unit values for the two numbers (if these larger numbers are not in the same two-digit number). Most of the examples in the literature of child-invented algorithms are limited to two-digit sums and differences (i.e., to sequence ten-units). There is little evidence concerning how easy it is for children to construct sequence hundred-units or thousand-units either spontaneously or with teacher or other-child support (except for the evidence already presented concerning problems children have at transition points) or concerning how difficult it is to use such sequence multiunits in invented or learned solutions for larger multidigit sums and differences. It seems to be much easier to invent addition procedures than subtraction procedures (see C. Kamii, 1989, concerning relative numbers of children doing so for addition and subtraction in a Piagetian classroom).

Some anecdotal evidence suggests that, when children who have sequence ten-unit structures for two-digit numbers begin to try to add and subtract three- or four-digit numbers, they may construct and use an amalgamated sequence/collected multiunit structure in which they use collected multiunits for the hundreds and sequence multiunits for the tens. Evidently, the regular English-named hundreds facilitate the construction of the collected hundred units. Some children in first- and second-grade classrooms in the Cognitively Guided Instruction Project (CGIP; see Carpenter, Fennema, Peterson, Chiang, & Loef, 1989, for a description of the project) carried out mental solutions for three-digit addition problems such as 486 + 379 by adding like values to find sums: "Four hundred and three hundred is seven hundred. Eighty and seventy is one hundred fifty so that's eight hundred fifty. Nine and six is fifteen so that is eight hundred sixty five." (Deborah Carey, personal communication, November 4, 1988). Addition of the tens did not use collected ten-units: These children did not say "eight tens and seven tens is fifteen tens." Such collected ten-units were used by U.S. second graders who had used base-ten blocks to add and subtract four-digit numbers; they explained such a problem by saying "eight tens plus seven tens is fifteen tens; I have to trade ten of those tens for one hundred, so that is one hundred and five tens" (Fuson & Briars, 1990). How the children in the CGIP classrooms knew that "eighty and seventy is one hundred fifty" is not clear. Such knowledge could conceivably derive from sequence ten-units in two ways: It could be a learned fact from earlier counting on by tens, or it could come from use of a thinking strategy such as "eighty plus twenty from the seventy is one hundred, and fifty is left from the seventy, so the sum is one hundred fifty." If the written marks/decade words associations were strong, the sum could also come from the visual pattern of written numerals "7 + 8 is 15, so 70 + 80 is 150." C. Kamii (personal communication, March 30, 1989) also reported that some second graders in a Piagetian classroom solved three-digit addition problems by collected multiunit addition of the hundreds (as in the previous example); these children used various means of adding the decade words. Thus, it may be that collected multiunits for hundreds and even thousands are fairly easy for some U.S. children to construct, even if they have used sequence multiunits for the decades.

The concatenated single-digit conceptual structure. Written marks for multiunit numbers are seductively like those for single-digit numbers. Multidigit number marks look as though they are concatenated single digit (CSD) numbers—single-digit numbers placed side by side. A 4, for example, looks the same whether it is in the ones, tens, or hundreds place (314 or 341 or 431). Evidently, school instruction for many children does not enable them to construct the conceptual structures below the second row in Table 2 or does not ensure that these conceptual structures are connected to the written marks and used when children are adding and subtracting multiunit numbers. The school instructional focus, as judged by textbooks and by children's performance, seems, rather, to be on procedural rules that dictate what one does to the written marks. Children make a range of errors—violations of the correct procedural rules—that are consistent with an interpretation of multidigit numbers as CSD numbers (i.e., as a visual layout of single-digit numbers unconnected to any other conceptual structure). The correct procedural rules cannot be derived just from a CSD conceptual structure. Further rules are required to constrain which digits are added to (or subtracted from) which digits. These rote procedural rules are given in Table 4, which also contains many partially correct but incomplete rules. These are not all possible rules used by children, but these rules do generate the common errors and several infrequent errors made by U.S. children. All these errors are consistent with an interpretation of written marks as CSDs.

The CSD structure is not just involved in addition and subtraction errors. It is used by many U.S. children on place-value tasks, and many children who calculate correctly also seem to use only a CSD conceptual structure along with the correct procedural rules in Table 4. Evidence concerning the widespread use of the CSD structure by U.S. children and the errors that result from the partially correct rules in Table 4 are discussed in the rest of this section.

Children indicate use of the CSD structure in several different place-value tasks. When shown, for example, the numeral *16* and sixteen objects and asked to show in the objects the *6,* the *1,* and the *16,* many elementary school children indicate six objects for the *6,* one object for the *1,* and all sixteen objects for the *16* (Behr, 1976; C. Kamii, 1985; M. Kamii, 1981; Ross, 1986, 1989; see also discussion in Labinowicz, 1985). For example, in a middle-class suburb of Chicago, the percentages of children doing so were 100% of the first-grade children after place-value instruction was completed, 49% of the fourth graders, 40% of the sixth graders, and 22% of the eighth graders (C. Kamii, 1985; C. Kamii & Joseph, 1988). When asked to read a three-digit number and then write the number that was one more than the given number, half the third graders studied gave answers reflecting a CSD structure. They increased by one the digit in one or more places other than the ones place, giving for 342 the answers 1342, 453, 442, or 352 (Labinowicz, 1985). Children also seem to use a CSD structure to decide which of two multidigit numbers is larger: 11 of 20 third graders sometimes ignored the position of digits and focused on a single digit in one number as being larger than a single digit in another number, choosing 198 as being larger than 231 (Labinowicz, 1985). Some were not able to justify their choices, but others did so with responses such as "The 9 is bigger than the 1" or "The 98 is higher than 23 or 31." Ginsburg's (1977) Stage 1 for children's understanding of written number (in which they exhibit no verbalizable meaning for the digits) fits the CSD structure. In the example protocol, the child says about the 123 he or she has just written for *one hundred twenty three,* the "1 is just 1, the 2 is just 2, and the 3 is just 3" (p. 86).

Ross (1986, 1988, 1989) found two early levels of place-value knowledge about two-digit numbers that reflect a CSD structure. In the first level, children are able to identify one digit as the tens and the other digit as the ones, but no quantities are associated with the tens or ones (they are just verbal labels), and many children at this level reversed the tens and ones. Ross noted that this nonquantitative labeling level was sufficient for these children to succeed on many school place-value tasks. She also identified a more advanced level in which children seem to possess quantitative meaning for the tens and ones (they relate object subgroupings of tens and of ones to two-digit numbers) but in which they actually are just relating digit values or digit positions to available groupings without regard for the size of the group (they will say that the 6 in 26 refers to six groups of four and the 2 refers to the two left-over objects if presented with this non-base-ten grouping of 26 objects). Bednarz and Janvier (1982) reported "digit-by-digit" strategies that ignored the values of the digits; these were used by many French-speaking Canadian third and fourth graders in two tasks in which three-digit numbers with certain properties (e.g., greater than a given number) were to be made.

Use of the CSD structure and lack of knowledge of rote rules may not be evident if children are given multidigit addition problems written vertically with like relative positions aligned and if the sum for each column does not exceed ten. Children add the digits in each column and write each sum in the space below the column. Errors appear if the problem is written horizontally, if the columns are aligned incorrectly, or if the multidigit numbers have different numbers of digits, because these situations all require knowledge that like multiunits (i.e., like positions relative to the rightmost position) must be added together. If this conceptually based understanding is lacking (i.e., if understanding of Component 1 in Table 3 is missing), correct vertical alignment of the multiunit numbers substitutes for this knowledge by placing like multiunits under each other so that the correct digits will be added to each other. Rote rule versions of vertical alignment are under Rule 1 in Table 4. The CSD structure by itself does not contain any information to direct the correct alignment or to correct an incorrect alignment. If asked to add two multidigit numbers written horizontally, children may not even keep the digits in each given number together (Rule 1a). A substantial number of second graders in Fuson and Briars's (1990) study, when asked on the pretest to rewrite the problem 67 + 1385 so that it would be easy to add these two numbers, dismembered the original numbers and rewrote vertically all six digits or wrote 671 above 358 or wrote 67 above 13 above 85. Children do seem to learn Rule 1b fairly readily. Children in that study rarely wrote the problem so that the 1 and the 5 stuck out on both sides of the 67, and no reports of such rewriting were found in the literature reviewed for this article. The correct Rule 1c, however, is much more difficult to learn and is violated by many children; the fact that multidigit numbers are written from left to right seems to lead many children to align the numbers on the left rather than on the right. The majority of the second graders in two different samples rewrote 67 + 1385 aligning them on

TABLE 4
Rules for Adding and Subtracting CSD Numbers

Number	*Rule Statement*
	Addition Rules
Rule 1	Write the numbers vertically, one above the other.
a	Do not break apart or reorder the single-digit numbers.
b	Align multidigit numbers on one side.
c*	Align multidigit numbers on the right.
Rule 2	Add the single digits in each vertical column.
Rule 3	If the sum is ten or more:
a	Do not write both digits of the two-digit sum.
b	The 1 from the two-digit sum must be written somewhere.
c	The 1 from the two-digit sum must be written in (added to) the adjacent column.
d*	The 1 from the two-digit sum must be written in (added to) the adjacent column to the left.
e*	If the sum exceeds 19, the tens digit is written.
f*	In the long addition algorithm, the rightmost digit in the two-digit sum is aligned under the added column and the leftmost digit is written in the next left position.
Rule 4	A blank digit (a blank in a vertical column)
a*	should be considered to be a 0.
b*	$x + 0 = x$.
	Subtraction Rules
Rule 1	Write the numbers vertically, with the bigger number on top.
a	Do not break apart or reorder the single-digit numbers.
b	Align multidigit numbers on one side.
c*	Align multidigit numbers on the right.
Rule 2a	Subtract the single digits in each vertical column.
Rule 2b*	Subtract the bottom number from the top number in each vertical column.
Rule 3	If the bottom number exceeds the top number:
a	You cannot do that subtraction (in whole numbers).
b	You must put more in the top number so that it will $\geq$ the bottom number.
c	You get more by taking what you need from another column.
d	You get more by taking one from another column and writing 1 in the needed column as a two-digit number.
e	You get more by taking one from the adjacent column and writing 1 in the needed column as a two-digit number.
f*	You get more by taking one from the adjacent column to the left and writing 1 in the needed column as a two-digit number.
g	If the adjacent column to the left is a zero, you move to the first nonzero column on the left and take one from it and write 1 in the needed column as a two-digit number and you also write a 1 in each zero column.
h	If the adjacent column to the left is a zero, you move to the first nonzero column on the left and take one from it and write 1 in the needed column as a two-digit number.

(Continued)

TABLE 4 *(Continued)*

i*	If the adjacent column to the left is a zero, you move to the first nonzero column on the left and take one from it and write 1 as a two-digit number in the adjacent column to the right (continue using this rule until 1 is written in the needed column).
Rule 4	A blank digit (a blank in a vertical column)
a*	should be considered to be a 0.
b*	$x - 0 = x$.

Note. Asterisk by rule number indicates correct procedural rule.

the left (Fuson & Briars, 1990). Labinowicz (1985) found that only 5 of 21 third graders identified numbers aligned on the left as misaligned and then either rewrote them or added them mentally. Ginsburg (1977) reported that many interviewed children wrote down numbers aligned on the left. Tougher (1981) found that less than half of a class of third graders aligned numbers correctly on the right when they wrote down dictated problems for addition. Thus, Rule 1c remains a considerable stumbling block for many U.S. children.

Rule 2 seems to be learned fairly readily; U.S. first graders are fairly successful at adding two-digit numbers with no multiunits over ten. Rule 2 may be relatively easy to learn, because it is supported by the physical appearance of the vertical alignment of single digits. However, some children experience difficulty even with Rule 2. Behr (1976) described one second grader who insisted on adding all the single digits in both two-digit numbers to find the sum (e.g., 31 + 42 = 10).

Rote rule solutions to problems in which the sum of any given multiunit exceeds nine are given in Rules 3a through 3f. These rules substitute for the understanding of the third component in Table 3: understanding that two digits cannot both be written for a multiunit sum and that ten of that multiunit must be traded for one of the next larger multiunits. Four examples of kinds of errors children make in such problems are given at the top of the addition errors in Table 5. Many children initially solve such problems by violating Rule 3a: They write the two-digit sum beneath its single-digit addends. Thus, the sum of 568 + 778 is 121316 (for problems written either horizontally or vertically). In a sample of almost all second graders in the Pittsburgh public schools, on the pretest, 66% of the children wrote at least one such two-digit sum; such errors comprised 65% of the errors made, and most children making such errors made them for all columns of a problem (Fuson & Briars, 1990). Proportions for a smaller sample from an economically heterogeneous school system bordering on Chicago were 55% and 52% (Fuson & Briars, 1990). Although such solutions violate Rule 3a (and the conceptual knowledge that such answers push digits into wrong positions), they are consistent with a CSD structure—Nothing in this structure indicates that one cannot insert single digits wherever one wants. Many

TABLE 5
Multidigit Addition and Subtraction Errors Reflecting a CSD Conceptual Structure

Addition Errors	*Subtraction Errors*
Write sum for each column (FB 189; F 47–48) 5 6 8 7 7 8 ——— 2 13 16	Smaller from larger (DMPE/D 115; FB 189; F 48–49; L 341; VL 174) 2 5 2 1 1 8 ——— 1 4 6
Vanish the one (FB 189; F 47–48) 5 6 8 7 7 8 ——— 2 3 6	Top smaller, write zero (VL 176) 2 5 2 1 1 8 ——— 1 4 0
Carry to the leftmost (B 229) 1 ¹1 6 8 1 5 6 ——— 4 1 4	Borrow unit difference (VL 150) 4 9 ~~5~~ ~~1~~ 1 9 ——— 3 0
Wrong alignment in long algorithm (G 116) 8 7 3 9 ——— 1 6 1 1 ——— 2 7	Always borrow left (VL 149) 2 ~~3~~ 6 ¹5 1 0 9 ——— 1 6 6
Add extra digit into column (Fr 35) 6 3 2 ——— 1 1	One borrow to multiple places (VL 175) 6 ~~7~~ ¹0 ¹0 ¹2 3 2 5 ——— 6 7 8 7
Reuse digit if uneven (Fr 35) 6 3 2 ——— 8 5	Multiple incorrect borrows across zero (D 325; VL 167) 5 ~~6~~ ~~7~~ 0 ¹0 ¹2 2 5 ——— 5 0 8 7
Ignore extra digits (Fr 35) 6 3 2 ——— 5	Reuse digit if uneven (Fr 35) 7 8 6 ——— 1 2

Note. Initials in parentheses are the source references and page numbers as follows: B is Baroody (1987); DMPE/D is Davis, McKnight, Parker, and Elrick (1979), cited in Davis (1984); Fr is Friend (1979); F is Fuson (1986); FB is Fuson and Briars (1990); G is Ginsburg (1977); L is Labinowicz (1985); VL is VanLehn (1986).

children do learn Rule 3a (that it is wrong to write two-digit numbers as a sum for a given column), but they do not have a conceptual structure to suggest what to do instead of writing both digits. They therefore violate Rule 3b: They write the digit for the ones in the ones position and do nothing with the 1 (the tens digit) in the two-digit sum. This second error in Table 5 was aptly described as "vanish the 1" (the 1 from the 16 vanishes) by a child in an interview in which children were to identify such a solution as correct or incorrect (Fuson & Briars, 1990). In the large urban sample, 13% of the pretest second graders made such vanishing errors, and 15% of the children did so in the heterogeneous suburban sample (Fuson & Briars, 1990). Cobb and Wheatley (1988) reported that wrong procedures for addition of two-digit numbers written vertically were evenly divided between writing both digits and vanishing the 1. The third addition error in Table 5 conforms to Rule 3b but violates Rule 3c. This error can arise from the CSD structure, because it places no constraints on an incorrect generalization of the ten-for-one trading procedure as requiring the 1's to be traded to the leftmost column rather than to the adjacent column. For two-digit problems, this leftmost rule and the correct adjacent rule are indistinguishable. Rule 3d is violated by children who forget the standard direction of adding and begin adding left to right; the 1 is then written in the column to the right. Several second graders receiving traditional school instruction made this error on a written pretest (Fuson & Burghardt, 1990b).

Rule 3e arises only when three or more multidigit numbers are added. In such cases, Rules 3b through 3d only tell what to do with a 1. Thus, if the sum is 21 or 31, for example, the rules will yield the wrong procedure. If the two-digit sum does not contain a 1, the rules will not tell what to do. There is little evidence in the literature about children's errors on sums of three addends or more; teachers have reported to the author anecdotal evidence concerning violations of Rule 3d in which the 1 is written to the left and the tens digit is written in the ones place. Friend (1979) reported with Spanish-speaking Nicaraguan children that problems requiring a trade of 2 or more were more difficult than those requiring a trade of 1; that is, it was more difficult to understand that the tens digit (or the number of tens) was traded to the column to the left than to learn to write a 1 in the column to the left. Rule 3f concerns an error that arises in the use of a modified algorithm that avoids carrying. In this algorithm, each sum is written on a separate line, and these partial sums are added. Children evidently use their predisposition to align numbers on the left in this algorithm and make the error listed in the fourth row of Table 5, violating Rule 3f.

Even many (perhaps in some settings, most) children who correctly solve multidigit addition problems requiring a trade seem to be using a CSD structure in which the 1 from the two-digit sum for a given column is written as a 1 in the next column, that is, they are using a CSD with Rule 3d. The 1 is not given a value either as a named value for the new column in which it is written or as a ten coming from the two-digit sum. All third graders interviewed identified the 1 written above the tens column as a *one* and not as a *ten* (Resnick, 1983; Resnick & Omanson, 1987). Half the interviewed third graders who correctly added a problem identified the traded 1 as a *one* rather than as a *ten* or as a *hundred* (for ten traded tens) in spite of probes such as "What does this 1 stand for?" and "What do the 3 and the 2 (the tens digits) stand for?" and "What is the 1 worth?" (Labinowicz, 1985). Silvern (1989) also found that most third, fourth, and fifth graders could not explain carrying using base-ten blocks; the carried 1 was not seen as a ten like the other tens in the problem.

Addition situations in which one addend has more digits than the other addend(s) seem to present special difficulties and require Rule 4. Friend (1979; cited in Davis, 1984) identified several different kinds of errors children make in such cases (see the last three errors in Table 5). These errors reflect a CSD structure, and each violates a different combination of rules from Table 4. The first error violates Rule 1a, Rule 2, and Rule 4: All three digits are added together. The second error also violates Rule 2 and Rule 4: The 2 is added to the tens column as well as added in its own ones column. The third error violates only Rule 4. The child does not know what to do in this case and so does nothing (violates Rules 4a and 4b) or knows Rule 4a but not Rule 4b and adds $6 + 0 = 0$ ("0 is nothing, so I don't need to write anything").

In subtraction, the use of a CSD structure and the violation of needed subtraction rules given in Table 4 are not usually evident for problems in which every digit in the top number exceeds that in the same relative position in the bottom number and in which the columns are already correctly aligned. Given a correctly aligned vertical subtraction multidigit problem requiring no trades, children using the CSD structure just subtract the bottom digit from the top digit in each column, obtaining the correct difference. As with addition, errors arise when uneven problems are given horizontally, when multiunit numbers are aligned incorrectly, when a trade is required in any problem (when the digit for any value in the minuend is less than the digit in that value for the subtrahend), or when the minuend has more digits than the subtrahend (see Table 5). Problems with zeros in the top number are particularly difficult. The rules for aligning multiunit numbers are the same for subtraction as for addition (see Table 4). Children who do not know the subtraction Rules 1a, 1b, and 1c presumably make the same errors in aligning numbers for subtraction as these violations cause for addition (the alignment data in the literature concern addition rather than subtraction problems). Rule 2 for subtraction is more complex than Rule 2 for addition, because addition of multiunits is commutative (the addends can be added in either order), whereas subtraction is not. The multiunit belonging to the overall smaller multiunit number (the addend/ subtrahend) must be subtracted from the same kind of multiunit belonging to the overall larger multiunit number (the sum/minuend). The CSD structure in conjunction with the commutative subtraction Rule 2a (the analogue to the addition Rule 2) leads to the very prevalent error of subtracting the top digit from the bottom digit when the top digit is smaller (the "smaller-from-larger" subtraction error in Table 5). This error does not even conserve the multidigit numbers, for parts of each are being subtracted from each other (the subtraction of the largest multiunits will not be a smaller-from-larger error, because the minuend multiunit will be larger than the subtrahend multiunit). Such errors are the most common subtraction errors made by U.S. children. In a sample of errors observed in the work of 1,147 students in Grades 2 through 5, 29% of the observed errors were of this type, and twice as many such errors were made as were made for the next most common type of error (VanLehn, 1986). Three fourths of the large urban sample of second graders made such errors consistently on the pretest, as did 85% of the heterogeneous suburban sample (Fuson & Briars, 1990). This was the most common error made on two-digit subtraction problems by the third graders interviewed by Labinowicz (1985). Knowledge of the noncommutative Rule 2b, but no knowledge of the how-to-get-more Rules 3b through 3i, leads to the second rather creative error in Table 5: Children write 0 as the answer for columns with smaller top numbers, indicating that the column cannot be subtracted or that they are left with 0 when they subtract as much as they can subtract.

Many children do learn some of the how-to-get-more Rules 3b through 3i. Knowledge of the partial Rules 3b through 3e and 3g and 3h lead to characteristic errors. All these partial rules are consistent with use of a CSD structure in which each written mark has only a single-digit meaning; no named value or relative position value of any digit is considered in these rules. Children may learn that they must put more in the top number and then just write a 1 in the needed column without taking it from any other digit (Fuson & Briars, 1990); this use of Rule 3b violates equivalence of trading and changes the value of the top number. Rule 3c generates an error that epitomizes the CSD structure: Just enough is taken from another digit to make the top number equal the bottom number (see the third error in Table 5). This error does maintain equivalence in trading (4 are taken from the 8 and added to the 5 to make 9), but it ignores any position multiunit value of the digit. The fourth error in Table 5 demonstrates Rule 3d and is the analogue to the similar error in addition (the third addition error in Table 5). Rule 3e is almost the correct Rule 3f, except that the 1 is taken from the right instead of from the left; children occasionally make such errors (e.g., Fuson & Burghardt, 1990b). Rule 3f works for all problems in which there is not a 0 in the minuend. Where trading must be carried out across one or more zeros, children demonstrating Rule 3f may not demonstrate Rule 3i. Trading across zeros is quite difficult for many children. The "one-borrow-to-multiple-places" error produced by Rule 3g recognizes that something must be done to all the 0 columns and "fixes" them all at the same time (makes them bigger than the subtrahend digits), but it ignores equivalence in trading because a single one from the first nonzero column is given to every column needing a 1. This was the second largest subtraction error category reported by VanLehn (1986). Davis and McKnight (1980) found that every third and fourth grader given the problem 7002 – 25 written in vertical form initially made the "multiple-incorrect-borrows-across-zero" error shown in Table 5, and a substantial proportion of the VanLehn errors were of this type. This error displays Rule 3h; it maintains equivalence of single units (a one is subtracted from some digit each time a 1 is added to—written by—another digit), but the values of the digits and the value of writing the 1 as a tens digit are not used. As with addition, subtraction problems in which the minuend has more digits than the subtrahend may lead to errors. Friend (1979, cited in Davis, 1984) reported a subtraction error of reusing a digit that is the same as the addition error of the same name (see Table 5).

Finally, as with addition, many children who do subtract correctly for problems that require trading demonstrate only a CSD structure in which the trading procedure is a rule-based procedure involving the trading of ones from one single digit to another single digit; no multiunit values are involved in the trading. For example, only 8% of third, fourth, and fifth graders who borrowed correctly on a test of written subtraction problems agreed with an explanation of borrowing from the hundreds to the tens position that described taking a hundred away from the hundreds position; many children saw borrowing as exchanging a one (Cauley, 1987). VanLehn (1986) analyzed the errors in a large corpus of multidigit subtraction problems and concluded that 85% of these errors could arise by induction based on visual-numeric features of correctly solved but possibly limited examples. An analysis of these features by this author indicated that all these visual-numeric features were consistent with a CSD structure used in conjunction with a visual layout structure (from Table 2). An examination of the remaining 15% of the errors indicated that all these errors were also consistent with a CSD structure. Many of the errors in this corpus violated the multiunit quantities and the regular ten-for-one or one-for-ten trades conceptual structures. Thus, the CSD structure may be almost universally used by children making subtraction errors, and multiunit conceptual structures either are not possessed or are not used by children making most of these errors. Several different computer programs that simulate multidigit subtraction and produce errors typically made by children have been written (see VanLehn, 1986, for a discussion of the several programs in the Brown and VanLehn research program; see Young & O'Shea, 1981, for a description of a different approach to this problem). All these programs use only a CSD structure in which digits have locations in certain columns but do not take on different quantitative multiunit values in these columns.

Trades may be described in articles as a ten-for-one or a one-for-ten trade, but the procedures the computer programs produce and the child error data the programs are written to imitate do not require any notion of ten but only rules about writing little 1's. Thus, the CSD structure is used by many individuals who analyze children's errors as well as by children themselves.

CLASSROOM EXPERIENCES THAT SUPPORT CHILDREN'S CONSTRUCTION OF CONCEPTUAL STRUCTURES FOR MULTIUNIT NUMBERS

How can U.S. English-speaking children be helped to construct the conceptual structures for multiunit numbers that are required to understand place value and multiunit addition and subtraction? Typical school instruction currently leads most children at best to use of a CSD conceptual structure with correct knowledge of all the rules in Table 4 and at worst to knowledge of only the incorrect rules in Table 4. Thus, the conceptual structures that are particularly lacking are the multiunit-quantities and regular ten-for-one and one-for-ten trades conceptual structures, but even the multiunit-names conceptual structure is problematic for some children at present. Materials that display physical collectible multiunits help children construct the conceptual collected multiunits that constitute the multiunit-quantities conceptual structure. Base-ten blocks were invented by Dienes (1960) precisely for this purpose; bundled sticks have been used fairly frequently in the classroom, and other materials can also be used. Considerable evidence exists that use of physical collectible multiunits helps children construct the multiunit-names, multiunit-quantities, and regular ten-for-one and one-for-ten trades conceptual structures. This evidence is briefly described. The following sections of this article explore relationships among the conceptual structures, aspects of linking all the conceptual structures, detrimental features of present school multidigit learning and teaching, and three alternative approaches to constructing multiunit conceptual structures. The final four multiunit structures listed in Table 2 will not be considered in this article, because those structures concern multiunit structures used in multiplication. All the other conceptual structures in Table 2 can be called the *additive multiunit conceptual structures,* because they are used in multiunit addition and subtraction.

Effects of Materials That Display Collectible Multiunits

Materials that display physical collectible multiunits can help children understand place-value concepts (i.e., construct and relate the first five conceptual structures in Table 2). Second graders who worked extensively adding and subtracting four-digit numbers using base-ten blocks demonstrated considerably better knowledge of place-value concepts than that reported in the literature for third and even fourth and fifth graders receiving standard instruction (Fuson & Briars, 1990). All second graders spontaneously and correctly identified the tens place by name when justifying their written addition or subtraction procedure, and 92% of them did so for the hundreds place. Second-grade classes averaged around 88% on tasks requiring translations of mixed-order named-value words and digits (e.g., 4 tens 6 hundreds 3 thousands 9 ones) to written marks and vice versa. Thus, most of these second graders attended to the named-value words as well as to the digits, in contrast to more than one third of the third and fourth graders who focused only on the digits, ignoring the words, on a more complex task (Bednarz & Janvier, 1982). Labinowicz (1985) found in one class that, of the 14 third graders unable to add one more than or ten more than 342, 10 were able to find one more than and 8 could find ten more than when using base-ten blocks. Behr (1976) also reported for second graders facilitative effects of using base-ten blocks on such tasks and on place-value tasks involving naming digit positions.

Base-ten blocks or collectible multiunits can also help children understand multiunit addition and subtraction (Cauley, 1987; Fuson, 1986; Fuson & Briars, 1990; Labinowicz, 1985; Resnick, 1983; Resnick & Omanson, 1987; Swart, 1985; Tucker, 1989). Tucker (1989) described a case in which simply showing one multiunit addition problem in blocks was sufficient to enable a girl who was previously familiar with the blocks to self-correct her error of aligning numbers on the left; looking at the blocks enabled her to see that like values must be added to like values and, thus, the multidigit marks must be aligned on the right. In one study, 100% of the second graders who had used the blocks aligned written uneven problems (e.g., 286 + 79) correctly on the right instead of making the common error of aligning such problems on the left (Fuson & Briars, 1990). First graders who used the blocks showed much better performance on two-digit and three-digit addition and subtraction than did first graders receiving standard instruction without the blocks (Swart, 1985). Many third graders who were making the subtraction error of subtracting the smaller top number from the larger bottom number corrected their subtraction procedure after working with base-ten blocks (Labinowicz, 1985). An introduction to multiunit subtraction with the blocks enabled older children making trading errors in written computation to correct these errors (Resnick, 1983; Resnick & Omanson, 1987). Reminding second graders who were making errors in written addition and subtraction procedures and who had initially learned multiunit addition and subtraction with base-ten blocks to "think about the blocks" enabled most of them to self-correct their own errors, even for problems with trading across zero (Fuson, 1986). Learning multiunit addition and subtraction with base-ten blocks enabled second graders to differentiate correct from the most common incorrect written procedures and to explain the addition and subtraction trading procedures in terms of trades between the named values of the marks (Fuson & Briars, 1990). Not a single interviewed child identified the traded 1 as a one, in marked contrast to children receiving usual instruction. Cauley (1987) found in one session that the use of bundled sticks with second and third graders who had not seen them before enabled several of them to conserve the value of the minuend in subtraction and enabled several others to establish correct trade rules for tens and for hundreds.

Although even one session of using collectible multiunits can be very helpful, such use does not guarantee the construction or use of collected multiunits for English words or written marks. Resnick and Omanson (1987) found that after the lesson with base-ten blocks, many of their instructed upper grade children regressed to making their old subtraction errors. Madell (1985) reported observing many children making unitary count/cardinal rather than multiunit use of base-ten longs and cubes in adding and subtracting two-digit numbers. Ross (1988) found that many second through fifth graders who had used base-ten blocks in their classrooms, as well as high proportions of children sampled from a wide range of schools receiving different kinds of mathematics instruction, had only her Level 3 "face-value" meanings for tens and ones in which these words referred to some aspect of grouped objects but not to quantities of tens or of ones. She emphasized that mere use of collectible multiunit embodiments does not ensure that children construct multiunit-quantities conceptual structures for written marks (Ross, 1989; her discussion did not use these terms, which are introduced in this article). The third- and fourth-grade children interviewed by Davis and McKnight (1980), all of whom showed errors on the problem 7002 – 25 and on similar problems with zeros in the minuend, had used base-ten blocks for place-value activities, although not for multiunit addition and subtraction. Cobb (1987) discussed the use of base-ten blocks and underscored the difference between the blocks possessing certain mathematical features and children seeing/conceptualizing these features in the desired ways; this is the distinction captured by the use in this article of "collectible" and "collected" multiunits. Labinowicz (1985) detailed problems some children had in using base-ten blocks in a wide range of place-value and addition and subtraction tasks. Not all children immediately see the same collectible multiunit items or relate them to other knowledge. However, many of the children's difficulties described in this book occurred on tasks requiring multiunit sequence counting rather than just collected multiunits—They involved counting by tens and hundreds as well as by ones. In spite of many difficulties noted, the overall thrust of Labinowicz's summaries of the use of the blocks in many tasks is that even brief use was often helpful.

Use of an embodiment that does not display physical collectible multiunits may confuse children and facilitate use of the inadequate CSD structure, because it gives no support for the construction of collected multiunits to give quantitative meaning to the named values. Labinowicz (1985) found that, when third graders who correctly identified 231 as larger than 198 were shown this same problem embodied in different-colored same-size chips (ones green, tens yellow, hundreds a third color), some of them were confused and changed their answer to reflect a CSD structure. In contrast, base-ten blocks were helpful in enabling children to focus on the largest value and change answers from incorrect to correct. Behr (1976) also found with second graders that base-ten blocks were more helpful on a range of place-value and multiunit addition and subtraction tasks than was an abacus. Extensive work making groupings of different sizes using colored chips or other nonsize embodiments would seem to foster in many children the face-value (superficial grouping) orientation discovered by Ross (1988): The temporary groupings of many kinds, combined with the lack of perceptual support (collectible multiunits) for the construction of collected multiunits, seem likely to engender in many children a superficial nonquantitative analysis of grouped situations. Money (dollars, dimes, pennies) is often used in textbooks to show hundreds, tens, and ones values. This embodiment has three problems. First, it may be confusing to some primary school children, because the dime is smaller than the penny and, thus, does not seem to be ten times as large as the penny. Second, the other coins (nickel, quarter) do not fit nicely within the ten-for-one trades in the positional marks system. Third, we ordinarily write dimes and pennies as tenths and hundredths, using a decimal point after the dollars ($5.62), so dimes and pennies are really an embodiment of decimal positions and not of whole number tens and ones.

Relationships Among the Conceptual Structures for Multiunit Numbers

The first five multiunit conceptual structures in Table 2 are not structurally isomorphic. The first two ("visual layout" and "positions ordered in increasing value from the right") are features of the unnamed positional marks, and the second two ("multiunit names" and "words ordered in decreasing value as they are said") are features of named-value English words. The multiunit-quantities conceptual structure gives the meaning to the multiunit-names structure, and it gives the values to be used in the position order of the marks and the word order of the words. These relationships among the conceptual structures need to be remembered when designing experiences to help children construct and relate all these conceptual structures. The relationships involved are displayed in Figure 1. The single arrows show the simpler isomorphic relationships that need to be constructed between elements of structurally similar systems (the vertical connections within the named-value systems and within the unnamed positional sys-

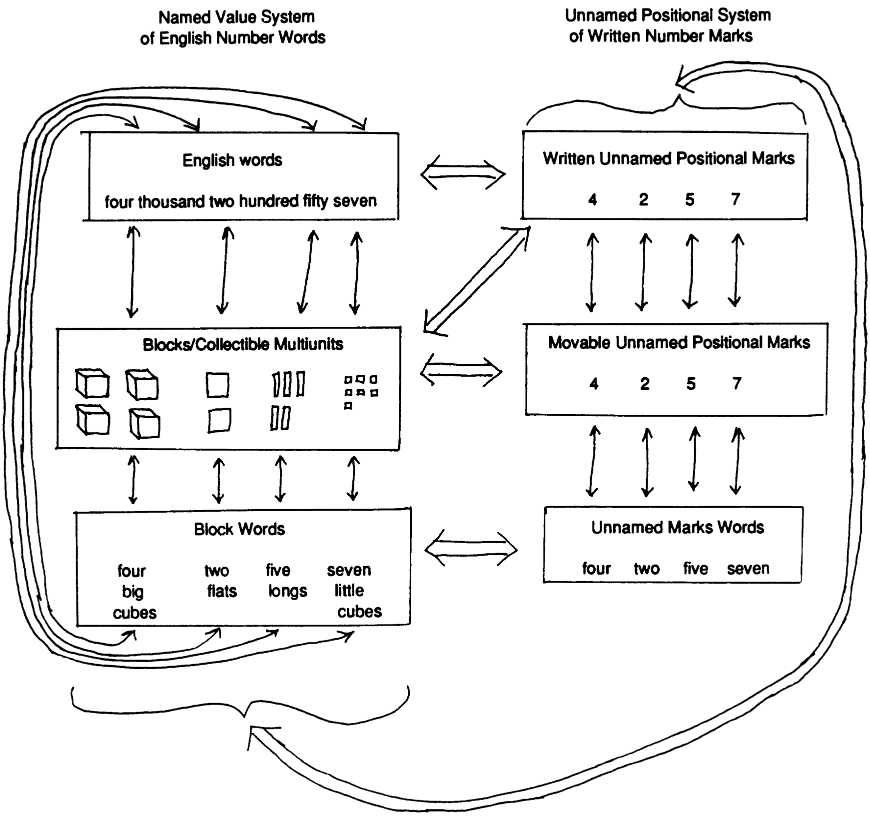

FIGURE 1 Relationships among words, marks, and multiunit conceptual structures.

tems), and the double arrows show the crucial, more difficult relationships that need to be constructed across the named-value and unnamed positional systems. The top boxes show the English words and the written marks: the multiunit-names and visual-layout conceptual structures. The top box on the left also shows "words ordered in decreasing value as said," because the English words shown are said in the correct value order. However, the collectible embodiment to support the construction of multiunit quantities shown in the middle on the left is required to give the value meanings to the English multiunit names and value order to "words ordered in decreasing value as said." It is also required to give position order to the movable unnamed positional embodiment in the middle on the right and to construct the "positions ordered in increasing value from the right" structure for the written marks on the top right. The bottom boxes show names for the embodiments in the second boxes; these provide verbal labels that permit users of the embodiments to discuss the embodiments in the second boxes. Base-ten blocks are shown for multiunit quantities, but any embodiment that displays collectible multiunits could be in that second box on the left. The embodiment on the right is index cards, each with a single digit written on it. This embodiment has several unnamed positional properties: (a) A digit can be put into any position; (b) a digit out of its position affects the relative position of other digits (digits can be easily put into or taken out of the row of digits); and (c) the digit row by itself does not name any values. Any embodiment that displayed these properties could be substituted in this box.

The collectible multiunits need to be arranged in the increasing right-to-left order shown so that the relative positions of the marks can take on their correct standard meaning. This right-to-left increasing order is an arbitrary feature of our written marks and must be shown to children; they cannot deduce it. Once this ordered relationship is established for the collectible multiunits, however, the digit cards can show the unnamed positional nature of the marks compared with the named value of words and multiunit quantities. Scrambling the blocks does not affect the quantity (the same multiunits are still there), but scrambling the digit cards does affect the quantity, because the multiunit values adhere to the positions and not to particular digits. The slanted arrow is the crucial relationship between multiunit quantities and the visual layout of the written marks; this is the relationship that seems to be so rarely constructed from usual classroom experiences.

The ten-for-one and one-for-ten trades conceptual structures arise from examining the quantitative relationship between contiguous multiunit quantities. Collectible multiunits enable children to ascertain that there are *ten* little cubes in *one* long, *ten* longs in *one* flat, and *ten* flats in *one* big cube (the same ten-for-one relationship holds for other ten-based collectible multiunits). Initially each ten-for-one relationship can be seen and related to the written marks. When a child has had experience with several such trades (e.g., at least the three trades involved in the first four named values), the child then may reflect on all the trades and see that they are all the same. In this way, the regular ten-for-one trades conceptual structure can be constructed and linked to English words and to written marks. The reverse regular one-for-ten trades conceptual structure can arise as a similar reflection on subtraction trades across several positions.

Some aspects of these multiunit conceptual structures require experience with several different multiunits to understand the attributes and relationships involved. It is possible to comprehend that large numbers are built up from different-size multiunits only if one sees larger and larger multiunits. The regular ten-for-one trades cannot possibly be understood as regular repeated trades unless one sees two or three such trades. The visual layout of written multiunit marks as several horizontal positions for single-digit marks requires seeing three or four such positions. For all these reasons, structures for multiunit numbers seem more easily constructed when children have an opportunity to experience and then reflect on several different multiunits. Furthermore, the many special difficulties of English words that interfere with the simple and regular relationships portrayed in Figure 1 are confined to multiunits of ten. Multiunits of hundreds and thousands are supported (named regularly) in the English words, and a regular relationship between English words and written marks also exists for the third and fourth marks positions. Thus, allowing children to work as soon as possible with collectible multiunits for three or four positions would seem to be advantageous for the construction of all the conceptual structures for multiunit numbers.

When one begins to add and subtract multiunits of ten, hundred, and thousand, one can think of the digits in a problem with a multiunit-quantities conceptual structure or with a regular one-for-ten trades structure. For example, if the tens digits are 8 and 6, one can think in English, "Eight tens plus six tens are fourteen tens. Ten of those tens make a hundred, so the sum is one hundred four tens. Write the four tens and trade the hundred over to the hundreds column." Or one can ignore the fact that the digits are tens and think "Eight and six are fourteen. That is one ten and four ones. Write the four ones and trade the ten over to the next column." The latter regular ten-for-one trades conception is an efficient one that will apply to digits in any position. Some evidence exists with U.S. and with Korean children that some children first generalize the ten-for-one trading with units to the tens and hundreds positions and use only a regular ten-for-one conceptual structure in thinking about addition. Other children fairly quickly think about this addition using a multiunit-quantities conceptual structure. Full understanding and efficient carrying out of multidigit addition and subtraction would seem to involve an integration of these two conceptual structures so that one might use the regular ten-for-one trades structure when carrying out addition but be able to use the multiunit-quantities structure when explaining or justifying the procedure. Most second graders in the Fuson base-ten block studies (Fuson, 1986; Fuson & Briars, 1990) used multiunit-quantities structures in explaining multidigit addition and subtraction, indicating that this is within reach of most second graders. Some lower achieving second graders and some average-achieving first graders did use only regular ten-for-one trades structures and did not ever say that a traded 1 in the hundreds column was a hundred but only that it was a ten. Many Korean second graders who had not yet learned three-digit addition carried out such addition correctly but described it only with regular ten-for-one trades conceptual structures; they also did not ever identify the traded 1 as a hundred but only as a ten. Almost all Korean third graders did identify the traded 1 as a hundred, even though many still also described addition with a regular ten-for-one trades structure; thus, they had evidently integrated these two structures and could use either when necessary. Thus, it may be that weaker and younger children cannot integrate the multiunit-quantities and regular ten-for-one trades conceptual structures initially. Alternatively, they may have just had insufficient support in the classroom for making this integration, or our interviews did not probe sufficiently to uncover this knowledge.

Classrooms need to provide activities that help children construct the conceptual structures and the arrow relationships in Figure 1. Two such activities that use all these conceptual structures and relationships are multiunit addition and multiunit subtraction requiring trading. The written marks give no clues about which digits to add or subtract; the English names and collectible multiunits do suggest adding or subtracting like multiunits. Addition and subtraction situations that require trades raise all the important understandings that must be constructed: Writing two digits for one value pushes other digits out of their correct relative positions; trading when there are too many or not enough requires ten-for-one or one-for-ten trades (trading requires one to notice the quantity in the traded multiunit); and situations with blanks or zeros raise and, thus, provide an opportunity to understand the difficult mappings between English words and written marks. Thus, all these conceptual structures and relationships can be constructed while learning to understand and carry out multiunit addition and subtraction if collectible multiunits are available during this learning.

The necessity for collectible multiunits, the nature of the relationships that must be constructed, the advantages of moving as soon as possible to four-digit numbers, and the value of multiunit addition and subtraction situations with trading as a setting for constructing the additive multiunit conceptual structures all arise from the attributes of the named-value English words and the unnamed positional system of written number marks. It is not clear a priori how early such an approach can be taken with U.S. English-speaking children and whether ordinary teachers can implement such an approach without extensive in-service training and in ordinary classroom conditions. A series of studies has indicated that such an approach can be successful with second graders (Fuson, 1986; Fuson & Briars, 1990). Second graders at all achievement levels, except those functioning at a low first-grade level, showed evidence of all the additive multiunit conceptual structures and relationships among them and demonstrated multiunit addition and subtraction competence considerably above that reported with usual school instruction (details are reported in the previous sections with the references to these studies).

Thus, U.S. English-speaking children are not doomed to use rote rules and a CSD conceptual structure and to inadequate understanding of place value and of multiunit addition and subtraction. Classroom learning and teaching focused on the construction of multiunit conceptual structures can lead to a radically different outcome.

Linking the Additive Multiunit Conceptual Structures

Frequent multiple verbalization using English words, block words, and unnamed-marks words can help children build all the necessary links among the conceptual structures. These words help to direct children's attention to critical features of the mathematical systems and embodiments and facilitate communication among the participants in a learning–teaching setting. In the Fuson base-ten block studies, operations on the blocks were described by the teacher or by a child as they were being carried out, frequently with at least two sets of words. For example, while trading ten small cubes for one long, the trader would say, "I'm trading ten of these baby cubes for a long, ten ones for one ten, and I'm putting the traded long here at the top of the longs column, the tens column." Or when reading the final answer for a problem, one might say, "So the sum is five six eight zero, five thousand six hundred eighty." These running commentaries served to help direct children's attention to the particular important features of a given situation, for example, that the trade was ten ones for one ten and not just many (unspecified) for one. This emphasis on verbalization is consistent with results reported by Resnick and Omanson (1987) concerning a positive relationship between the amount of verbalizing and the amount of learning with the blocks and with recommendations by Thornton and Wilmot (1986) concerning the necessity for learning-handicapped children to hear themselves speak in order to learn. Unnamed-marks words are particularly useful in the beginning in facilitating the participation of children who have not yet learned the English words, and later they facilitate discussion of the importance of the position of the word or digit.

When adding and subtracting with the blocks, the blocks-to-written-marks links need to be made strongly and tightly: Each step with the blocks needs to be immediately recorded with the written marks. In my own early work with multibase blocks with children and with teachers (work that preceded the tight-link approach used in Fuson, 1975, and in Bell et al., 1976), it became clear that adults' and children's usual tendency in using the blocks is to solve a whole problem with the blocks and then just write down the answer obtained with the blocks. If children or teachers are allowed to use blocks in this way, written marks do not "take on" the named-value collected multiunit meanings of the blocks, and written trading procedures do not become meaningful. Most children (and even most teachers) do not possess the cognitive resources to remember the whole procedure with the blocks and connect this whole procedure to the written procedure. Children are able to reflect on one block step at a time, however, and then describe/remember this block step as they record this step with the written marks. Recording of block operations with the marks also facilitates reflection on what has been done with the blocks, because one has to write down a certain number in a certain location. Many teachers evidently underestimate this need to make very tight connections—or even any connections at all—between the blocks and the mathematical marks. Hart (1987) reported that this need is not necessarily seen in England by teachers using the blocks, who frequently make minimal or even no links between these two, abandoning the blocks very soon after introducing the written marks.

Many discussions of the use of physical embodiments to support the construction of meaning for mathematical marks emphasize the importance of helping children construct initial links from embodiments (multiunit quantities) to written marks (e.g., Beattie, 1986; Bell et al., 1976; Davis, 1984; Dienes, 1960, 1963; Hiebert, 1984; Moser, 1988; Resnick, 1982; Skemp, 1981; Underhill, 1977). However, the reverse marks-to-multiunit-quantities links were also found to be very important in Fuson (1986). After some period of time, errors began to creep into written-marks procedures for some children. For most interviewed second graders with errors, asking them to "think about the blocks" was sufficient for them to self-correct their own errors, even with problems with zeros in the top number. Thus, their mental conceptual multiunit structure for the blocks was strong enough to direct correct marks procedures, but they did not always access this conceptual structure when doing a written problem. Frequent distributed practice of a few multiunit addition and subtraction problems during which children are asked to think about the blocks and check the accuracy of their procedures may serve to help prevent errors from creeping into written procedures and increase the frequency with which children will spontaneously make the marks-to-blocks link to check on their own procedures. The necessity of emphasizing such marks-to-blocks links was recognized by Dienes (1963) when discussing how easy it is for symbolism to become autonomous without occasional feedback to the experiences from which the symbolism has been derived, by Resnick and Omanson (1987) in discussing the need for interjecting semantic critics into the computational procedure, and by Thornton and Wilmot (1986) in discussing the use of manipulatives with learning-handicapped students.

Many discussions of using embodiments generally and of using base-ten blocks specifically assume the validity of Bruner's (1966) suggestion of the importance of a pictorial (iconic) level connecting concrete and abstract work. They recommend that a period of drawing pictures of the base-ten blocks or other pictorial kinds of activities be interspersed between work with the blocks and work with the written standard positional base-ten marks (e.g., Davis, 1984; Heddens, 1986; Underhill, 1977). The use of Bruner's concrete–pictorial–abstract continuum in this context ignores the fact that the blocks and the written marks are not endpoints on a single continuum: They are structurally different systems that must be connected. Pictures have the same properties as the blocks (and different properties from the marks), but they are very complicated and time consuming for children to produce as drawings. Pictures may even be much more difficult for children to comprehend, let alone produce. Behr (1976) found that second graders had great difficulty with problems asking a child to find one more or less than, ten more or less than, or one hundred more or less than a given number when the problems were presented in the iconic mode (with pictures of base-ten blocks), but many children answered successfully when given the blocks themselves. In the Fuson base-ten block studies, children easily made links directly between the written marks and the base-ten blocks, as indicated in Figure 1. It is not clear at this time what, if any, advantages are provided by pictures, and there are definite disadvantages.

It may take a long time for a given child to construct conceptual collected multiunits and link them to written marks and to English words. Children in the Fuson block studies varied from a few days to several weeks before they felt comfortable with the written marks alone and could discuss addition or subtraction of the marks using English words or block words. Children readily learned to add and subtract with the blocks: The collectible multiunits presented by the blocks suggested combining like multiunits and trading. What took time was associating multiunit quantities and English names to written marks and to addition and subtraction with written marks.

It may be more difficult for children to make the links between collected multiunits and written marks procedures after they have highly automatic written procedures not connected to multiunit quantities. Although base-ten block instruction can help older children correct their errors (Resnick, 1983), it does not always do so (Resnick & Omanson, 1987). Hiebert and Wearne (1987) also reported that for decimal fractions the blocks were less effective when used remedially to correct already present errors and misconceptions than when they were used for the initial learning–teaching experience. However, both these remedial uses were of short duration, so some children may just not have had sufficient time to construct collected multiunit quantities and connect them to written marks procedures.

Many different classroom organizations can support children's construction of multiunit conceptual structures, and many different addition and subtraction procedures can be discovered or demonstrated. Teachers can lead children to particular procedures, or children can invent their own procedures. The classroom organization used by teachers in the Fuson block studies varied. Some teachers used one set of blocks with the whole classroom at the same time. Others used one set with one small group at a time while other groups worked on other kinds of problems. Some teachers had many small groups working simultaneously, each with their own set of blocks. Teachers led children to particular multiunit procedures in these studies; in research just completed, however, children used the learning–teaching setting in Figure 1 to invent their own procedures. Discussion of advantages and disadvantages of various classroom organizations and of various multiunit procedures is beyond the scope of this article (see Fuson & Burghardt, 1990a).

Use of an embodiment that displays collectible multiunits to support the construction of conceptual understanding of a written marks procedure exemplifies what VanLehn (1986) called "learning by analogy," Resnick (1982, 1983) termed "relating semantics and syntax," and Schoenfeld (1986) discussed as making an abstraction between a reference domain (the blocks) and a symbol system (the written marks). All these distinctions can be elucidated by understanding the differences among the English words, quantitative multiunit conceptual structures, and the written marks, because it is these differences that lead to the relationships specified in Figure 1. For learning by analogy to occur, all the relationships in Figure 1 need to be constructed. These differences are the source of what Schoenfeld, following a personal communication with J. S. Brown in 1986, discussed as a breakdown of the isomorphism between the blocks and the marks. As discussed earlier and portrayed in Figure 1, however, the natural isomorphism is not between the blocks and the marks: It is between the English words and the blocks. The words and the blocks can provide conceptual structures that can give meanings to the marks (via the relationships in Figure 1) and can then direct the multidigit addition and subtraction marks procedures. Resnick's (1982, 1983) discussion of the importance of relating semantics (conceptual understanding) and syntax (the written marks procedure) reflects the same viewpoint that emerged in the late 1960s in my own work with the base-ten blocks and is portrayed in the relationships in Figure 1: Resnick's semantics is the left column and the syntax is the right column. Although the distinction is crucial, the particular words chosen by Resnick to describe the distinction seem misleading. In most natural languages, the syntax is arbitrary and is generally unrelated to the semantics. Rarely if ever can the syntax be reliably induced from the semantics of the language. It is this inductive relationship, however, that is the whole point of Figure 1 and of the instructional approach based on the relationships portrayed in Figure 1. Written marks procedures are fully derivable from and explainable by the multiunit-quantities structure. Thus, it seems preferable to drop the words *semantics* and *syntax* and stick with the actual underlying meanings of these terms: conceptual (or quantitative) understanding versus the written marks procedure.

Resnick (1983) described three stages of decimal (multiunit number) knowledge: (a) unique partitioning of multidigit numbers (two-digit numbers as a part–whole schema with the special restriction that one part is a multiple of ten), (b) multiple partitionings of multidigit numbers (using ten-for-one or one-for-ten trades to make an equivalent multiunit number), and (c) application of part–whole to written arithmetic (carrying out the trades in written marks problems). Knowledge in the first stage is organized into nodes related to each mark's position. Aspects of the visual-layout, multiunit-names, and multiunit-quantities (as base-ten blocks but not as conceptual structures) conceptual structures, as well as knowledge of sequence multiunits as lists of counting by ten, hundred, and thousand appear in these nodes. The second and third stages involve aspects of the regular ten-for-one and one-for-ten trades conceptual structures. The discussion of these three stages includes many important aspects of multiunit numbers. However, the node structure does not permit relationships to be portrayed between elements at different nodes (as Table 2 does), and the differences between English words and written marks are not discussed. The emphasis on a multiunit number as a part–whole schema (albeit a special one) also seems a bit misplaced. The term *part–whole* implies that the parts are all the same as each other (ruling out, or at the least not conveying, the possibility of different kinds of multiunits), and it emphasizes that the whole multiunit number is made up of its parts (the easy aspect of large whole numbers) while omitting any reference to the nature of these parts (the difficult and central aspect of large whole numbers in any language and in any written marks). The multiunits language conveys both these aspects while being explicit about the nature of the parts of which a large whole number is composed.

Detrimental Characteristics of Current Textbook Treatment of Multiunit Topics

Several characteristics of current-textbook grade placements and treatment of multidigit addition and subtraction topics in the United States interfere with rather than support children's construction of linked multiunit conceptual structures. These are discussed in Fuson et al. (1988) and in Fuson (1990, in press) but are described briefly here to indicate changes required for classrooms that support such construction. Experiences with multidigit number words and number marks and with multiunit addition and subtraction are distributed across four and, sometimes, even five grades (Fuson et al., 1988). Children work many two-digit addition and subtraction problems with no trading in first grade (e.g., 23 + 45); two-digit problems requiring trading are not introduced until second grade—8 months to 1 year later (Fuson, in press). Three-digit problems in several text series are not given until third grade, four-digit problems occur first in either third or fourth grade, and larger problems usually appear in fourth or later grades. This contrasts with practice in the Soviet Union, Japan, Taiwan, and Mainland China in which problems up through six places appear in the third-grade text. In U.S. texts, particular subtypes of addition and subtraction problems are chunked rather than intermixed: Problems with a trade from ones to tens precede those with a trade from the tens to hundreds, which precede those with both of these trades. This practice increases the number and kinds of errors children make, for they invent rules that work in restricted situations but create errors when used with unfamiliar problem types (see the discussion by VanLehn, 1986). This piecemeal introduction of multidigit addition and subtraction over a long period of time interferes with the reflective abstraction of the common features of the procedures across different positions and conveys to children the mistaken idea that there are many different procedures used for different types of problems.

The prolonged focus of multiunit work only on two-digit numbers and especially the initial restriction to two-digit problems with no trading (and thus no necessary opportunity to see that the second position is a ten) mean that children must face all the consequences of the English irregularities for two-digit words for a prolonged period without the linguistic support of the regular named English hundreds and thousands and without seeing ten-for-one trades that could help them construct multiunits of ten. This effort to simplify children's learning by restricting their experience to two-digit numbers not only underestimates their abilities but also makes it difficult for them to construct adequate conceptual structures. The push of children in CGIP classrooms to pose many problems to themselves with three-digit and four-digit numbers suggests that such problems can be highly motivational as well as provide support for regular named-value multiunits of hundreds and thousands (Jenkins, 1989).

There is in textbooks a muddled and reversed order of using multiunit and unitary conceptual structures for two-digit marks, sums, and differences. Two-digit written marks are initially related to multiunit tens and ones and not to unitary patterns in counting words (which all children experience first). Two-digit sums and differences without trading precede single-digit sums and differences to 18 (e.g., problems such as 41 + 57 precede problems such as 8 + 8). The former are treated as multiunits of tens and ones, but the latter are treated as unitary numbers. In most texts, there is little or no attempt to show single-digit sums or differences greater than ten in ways that are structured around ten. Thus, children are expected to construct multiunits of ten and one early in first grade and to use these to add and subtract two-digit numbers before they have even had any experience with unitary addition and subtraction of smaller unitary single-digit numbers that sum only to 18. They are expected to see several tens in a two-digit number such as 47 before they see one ten in 17.

Finally, in textbooks, rote rules are stated with the very first multiunit addition and subtraction problems, and inadequate support is provided for constructing collected multiunits or for linking these to written marks or written marks procedures. Pictures of base-ten blocks may be given for one or several pages, but these are discontinued far too rapidly for most children to construct conceptual collected multiunits. Frequently, pages showing blocks are messy and busy, and the links between the marks and the blocks are not clear. Finally, it is more difficult to show the steps involved in addition and subtraction with the blocks in a series of pictures than it is to use actual blocks in real time in the classroom. Few texts successfully overcome this problem.

The conceptual structures required for understanding addition and subtraction of multiunit numbers suggest different characteristics of topic order and grade placement. These characteristics have been discussed in earlier sections and in Fuson (1990). These proposed new characteristics are:

1. Reading and writing two-digit marks are initially related to unitary sequence/counting structures and only later are related to multiunit conceptual structures.
2. Addition and subtraction of all single-digit numbers precede all multiunit addition and subtraction.
3. Understanding of place value is multifaceted and prolonged and accompanies and follows understanding of multiunit addition and subtraction.
4. Multiunit addition and subtraction can be presented all at once whenever children are ready to construct multiunit conceptual structures (problems with and without trades are presented from the beginning, and all possible combinations of trades are done from the beginning).
5. Multiunit addition and subtraction procedures arise from multiunit conceptual structures, and adequate support is provided for constructing such conceptual structures.

Three Alternative Routes to Using Multiunit Conceptual Structures in Multidigit Addition and Subtraction

Unitary single-digit and sequence multiunit solution procedures. This alternative was discussed in an earlier section. Children invent the usual unitary solution procedures to solve single-digit sums and differences and then extend their unitary-sequence counting procedures to multiunit-sequence counting of tens and ones. Multiunit-sequence counting can also be extended to hundreds and to thousands. Difficulties with this approach are that the multiunit-sequence counting procedures even for two-digit numbers are difficult for many children to learn, and those for three-digits are even more difficult. Such learning typically takes until third or even fourth grade, especially the subtraction counting-down sequence procedures that seem to be more difficult than addition procedures.

Unitary single-digit and collected multiunit solution procedures. This is the alternative used in the Fuson block studies. Children were first helped through their usual unitary solution procedures for single-digit numbers to unitary counting on for addition and counting up for subtraction (Fuson & Secada, 1986; Fuson & Willis, 1988). Children then engaged in activities focused on constructing the links among the additive multiunit conceptual structures, and they concentrated especially on multiunit addition and subtraction. This approach required that children shift from using a unitary conceptual structure for finding single-digit sums or differences to using a multiunit conceptual structure to carry out trading. Some children demonstrated this shift in conceptions by using an extra step in addition. They would find a two-digit sum by counting on (e.g., $7 + 5 =$ "seven, eight, nine, ten, eleven, twelve"), write the two-digit marks for the English word *twelve* by using a rote unitary counting association (write 12 for *twelve*), and then shift to a collected multiunit meaning for these written marks to see them as 1 ten and 2 ones. This multiunit meaning told them what to trade (put the 1 ten with the tens and write the 2 ones in the ones position). These children had available the multiunit meaning of 12, but they initially needed the visual support of the written 12 to generate this meaning. They clearly were using a unitary conception to find the sum "twelve"; otherwise the multiunit meaning would have been directly available from the word *twelve,* and the written step out at the side would have been unnecessary. Eventually, the multiunit meaning became attached more easily to the word *twelve* for many children, and they abandoned the extra writing step. Unless they also shifted to a new multiunit method of finding single-digit sums or differences, however, they still needed to shift from a unitary to a multiunit conception. To avoid this conceptual shifting for each position in subtraction, children were shown a procedure in which all trading was done first (each position was checked to see if trading was necessary and trading was done if required), and then all positions were subtracted. Thus, multiunit quantities conceptions could be used for trading, and then unitary conceptions could be used for all the subtracting.

Children can learn to carry out the unitary-to-multiunit shifts fairly readily, and this approach can be quite successful (Fuson, 1986; Fuson & Briars, 1990). Second graders of all achievement levels did learn multiunit addition and subtraction of at least four-digit numbers, and most could explain trading using multiunit quantities in both addition and subtraction. For addition, an example of such an explanation is:

> That's eight tens and eight tens is sixteen tens and ten of those tens make one hundred and then six tens left. So trade the hundred to the hundreds place and write the six tens here. It's one hundred and six tens.

This second alternative of adding and subtracting collected multiunits for three- or four-digit numbers seems to be easier than the first alternative of doing so with multiunit sequence procedures, because children do not have to learn the difficult sequence-counting skills counting by tens, hundreds, and thousands and coordinate these with written numerals and multiunit-quantities conceptual structures.

Collected multiunit single-digit and multidigit solution procedures. Asian children learn multiunit addition and subtraction procedures for adding and subtracting single-digit numbers with sums between eleven and eighteen (these procedures were described in an earlier section); the named-ten in the language supports these procedures structured around ten. Because the language and these procedures turn the single-digit sum into multiunits (a ten and some ones), trading is not necessary. Trading (changing ten ones into one ten or ten tens into one hundred) is only required for unitary collections that exceed ten. As with the unitary sums, children can find multiunit single-digit sums within the correct value/position by using multiunit quantities conceptual structures or can ignore the value/position and treat the single digits as if they were ones. Thus, Korean children explaining addition of tens used two different approaches (Fuson & Kwon, 1990b). Some said "eight ten plus eight ten is one hundred six ten, and I put the one hundred over here in the hundred place," whereas others said "eight plus eight is ten six, and I put the ten over here (pointing to the hundred position but not naming it)." More of the second than of the third graders made the latter kind of explanation, without ever indicating that they realized the 1 was also a hundred, so, as with U.S. children, it may take some children some time to relate the single-digit-marks addition to the multiunits in each position. Sums or differences over ten can be found for any value/position by using generalizations of the ten-structured methods: "eight ten plus six ten is one hundred (putting two ten from the six ten with the eight ten) and four ten (leftover from the six ten)."

These Asian solutions, thus, suggest a third alternative for English-speaking U.S. children: Single-digit solution methods structured around ten (adding up over ten, subtracting down over ten, and subtracting from ten) could be supported with materials showing collectible multiunits. These methods could be used for single-digit sums and differences (e.g., 8 + 7) and for such sums and differences in two-digit addition and subtraction (28 + 37). Madell (1985) reported that U.S. children allowed to invent two-digit addition and subtraction procedures using base-ten blocks eventually invent exactly these ten-structured multiunit procedures using the blocks. In subtracting 53 – 24, for example, they subtract from ten by taking the 4 from one of the tens, leaving six, which is added to the original 3, making nine ones, or they subtract down over ten by taking three of the 4 ones away from the 3 ones and then taking the remaining one from a ten, leaving nine. Collectible multiunits of tens, hundreds, and thousands could support use of these ten-structured addition and subtraction methods within those values, as in the Korean example given earlier.

How easy this approach would be to implement is not clear. Baroody

(1990) suggested that multiunit structures (tens and ones) be taught as early as kindergarten and used for single-digit sums and differences. It is not clear whether children need to go through the usual developmental sequence of unitary solution procedures before reaching multiunit ten-structured conceptions. If they do, it is unlikely that children could learn ten-structured methods in kindergarten or early first grade. U.S. children might find the support of regular named tens to be of considerable help in this approach, and using English forms of Chinese words for all two-digit numbers might enable children to construct multiunits of ten and one considerably more easily and earlier than they do at present. Using fingers in the Korean way so that fingers can be reused for numbers over ten may also prove to be helpful to U.S. children in understanding ten-structured methods of single-digit addition and subtraction (see Fuson & Kwon, 1990a, in press). Madell (1985) reported that U.S. children used unitary conceptual structures with the base-ten blocks for a long time (sometimes 2 years) before inventing the Asian single-digit methods structured around ten. Perhaps these other supports, or using blocks for four-digit instead of just two-digit numbers, or explicitly discussing ten-structured methods, would enable children to understand these methods more easily.

CONCLUSION

It is clear that U.S. elementary school mathematics classrooms are failing badly in their current attempts to help children construct conceptual structures for multiunit whole numbers. This failure seems largely attributable to inadequate support within the classroom for children's construction of linked multiunit conceptual structures and for the use of these conceptual structures in multiunit addition and subtraction. The devastating effects of this inadequate support are exacerbated by two factors that interfere with U.S. children's ability to generalize the features of the English words, the written marks, and multiunit addition and subtraction procedures across several places: (a) the prolonged and piecemeal introduction of multiunit addition and subtraction over several grades and (b) the irregularities in the English system of number words for two-digit numbers that induce children to use unitary and CSD conceptual structures rather than multiunit conceptual structures for multidigit numbers.

Cross-cultural research indicates that the U.S. grade placement of multiunit addition and subtraction topics is later than that of several other nations and that U.S. children's comprehension of these topics lags behind that of children in these other countries. Thus, the failures in these areas are not due to U.S. expectations that are beyond the developmental capabilities of children. Some research indicates that, with proper classroom support for constructing multiunit conceptual structures, U.S. second graders can understand multiunit addition and subtraction and calculate accurately even for large multiunit problems. This support requires a learning and teaching setting that facilitates learning the named-value features of the English number words and the unnamed positional features of the written multidigit marks by constructing and linking six additive multiunit conceptual structures—visual layout, positions ordered in increasing value from the right, multiunit names, words ordered in decreasing value as they are said, multiunit quantities, and regular ten-for-one and one-for-ten trades—and by using these conceptual structures in multiunit addition and subtraction. Using materials that display physical collectible multiunits, from which children can construct conceptual collected multiunits for the multiunit-quantities conceptual structure, and linking collectible multiunits to written marks and English words play an especially crucial role in meaningful multiunit addition and subtraction and place value.

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