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USING A BASE-TEN BLOCKS LEARNING/TEACHING APPROACH FOR FIRST- AND SECOND-GRADE PLACE-VALUE AND MULTIDIGIT ADDITION AND SUBTRACTION

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A learning/teaching approach used base-ten blocks to embody the English named-value system of number words and digit cards to embody the positional base-ten system of numeration. Steps in addition and subtraction of four-digit numbers were motivated by the size of the blocks and then were carried out with the blocks; each step was immediately recorded with base-ten numerals. Children practiced multidigit problems of from five to eight places after they could successfully add or subtract smaller problems without using the blocks. In Study 1 six of the eight classes of first and second graders (N = 169) demonstrated meaningful multidigit addition and place-value concepts up to at least four-digit numbers; average-achieving first graders showed more limited understanding. Three classes of second graders (N = 75) completed the initial subtraction learning and demonstrated meaningful subtraction concepts. In Study 2 most second graders in 42 participating classes (N = 783) in a large urban school district learned at least four-digit addition, and many children in the 35 classes (N = 707) completing subtraction work learned at least four-digit subtraction.

The English spoken system of number words is a named-value system for the values of hundred, thousand, and higher; a number word is said and then the value of that number word is named. For example, with five *thousand* seven *hundred* twelve, the "thousand" names the value of the "five" to clarify that it is not five ones (= five) but is five thousands. In contrast, the system of written multidigit number marks is a positional base-ten system in which the values are implicit and are indicated only by the relative positions of the number marks. In order to understand these systems of English words and written number marks for large multidigit numbers, children must construct named-value and positional base-ten conceptual structures for the words and the marks and relate these conceptual structures to each other and to the words and the marks.

English words for two-digit numbers are irregular in several ways and are not named-value, in contrast to Chinese (and Burmese, Japanese, Korean, Thai, and Vietnamese) words in which twelve is said "ten two" and fifty seven is said "five ten seven." These irregularities make it much more difficult for English-speaking

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children than for Chinese, Japanese, or Korean children to construct named-value meanings for multidigit numbers (Fuson, in press a; Fuson & Kwon, in press; Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989). English-speaking children use for a long time unitary conceptual structures for two-digit numbers as counted collections of single objects or as collections of spoken words (Fuson, Richards, & Briars, 1982; Fuson, 1988a; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988); these early conceptual structures can interfere with children's later construction of named-value meanings. The lack of verbal support in the English language for named-value or base-ten concepts of ten makes it particularly important that support for constructing such ten-structured conceptions be provided in other ways to English-speaking children.

In the United States such support is rarely given or is insufficient. Children more commonly are taught multidigit addition and subtraction as sequential procedures of adding and subtracting single-digit numbers and writing digits in certain locations (Fuson, in press c). These experiences result in many U.S. children constructing conceptual structures for multidigit numbers as concatenated single-digit numbers, a view that is inadequate in many ways and results in many errors in place-value tasks and in multidigit addition and subtraction (Fuson, in press a; Kouba et al., 1988). Even many children who carry out the algorithms correctly do so procedurally and do not understand reasons for crucial aspects of the procedure or cannot give the values of the trades they are writing down (Cauley, 1988; Cobb & Wheatley, 1988; Davis & McKnight, 1980; Labinowicz, 1985; Resnick & Omanson, 1987). U.S. children also show quite delayed understanding of place-value concepts (Kamii, 1986; Kouba et al., 1988; Labinowicz, 1985; Miura et al., 1988; Ross, 1989; Song & Ginsburg, 1987).

Furthermore, in the United States, instruction in the addition and subtraction of whole numbers typically is both delayed and extended across grades more than in countries like China, Japan, Taiwan, and the Soviet Union that have been characterized as fostering high mathematics achievement (Fuson, Stigler, & Bartsch, 1988). In the United States the single-digit sums and differences to 18 consume much of the first two grades, and work on the multidigit algorithms with trading (carrying and borrowing) is distributed over 4 or 5 years beginning with two-digit problems in second grade followed by the introduction of problems one or two digits larger each year. In contrast, other countries stress mastery of sums and differences to 18 in the first grade, and they complete multidigit instruction by the third grade.

In order to use and understand English words and base-ten written marks and add and subtract multidigit numbers, children need to link the words and the written marks to each other and need to give meaning to both the words and the marks. The learning/teaching approach used in the present studies was developed to meet these goals. It is an adaptation of an approach used by the first author with teachers and children for 20 years (the teacher version is in Bell, Fuson, & Lesh, 1976). It provides children an opportunity to construct the necessary meanings by using for each system a physical embodiment that can direct their attention to crucial meanings and help to constrain their actions with the embodiments to those consistent with the mathematical features of the systems. The English named-value system of words is embodied by a set of base-ten blocks (Dienes, 1960), and the positional base-ten written marks are embodied by digit cards (numerals written on small individual cards). English words, words for the block embodiment, and words for the digit cards (see Figure 1) were used to help direct children's attention to critical features of the mathematical systems and embodiments, facilitate communication among the participants in the learning/teaching approach, and support the construction of links among the different systems and embodiments.

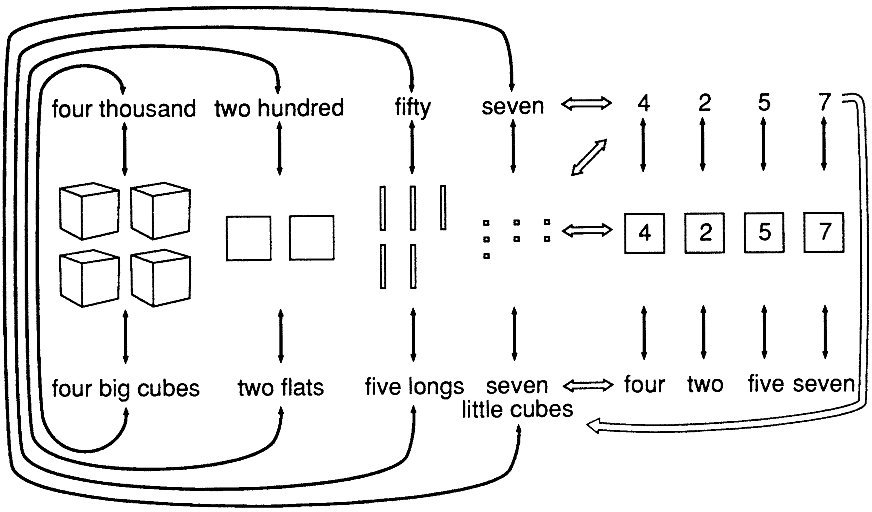

Figure 1. The learning/teaching approach.

Features of the approach in action are as follows:

- When adding and subtracting with the blocks, the blocks-to-written-marks links are made strongly and tightly: Each step with the blocks is immediately recorded with the written marks.
- Links among the English words, base-ten blocks, digit cards, and base-ten written marks are strengthened by the constant use of the three sets of words.
- Children work with the learning/teaching approach for many days; they are allowed to leave the embodiments and do problems just in written form whenever they feel comfortable doing so.
- When children begin to do written problems without blocks, their performance is monitored to ensure that they are not practicing errors.
- Addition and subtraction both begin with four-digit problems (or in some cases, these problems immediately follow initial work with two-digit problems).

- Children spend only 1 to 4 days on place-value concepts initially; much place-value learning is combined with the work on multidigit addition and subtraction.
- A modification of the usual algorithm is used for subtraction (see the methods section for Study 1).

These features, and the reasoning behind them, are discussed in Fuson (in press a), where distinctions between named-value and positional base-ten systems are discussed more fully and literature pertaining to both adequate and inadequate conceptual structures children construct for multidigit numbers are reviewed.

Results of an earlier study with this learning/teaching approach were reported in Fuson (1986a). In that study second graders and some first graders learned to add and subtract multidigit numbers much more accurately than reported for usual school instruction. Most of these children successfully and independently extended the procedures learned with the blocks to five- through ten-digit symbolic problems done without the embodiment. Children who made errors were interviewed, and those still making errors were told to think about the blocks as they solved problems. Most of these children were able to use a mental representation of the blocks to self-correct their written errors, and this use of the blocks showed understanding of place-value concepts.

This study left unanswered several important questions that were addressed by the two studies reported here. First, the grade level, achievement level, and socioeconomic level of the students who could benefit from the learning/teaching approach was not clear from the limited sample used in that initial study. Study 1 reported here extended the sample to second graders of all achievement levels and to first graders of above-average and average mathematics achievement. Study 2 extended the sample to second graders in a large urban school district. The goal for both the age/achievement and the residential extensions was not to manipulate these various background variables in order to determine their differential effects on performance. It was simply to examine whether the effects of the learning/teaching approach could be considered to generalize across a heterogeneous population.

Second, there were the practical questions of whether the learning/teaching approach could be distanced from its designer, communicated in a fairly small amount of in-service time, and implemented by teachers with little field support. These seem to be crucial issues determining the feasibility of wide-scale use of the learning/teaching approach. Distancing focused on three major aspects of this learning/teaching intervention: the classroom teaching, the in-service teaching of the involved teachers, and teaching and supervision of field support personnel. In Fuson (1986a), project staff members did some of the teaching, the project designer conducted the teacher in-service, and the field support person was taught and supervised closely by the project designer. In both of the studies reported here, all of the teaching was done by classroom teachers using lesson plans and student worksheets developed by the project designer. In the second study, the project designer did not conduct the in-service sessions nor supervise the field support persons. The amount of in-service time was fairly small for both studies: a 1-hour overview of the learning/teaching approach in the first study and one or two 2½-hour in-service sessions in the second study. Field support was provided in the first study by two teachers in each school who had taught the learning/teaching approach in the first year. In the second study, three elementary (K–8) mathematics supervisors were available to provide field support for the 132 second-grade teachers targeted for the learning/teaching approach, but these supervisors also had many other duties.

The results of the two studies reported here are analyzed with respect to three goals of the learning/teaching approach:

1. understanding multidigit addition and subtraction and justifying procedures with named-value/base-ten concepts;
2. understanding place-value concepts;
3. being able to add and subtract multidigit numbers of several places, including subtraction problems with zeros in the top number.

The literature concerning performance in these areas by children receiving usual instruction is briefly summarized in the discussion of the results of each study in order to provide a context within which to interpret the results.

STUDY 1

Method

Subjects

Children from two schools in a small city on the northern border of Chicago served as subjects. Teachers grouped children by mathematics achievement in these schools depending upon recommendations of the previous teacher; children were moved to a different room at any time a teacher thought that a move should be made. In each school there were sufficient first graders for three math classes, one each of low, average, and high math achievement. The high-achieving first-grade classes from both schools were asked to participate in the study. The teachers of the average-achieving first graders in both schools asked later in the year to participate and were allowed to do so. In one school there were three second-grade math classes, one each of low, average, and high math achievement. Many of the children in the high-achieving class had received addition multidigit instruction as first graders in the study reported in Fuson (1986a), so only the low- and average-achieving classes participated. In the other school there were only enough second graders to form two classes. The five lowest achieving second graders were grouped with a low-achieving first-grade class, and the remaining children were grouped into a high/average and an average/low class. Many of the children in the high/average class had received addition multidigit instruction as first graders, but this class was retained in the present study in order to study subtraction learning for all children and addition learning for the new children. All eight classes (N =

169) participated in the addition instruction, and three second-grade classes (N = 75) received the subtraction instruction.

Teachers

Four teachers (two from each school) had participated in the multidigit instruction in the Fuson (1986a) study. The other teachers were given a brief overview of the instruction, lesson plans, student worksheets, and tests. For questions and further help, they were to rely on the two teachers in their school who had taught the materials before. A research project assistant also visited the schools weekly to check on teaching progress.

Instruction

All children first learned to find sums and differences to 18 by counting on and counting up with one-handed finger patterns (see Fuson, 1986b, 1987, 1988b; Fuson & Secada, 1986; Fuson & Willis, 1988). These counting procedures could be used for any addition and subtraction facts children did not know. They have been found to be efficient and accurate enough for use in the multidigit algorithms (Fuson, 1986a).

Each class had at least one set of base-ten blocks. The first phase of instruction focused on exploration of the relationships between the different blocks and on use of the blocks words (little cubes, longs, flats, big cubes, or names chosen by children) and English words (ones, tens, hundreds, thousands). Both the consistent one-for-ten and ten-for-one trades between adjacent places and the nonadjacent trades (one-for-hundred and one-for-thousand) were discussed and demonstrated. Then the blocks were used to make different three- and four-digit numbers (e.g., 3725), and index cards each containing one numeral were used to make the base-ten version of the number beside the blocks (e.g., four cards containing the numerals 3, 7, 2, and 5 were selected and were put down in order to the right of the blocks). These cards, and numerals written on children's worksheets, were read by base-ten words (e.g., "three seven two five"). These activities were accompanied by much verbalization of the block words, the English words, and the base-ten words.

Addition and subtraction with the blocks were done on a large cardboard calculating sheet (see Figure 2). Addition was considered first. A written problem was given. Blocks for the top number were placed in the top row of the calculating sheet, and then blocks for the bottom number were placed in the second row (see Figure 2). Addition was done column by column, beginning on the right. The blocks in a given column were added together (pulled down) into the bottom row. If the sum was nine or less, it was recorded with the digit cards. Each child also recorded each step on his or her own worksheet. If the sum was over nine, ten of the smaller pieces were traded for one of the next larger pieces, and the result recorded with digit cards and on individual worksheets. Much verbalization of all three sets of words accompanied all addition and subtraction, and recording with written marks was done after each action with the blocks. The necessity of trading

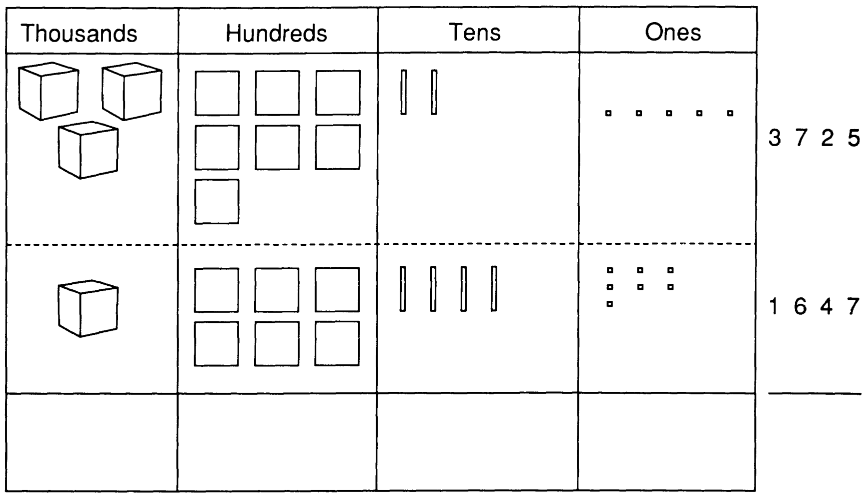

Figure 2. Calculating board with an addition problem.

was raised by showing what happens with the digit cards if a two-digit number is written in any column (the other digit cards get moved over to the left, making a bigger number). The fairness of the ten/one trades, and the idea of trading to get more (in subtraction) or trading when you had too many (in addition), arose from the size of the blocks: ten of the blocks in any column were equivalent to one block in the column to the left.

Multidigit subtraction can be shown in various ways with the blocks, and the subtraction within each value can be phrased in different ways in words. The children in this study had multiple interpretations of subtraction available (as take-away, comparison, and equalize, see Fuson, 1986b, 1988b; Fuson & Willis, 1988). We suggested that teachers verbalize the subtraction within values as "Seven plus how many to make twelve?" or "Twelve minus seven is how many?" (because these fit children's use of counting up to find these differences better than using the words "take-away") and that they separate the blocks for the top number into those that match the bottom number (the subtrahend) and the leftover blocks (the difference) and then move the difference nonmatching blocks to the bottom row as the answer.

A simplification of the usual algorithm was also used. Children first checked each column of the top number to be sure that it was larger than the bottom number in that column. If a top digit was not as large, a one-for-ten trade (borrow, regrouping) was made from the column on the left. After all the necessary trading had been done to the top number so that each top number was as large as or larger than each bottom number, subtraction was done column by column. Both the trading and the subtracting can be done from either direction, but teachers usually modeled the typical U.S. right-to-left approach. This trade-first algorithm reduces the difficult alternation of trading and subtracting used in the common algorithm and thus eliminates the need for children to switch repeatedly from a named-value representation for trading to a unitary representation for subtracting (Fuson & Kwon, in press). The initial sustained focus on making all the top columns larger also helps to avoid the common error of subtracting the top number from the bottom number when the top number is smaller.

Teachers organized their classrooms in different ways for this instruction. Some worked with the whole class, having children participate in solving problems with the blocks and the index cards. Others divided their class into small groups and either worked with groups simultaneously or serially while the other groups worked on other topics. In the former case, children who had learned the blocks procedure the year before or older children shown how to use the blocks worked with each group initially to ensure that the blocks and written-marks procedures were correct and that children in the group were understanding the relationships involved. In all cases all children had worksheets, and all recorded each problem as it was worked with the blocks.

Children in the average and high-achieving second-grade classes were able to do three- and four-digit addition and subtraction problems with the blocks initially. In the low second-grade class and first-grade classes, children had difficulty relating the four columns of blocks to the four columns of written marks. Therefore, in these classes two-digit problems were done first, and then three- and four-digit problems were done with the blocks and written marks. Whenever children said they understood the written-marks procedure and did not need the blocks any more, they were allowed to go to their seats to work on worksheets containing three- and four-digit problems. Their procedure was checked by someone before they were allowed to leave the blocks. Worksheets with larger problems (up to eight digits) were available for children who wished to try them.

Work on subtraction was followed by very short units focusing on aspects of meaningful addition (alignment of problems with different numbers of digits, adding 3 two-digit numbers requiring a trade of 2) and place value (translating from mixed order words to numerals and vice versa with no trades, doing the same with trades required, and choosing the larger of two multidigit numbers). The lesson plans described how attention could be directed within the learning/teaching approach to facilitate the learning of these concepts.

The time necessary to complete each unit varied considerably from class to class. The initial introduction/addition unit took from 3 to 6 weeks, and the subtraction unit took from 2 to 4 weeks. Each meaningful addition and place-value concept took about a day.

All of these classes were also participating in an instructional research project focused on teaching addition and subtraction word problems. These topics and the multidigit topics went far beyond the district goals. Teachers had to meet district goals as well as teaching these extra topics. In some classes teachers also had to cover considerable ground before the multidigit work could begin (e.g., learning about single-digit sums and differences to 18). Therefore different classes completed different topics. The low-achieving second-grade class and one average first-grade class only completed addition of two- and three-place numbers. The teacher of the other average first-grade class only taught multidigit addition to 10 of the 24 children in her class, but she did complete the generalization of the algorithm past four places with the participating children. All other classes completed the generalization of the addition algorithm to problems with as many as seven digits. The high and average second-grade classes completed subtraction, and the average/low second-grade class completed ordinary subtraction and began work on problems with zeros in the minuend. The work on meaningful addition and place-value concepts was completed only by the high- and average-achieving second-grade classes.

Measures of Skill and Understanding

Addition and subtraction calculation tests. All children were given two addition pretests. The Timed Addition Test contained 12 problems, with 2 two-digit, 2 three-digit, 3 four-digit, 1 five-digit, 3 six-digit, and 1 seven-digit problem; children worked on these problems for 2 minutes. All problems required trading in one or more places (the number of trades ranged from one to five). The Ten-Digit Addition Test was a single ten-digit problem (6385740918 + 8557586736). All problems were written aligned in vertical form. These same two tests were also given as posttests. The lower achieving and younger classes were also given an Untimed Addition Minitest of four problems (2 two-digit and 2 three-digit problems, each requiring one trade). Parallel subtraction tests (Timed Subtraction Test, Ten-Digit Subtraction Test, Untimed Subtraction Minitest) were made by using inverse problems from the addition tests; children were given 3 minutes for the Timed Subtraction Test because subtraction had been slower than addition in the earlier study. A fourth subtraction test (Zeros Subtraction Test) consisted of four problems with zeros in the top number: 1 two-digit, 2 three-digit, and 1 four-digit problem with one, one, two, and three zeros, respectively.

The tests for each child were first evaluated to determine whether the child showed any evidence of correct trading; two correctly traded columns were required for the child to be judged as showing some indication of trading.

Each test was then scored to permit a finer evaluation of performance. Scoring was based on each digit in the answer: one point was given for each correct digit. This procedure was adopted because scoring each problem only as correct or incorrect does not differentiate a solution in which all columns but one are correct from a solution in which a child demonstrated no notion of multidigit addition or subtraction.

An analysis of the kinds of errors was made on the ten-digit problem. The errors identified in Fuson (1986a) were classified into four categories reflecting increasing amounts of knowledge about multidigit addition or subtraction as follows:

1. Preaddition/presubtraction error: Columns were left blank or filled in with seemingly random numbers; presubtraction errors also included adding.

2. Column addition/subtraction error: Addition/subtraction problems were approached column by column: In addition the sum of each column was written below that column even when the sum was a two-digit number (e.g., 28 + 36 = 514); in subtraction the smaller number in each column was subtracted from the larger number (e.g., 36 - 28 = 12).

3. Trading error: Trading errors involved some partially successful attempt to trade (carry, borrow); in addition problems these errors included the following: the trade was not written or added in anywhere, a trade was made when the sum was not over 9, the tens digit rather than the ones digit was traded, a trade was made but ignored when that column was added (this error might have been a fact error—such errors were counted as both trade and fact errors), the trade was subtracted from rather than added to the top number; in subtraction problems these errors included the following: the left column was not reduced by one even though a trade was recorded in the right column, a trade was made even though the top number was already larger, more than one trade was made from a given column, 1 was subtracted from the traded-to column, the right column received 11 rather than 10, 1 was subtracted from a left column even though no trade was recorded to the right.

4. Fact error: Fact errors involved correct trading but incorrect adding or subtracting in a column.

Two coders coded all errors. Coder agreement was 97%.

Because not every column in every problem required a trade, children making consistent column addition/subtraction errors could get 20% correct digit scores on both Untimed Minitests and 9% on the addition Ten-Digit Test, and children making trading errors that were incorrect in only one column could get digit scores ranging between 36% (on the Ten-Digit Tests) and 60% (on the Untimed Minitests).

Place-value and meaningful multidigit addition written tests. Three aspects of place-value understanding and two aspects of meaningful multidigit addition were assessed through written tests. The Mixed Words to Numerals Test required a child to write a three- or four-digit numeral for numeral/word named-value combinations given in mixed order (e.g., 6 hundreds, 4 tens, 5 thousands, and 7 ones). The Traded Word/Numeral Test required a child to write a three- or four-digit numeral for numeral/word named-value combinations given in standard order (e.g., 2 thousands, 16 hundreds, 1 ten, and 4 ones) or to fill in a numeral blank when the three- or four-digit numeral was given with the numeral/word named-value combination (e.g., 2643 is 2 thousands, ___ hundreds, 14 tens, and 3 ones). All of these items had one numeral/word pair that exceeded 10 and thus had to be traded to the left in the former items or to the right in the latter items to make the correct answer; these items were modeled after those in Underhill (1984). The Choose the Larger Number Test required a child to choose the larger of a pair of three- through seven-digit numbers by circling the larger number and by inserting a < or > between the pair of numbers. The five pairs of numbers were all misleading in that all digits in the smaller number except one were equal to or greater than the corresponding digits in the larger number. The Alignment Test presented horizontally-written problems whose addends had different numbers of digits; children were told to write the problem so that it could be added easily. This tested a combination of place value and addition understanding—understanding that one added and therefore aligned like places; the different numbers of digits were chosen to maximize the frequent error of aligning such problems on the left rather than on the right. The Trading 2 Instead of 1 Test consisted of problems with three addends that required a trade of 2 tens rather than 1 ten because the sum in the ones column exceeded 20; the first item had a sum of 21 to maximize the possibility that children would rotely trade the 1 as they had been doing for problems with two addends rather than trading the number of tens (2, in these problems). These tests had between two and six items. Each test item was marked as correct or incorrect, and test means were converted to percentages for ease of comprehension of the test results.

Understanding of Addition, Subtraction, and Place Value

Individual interviews were carried out to assess children's understanding of addition, subtraction, and place value. Eight children from one class at each achievement level were randomly selected to be interviewed (the average-achieving first graders were from the class in which all children participated). Therefore, the addition interview sample contained 40 children, and the subtraction interview sample consisted of the 24 second graders in the addition interview sample. Interviews were conducted individually in a room outside the classroom. Children were shown solved multidigit problems, each written on a separate index card. Each problem solution was written in a color different from the color of the original problem. Two addition problems were solved correctly: a two-digit problem with a trade from the ones to the tens and a four-digit problem with a trade from the hundreds to the thousands. Two addition problems were solved incorrectly. The two most common addition errors before instruction were used: (1) column addition, for example, for 8 + 6 writing 14 in the ones column and (2) ignoring the tens digit of a two-digit sum and just writing the ones digit. Five subtraction problems were given. Two were solved correctly and paralleled the correctly solved addition problems, except that different numbers were used. A third showed the common error of column subtraction—subtracting the smaller from the larger number even when the smaller number is on the top. Two three-digit problems with two zeros in the top number were given. One was solved correctly, and the other showed 1 hundred traded for 10 ones.

Children were told that they would be shown problems that somebody else had solved and that some problems were correct and some were wrong. They were then shown an index card with a problem written on it and asked if that problem was right or wrong. After a judgment was made, they were asked why it was right or wrong. The interviewer wrote down verbatim the child's responses and any interviewer prompts. Children were randomly assigned to one of two different orders of problems. One sequence began with a correct problem, and the other began with an incorrect problem. The more difficult problems (the four-digit addition problem and the subtraction problems with zeros) were given in the last half of the interview.

The interview records were classified by the interviewer and one of the authors. The classification of a problem as correct or incorrect was evaluated first. If a child changed his or her answer, the last assignment was coded. The raters agreed on 100% of these classifications. The interviews were coded for place-value understanding of the tens or hundreds values of written numerals within an explanation of addition or subtraction; to receive credit, a child had to use the word "ten" or "hundred" to identify a numeral correctly sometime during an explanation. The addition interviews were coded for two aspects of addition and place-value understanding: (a) explaining the written procedure as trading 10 ones for 1 ten or 10 tens for 1 hundred, and (b) identifying the traded 1 as a ten or as a hundred. For (a) a child had to explain explicitly the trading or say that the ten came from the 13 ones or the hundred came from the 16 tens. The subtraction interviews were coded for three aspects of subtraction and place value understanding: (a) explaining the written procedure as trading 1 ten for 10 ones or 1 hundred for 10 tens; (b) identifying the traded 1 as a ten or as a hundred; and (c) explaining the double trading over two top zeros, i.e., the trade of 1 hundred for 10 tens and the trade of 1 ten for 10 ones.

All of these aspects were evaluated for tens and for hundreds. Coder agreement was 95%. Children's explanations did not always spontaneously cover all of the coded aspects of the interview. A series of prompts was used to try to ascertain such knowledge. These included questions about the traded 1 ("What's the one?" or "One what?") and a question about the 8 tens in the four-digit addition problem ("Eight what?"). The most explicit prompt was to ask a child to think about the blocks; this was used when a child failed to give any answer to other prompts. However, due to the complexity of the interview and the fact that the attributes of the responses to be coded were finalized after the interviews were completed, needed prompts were not always given. Thus, the data may underestimate children's knowledge.

Results

Addition Multidigit Computation

On the pretests only 9 of the 169 children showed any indication of correct trading, whereas on the posttests 160 of the 169 children showed such evidence, a very large and statistically significant change (McNemar's test chi-square = 151, $p < .0001$). Of these 160 children, 156 correctly traded on a four-digit or larger problem. Of the 13 children failing to demonstrate correct trading or doing so only for two- or three-digit problems, 7 were in the average-achieving first-grade class and 5 were in the low-achieving second-grade class. Paired t-test analyses of pretest-posttest differences on the digit scores for each test for each class separately revealed significant improvement for every test for every class, $p < .001$ in all cases.

Posttest digit scores are shown in Table 1. These indicate excellent performance for all classes except the low-achieving second-grade class and the average first-grade class in which all children participated in the learning/teaching approach. Even the latter two classes demonstrated some learning, because their Untimed Minitest scores were well above those obtainable by carrying out column errors (75% and 69% compared to 20% for column errors). Teachers reported that children were enthusiastic about the multidigit instruction and enjoyed solving large problems and that many of the higher-achieving second graders knew most of their addition facts or used thinking strategies to find sums they did not know and most of the other children counted on with one-handed finger patterns to solve sums they did not know.

Table 1
Addition Computation Posttest Digit Score Means for Each Class and Achievement Level in Study 1

	Grade/achievement level							
Tests	2 High/av	2 Av	2 Av/low	2 Low	1 High	1 High	1 Av	1 Av
n	29	23	21	14	26	25	10	21
Percentage of correct digits in answers								
Untimed Minitest	ng	ng	ng	75	92	98	92	69
Ten-Digit Test	99	93	90	58[a]	88	91	93	ng
Timed Test	98	91	94	74	92	94	91	ng
Mean number of correct digits completed in 2 minutes on Timed Test	28	25	26	15	17	24	12	ng

Note. Percentage of correct digits in the answer is out of all digits in the Untimed Minitest and Ten-Digit Test and out of the columns attempted by a given child in the Timed Test. *ng* means the test was not given.
[a] The low-achieving second-grade class only completed 2- and 3-digit addition.

The errors made on the Ten-Digit pretests and posttests are given in Table 2. These analyses show a large reduction in the number of errors made. Few of the primitive preaddition and column addition errors were made on the posttest. There was a reduction in the trading errors and no increase in the fact errors in spite of the fact that almost all children were adding and trading on almost all problems on the posttest.

Table 2
Number and Kinds of Pretest and Posttest Addition and Subtraction Errors in Study 1

	Preaddition/ presubtraction		Column add/sub		Trading error		Fact error	
Tests	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Addition Ten-Digit Test	527	28	837	18	109	79	57	45
Subtraction Ten-Digit Test	135	4	650	14	0	96	8	22

Note. There were a possible 1859 errors in addition and 825 errors in subtraction calculated by multiplying the number of digits in the answer (11) by the number of subjects (N = 169 for the Addition Test, N = 75 for the Subtraction Test).

Subtraction Multidigit Computation

On the pretests 2 of the 75 children participating in the subtraction learning/teaching approach showed some evidence of trading; on the posttests 72 of the 75 children showed such evidence, a very large and statistically significant change (McNemar's test chi-square = 70, $p < .0001$). Five of these 72 children demonstrated such trading for two- or three-digit problems but not for larger problems. Paired t-test analyses of pretest-posttest differences on the digit scores for each test for each class separately revealed significant improvement for every test for every class, $p < .001$ in all cases.

Mean digit scores for each test for each class are given in Table 3. Performance by the high/average class was excellent on all tests, and for the other two classes performance was good on the Timed Test and the Untimed Minitest. Scores for the average and average/low classes on the Ten-Digit Test and on the Zeros Test revealed weaker performance that was nevertheless above the level of consistent trading errors (36% and 33%, respectively). Teachers reported that some children knew subtraction facts or used thinking strategies to determine difficult differences but that most counted up with one-handed finger patterns to determine facts they did not know.

Table 3
Subtraction Computation Posttest Class Means by Achievement Level in Study 1

	Achievement level		
Tests	High/av	Av	Av/low
n	29	23	23
Percentage of correct digits in answers			
Untimed Minitest	ng	89	87
Ten-Digit Test	95	72	75
Timed Test	95	84	84
Zeros Test	92	78	(49)
Mean number of correct digits completed in 3 minutes on Timed Test	22	15	16

Note. Percentage of correct digits in the answer is out of all digits in the Untimed Minitest, Ten-Digit Test, and Zeros Test and out of the columns attempted in the Timed Test. The Zeros Test for the Av/low class is in parentheses because this class only began work on zero problems. *ng* means the test was not given.

The error analyses presented in Table 2 indicate an almost complete elimination on the posttest of the large number of presubtraction and column subtraction errors made on the pretest. Substantial numbers of trading errors were made on the posttest, but most posttest trading was correct (over 80% of the trades involved no error in either column). Few fact errors were made on the posttest, only half as many fact errors as were made in addition.

Place-Value and Meaningful Multidigit Addition Written Tests

Results of the written test measures of place-value and meaningful multidigit addition are given in Table 4. Almost all children taking these tests were misled by these items on the pretest (except for the Circle the Larger Number Test). On the posttest, children in both classes showed very considerable gains on the place-value tests, all children correctly aligned problems, and most children traded 2 when they had 20-some ones or tens.

Table 4
Percentage Correct on Place-Value and Meaningful Addition Written Test in Study 1

	Grade/achievement level			
	2 High/av		2 Av	
Tests	Pre	Post	Pre	Post
Place-value tests				
Mixed Words to Numerals Test	3	98	8	83
Traded Word/Numeral Test	2	90	3	72
Choose the Larger Number Test				
Circle the larger number	50	96	34	84
Insert > and < symbols in the number pairs	0	96	44	98
Meaningful addition tests				
Alignment Test	0	100	5	100
Trading 2 Instead of 1 Test	0	100	12	73

Note. The Class 2 High/av pretests were given early in the year, and the Class 2 Av pretests were given midyear.

Understanding of Place-Value, Addition, and Subtraction

Every interviewed child correctly classified all four addition problems as having been solved correctly or incorrectly, 94% correctly classified the subtraction problems with no zeros, and 94% of the children completing instruction on the subtraction problems with zeros classified such problems correctly. Results of the interview measures are given in Table 5. Every child but one identified a numeral in the tens place as x tens at least once during their explanations. Similar identification of a hundreds numeral was done by 92% of the second graders but by only 50% of the first graders. Almost every child explained the ten-for-ones trading and identified the traded 1 as a ten for both addition and subtraction; three-fourths of these explanations were spontaneous without any prompts. The problems with errors were much more effective than were correct problems in eliciting spontaneous explanations, indicating that the children were not just repeating memorized verbal explanations for correct problems. For the hundreds concepts prompts were required for about three-fourths of the responses, but this seemed to stem as much from the fact that only a correct problem was given for the hundred trade as from hundreds being more difficult. Children in the second-grade average/low-achieving class and especially in the average-achieving first-grade class showed more limited understanding of the ten/hundred trade than did the children in the other three classes. Most children failing to identify the traded 1 as a hundred identified it as a ten, and most of these identified that 1 as coming from the "8 tens plus 8 tens is 16 tens." Thus, they had learned a general aspect of multidigit trading, to trade the tens digit from any two-digit sum, but they could not simultaneously fit this general view of trading within the named-value places to name the new value

Table 5
Percentage of Students Demonstrating Understanding of Place Value, Addition, and Subtraction in Study 1

	Grade/achievement level									
	2 Hi/av		2 Av		2 Av/lo		1 Hi		1 Av	
Tests	Ten	Hun	Ten	Hun	Ten	Hun	Ten	Hun	Ten	Hun
Place-value understanding										
Identify the tens and hundreds values of written numerals	100	100	100 (13)	100	100	75	100	75	88 (25)	25
Addition and place-value understanding										
Explain written procedure as trading 10 ones for 1 ten or 10 tens for 1 hundred	100	88	100 (13)	100 (25)	88 (50)	50	100	100 (13)	88 (13)	13
Identify the traded 1 as a ten or a hundred	100	88	100 (38)	88 (25)	100 (38)	63	100	100 (13)	88 (25)	13
Subtraction and place-value understanding										
Explain written procedure as trading 1 ten for 10 ones or 1 hundred for 10 tens	100	100	100	100 (13)	100	75	ng	ng	ng	ng
Identify the traded 1 as a ten or a hundred	100	100	100	100	100	75	ng	ng	ng	ng
Explain the double trading over two top zeros: hundreds to tens and tens to ones	100	100	75	88	38	38	ng	ng	ng	ng

Note. Percentages in parentheses are children who responded only after they were prompted to think about the blocks. *ng* means the test was not given.

of the traded 1. Not a single interviewed child identified the traded 1 as a one, in sharp contrast to children receiving traditional instruction.

Discussion

The second graders and high-ability first graders showed multidigit addition and subtraction computation performance that was very considerably above that shown by third graders receiving traditional instruction (cf. Kouba et al., 1988). The subtracting-smaller-from-larger error that is so common in multidigit subtraction was almost completely eliminated. These children also showed competence far above that usually demonstrated by third graders in verbally labelling tens and hundreds places, in changing words to numerals and vice versa even when these were given in mixed order or required trading, in choosing the larger number, in aligning uneven problems on the right rather than on the left, in showing the quantitative meaning of tens and ones, and in identifying the traded 1 in addition and subtraction as a ten or as a hundred rather than as a one (cf. Cauley, 1988; Ginsburg, 1977; Kamii, 1985; Kamii & Joseph, 1988; Labinowicz, 1985; Resnick, 1983; Resnick and Omanson, 1987; Ross, 1986, 1989; Tougher, 1981).

Kamii (1985; Kamii & Joseph, 1988) and Ross (1986, 1989) reported that on digit correspondence tasks most second graders and many third and fourth graders receiving traditional instructional show no understanding that the tens digit means ten things (these children show one chip—rather than ten chips—to demonstrate what the 1 in 16 means), or they are misled by nonten groupings and show only a grouping face-value meaning (for 13 objects arranged as three groups of four objects and one left-over object, they say that the 3 means the three groups and the 1 means the one left-over object). These tasks were not available at the time this study was carried out, but reviewers raised the question of whether children in the study would have demonstrated place-value understanding on these tasks. At that time two teachers were still carrying out reasonable facsimiles of the instruction with their above-average and average-achieving second-grade classes. In an attempt to provide some information on this issue, half the children from each achievement-level grouping within each class were randomly chosen to be individually interviewed ($n = 22$). They were given these two tasks and a subtraction problem with zeros in the top number.

On the Kamii task (showing with chips what the 6 and the 1 in 16 mean), 12 children immediately showed ten chips as the meaning of the 1, another 4 first showed one chip but showed ten chips when asked to show with the chips "what else could this part (the 1) mean?" another child showed ten chips when given the task again after working the four-digit subtraction problem, and 3 children first showed one chip but showed ten chips when asked to "look at the places" in 16 (tens and ones were not mentioned). Thus, more than half of these children had tens and ones available as their first meaning for a two-digit numeral and four others had it readily available as a second choice, while four more first showed their unitary meaning but showed a tens and ones meaning when a multidigit context was elicited for them; overall, 91% of the interviewed children showed that the 1 meant ten objects. Not a single child showed a grouping face-value meaning on the Ross task; performance was the same as performance on the Kamii task. Thus, on these tasks also, second graders using the base-ten blocks showed performance considerably above that ordinarily shown by second graders receiving traditional instruction.

STUDY 2

Method

Subjects and Teachers

Potential subjects were all second graders in the 132 second-grade classrooms in the Pittsburgh Public School system. A 2½-hour in-service training session on using base-ten blocks to teach multidigit addition and subtraction was offered to all second-grade teachers in August. This in-service session was voluntary; teachers were paid salary to attend. The workshop went through the teacher plans for the learning/teaching approach, focusing particularly on using the blocks and linking actions on the blocks to steps in the written multidigit addition and subtraction procedures. In November, a follow-up 2½-hour session focusing more intensely on subtraction (including the new trade-first algorithm) was given to these teachers, and a 2½-hour session on both addition and subtraction was given for teachers who had not attended the August session. These sessions were given by the second author, who is experienced in using the base-ten blocks to teach the multidigit algorithms. A math supervisor who had no previous experience with the base-ten blocks gave another 2½-hour in-service session in December for those not able to attend earlier sessions. Most second-grade teachers (91%) attended at least one of these sessions.

Teachers were urged to use the base-ten blocks and lesson plans to teach the multidigit algorithms. Three elementary mathematics supervisors were available as questions arose, though they also had many other duties concerning teachers at other grade levels. The supervisors encouraged teachers to try the approach, but because the goals went considerably beyond the district second-grade goals, participation was voluntary. Many teachers started teaching multidigit addition and subtraction somewhat late in the year and expressed doubts that they would be able to finish all of the units. In order to increase the number of teachers finishing at least the addition and subtraction work, the supervisors suggested not covering the meaningful addition and place-value units but finishing the subtraction work at least up to the problems with zeros. The number of teachers and children who participated in various aspects of the study are discussed in the final section of the methods section.

Instruction

Teacher lesson plans and a class set of student worksheets in individual student booklets (both as described in Study 1) were sent to each second-grade teacher in the district. At the in-service sessions some teachers expressed a preference for using the blocks to show subtraction as take-away instead of as comparison because the take-away method fitted better their conception of subtraction as take-away. Teachers were allowed to use take-away if they wished: The top number (the minuend) was made with blocks and blocks were taken away for the bottom number. One class set of base-ten blocks (the Educational Teaching Aids neutral-colored blocks, metric version) was available in each school.

Testing

Tests. The addition and subtraction calculation tests and the place-value and meaningful-addition written tests used in Study 1 were used in this study. All tests were given as pretests at the beginning of the year. The same tests were given as posttests as each phase of the learning/teaching approach was finished (e.g., the addition calculation tests were given at the completion of the addition teaching). Teachers graded all tests according to written directions. They returned to the central district office the pretests accompanied by a class list containing pretest scores. Posttests accompanied by a class list with posttest scores were returned to the central office as teachers gave them. For both the pretests and the posttests, the tests of four children in each classroom, two boys and two girls, were randomly selected and graded by research staff members in order to check the teacher grading. The few teachers with systematic grading errors had their scores corrected.

Criterion scores and error classification. Criterion scores were adopted for the addition and subtraction Untimed Minitests, the addition and subtraction Ten-Digit Tests, and the subtraction Zeros Test. These were based on the digit scores described for Study 1. The trading criterion score was 8 or more for the addition and subtraction Untimed Minitests and the addition and subtraction Ten-Digit Tests because a child making trading errors that were incorrect in only one column could obtain scores of 6 out of 10 on the Untimed Minitests and 4 out of 11 on the Ten-Digit Tests. A score of 8 required a child to make at least two correct trades with no fact errors on the Untimed Minitests and four correct trades with no fact errors on the Ten-Digit Tests. For the subtraction Zeros Test, a criterion score of 9 (of the 12 digits correct) was selected because this score meant that the child demonstrated correct trading for at least two of the three zero aspects tested.

Error analyses were carried out on four Ten-Digit Tests drawn at random from each of 30 classrooms randomly selected for each test and time (pretest, posttest). Errors were classified into the four categories used in Study 1. The classification was done by the same two coders used in Study 1; coder agreement was 96%.

The Pretest and Posttest Samples

Of the 132 teachers, 125 (95%) returned pretests for 2723 children. Pretests were returned from at least one classroom for every school in the district. Across all of the tests the number of completed pretests ranged between 2531 and 2378. To ascertain whether the pretests represented the whole sample of children with one or more returned pretests, on each test the scores of children who had complete data on all tests were compared to scores of children who had one or more missing scores on other tests. There were no significant differences between these groups on any tests.

Only part of the potential sample of classrooms completed the work with the learning/teaching approach and returned the posttests. The number of children with returned posttests is given for each test in Table 6. The number of teachers who returned addition calculation, subtraction calculation, and place-value/meaningful addition posttests was 42, 35, and 16, respectively. These teachers came from 18, 18, and 9 different schools, respectively. This partial return raised the obvious question of whether the children for whom posttests were returned differed from the children without returned posttests. The addition calculation pretests were the focus of the difference analyses because all other pretests showed floor effects. Several aspects of the addition pretests for the children with no returned posttests were compared to pretests for the children with returned posttests. The percentage of children with pretest scores on the Untimed Minitest at or above criterion was a bit higher for the children with no posttests than for those with posttests, the mean digit scores on the Untimed Minitest and the Timed Test were about the same for both groups, and children with no posttests showed somewhat more advanced errors than did the children with posttests (more of the former made at least one trading error while more of the latter made preaddition or column addition errors). Thus, the posttest sample children were, if anything, initially a bit worse at multidigit addition calculation than the children not participating, and all children showed floor effects on the multidigit subtraction calculation and place-value and meaningful addition pretests; that is, both groups had the same low level of initial knowledge.

Most (90%) of the posttest teachers came from a school in which all the second-grade teachers returned posttests. Teachers within a given school almost always returned exactly the same posttests. Thus, the performance data to be reported come from all achievement levels of second graders. The schools with all teachers participating were distributed across the whole range of schools in the city with respect to location, ethnicity, and socioeconomic level. Participating teachers did not seem to differ much from nonparticipating teachers in their rate of attendance at the in-service sessions: 76%, 15%, and 10% of the participating teachers attended two, one, and zero sessions, respectively, while these percentages for the nonparticipating teachers were 68%, 23%, and 9%.

Two classes from a magnet school were dropped from the addition sample because more than half the children were above criterion on the pretest, indicating previous addition instruction. One of these classes was also dropped from the subtraction sample for the same reason.

Results

Addition Multidigit Calculation Performance

On the pretest 10% of the instructed sample met the criterion on the Untimed Minitest, and on the posttest 96% of the children met this criterion. This shift for the Ten-Digit Test was from 5% on the pretest to 90% meeting criterion on the posttest. Both of these changes were significant, McNemar's test of correlated proportions chi-square = 674 and 659, $p < .0001$. The children were quite accurate adders, with digit scores on the three tests showing that they solved between 89% and 96% of the columns correctly (Table 6), and they solved a mean of 24.3 columns of multidigit problems correctly in 2 minutes.

These children showed the same large reduction in preaddition and column addition errors from the pretest to the posttest as shown by the children in Study 1 (see Table 7). Trading errors were also reduced considerably, even though almost all children were trading on the posttest.

Subtraction Multidigit Calculation Performance

Hardly any children met the trading criteria on the subtraction pretests (2%, 1%, and 0.4% on the Untimed Minitest, Ten-Digit Test, and Zeros Test, respectively), but 84%, 70%, and 81% of the instructed children met the criterion on the respective posttests. These changes were all significant, McNemar's chi-square = 580, 487, 486, $p < .001$. Children obtained digit scores on the various tests ranging between 80% and 90% correct (see Table 6). Children solved subtraction problems more slowly than addition problems, solving a mean correct 18.4 columns in 3 minutes.

Table 6
Percentage Correct on the Addition and Subtraction Computation Posttests and the Place-Value and Meaningful Addition Posttests in Study 2

	n	% Correct
Addition computation tests		
Untimed Minitest	783	96
Ten-Digit Test	776	89
Timed Test	780	92
Subtraction computation tests		
Untimed Minitest	707	90
Ten-Digit Test	705	80
Timed Test	669	85
Zeros Test	602	85
Place-value tests		
Mixed Words to Numerals Test	360	88
Traded Word/Numeral Test	360	53
Choose the Larger Number Test		
Circle the larger number	360	67
Insert > and < symbols in the number pairs	363	65
Meaningful addition tests		
Alignment Test	300	85
Trading 2 Instead of 1 Test	278	80

Note. The % correct for the addition and subtraction computation tests are the percentage of correct digits out of the total digits in the Untimed Minitests, Ten-Digit Tests, and Zeros Test and out of the digits attempted by a given child in the Timed Tests.

Table 7
Number and Kinds of Pretest and Posttest Addition and Subtraction Errors in Study 2

	Preaddition/ presubtraction		Column add/sub		Trading error		Fact error	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Addition Ten-Digit Test	341	7	798	3	79	45	11	83
Subtraction Ten-Digit Test	282	8	984	26	6	187	1	58

Note. There were a possible 1320 errors for each test calculated by multiplying the number of digits in the answer (11) by the number of subjects (N = 120).

The subtraction-error analyses indicated a substantial movement from the presubtraction and column subtraction errors to the more advanced trading and fact errors (see Table 7). The percentages of posttest errors falling within each error category are similar for Study 1 and Study 2.

Place-Value and Meaningful Addition Tests

The pretest scores on most of the place-value and meaningful addition tests were very low, indicating that children were responding to the misleading nature of the items. For example, on the Alignment Test, most children aligned the numbers on the left, recopied the problems horizontally, or treated each digit as a separate number and formed new problems (e.g., 67 + 1385 was written vertically as 67 + 13 + 85). On the test giving mixed order words, 38% ignored the words and wrote the numerals in their given order and 39% left blanks or wrote seemingly random responses; only 23% showed even any partial knowledge. About a sixth of the children did get three of the five items correct on the Choose the Larger Number Test and another sixth got four or five items correct, indicating some pretest ability to compare multidigit numbers.

Performance on the posttest Mixed Words to Symbols Test, the Alignment Test, and the Trading 2 Instead of 1 Test was good, ranging from 80% to 88% (see Table 6). Individual class means on these tests ranged from lows of 59% to 66% to highs of 100%. Performance on the Choose the Larger Number Test improved to moderate levels of accuracy, with little difference between scores obtained by circling the larger number or inserting < or > between the numbers (67% and 65%). Class means on the Traded Word/Numeral Test were extremely variable, ranging from 3% to 88%, with an overall mean performance of 53% of the items correct.

Discussion

Informal teacher reports via the supervisors and direct communication to the district mathematics director indicated considerable enthusiasm and enjoyment of the learning/teaching approach by both teachers and children. Being able to solve large problems seemed to empower children and make them feel good about themselves and about mathematics. Children learned multidigit addition quite well, though they still made some addition fact errors and occasional trading errors. The subtraction test scores and error analyses indicated that most children could trade correctly and that few continued to make the presubtraction and the subtract-smaller-from-larger errors so common on the pretests. However, many children did not completely master subtraction computation and continued to make some trading and fact errors, especially on the ten-digit problem. Both addition and subtraction performance was considerably above that ordinarily reported for third graders, as was performance on the Alignment Test, the Mixed Words to Symbols Test, and the Choose the Larger Number Test. Children showed more limited ability to generalize trading to the new Traded Word/Numeral Test.

There were obvious limitations to this study. Because systematic classroom observations were not made, it is not clear how closely the work with the blocks followed the lesson plans. Thus, no inferences can be made about which features of the learning/teaching approach might be crucial and whether any might be expendable. It is not clear why teachers in some schools participated while those in other schools did not. Informal reports to field supervisors indicated that the school-based decisions to participate were sometimes initiated by the principal and sometimes by the teachers. The field supervisors reported that some teachers expressed skepticism that second graders could learn material so much above grade level even though the success of the approach with the children in Study 1 was discussed in the in-service sessions; this skepticism may have contributed to decisions not to use the approach. The partial participation by teachers did not seem to bias the sample with respect to initial knowledge of the participating children. The teacher assignment and transfer policies of the district make it unlikely that the best teachers are heavily concentrated in certain schools (i.e., only in the participating schools), but there still might have been some bias toward participation by the better teachers in the district. Finally, although the scores on the addition and subtraction computation tests and the shifts in errors from pretest to posttest were similar in Study 1 and Study 2, the lack of interview data in Study 2 means that it is not clear whether the children in Study 2 understood and could explain multidigit addition and subtraction as well as could the children in Study 1.

GENERAL DISCUSSION

On all tests and interview measures, performance by second graders of all achievement levels considerably exceeded that reported in the literature for third graders receiving usual instruction. Most children learned to trade in four-digit addition and subtraction problems, column errors frequently resulting from usual instruction were virtually eliminated, and children showed considerable generalization of multidigit addition and subtraction to multidigit problems larger than four digits. Most children aligned uneven addition problems on the right, traded 2 instead of 1 when necessary, and could translate from mixed words and numerals to multidigit numerals. Children in Study 1 showed in the interview quantitative understanding of written multidigit numerals and used this understanding to explain one/ten and ten/hundred trading procedures in both addition and subtraction.

These results indicate that second-grade classroom teachers can use the learning/teaching approach effectively to support high levels of meaningful learning in many of their children. Children from a small city/suburban heterogeneous population and children from a wide range of schools in a large urban school district demonstrated such learning, so the learning/teaching approach can be used successfully with a fairly wide range of children. The successful learning in both studies indicated that the learning/teaching approach could be implemented on a broad scale with a moderate amount of in-service time, materials, and teacher support. Many participating teachers in Study 2 did ask for their own set of blocks for the coming year, so one set of blocks per building is clearly not ideal. In particular, more sets of blocks may facilitate the use of place-value units in the crowded end-of-the-year schedule.

The approach did not result in maximal learning in all areas by all children. Some children continued to make occasional trading and fact errors, particularly in subtraction with the ten-digit problem. Some children were not able consistently to choose the larger of two three-digit through seven-digit pairs of numbers, and many children in Study 2 did not generalize trading to all of the items on the Traded Word/Numeral Test. Whether these limitations are inherent in this approach or are due to inadequate implementation of certain features of the approach or simply to insufficient time with the approach for some children is not clear. In the first study of the approach (Fuson, 1986a), telling children to "think about the blocks" was sufficient for most of them to self-correct errors they were still making after the initial learning or to self-correct errors that began to appear on delayed posttests after correct initial learning. Thus, the blocks can be a powerful support for children's thinking, but many children do not seem spontaneously to use their knowledge of the blocks to monitor their written multidigit addition or subtraction. This suggests that frequent solving of one multidigit addition or subtraction problem accompanied by children's thinking about the blocks and evaluating their written-marks procedure might be a powerful means to reduce the occasional trading errors made by children.

A limitation of both of these studies is that their designs did not permit an evaluation of any of the specific features of the learning/teaching approach. The approach had many features, not all of which may be crucial to its success. These features stemmed from the need to provide children an opportunity to construct conceptual structures for the mathematically different English named-value system of number words and the positional base-ten system of written marks and to think about how these systems work in multidigit addition and subtraction; how the features relate to children's learning are discussed in Fuson (in press a). These studies are also limited because they were not intended to provide a complete addition and subtraction or place-value experience. Obviously important topics were omitted that relate to the goals of understanding multidigit addition and subtraction (e.g., estimation, alternative methods of adding and subtracting). Future work might explore how well the learning/teaching approach could support these more extensive goals.

These two studies raise several issues for future research concerning the use of embodiments in learning multidigit addition and subtraction and place value (see also Baroody, in press; Fuson, in press b). First, we took no position concerning whether the teacher or the children moved the blocks or whether learning proceeded within a total class approach, within simultaneous small groups, or within serial small groups. In Study 1 different teachers used all of these, and they all seemed to be effective. Other possible outcomes of these different approaches (for example, beliefs that success depends on effort, attempts to understand, and cooperation with peers as reported in Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990, for a small-group problem-solving classroom organization) might be explored. Second, relative benefits of using the learning/teaching approach to support prechosen multidigit addition and subtraction procedures, as in the present studies, versus using the approach to support procedures invented by children, might be examined. Thus, the focus of the present studies on computation as meaning (on understanding multidigit addition and subtraction and place value) might be contrasted with computation as problem solving (Labinowicz, 1985). The latter does not necessarily result in more competence (for example, only 34% of the third graders who had reinvented arithmetic without traditional instruction solved 43 – 16 correctly, Kamii, 1989), but the support of the learning/teaching approach in Figure 1 might help children invent multidigit addition and subtraction procedures. Third, several aspects of a more gradual use of base-ten blocks as proposed elsewhere do not seem to be necessary for high levels of skill and understanding, because they were not implemented in our approach. These include prolonged work with two-digit numbers, followed considerably later by work with three-digit and even later by four-digit numbers; rather extensive experience with trading before trading is set within addition problems; extensive practice just with the blocks with no recording; pictorial recording before recording with base-ten written marks; and use of the blocks to count on by tens and hundreds (e.g., Baroody, 1987; Davis, 1984; Labinowicz, 1985; Wynroth, 1980). Future research may establish that these aspects do bring particular benefits, but it seems wise to undertake such research rather than merely to assert these benefits.

The results suggest a grade placement for multidigit addition and subtraction and place-value concepts with this approach. Even though many average-achieving first graders were able to learn the multidigit addition algorithm, their relatively poorer performance on some aspects of the interview suggests that the approach in these studies risks pushing children beyond their comfortable learning range. Some of these children may still require perceptual unit items for thinking about single-digit numbers and thus may have trouble using the blocks to construct conceptual ten-unit, hundred-unit, and thousand-unit items made out of collected ones. Therefore, for first graders of average and below-average mathematics achievement and perhaps even for many high-achieving first graders, it may be better to concentrate in the first grade on helping children to build and use their unitary sequence/counting conceptual structures for adding and subtracting single-digit numbers (i.e., sums and differences to 18). Trying to build simultaneously these unitary conceptual structures and the multiunit named-value/base-ten conceptual structures needed for multidigit addition and subtraction, especially given the interference the irregular English number words create for this task (cf. Fuson & Kwon, in press), may be too difficult for many first graders. The learning/teaching activities tested in these studies do seem to be developmentally appropriate for second-grade children of all achievement levels except perhaps those with special difficulties. Teachers reported that second-grade children in both studies enjoyed the learning activities and felt good about themselves and their ability to do such problems with understanding. Thus, the typical textbook extension of multidigit addition and subtraction problems over Grades 2 through 4 or 5, adding one or two digits each year (Fuson, in press c), underestimates what our children can learn. The conceptual bases for general multidigit addition and subtraction algorithms are well within the capacity of most second graders if they are learned with the support of physical materials that embody the relative size of the base-ten places and demonstrate the positional nature of the multidigit written marks and if the focus of such learning is understanding and not just procedural competence.

REFERENCES

Baroody, A. J. (1987). *Children's mathematical thinking: A developmental framework for preschool, primary, and special education teachers.* New York: Teachers College Press.

Baroody, A. (in press). How and when should place-value concepts and skills be taught? *Journal for Research in Mathematics Education.*

Bell, M. S., Fuson, K. C., & Lesh, R. A. (1976). *Algebraic and arithmetic structures: A concrete approach for elementary school teachers.* New York: The Free Press.

Cauley, K. M. (1988). Construction of logical knowledge: Study of borrowing in subtraction. *Journal of Educational Psychology, 80,* 202–205.

Cobb, P:, & Wheatley, G. (1988). Children's initial understandings of ten. *Focus on Learning Problems in Mathematics, 10,* 1–28.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex.

Davis, R. B., & McKnight, C. C. (1980). The influence of semantic content on algorithmic behavior. *Journal of Children's Mathematical Behavior, 3,* 39–87.

Dienes, Z. P. (1960). *Building up mathematics.* London: Hutchinson Educational Ltd.

Fuson, K. C. (1986a). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of Psychology of Education, 1,* 35–56.

Fuson, K. C. (1986b). Teaching children to subtract by counting up. *Journal for Research in Mathematics Education, 17,* 172–189.

Fuson, K. C. (1987). Research into practice: Adding by counting on with one-handed finger patterns. *Arithmetic Teacher, 35*(1), 38–41.

Fuson, K. C. (1988a). *Children's counting and concepts of number.* New York: Springer-Verlag.

Fuson, K. C. (1988b). Research into practice: Subtracting by counting up with one-handed finger patterns. *Arithmetic Teacher, 35*(5), 29–31.

Fuson, K. C. (in press a). Children's conceptual structures for multidigit numbers: Implications for addition and subtraction and place-value learning and teaching. *Cognition and Instruction.*

Fuson, K. C. (in press b). Issues in place-value and multidigit addition and subtraction learning. *Journal for Research in Mathematics Education.*

Fuson, K. C. (in press c). Research on learning and teaching addition and subtraction. In G. Leinhardt & R. T. Putnam (Eds.), *Cognitive research: Mathematics learning and instruction.*

Fuson, K. C., & Kwon, Y. (in press). Chinese-based regular and European irregular systems of number words: The disadvantages for English-speaking children. In K. Durkin & B. Shire (Eds.), *Language and Mathematical Education.* Milton Keynes, Great Britain: Open University Press.

Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. Brainerd (Ed.), *Progress in cognitive development: Children's logical and mathematical cognition,* Vol. 1 (pp. 33–92). New York: Springer-Verlag.

Fuson, K. C., & Secada, W. G. (1986). Teaching children to add by counting on with finger patterns. *Cognition and Instruction, 3,* 229–260.

Fuson, K. C., Stigler, J. W., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in China, Japan, the Soviet Union, Taiwan, and the United States. *Journal for Research in Mathematics Education, 19,* 449–458.

Fuson, K. C., & Willis, G. B. (1988). Subtracting by counting up: More evidence. *Journal for Research in Mathematics Education, 19,* 402–420.

Ginsburg, H. (1977). *Children's arithmetic: How they learn it and how you teach it.* Austin, TX: Pro-Ed.

Kamii, C. (1985). *Young children reinvent arithmetic.* New York: Teachers College Press.

Kamii, C. (1986). Place value: An explanation of its difficulty and educational implications for the primary grades. *Journal of Research in Childhood Education, 1,* 75–86.

Kamii, C. (1989). *Young children continue to reinvent arithmetic—2nd grade: Implications of Piaget's theory.* New York: Teachers College Press.

Kamii, C., & Joseph, L. (1988). Teaching place value and double-column addition. *Arithmetic Teacher, 35*(6), 48–52.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). *Arithmetic Teacher, 35*(8), 14–19.

Labinowicz, E. (1985). *Learning from children: New beginnings for teaching numerical thinking.* Menlo Park, CA: Addison-Wesley Publishing Company.

Miura, I. (1987). Mathematics achievement as a function of language. *Journal of Educational Psychology, 79,* 79–82.

Miura, I. T., Kim, C. C., Chang, C., & Okamoto, Y. (1988). Effects of language characteristics on children's cognitive representation of number: Cross-national comparisons. *Child Development, 59,* 1445–1450.

Miura, I. T., & Okamoto, Y. (1989). Comparisons of American and Japanese first graders' cognitive representation of number and understanding of place value. *Journal of Educational Psychology, 81,* 109–113.

Nicholls, J. G., Cobb, P., Wood, T., Yackel, E., & Patashnick, M. (1990). Assessing students' theories of success in mathematics: Individual and classroom differences. *Journal for Research in Mathematics Education, 21,* 109–122.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking.* New York: Academic.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3). Hillsdale, NJ: Erlbaum.

Ross, S. H. (1986, April). *The development of children's place-value numeration concepts in grades two through five.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Ross, S. H. (1989). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher, 36*(6), 47–51.

Song, M., & Ginsburg, H. P. (1987). The development of informal and formal mathematical thinking in Korean and U.S. children. *Child Development, 57,* 1286–1296.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger Scientific.

Tougher, H. E. (1981). Too many blanks! What workbooks don't teach. *Arithmetic Teacher, 28*(6), 67.

Underhill, R. (1984). External retention and transfer effects of special place value curriculum activities. *Focus on Learning Problems in Mathematics, 1 & 2,* 108–130.

Wynroth, L. (1969/1980). *Wynroth math program: The natural numbers sequence.* Ithaca, NY: Wynroth Math Program.

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