Fuson, K. C. (1988). Children's counting and concepts of number.
New York: Springer-Verlag.
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Part I Number Words

1. Introduction and Overview of Different Uses of Number Words

1.1. Introduction

Number words are special kinds of words. The aspect of a situation described by a number word is not obvious. Understanding the many uses of number words requires many different conceptual structures. These conceptual structures are built over a number of years and move from quite simple to quite complex. Children's increasing ability to use number words correctly provides an interesting domain of study in which cognitive development, language, and procedural activity overlap.

Partly for this reason there has been a considerable amount of research in the past 15 years on children's understanding of numerical concepts and on their ability to carry out numerical procedures such as counting. This research has been conducted by people from different fields: cognitive science, developmental psychology, educational psychology, experimental psychology, and mathematics education. Many talented people have worked in this area, and there is now a considerable accumulation of information. However, the literature presently suffers from two limitations. First, the different kinds of literature remain somewhat separate, with increasing but still incomplete cross-referencing among them. Second, some terms are used in mathematically incorrect ways, particularly the terms "ordinal," "ordering," and "order relations." These differing uses of terms add quite unnecessary confusion to this area, limiting the extent to which studies can build upon one another. One goal of this book is to alleviate these limitations. First, throughout the book, the literature reviewed is drawn from all areas and findings from these areas are integrated. Also, new data and new conceptual analyses are presented, and these reflect a range of methodologies drawn from these areas: experiments, interviews, task analyses, and brief instructional intervention studies. Second, much of this chapter is devoted to providing a framework of possible uses of number words within which future research can function. Mathematically different number-word situations are identified, and the attributes of these situations are discussed. This framework will then allow research results to build on one another and provide cumulative and related knowledge.

The ultimate aim of my own work in this area is to improve the numerical learning experiences of young children in the home, preschools, and schools. However, in order to do this, I believe that we must understand the ordinary developmental sequence of concepts and procedures children come to use in different number-word situations. Once this is known we can then try to provide experiences that will maximize the opportunity of children to learn this sequence. Toward this aim this book traces developmental sequences in several different number-word situations. Such a sequential emphasis necessarily must isolate what younger children do not understand and/or cannot do while demonstrating that older children have such competencies. Therefore, it seems important to emphasize that reflection on my own data and on the data available elsewhere indicates that young children throughout the age range covered in this book, from age 2 to 8, demonstrate very considerable competence in the numerical domain. Numerical concepts are very abstract and become quite complex; children of these ages who have reasonable opportunities to learn these concepts show really quite amazing competence in their use of number words.

1.2. Number-Word Situations: Different Uses of Number Words

Number words are used in a variety of mathematically different situations with a concomitant variety of different external referents accompanied by different internal meanings. Children must come to learn both the number words themselves and these various situational uses of number words. Assessing children's understanding of these different uses is complicated by the fact that in English the same number word is used to refer to several different meanings. Adults understand all of these meanings and can shift among them with ease. However, a child may use a number word with only one of its meanings and may not know other meanings of that word or may not be able to shift easily among various meanings of that word. An important development throughout the age range traced by this book (age 2 to 8) is the inceasing ability to shift among meanings and, finally, to integrate several of these meanings. Adults shift so easily and have such integrated meanings that it is difficult for adults even to comprehend how separate these meanings are for young children.

In order to understand the learning task that faces the young English-speaking child in the domain of number words, it is necessary to ascertain what these various uses of number words are. These uses, the nature of the object situation in which each use occurs, and relations or operations for each use are presented in Table 1-1; the uses are not listed in the order in which they are learned. The first three uses of number words given in Table 1-1 are numerical uses: the meaning of the number word is a cardinal number, an ordinal number, or a measure number. The next two uses require the utterance of the conventional sequence of number words, either without (sequence situations) or with correspondences with entities (counting situations). Another kind of use is for reading symbols (numerals); these symbols may refer to any of the first five uses. Finally, number words are also used as labels in nonnumerical or quasi-numerical situations. Each of these uses are discussed in this chapter.

1.2.1. Cardinal, Ordinal, and Measure Uses

The entities in a cardinal situation are discrete. The cardinal number word refers to the whole set of entities and tells how many entities there are, that is, describes the manyness of the set. In an ordinal situation the entities are discrete and ordered (a linear ordering exists on the entities); the ordinal number word tells the relative position of one entity with respect to the other ordered entities. In a measure situation the entity to be measured is a continuous quantity (e.g., length, area, volume, time). A unit for that kind of quantity must be selected (e.g., a centimeter, a square mile, a cubic decimeter, a swing of a pendulum 1 m in length) and repeatedly applied to the particular

Table 1-1. Uses of Number Words

Uses of number words	Object situation	Ordering already exist?	Units exist?	Referent	Describes	Equivalence relation	Order relations	Operations[a]
Cardinal	Discrete entities	No	Perceptual unit items[b]	Whole set of entities	The manyness of the set	Same number as	More than, fewer than	+, − of whole numbers
Ordinal	Discrete entities	Yes	Perceptual unit items[b]	One entity within a set	The relative position of that entity[c]	Same relative position[d]	Ahead of, behind in the ordering[d]	+, − of ordinal numbers[e]
Measure	A continuous quantity	No	Unit of that continuous quantity	The continuous quantity	The manyness of the units that cover (fill) the quantity	Same amount	More than, less than	+, − of whole numbers, decimals, fractions
Sequence	None	Yes	Develop-mental changes in these (see chapter 2)	No reference	Nothing	The same word	After/before just after/ just before	Sequence words taken as the sets for +, − of whole numbers
Counting	Discrete entities[f]	No[g]	Perceptual unit items[h]	Each entity	Nothing	None[i]	None[i]	None

Symbolic (numeral)	A symbol	No	No	That symbol; may also refer to one of the other uses	That symbol; whatever the use of the symbol is[j]	Only if interpreted as cardinal or measure	Only if interpreted as cardinal or measure	Only if interpreted as cardinal or measure
Nonnumerical labels or series	A single entity	No[k]	No	That entity	Various attributes of an entity	No[k]	No[k]	No[k]

[a] We deal here only with the two simplest operations, addition and subtraction.

[b] Perceptual unit items are the first unit items used in this situation. Children eventually come to use other unit items in this situation (see chapters 8, 9, and 11).

[c] An ordinal number word describes the position of a given entity with respect to the other entities in the set; this relative position is derived from the ordering on the set.

[d] The particular words used to describe the relation will depend on the kind of ordering on the entities.

[e] See text for explanations.

[f] These can be the units used in a measure situation.

[g] An ordering must be established by the counter during the activity of counting, but this ordering may be established over time and may not be able to be reconstructed at the end of counting.

[h] An early "precounting" may occur in which objects are not taken as perceptual unit items, but the number words do refer in a general unspecified way to the objects.

[i] Equivalence and order relations could be established, based on the ordering of the objects, but such relations are rarely established. The relations on cardinal, ordinal, measure, or sequence number words are used instead.

[j] For example, if the symbol has a cardinal use, the symbol will describe the manyness of the cardinal set.

[k] An ordering, relation, or operation may exist, but it is not crucial to the nonnumerical use (see text).

continuous quantity until the quantity is used up. The measure number word tells how many units are required to cover (fill) the continuous quantity.

Units also are required in cardinal and ordinal situations, although they are less obvious than in a measure situation. In a given cardinal or ordinal situation, one must decide which discrete entities are to be taken as part of that situation. These entities can be quite heterogeneous. For example one can consider a container of yogurt, a plastic spoon, an orange, a napkin, and two peanut butter cookies to be the entities and ask a question about those entities as a cardinal situation (e.g., "How many things did I put in the lunch sack?") or as an ordinal situation (e.g., "Which thing did I put in third?"). To convert the quite different entities in the lunch sack to a cardinal or an ordinal situation, one has to consider each entity as a separate and equivalent item, ignoring their different characteristics and focusing only on their characteristic of being a discrete unity. Steffe, von Glasersfeld, Richards, and Cobb (1983) have made this distinction for entities that are counted and call such a conceptual discrete unit taken from a perceived entity a "perceptual unit item." We will adopt this term and use it for cardinal and ordinal situations also. This discussion about units in cardinal and ordinal situations makes clear the fact that, although the units in a measure situation may have to be used to cover (or fill up) the actual quantity, these units must also be mentally considered as units by a perceiver in order for that situation to be a measure situation for that perceiver.

The cardinal and measure situations each also require another conceptual operation in order to be considered a cardinal or measure situation. In a cardinal situation the separate entities (the perceptual unit items) in the situation must be conceptually unified into a single whole in order for the cardinal number word to refer to that single whole. We will call this conceptual operation "cardinal integration."[1] Likewise, a measure situation requires a conceptual unifying of all the measure units into a single continuous quantity; we will call this conceptual operation "measure integration."

Any two cardinal situations, two ordinal situations, or two measure situations can be related by an equivalence relation or by one of two order relations. The equivalence and order relations differ for each of these situations (although most of the words used for the cardinal and measure order relations are the same). For any two cardinal situations, one situation has the same number as, more than, or fewer than the other situation. For any two measure situations, one situation has the same amount as, more than, or less than the other situation. The actual words used to describe the relation in an ordinal situation will depend on the nature of the ordering on that situation. Equivalence and order relations are not used very frequently to compare ordinal situations in ordinary life, and the words are sometimes a bit awk-

[1] Steffe et al. (1983) used the term "integration" for a cardinal unifying in a more complex addition situation discussed in chapter 8. We have adopted this term but use it with prefixes throughout the book in order to differentiate it from the original special use.

ward. For example, if the ordinal situations were the order in which two children put entities into their lunch sacks, a child could say, "We put our yogurts in our sacks at the same place in our orders" or "I put my yogurt in my sack earlier [in my order] than you did [in your order]," or "I put my yogurt in my sack later [in my order] than you did [in your order]." These uses "earlier than" and "later than" do not, of course, refer to the actual time at which the yogurt was put in the sack, because time is a continuous quantity and a comparison of times thus involves an order relation on two measures situations. If the ordinal situations involve two older siblings, each standing in a different line of people at the grocery store, one sibling could compare their relative positions in the two lines by saying. "We're in the same places in our lines," or, "I am closer to the front of my line than you are to the front of your line," or "I am farther behind in my line than you are in your line." Many different kinds of orderings are used in real life (e.g., shortest to tallest in a graduating class, highest to lowest grade-point average, first arrival to last arrival at a fish counter), so there are many different possible wordings for equivalence and order relations on ordinal situations. However, the rather forced nature of the examples given indicates how relatively infrequently we use equivalence or order relations on ordinal situations in real life. Of course, as with the situations themselves, the relations on two cardinal, measure, or ordinal situations must be established and understood by a given user of such relational terms. How this understanding is gradually obtained by children for cardinal situations is the focus of chapter 9.

Mathematical operations of various kinds can be performed on cardinal situations, measure situations, and ordinal situations (although, as with relations, the last are not usual and are quite awkward). Mathematically, cardinal situations are those to which the natural numbers, or the positive whole numbers, refer. Measure situations can be described by natural or whole numbers, decimals, and fractions. The latter two are used when a given until does not cover (or fill up) all of the continuous quantity; then a smaller unit is used to cover the part of the quantity not covered by the original units. There are several mathematical operations that can be defined on cardinal, measure, and ordinal situations. In this book we discuss only the two simplest such operations: addition and subtraction. Addition and subtraction of cardinal and measure situations are the familiar addition and subtraction of whole numbers that are taught in school arithmetic (see chapter 8). Addition of ordinal numbers (or addition in ordinal situations) must involve some notion of, to use a specific example, "sixth plus third equals what?" To return to the lunch sack example, if I put in the yogurt, orange, two cookies, plastic spoon, napkin, and then was interrupted, and then returned to put in two dimes and a nickel and another dime for milk money, I could (although my family would undoubtedly eye me warily) announce, "I put the napkin in sixth and then put the nickel in third, so overall the nickel was put in ninth." The artificiality of this example demonstrates how rarely we use addition of two ordinal situations, and we probably use subtraction of ordinal situations even less often.

How are number words assigned to cardinal, ordinal, and measure situations? Counting is a very important way of doing so; it is discussed in the next section. There are two other ways in which number words are assigned to cardinal and measure situations; it is possible that these methods can also be used in ordinal situations, but this seems not to have been investigated. For very small numbers (two, three, and possibly four and five), people seem to be able to apprehend directly the appropriate cardinal or measure word, that is, to *subitize* it. How this is done is still a matter of dispute (e.g., Cooper, 1984; Kaufman, Lord, Reese, & Volkmann, 1949; Klahr & Wallace, 1973, 1976; Mandler & Shebo, 1982; von Glasersfeld, 1982). Subitizing seems to be used by very young children and may even be used by infants (e.g., Cooper, 1984; Starkey & Cooper, 1980; Strauss & Curtis, 1984). However, there still is disagreement concerning the developmental relationship between labeling a subitized situation with the correct number word and counting in order to label that situation (see Chapters 7 and 9 for further discussion of this debate). The other way in which a number word is chosen for a given cardinal or measure situation is by estimation (e.g., Gelman, 1972; Klahr & Wallace, 1973, 1976). The bases for making numerical estimates are not clear, and the term is used to describe different kinds of behavior ranging from a very rapid appraisal of a situation to a more systematic use of a bench mark as a comparison or unit of measure (e.g., Siegel, Goldsmith, & Madsen, 1982). In short there are few data on children's use of estimation in either cardinal or measure situations.

1.2.2. Sequence and Counting Uses

The sequence and counting uses of number words are differentiated by whether or not the number words refer to objects. If number words are just said aloud (or later, silently) in their usual English sequence with no reference to objects, they are being used in a sequence situation. In this case an ordering exists on the number words (the standard English ordering of these words), and the words have no referents and describe nothing. If the number words are said in an entity situation and each number word refers to an entity,[2] then the situation is a counting situation. The counting words have reference to the objects to which they are attached by the activity of counting, but these words do not describe the objects. As in cardinal and ordinal situations previously discussed, in a counting situation involving physically present entities, the person counting must unitize the entities to be counted. Each of them must be taken as a single countable unit to which one word will be applied regardless of whether that entity is actually much larger than (or is in some other way different from) other entities to be counted, that is, each

[2] For simplicity this description assumes that no counting errors are made. Of course, an error can be made in which one word refers to more than one object or a word refers to no object (see chapters 3 through 6), but we would still wish to call this a counting situation.

must be taken as a "perceptual unit item" (Steffe et al., 1983). Counting in cardinal situations involves a developmental sequence of conceptual unit items that become more abstract and complex (chapters 8 and 9; Steffe et al., 1983).

For a number word to take on a numerical meaning, counting must occur within a situation that also can be thought of as a cardinal, ordinal, or measure situation. In this case the situation is first considered by the person counting to be a counting situation, and the counting is done. Then the situation is thought of as a cardinal, ordinal, or measure situation rather than as a counting situation. For this shift to occur in a cardinal situation, the counter, after counting, must make a cardinal integration of the counted entities and then think of the last counted word as referring to the manyness of this whole set of counted entities. Similarly, a measure situation requires a measure integration and a shift of the last counted word from its counting reference to the last counted entity to its measure reference to the manyness of the units covering (filling) the continuous quantity. An ordinal situation does not require an integration of the counted entities, but it does require a shift from the count reference to an entity during counting to the ordinal reference to the relative position of that entity with respect to the other entities in the situation; this ordinal reference is a much more complex one in which the entity must be considered in relation to all of the other entities. Counting is thus a method of deciding which number word should be used to describe a particular cardinal, ordinal, or measure situation. The use of counting in these situations requires the person doing the counting to consider the situation in two different ways: first as a counting situation and then, after the counting is completed, as a cardinal, measure, or ordinal situation in which the last counted word then describes the cardinal, measure, or ordinal number of that situation. Thus, the counter must make a count-to-cardinal, a count-to-ordinal, or a count-to-measure shift in conceptualization of the same situation.[3]

There is a standard ordering on the number words in the number-word sequence of every language. Children appreciate this ordering in English at a very young age (see chapters 2 and 10). This ordering permits equivalence and order relations to be established on sequence number words. The equivalence relation is just the trivial relation "is the same word as." Two pairs of order relations can be derived from the sequence ordering: After/Before and Just After/Just Before.[4] It is easier to see how these relations come only from the sequence ordering by considering the example of the alphabet or of an

[3] We called these count-cardinal, count-ordinal, and count-measure shifts in Fuson and Hall (1983). We have added the *to* here because we want to discuss shifts in both directions, and the "to" makes the directionality more explicit and easier to process.

[4] The After/Before relations were called Comes After/Comes Before in Fuson, Richards, and Briars (1982) and were simplified here to After and Before. The relations Just After and Just Before were called And Then and And Then Before in Fuson et al. (1982). The reason for the latter change is discussed in chapter 2.

unfamiliar number-word sequence: il ee sam sa o yuk chil pal koo.[5] Examples of the first pair of relations are the following: *G* is before *P*, *sam* is before *yuk*, *D* is after *A*, *koo* is after *ee*. Examples of the second pair of relations are the following: *X* is just before *Y*, *chil* is just before *pal*, *T* is just after *S*, *sa* is just after *sam*. Children who learn to say the English sequence of number words cannot immediately use the relations on the sequence. They go through a fairly long period of saying the sequence alone and of using the words in counting before these relations can be used. Changes in the internal representation of the sequence accompany the ability to construct these relations on sequence words. These changes are outlined in Fuson, Richards, and Briars (1982) and are summarized here in chapter 2. These sequence relations for larger number words (ten and higher) come to be used to make the order relations on cardinal and measure situations (see chapter 2). Changes in the sequence reflect changes in the units represented by the sequence number words. Children finally arrive at a level in which the sequence words themselves are taken as the entities, that is, as sequence unit items, for the addition and subtraction of whole numbers. When this occurs but not before, one can consider operations to occur with sequence number words. The uses of sequence words in this way are discussed briefly in chapter 2 and in more detail in chapter 8.

When the sequence words are used in counting objects, the standard sequence ordering is used. However, this ordering is a property of the sequence rather than of the counting (the ordering exists when the sequence is said alone). During the activity of counting, an ordering is also established on the objects. However, this ordering may only unfold over time as each object is moved from an uncounted to a counted set or as each object is indicated in turn; the ordering may be neither established before the counting begins nor able to be reconstructed after the counting ends. This is in contrast to an ordering such as the spatial ordering involved in objects arranged in a row; here the ordering is evident before counting starts, although one still must choose between the two orderings beginning at each end of the row. Because such a preexisting ordering does not have to exist in a counting situation, a "no" is entered in the ordering column of Table 1-1 for the counting use of number words.

1.2.3. Symbolic Uses

Reading numerals (written symbols such as 8, 2, or 16) is another use of number words. Initially, for young children this is just a simple associative learning task: See the numeral and recall its number word. However, sometimes numerals are intended to have cardinal, ordinal, or measure meanings. In these cases the fact that a child says the correct number word does not en-

[5] These are Korean number words. The spellings are those of a graduate student native Korean speaker, Youngshim Kwon, and myself and may not be the standard English transliterations.

sure that the child has understood or is using the cardinal, ordinal, or measure numerical reference of the symbol.

In schools concrete embodiments (physical materials) are sometimes used to give meaning to symbols, especially to operations on symbols. Such embodiments may be cardinal situations, such as poker chips, or measure situations, such as Cuisenaire rods (ten different lengths of rods, each length colored a different color). Young children who must use perceptual unit items in order to give meaning to cardinal and measure number words require some such embodiment that provides them with perceptual until items.

1.2.4. Nonnumerical Uses

A final use of number words is as a label, an identifying attribute of an object. These uses are quite common in the lives of adults, for example, the two-o-one bus (meaning the bus with 201 on the front, not the bus that arrives at 2:01), flight number six-ninety-seven, social security numbers, telephone numbers, house numbers, office numbers, area codes, and zip codes. These uses primarily involve the use of numerals, of symbols; number words are used to read these symbols. Because the nonumerical uses of number words are not cardinal or measure uses, the whole English system of number words is not used (e.g., the telephone number 475-0386 is not read as four million seven hundred fifty thousand three hundred eighty six). Rather, each numeral is read as a separate number word, or sometimes pairs of symbols are used as number words under one hundred (e.g., the house address 1918 will often be said as nineteen eighteen rather than as one nine one eight).

Some of the nonnumerical uses are actually quasinumerical; parts of them have some simple numerical interpretation but parts do not. For example the first two numbers for a street address often tell the number of blocks that house is from some starting point; however, in many places the system for assigning the last two numbers is more mysterious (why is 1918 just next to 1924 and to 1914?). Some of these nonnumerical uses may also have had some numerical basis when they were first invented. For example the area code is a sort of roughly derived measure of the population of the area: The greater the population the smaller the numbers (considered as cardinal numbers) in the area code. With pushbutton phones rather than dial phones, this reason for the assignment of area codes no longer holds; it does not take longer to push the larger numbers, and pushing the larger numbers is not more subject to error. Similarly, other nonnumerical uses may be nonnumerical only to people who are not privy to the underlying system or may become nonnumerical to everyone once the reason for the underlying system no longer holds.

1.2.5. Which Number Words Are Used When?

In English the same number words are used in sequence, counting, cardinal, measure, and nonnumerical situations, although the use in measure situations does involve saying the word for the measure unit after saying the number word (e.g., eight centimeters, fifteen seconds). The uses are differentiated only by the referents of the number word. These same number words are used for symbols (numerals) up to 10. Beyond that, the number words used depend on the referent of the symbols (numerals). The conventional English system of number words is used for sequence, counting, cardinal, and measure referents of the symbols, and specific examples may be said as single or double digits for nonnumerical referents, as discussed previously. In ordinal situations there are special number words. Some of these are irregular (first, second, third, fifth), but most are constructed by adding *th* to the regular English number words (fourth, sixth, seventh, etc.). These number words do then differentiate the ordinal use from the other uses of number words.

1.2.6. Distinctions Among Ordinal, Order Relations, and an Ordering

We would like to emphasize some distinctions we have made here that are often confused in the literature on children's concepts of number and of counting. These are the differentiation of ordinal, order relations, and an ordering. An *ordinal situation* is very easy to distinguish. It is one to which an ordinal number word (first, second, third, etc.) will apply. An ordinal situation concerns the relative position of one entity with respect to all the other entities in an ordered situation. *Order relations* can be established on cardinal, ordinal, measure, and sequence uses of number words (see Table 1-1). Order relations involve a comparison between two nonequivalent cardinal, ordinal, measure, or sequence situations. The use of order relations on cardinal, measure, or sequence situations (e.g., seven cookies are more than five cookies or seven comes after five in the sequence) are not "ordinal" situations; they are still cardinal, measure, or sequence situations in which *an order relation* is being applied. Repeated use of a given order relation on all the elements of a given finite set results in *an ordering*. An ordering can be defined in a cardinal or a measure situation, but these situations do not require an ordering. An ordering *must* be defined on a situation for it to be an ordinal situation, because otherwise one could not use the ordinal number words to refer to the relative position of one item with respect to the ordering. However, the existence of an ordering in a situation does not ensure that the situation is ordinal. For example, asking how many people are in the ticket line requires that situation to be considered as a cardinal, not an ordinal, situation. This confusion is particularly acute with the ordering present in the sequence of number words. The existence of this ordering on the number words does not mean that saying the number words or using the number words in counting implies an ordinal situation. Because the number words and counting are both used in cardinal, ordinal, and measure situations, it seems particularly important to refrain from considering the number words and counting to be ordinal situations. If sequence and counting situations actually were ordinal situations, one would use the sequence of ordinal words for counting. We do not do so, because in sequence and counting situations we are not interested in the relative position of every entity with respect to every other entity. We only wish to say the sequence of words or to make a correspondence of the words with entities using the usual ordering on the sequence of number words.

1.3. Learning Number Words in the Home: Children's Early Experiences With Number Words

1.3.1. Results From Three Studies

There is little research on the kind and range of experiences young children have with number words. However, recently three studies provided some information about this. The results of these studies are briefly summarized here.

Because the same number words are used in cardinal, measure, sequence, counting, symbolic, and nonnumerical situations, a young English-speaking child will hear the same number words used in many different kinds of situations. Some research results indicate that this early experience may be heavily weighted toward the words *two* and *three* and that the spontaneous use of number words by young children is similarly weighted. The results of a longitudinal study of mothers interacting in a standard setting with their child over the age range from 9 to 36 months indicate that both mothers and children use the word *two* much more frequently than the word *three* and use *three* much more frequently than *four* (Durkin, Shire, Riem, Crowther, & Rutter, 1986). This result was also reported for a longitudinal study of nine children from their first birthday until they entered school (Wagner & Walters, 1982; Walters & Wagner, 1981). These latter observations were made in the home and involved both free play and structured tasks focused on number. Between the ages of 1–9 (one year nine months) and 3–9, the children used *two* 158 times, *three* 47 times, *four* 18 times, and *five* only 4 times (the type of situation was not specified). The use of counting was strongly and inversely related to the size of the number word. In a sample of cardinal situations that could be evaluated with respect to whether counting did or did not occur, counting occurred for 9%, 53%, 71%, and 100% of the situations in which the words *two*, *three*, *four*, and *five* through *twelve* were used, respectively. Thus, even for *four*, and certainly for words above that in the sequence, uses of number words are strongly connected to counting and sequence uses of the number words. Many of the uses of the word *two* seem to be in cardinal situations without the use of counting.

Mothers evidently provide experiences with number words that range over almost all of the uses listed in Table 1-1. For example in the simple structured situation studied longitudinally by Durkin et al. (1986), about 60% of mothers' uses of number words consisted of routine and/or pedagogical behavior that included nursery rhymes, stories, songs, saying the sequence of number words with the child, and counting objects (i.e., sequence and counting uses). The remaining 40% of mothers' uses of number words consisted of spontaneous incidental descriptions of quantifiable aspects of the environment (presumably labeling cardinal, ordinal, or measure situations). These uses also included order relations and addition and subtraction operations, at least for cardinal situations. Saxe, Guberman, and Gearhart (in press) found that mothers of 2- and of 4-year-olds reported using certain situations (e.g., playing a board game or a card game) to teach different uses of number and that over time the mothers structured the activity of the child in more complex ways. This restructuring of an activity as the child got older and learned certain numerical skills or concepts ranged over the four levels of complexity into which Saxe et al. divided the number domain: use of number words without correspondences (e.g., saying the sequence or nursery rhymes, reading numerals); single-array correspondences (e.g., counting in a cardinal situation, giving a cardinal estimate without counting); relating summations of correspondences (e.g., reproducing a cardinal situation or comparing two cardinal situations—what we have identified as establishing equivalence or order relations on cardinal situations; some examples were also given of sequence order relations); and relating and manipulating summations of correspondences in arithmetical reasonings (what we have called the addition and subtraction operations on cardinal situations). Although more data are needed on the kinds of experiences young children have with number words, these pioneering studies indicate that the experiences do range over sequence, counting, and cardinal situations and extend to order relations and addition and subtraction operations for these uses of number words.

Several aspects of the experiences with number words provided to children by mothers fall within a Vygotskiian framework of a common goal-directed activity structured by the mother with age-appropriate support for the child (e.g., Rogoff & Wertsch, 1984; Saxe et al., in press; Vygotsky, 1962; 1978; 1986). First, mothers seem to use number words more with very young children and then decrease their use as the children begin to use number words more frequently. Durkin et al. (1986) found that in mother–child interactions mothers used number words in the observed situation more frequently than did children from 9 to 33 months but that children's use increased considerably with age and surpassed that of mothers from 33 to 36 months. Second, mothers in both the Durkin et al. and the Saxe et al. studies frequently structured a given situation by the use of questions, directions, and gestures into one of the uses in Table 1-1. That is, they did not always just use the number word, but rather directed the child's attention to the appropriate attributes of a situation. Third, after such an initial structuring, the mother might then continue to restructure the same situation into other uses in a sequential goal-directed fashion (Saxe et al., in press). For example in order to help a child make a set equivalent to a given set, the mother might ask a 4-year-old to count the given set, then ask how many are in that set, then tell the child to go over to the chips and count out the same number of chips (or say the explicit number of chips: e.g., count out seven chips), and then check that there are the same number of chips by matching the two sets (the original and the one obtained by the child). Mothers made judgments about what steps their child could carry out and then structured the steps into greater or lesser difficulty accordingly. Thus, for a large set with a 2-year-old, they might carry out each step themselves but have the child count with the mother for each of the counting steps.

Saxe et al. reported some social class differences in which middle-class mothers structured a given situation so as to require more advanced numerical activities by the child than did working-class mothers, but in many aspects mothers did not differ across social class. Ginsburg and Russell (1981) also reported that social class and race affected few aspects of early mathematical thinking. However, their data pooled over kindergarten and preschool did indicate that children from middle-class intact families produced significantly longer accurate number-word sequences than did children from middle-class nonintact families or from lower class families and that middle-class preschool children produced significantly longer accurate number-word sequences than did lower-class preschool children.[6]

The Durkin et al. (1986) study did indicate that, with young children (up to age 3), parents often oversimplify number word situations and that such situations are replete with ambiguities, conflicts, and contradictions. In one example the mother cued the child to consider a situation in a given way (e.g., "How many eyes?") but then also used a cue that is often used to cue the saying of the number-word sequence ("one"). The child responded as in a sequence situation ("Doo fwee"), to the consternation of the mother. Another source of ambiguity is that the word *one* is often used by mothers both as a pronoun and as a number word, sometimes very close together: "The other one—look, one. . .two" (p. 283). The function words *to*, *too*, and *for* also are sources of confusion, conflicting with *two* and *four*. Durkin et al. speculate that the conflicts and contractions promote development, but it also seems possible that clearer differentiation of different uses of number words might accelerate children's understanding of these different uses.

1.3.2. Examples From a Mother's Diary

Table 1-1 provides a frame of reference for thinking about and carrying out research concerning children's increasing understanding of different uses of number words. However, neither it nor the research summarized in 1.3.1 indicate the richness and even precocity of the numerical thinking of which

[6] The data from the kindergarten children probably suffer from a ceiling effect because children were stopped at 50, and many kindergarten children have accurate sequences above that. Thus, there may also be effects of class or of family intactness at this age.

young children are capable when small number words are involved. Nor does it reveal how very dependent such learning is on specific experiences provided in the daily life of the young child. It seems very useful to give some flavor of each of these vital aspects of young children's uses of number words. Tables 1-2 and 1-3 are included for this purpose. They consist of all entries concerning number words in the diaries I wrote for my two children when they were young. During much of the diary period, I was not engaged in research on young children's number concepts. Thus, the entries were not systematic or even focused on what I might consider crucial if I were keeping such diaries now. However, having majored in mathematics as an undergraduate and having (or, initially, getting) a doctorate with graduate work in mathematics and in mathematics education, I did have an interest in these areas. The diaries were written to each child and consisted of descriptions of accomplishments, activities, and characteristics that I thought interesting, amusing, or noteworthy. They were intended to provide each child with some sense of herself when she was little and to capture that irreproducible charm and inadvertent wit exhibited by each young child as she constructs (and delightfully at times misconstructs) in her own way the human cultural mileu into which she is born. The second table is noticeably shorter than the first only because the amount of time available to the diary keeper unfortunately did not double with the birth of the second child; this second table does indicate some ways in which the experiences of a second child may be influenced by an older sibling.

Table 1-2. Mother's Diary Entries of Adrienne's Uses of Number Words

Age[a]	Diary entry	Use
1–8	You walking up the stairs: "A B C four five six."	S
	My phrases, your responses: "A–A, B–B, C-three."	S
	Five is by far your favorite number.	?
	You know *two* and use it correctly in new situations; *two* means one in each hand.	Cd
1–10	You were petting Sam (cat) and said "Sam tape [tail]." Then "Tshad tape". (Tshad, the dog, was not in the room). Then "Two tapes [tails]."	Cd
	Before now you had frequently said two cups, two cookies, but always when they were both physically present.	Cd
1–11	"I got two shobels [shovels]."	Cd
	You occasionally count things, especially pictures. You say one number each time you lift your finger and point it down again, but you count some things several times and others not at all. Typical counting series: "one, four, five, eight, four, five, two, six."	Ct
	Tonight we counted steps going up to bed (we usually do). We were on nine and you said, "One two fee four five six." First time so many correct.	Ct

2–0	You can count three things correctly. That is, you point to objects and say "one two three." You also say three correctly when asked "how many" (without counting).	Ct Cd
	"One two three eight jump." (You are into counting and jumping off the hassock or even the couch.)	S
2–1	You counting on your muffin (a toy): "One two three eight seven three."	Ct
	Several times today: "I two years old."	M
	"Make a *B*. Do it again. Two *B*'s."	Cd +
2–4	You fixed my fingers for "two years old" (the pointing and index fingers up and the thumb holding down the third and fourth fingers).	M
2–6	Two tomatoes were on the table. You said and acted out the following: "One tomato from two tomatoes leaves one. Two tomatoes from two tomatoes leaves no." I asked you what no tomatoes from two tomatoes was: "Two tomatoes." "Sesame Street" does things similar to the first two sentences.	Cd – Cd – Cd –
2–7	Putting prunes back into a box, you correctly counted them up to nine. When asked how many prunes: "three" (your standard "how many" answer at this point: three eyes, etc.)	Ct No Ct–Cd
2–8	"Three of us are sitting down. You, Erica, and me [pointing to each]. If Daddy sits down, there will be four of us."	Cd +
2–9	You to father: "Do you want to have a picnic?" Father: "Not this time. I'll go next time." You: "We'll have two picnics. You'll go to the second picnic."	Cd, O
	Father, asking you how you want to be swung around: "Do you want one and one?" You: "No, I want two and two [two arms and two legs held]."	Cd
2–10	I cut your peanut butter sandwich in half and then into half again. You watched and said, "Two and two make four."	Cd +
	You just asked for four olives (you love olives!). Your father gave them to you, and you said, "Two and two make four."	Cd Cd +
2–11	"Daddy, I hear three cracker-fires." (It was the fourth of July.)	Cd
3–0	You were quite excited at your birthday party and immediately understood and remembered that you were three years old.	M
	You count correctly putting fingers up (index first, then big finger, then ring finger, then pinky finger) up to four.	Ct
	Several times you have said things like: "I can open the door from the outside now. When I was two and a half I couldn't open it, but now I am three years old and I can open the door."	M
3–2	"Eight nine ten eleventeen twelveteen thirteen." Later, raising the correct number of (unit) fingers and concentrating very hard: "twenty-one, twenty-two, twenty-three, twenty-four, twenty-five." Each *twen* was very long. You count everything. You love to count.	S Ct

Table 1-2. *Continued*

Age[a]	Diary entry	Use
3–4	There is a bit on "Sesame Street" in which a child counts backward from ten. Lately you have said (to me and your father), "I can count backwards. Ten nine eight seven six four" and then needed questions to finish off. Me: "What comes before four?" You: "Three." "What comes before three?" You: "Two." I didn't know if it was a rote thing or not. Later you got the clock and said: "I can count backwards: twelve eleven ten nine eight seven six five four three two one [pointing to each number in turn]."	S bk S rel S bk Num
	You love to type. You ask me to spell words, and you type them. After we had done GO, you said, "Why is go a two word?"	Cd
	You were bothering your father while he was trying to tie a bow on a Christmas present. Father: "I can't do two things at once; I can only do *one* thing at a time." You, in a pout: "I can't do *nothing* at a time." (You had been trying to tie a bow and had been unsuccessful.)	Cd Zero
3–5	You were typing and saying as you typed: "one, two, three, four, five, six, seven, eight, nine [pause]. I need a ten."	S Num
3–6	You were counting pictures. Got (with a couple of omissions) to twenty. I said, "Twenty-one, twenty-two." You said, "No, this one [pointing to the second picture] was two." Evidently you thought twenty-two was the same as two. I explained that twenty-two meant twenty and two more. Later you counted hangings on the wall: "Twenty-one, twenty-two, twenty-three, twenty-four [there are four hangings]."	Ct S, Cd S, Cd Ct
3–8	Pretending to water and feed the plants: "In the morning they get two amounts of food and in the evening they get three amounts of food."	M
	You frequently ask us: "Do you know how old Heather [your imaginary older sister] is?" The answer ranges from 5 to 16.	M
3–9	"Heather is 16. She goes to high school."	M
	"Mom. Three-quarters orange juice plus three-quarters orange juice equals what?" (I have been giving you "three quarters" of a glass of juice in the mornings and saying so). I asked you what would happen if we poured three quarters and then another three quarters in the glass. You: "Splat!" Then you asked: "What would sixteen three-quarters make?" I asked you, but you wanted me to answer. I said: "A very messy kitchen." You grinned and agreed.	M+
3–10	"White and yellow make light yellow. pause. White makes any color light."	Principle[b] Cd+
	"What's four and four?" I said to count on your fingers. "One, two, three, four. One, two, three, four. One, two, three, four, five, six, seven, eight. Four and four make eight."	Ct–Cd

3–11	"When Erica [younger sister] is four and I am four, we'll be twins." At other times you say things like: "When I'm five, Erica will be two." (The arithmetic is not always accurate.)	M M+
4–0	You noticed the string from my tampon and asked me about it. I said that "every month a tiny egg comes to the uterus (you have known about babies growing in the uterus and used the word for 1½ years—since Erica was noticeable). If no baby is made, the walls of the uterus wash off to make a nice clean place for a baby to grow the next time. A baby is made if an egg from the mother and a sperm from the daddy come together." You said, "And that happened two times?" I said, "Yes. One time and the baby was you, and another time and the baby was Erica." You had a funny look on your face when I mentioned the egg. I asked if you thought it was funny that people came from eggs. You smiled in a relieved way and said yes. I explained that they were different from regular eggs.	Cd Cd+
4–4	"seventy sixty" "eighty zero"	S S
4–6	You have learned about zero and infinity from Leslie (in carpool) and use them a lot. One very cold morning I said to you, "Today is very cold. The temperature outside is zero." You said, "That means there is *no* temperature." I explained about the thermometer and showed you the 0 and the numbers below zero. A couple of days later I heard you say in an animated voice to a neighbor boy "Did you know that there are numbers underneath zero? Minus one, minus two, minus three . . . minus ten." (I had not listed all of those; just some of them.)	Zero ∞ M Integers
4–7	"How many puppies does a dog have? How many kittens does a cat have?" "Mommy, I can count backwards. Ten, nine, eight . . . three, two, one, zero, minus one, minus two, minus three, minus four . . . minus ten." I was floored. "Sesame Street" does ten to zero and you had done zero to minus ten before but you had never put them together before.	Cd S bkwd Integers
4–9	To Erica (younger sister): "No, you can't have it. It has my germs on it. See [pointing] one, two, three, four, five." When we were preparing bags of popcorn and peanuts for your (early) school birthday, you counted kernals of colored popcorn. In great excitement and wonder: "Ooh! I counted up to one hundred and two!" You refused to make piles of ten. You finally quit at 150. You played this game with Erica (then 2–4): You put colored beads or other objects out on a cloth and then asked her questions. "Erica, what does it make adding one and one?" Erica (without looking at the beads): "Two." You would put out the correct number of objects. Erica answered somewhat randomly, rarely looking at the objects.	Ct Ct–Cd Cd+

Table 1-2. *Continued*

Age[a]	Diary entry	Use
5–1	I asked you how many pieces I had been cutting apples into. You said: "Six." I asked you how you knew. You said: "Because upstairs we had 3 and 3" (you,. Erica, and friend each had two pieces).	Cd+ mental represen-tation
	To someone who asked when your birthday was: "Summer the ninth" (it is August the ninth; we have been doing seasons and months lately).	M, O
5–7	Conversation with you with in the bathtub: You: "How much is seven and seven?" I held up seven fingers and had you hold up seven fingers. You counted. You still usually will not count on from the first number; you need to count the fingers for the first number. You asked and we did nine and nine, four and four, six and six, five and five, and then ten and ten (you did that with your fingers and toes). You said twenty. I said it was called that because it was twin tens—two tens. You said, "I know" and thought for a bit. Then you said, "There's a zero to make the ten and the 2 to make the [pause] two tens." You asked what twenty and twenty were. We used all our fingers and toes. You counted on from twenty. Got forty. I asked you how many tens were in forty (pointing to our fingers and toes—took a bit of focusing for you to see the tens). You said, "Oh, the zero for ten and the 4 for four tens."	Cd+ Cd S Num Base ten num Cd–Ct Base ten num
	You drew a 2 in the air and asked, "Is it this way or the other way [backwards]?" You still do many of your numbers backward—but not letters. I asked you how you knew where to move your arm to draw a 2. "I don't know." "Do you see a picture of the 2?" "Yes, I see an invisible 2, and I just draw it."	Num
	One night before school (Montessori preschool) was out, I was getting you both ready for bed. Erica was crying at every little thing. We went into your room where you could see your digital clock and you said, "No wonder Erica has been acting like a baby. It's 8:00." (We usually get ready at 7:30). You and Erica both have digital clocks, and you are really getting good at telling time. You tell me that 7:50 is ten minutes before 8:00, etc. You know all the ten minutes that way. I asked, "Eight oh eight," the other day and you said, "Eight minutes after eight?"	M M M
	You brought home a Montessori numeral sheet today. It was divided into 1-inch squares and was ten squares by eight squares. The teacher had written the symbols in order from 1 through 10 across the top row. You then wrote seven more rows of numerals under them. Your 1, 3, 4, 5, and 7s were mostly quite good. You were struggling with 2s; they turned more and more into *Z*s as they went down the page. You had trouble with the loop in the 6. The reverse direction cross overs in the 8 were really difficult for you, so you tried different strategies on the 8s. You tried partial	Num

	cross overs and overlapping loops and then settled for a circular top and a U fastened to the bottom of the loop. Your 10s were too scrunched because you started the zero too close to the 1; you went to the right and then when you looped back to the left you ran into the 1. You also started the zero at the bottom. But overall quite a good job; everything was recognizable.	
6–0	You know most of the double sums (3 + 3, 7 + 7, etc.). I asked you some tonight and then asked nine plus nine. You closed your eyes and scrunched up your face and thought and then said, "Eighteen." I asked how you had figured it out. "Well, I knew eight plus eight was sixteen, and I knew there had to be one in the middle, so it was eighteen." (i.e., you knew the double sums went up skipping every other number). I then asked you five plus seven. You thought for a while and then said, "Now don't ask me to describe it because it is very difficult." You closed eyes, etc., and after a while said. "Twelve." I was surprised. "Could you give me a clue?" "Well, I had one five and there was another five in seven with two left over, so that made one was eleven and two was twelve." Big smile. Me too.	Cd+ Thinking Strategies
6–7	You were doing a sheet of two-digit addition of numbers without any trading (carrying) for first-grade homework. I said, "Look, You can do problems with more digits in the same way," and started to show you three-digit and four-digit problems. You got almost hysterical, "I have to do it the way the teacher said. She will get really mad if I do bigger problems. I can't do anything else until she shows me. Don't make me do it, she'll get mad." I had never seen you have this attitude toward anything.	Base ten

Cd, cardinal; Cd+, cardinal addition; Cd−, cardinal subtraction; Ct, count; M, measure; O, ordinal; Num, numerals; S, sequence; S bkwd, sequence backwards; S rel, sequence relations; Ct–Cd, Count-to-cardinal transition; Cd–Ct, cardinal-to-count transition; ∞, infinity.

[a] 1–8 is 1 year 8 months, 1–10 is 1 year 10 months, and so on.

[b] See chapter 10 for discussion of principles.

Table 1-3. Mother's Diary Entries of Erica's Uses of Number Words

Age[a]	Diary Entry	Use
1–11	"Eight, nine, come." (Hide-and-go-seek. It's a challenge to hide before you say come!)	S
	Building with blocks "Three, four, five, six, seven, eight, nine, ten." (I was astounded; you must have been counting with babysitter and son same age as you.) Also later "ten, eleven, twelve."	Ct Ct
	"Adi home six o'clock" said half an hour after Adrienne went to a friend's house. I had told Adrienne she must be home at six o'clock, and she and her friend left chanting it.	M
2–0	"Mommy read for a little couple minutes."	M

Table 1-3. *Continued*

Age[a]	Diary Entry	Use
2–4	"I don't got some lots of milk" in a hurt tone when I gave you a small amount of milk. You wanted a lot and couldn't have it because you had diarrhea.	M
	Adrienne played this game with you several times: She put colored beads or other objects out on a cloth and then asked questions. "Erica, what does it make adding one and one?" You (without looking at the beads): "Two," Adrienne would put out the correct number of objects. You answered somewhat randomly, rarely looking at the objects. (This game obviously stemmed from "Sesame Street.")	Cd+
2–6	You took 2 crackers. I asked, "Do you want 4?" You said, "Yes. Two more."	Cd+
2–8	"Mommy. A lady gave us three pieces of candy [for you and two friends]. I ate mine, and I didn't get *any* cavities."	Cd
2–11	"When I was 4 years old, I went to first grade in California."	M, O
	"Mickey Mouse is the same age as I am." "How do you know?" "They said it on TV."	M=
	"Grandma lives very far away. She lives four months away." (We have been saying your birthday is one month away.)	M
	"I want manier than five."	Cd >
3–6	"Children can have babies. I had two babies. These babies were in my uterus."	Cd
	"I am 3 and a quarter and a penny." (We have been saying that you are 3 and a quarter, but you will soon be 3 and a half. Lovely confusion between fraction and money meanings of "quarter.")	M
	"I am 8 and 9 quarters." (In a pretend game.)	M
3–8	You brought home a Montessori number sheet today. It had a 9 by 6 array of 2-cm squares touching each other. The teacher had written the numerals 1 through 9 in the top row, and you were supposed to write numerals in the five rows below. You did reasonable 1s, 4s, and 7s. Your 8s were two overlapping or sometimes just touching circles: pretty good. All of the 9s were a circle with a stick at the bottom—just like a lollipop. You had a great deal of trouble with the 2, 3, 5, and 6. One or two of these at the top were fairly good—I imagine you had your usual group of 5-year-old boys helping/showing you (you go to the math corner with them and they go to the messy kichen stuff with you). There were some curvy squiggles in the other squares, except that in some of the 6 boxes you had written 7s and 4s (you could make those well!). All in all, very impressive. It must have taken you a very long time to do all of these.	Num
4–2	Walking to my office with you, I asked you how many chairs were in my office. You said, "Four." (This was the use of mental	

	representation; you said you counted the chairs in your head. There were four chairs in my office, one at my desk and three at a table.)	Ct–Cd
5–10	At a conference with your first-grade teacher, I asked to see your work when I was told that you were making a lot of mistakes and were in the low group. Every problem was correct, but you were reversing many of your numbers in writing. These problems were all marked wrong. I pointed out this distinction between the correct answer and writing the symbols correctly (barely able to contain my anger and incredulity at this teacher's incompetence) and further observed that it is quite common for children to reverse numbers or letters in the primary grades. And this is supposed to be a good school system!	Cd+ Num
6–2	You had been doing some two-digit problems without trading (carrying) in first grade. I wrote down the problem 42 + 35 (with the 35 under the 42) and asked you to do it. You wrote 77. I asked you, "How does this work?" You said, "I don't know. The teacher just told us how to do it." "But why does it work?" "Mom. I don't. . . Oh, because you are adding the tens and adding the ones." You drew a 1 with a circle around it above the ones column and a 10 with a circle around it above the tens column. You then said, "I can carry on over. Adri taught me how to do it. Give me a problem." So I wrote down 46 + 38 (in vertical form). You crossed out the 6 in the 46 and wrote it above the tens column and added the 6, 4, and 3 to get 138; you obviously remembered that you had to carry something on over—you just did not remember what. I wrote the problem again and said that you should try again. I asked how much six and eight were. You decided that 6 + 8 was 14, and you wrote the 14 at the side of the problem. You then said, "Oh, yes, I am supposed to carry the ten on over." You then did that, writing down the 4 and carrying the 1 ten over. You wanted me to give you more problems. I gave you three more problems and you "carried on over" correctly on all of them. You wrote the sum of the ones out to the side every time, and then carried the ten over and wrote down the ones in the ones column. You still wrote your 3s and 5s backward. You were quite proud of being able to do these difficult problems.	Base ten Base ten+ Base ten+

Cd, cardinal; Cd+, cardinal addition; Ct, count; Ct–Cd, count-to-cardinal transition; M, measure; Num, numeral; O, ordinal; S, sequence.
[a] 1–11 is 1 year 11 months, 2–0 is 2 years 0 months, and so on.

The effect of specific experiences, that is, of opportunity to learn, is quite clear from these diary entries. The influence of "Sesame Street" number activities was quite strong and clearly led to "seeing" and talking about addition and subtraction of small numbers in many situations at quite a young age. Some understanding of fractions (a half, one quarter, three quarters) and of integers (the thermometer example) seems possible even before school age, and this understanding might have been extended more than it was (e.g., a demonstration of a three-quarter glass of juice plus a three-quarter glass of juice would have fairly easily led to the conclusion "a glass and a half of juice"). The availability of number symbols in the home (TV dials, clock faces with numerals, typewriter or computer, playing cards) and of adults or older siblings who will read these number symbols to a child obviously can influence enormously the age at which a child relates number words and number symbols.

Some other aspects of early number-word learning also are evident. First, the interest in and enthusiasm about knowing numerical concepts is reflected in many entries, including conversations between children. Second, the effect of living in a house containing an older sibling would seem to be to accelerate the exposure to number words, because the younger child will hear uses directed by the adult to the older child (e.g., in Table 1-3 "six o'clock") and will hear and be involved in games and fantasy play with the older sibling that involve number words (e.g., in Table 1-3 hide-and-go-seek at age 1–11, the adding game at 2–4, the statement about being 4 years old and being in first grade at age 2–11). This may or may not be accompanied by a decrease in the amount of adult use of number words to the second or later child (similar to the decrease in amount of reading to a later child). The accelerated exposure also may or may not be useful to the younger child. A particular experience may or may not be within the zone of proximal development of the child (Vygotsky, 1978), that is it may or may not be learnable even with the help of the older sibling, whose sensitivity to what may be understood and how to help with this understanding undoubtedly varies considerably. Third, it is quite evident that number-word situations include mental representations of entities not physically present (e.g., the comment "two tapes (tails)" in Table 1-2 at age 1–10 when the owner of one tail was not in the room and the counting of chairs in a mental representation of my office in Table 1-3 at age 4–2) and include imaginary entities (counting germs in Table 1-2 at age 4–9, presumably to keep the younger sibling from eating some desirable food). Fourth, the words used for different kinds of measures can be confusing. For example time and distance were confused in Table 1-3: "Grandma. . .lives four months away."; seasons and months were confused in Table 1-2: My birthday is "summer the ninth." Finally, the stress in school on doing things the way the teacher said to do it regardless of comprehension is evident in the later entries in both tables; neither thinking about nor extending what has been taught is perceived as necessary, and the latter even sometimes evokes almost terror. The contrast between the two interactions in Table 1-2 at ages 5–7 and 6–7 is very striking and clearly is not a recommendation for American elementary school mathematics instruction (in a "very good" school district).

The pervasive influence of the nature of the experiences young children have in different kinds of number-word situations and the rather considerable difference one might expect to find in the situations emphasized or provided in different home environments suggest that there may be considerable individual differences in children's ability to use number words in different situations. Thus, one child might have a very well-developed symbol domain by age 4 while another child may be able to recognize and write few symbols. One child may frequently play with cards or with dominoes and thus recognize figural patterns for the number words up to ten, while another child might have no such experiences and thus recognize few or no such patterns. In fact in much of the data we report in this book, there is a very considerable range in the age at which children respond correctly to most number-word situations. A considerable amount of this range would seem to result from differences in children's opportunity to learn about these number-word situations.

1.4. The Precocity of Use of Small Numbers

Infants and young children demonstrate considerable precocity in situations involving very small numbers (e.g., Table 1-2 and 1-3; Cooper, 1984; Starkey, 1987; Starkey, Spelke, & Gelman, 1983; Strauss & Curtis, 1984). Older preschool and school-aged children continue to show competence in more difficult situations that involve very small numbers (1, 2, 3) considerably before they do so with larger numbers. For example children can solve problems such as 7 + 1 or 7 + 2 before solving 7 + 6, and children establish equivalence or order relations on very small sets before doing so on large sets (see chapters 7, 8, and 9 for details of these examples and for other examples). Gelman and Gallistel (1978) argued that preschoolers showed competencies with small numbers before larger numbers because they can reason about specific numerosities but not about unspecified general "algebraic" numerosities.[7] Gelman and Gallistel furthermore argued that counting provides young children with specific numerosities for small but not larger numbers (because the counting of larger numbers is not accurate), and thus counting ability is responsible for this ability to deal with small but not larger numbers. We agree with the first part of this position (that concerning children's competence with specific numerosities) and in later chapters provide and discuss evidence that supports this argument. Our position, however, differs with the second part of the Gelman and Gallistel position in at least three ways. First, the evidence presented in this book indicates that even 3-year-olds show considerable competence in counting numbers above 3 and that by age 4½, most children can count large numbers of objects in rows (up to 20) with consi-

[7] This position was later modified in Gelman (1982a) to include young children's ability to use one-to-one correspondence (matching) without ascertaining specific numerosities. This position is consistent with that outlined in this book (e.g., data are provided here concerning children's use of correspondence in equivalence and order situations) and is not one of the three differences discussed later.

derable accuracy. Second, 4-year-olds show some ability to use counting of numbers larger than 3 to establish equivalence and order relations on cardinal situations and to add and subtract cardinal situations and 5-year-olds show considerable such ability. Third, children's precocious competence with very small numbers (i.e., that which outruns competence with larger numbers by a year or more) depends not on counting but on special perceptual methods of obtaining or representing specific numerosities in these situations. These special perceptual methods include subitizing and the use of auditory, visual, or kinesthetic figural patterns[8] (see examples and evidence in chapters 7 through 10). Thus, our position suggests that preschool children have both more and less competence than indicated by the Gelman and Gallistel position: they have more because they do have considerable competence even with larger numbers when they obtain or use specific numerosities from counting, and they have less because their competence with very small numbers derives from special nongeneralizable processes that will not extend to larger numbers.

The special perceptual processes used by infants and young children with very small numbers may be similar to those used by animals in numerical tasks. Primates, birds, rate, mice, cats, and raccoons have demonstrated competence in various kinds of numerical situations (e.g., see reviews in Davis & Memmott, 1982, and in Davis & Perusse, 1987; see also Matsuzawa, 1985; Pepperberg, in press; Thomas, Fowlkes, & Vickery, 1980). Some authors have claimed that such competence suggests counting competence in that given species. However, Davis and Perusse (1987) argued convincingly that such claims exceeded the presented evidence. Most such studies provide evidence only of use of relative numerousness judgments (which involve order relations on cardinal situations) and of subitizing or of figural patterns. Most of this literature concerns cardinal situations, but some recent work concerns ordinal situations (e.g., Davis & Bradford, 1986). Competence has been demonstrated with both simultaneous and sequential visual entities and with sequential auditory, tactile, and kinesthetic entities (Davis & Perusse, 1987). In contrast, in the literature on humans, there has been relatively little work with infants or young preschoolers with sequential entities. Almost all the evidence reviewed in this book, including the new evidence presented here, concerns entities that are all simultaneously physically present. This animal literature is similar to that on human infants because most of the animals and all of the infants have not been taught number words. Thus, subitizing used with respect to this literature refers only to the perceptual process underlying discriminations among very small numbers and does not include the ability to label these small number situations with distinctive "words." Human 1- and

[8] Because the debate concerning whether subitizing involves figural patterns is not resolved, we will include the use of figural patterns as a possible separate procedure.

2-year-olds do possess such abilities, as evidently do some primates[9] and the grey parrot trained by Pepperberg (in press).

The extent to which competence with small numbers facilitates later competence with larger numbers is not clear. This may even vary across kinds of number-word situations. However, as will become clear from chapters 7 through 9, competence in cardinal situations involving larger numbers frequently requires procedures and understandings that are not required in situations involving very small numbers. Thus, to understand children's developing competence in numerical situations from age 2 through 8, one must study children's ability in situations involving larger numbers. Matching, establishing a one-to-one correspondence between pairs of elements in two sets, is one procedure that is useful and comes to be used in situations with larger numbers (see chapter 9). Counting to establish specific numerosity is useful in even more situations. Thus, if one wants to understand the general development of children's understanding of concepts of number from age 2 through 8, one must study children's use of matching and especially of counting in numerical situations.

1.5. Focus and Outline of the Book

This book focuses on the very considerable increase in children's competence in cardinal, counting, and sequence situations from age 2 through 8. The increase in competence that occurs over this age span is quite impressive, with children moving from very little understanding of any of these situations to an ability to add, subtract, and compare (establish equivalence and order relations on) very large whole numbers. This increase is marked by changes in children's conceptions of the cardinal, counting, and sequence situations and of the relations that exist between and among them. This book traces these changes.

Measure and ordinal situations are not treated for three reasons. First, they are both more complex and begin to be understood later than are the other three situations. Measure situations involve continuous quantities and thus the repeated use of a unit. Because repeated use of a unit is more complex than just using a perceptual unit item (e.g., Carpenter, 1975; K. Miller, 1984), measure situations are more difficult for children than are cardinal situations (see Fuson & Hall, 1983, for a review). The necessity to learn special ordinal number words complicates performance in ordinal situations, and this learn-

[9] A friend's initial experience with Koko, the gorilla trained to sign, was a request by Koko, "Koko two flowers." My friend complied, picking two small flowers and handing them to Koko. Koko took one in each hand, as small children do, sniffed each of them delicately, and then ate each in turn.

ing clearly lags behind the learning of the sequence/counting/cardinal number words by several years (Beilin, 1975). For the first five ordinal words and the first eight cardinal words, 2-, 3-, and 5-year-olds were correct on 4% versus 48%, 0% versus 69%, and 57% versus 95% of the ordinal versus the cardinal number words, respectively. For words up through the first 23 words, the 5-, 6-, and 7-year-olds were correct on 2% versus 82%, 10% versus 90%, and 46% versus 100% of the ordinal versus the cardinal number words, respectively. Thus, even at age 7 many children either do not know the rule by which ordinal words are generated or cannot successfully use that rule. Second, there is relatively little literature on measure and ordinal situations compared to the other three situations. Third, the pattern of relationships among sequence, counting, and cardinal uses of number words would seem to extend fairly readily to both ordinal and measure uses. Thus, it seems sensible to obtain some understanding of this pattern of relationships and then to explore how the relationships might extend to ordinal and measure uses of number words.

This book sketches the development of children's competence in cardinal, counting, and sequence situations from age 2 through 8. New data concerning these areas are presented, and data from other investigators are reviewed. Chapter 2 briefly summarizes research on children's learning of the English sequence of number words and on later useful advances in children's representation of this sequence of number words. Chapters 3 through 6 focus on the kinds of correspondence errors children make when counting and on variables that affect these different kinds of errors. Chapter 3 contains an analysis of the activity of counting that generates a category system of possible errors in the correspondences among the words used in counting, the points used in counting, and the objects being counted (i.e., the perceptual unit items "seen" by the counter in the counting situation). Data are presented on age-related changes in different kinds of errors for counting objects arranged in a row and on the effects of location of object, effort, sex of counter, and number of objects counted. Chapter 4 reports on correspondence errors made when objects are presented in different spatial arrangements (rows, disorganized arrays, circles) and when indicating acts other than pointing are used. Chapter 5 returns to objects in rows and presents three studies in which the effects of certain variables on the occurrence of counting correspondence errors are examined; the variables include homogeneity of the objects, distance between the objects, location of the objects at the beginning, middle, or end of the row as well as within the first or second half of the row, number of objects counted, and age of counter. The results across the studies presented in chapters 3, 4, and 5 are summarized and discussed in chapter 6.

Chapters 7, 8, and 9 concentrate on relationships between counting and concepts of cardinal number that children learn over the age range 2 through 8. Chapter 7 focuses on the early relationships constructed between the ages of 2 and 5. Chapter 8 discusses the later, more complex relationships that underlie children's understanding of and ability to carry out the operations of addition and subtraction with whole (cardinal) numbers. Developmental progressions are proposed for children's representations of addition and subtraction situations, their addition and subtraction solution procedures, and their concepts of cardinal number and conceptual relationships between counting and cardinal number. Chapter 9 presents data on children's use of counting and of matching in establishing equivalence and order relations on cardinal number situations (i.e., the same number as, more than, fewer than relations) and then outlines a developmental sequence of strategies children use in equivalence and order situations that culminates in Piaget's conservation of cardinal equivalence and a postconservation stage of truly numerical counting. In all three of these chapters, the emphasis is on changes in children's concepts and representations of number words, counting activity, and cardinality. Together these chapters indicate how sequence, counting, and cardinal situations move from being separate situations to becoming integrated as the number-word sequence itself becomes a cardinalized unitized seriated embedded conceptual structure.

Chapter 10 relates chapters 2, 6, and 7 to each other by reporting data on sequence, correspondence, and cardinal aspects of counting within the same children. Developmental relationships among Gelman and Gallistel's (1978) three how-to-count principles are considered, and these principles are discussed in some detail. Because so much is covered in each chapter of this book, the final chapter does not attempt a final overall summary of results. Such a summary overview of the conceptual and empirical results can be obtained by reading the present chapter, chapter 6 (in which chapters 3, 4, and 5 are summarized and discussed), and the chapter summaries for chapters 2 and 7 through 10. The final chapter provides an overview of changes in children's number word concepts, over the age span age 2 to 8, including the increasing integration of sequence, count, and cardinal meanings of number words.