11. An Overview of Changes in Children's Number Word Concepts From Age 2 Through 8

Young children's early experiences with number words occur in many different specific situations. These situations fall into seven mathematically different kinds of situations: cardinal, measure, ordinal, counting, sequence, symbolic (numeral), and nonnumerical situations. These uses are described in chapter 1. In English the same number words are used for six of these kinds of situations. For ordinal situations, special ordinal modifications of these number words are used; most of these ordinal number words sound quite similar to the standard numbers words (having *th* on the end). Because the words are identical for most situations, and similar even in ordinal situations, it is fairly easy to relate uses of the same number word across these mathematically different situations. However, very young children show little understanding of any such relationships. Their initial uses of number words are confined to use within a given kind of situation. A major aspect of the learning that occurs throughout the period from age 2 through 8 concerns the increasingly complex relationships between different kinds of situations.

The separate situations and some relationships between various pairs of situations are sketched in Figure 11-1. Each kind of situation has very many specific entries, which are probably organized into semantic networks of various kinds. The situations in Figure 11-1 contain only a few suggestive entries (some taken from the diary entries in Tables 1-2 and 1-3). Addition and subtraction operations can be carried out in cardinal, measure, and sequence situations; these are placed in an inverted triangle within each of these situations. Equivalence and order relations can be established on cardinal, measure, sequence, and ordinal situations; these are placed in a square

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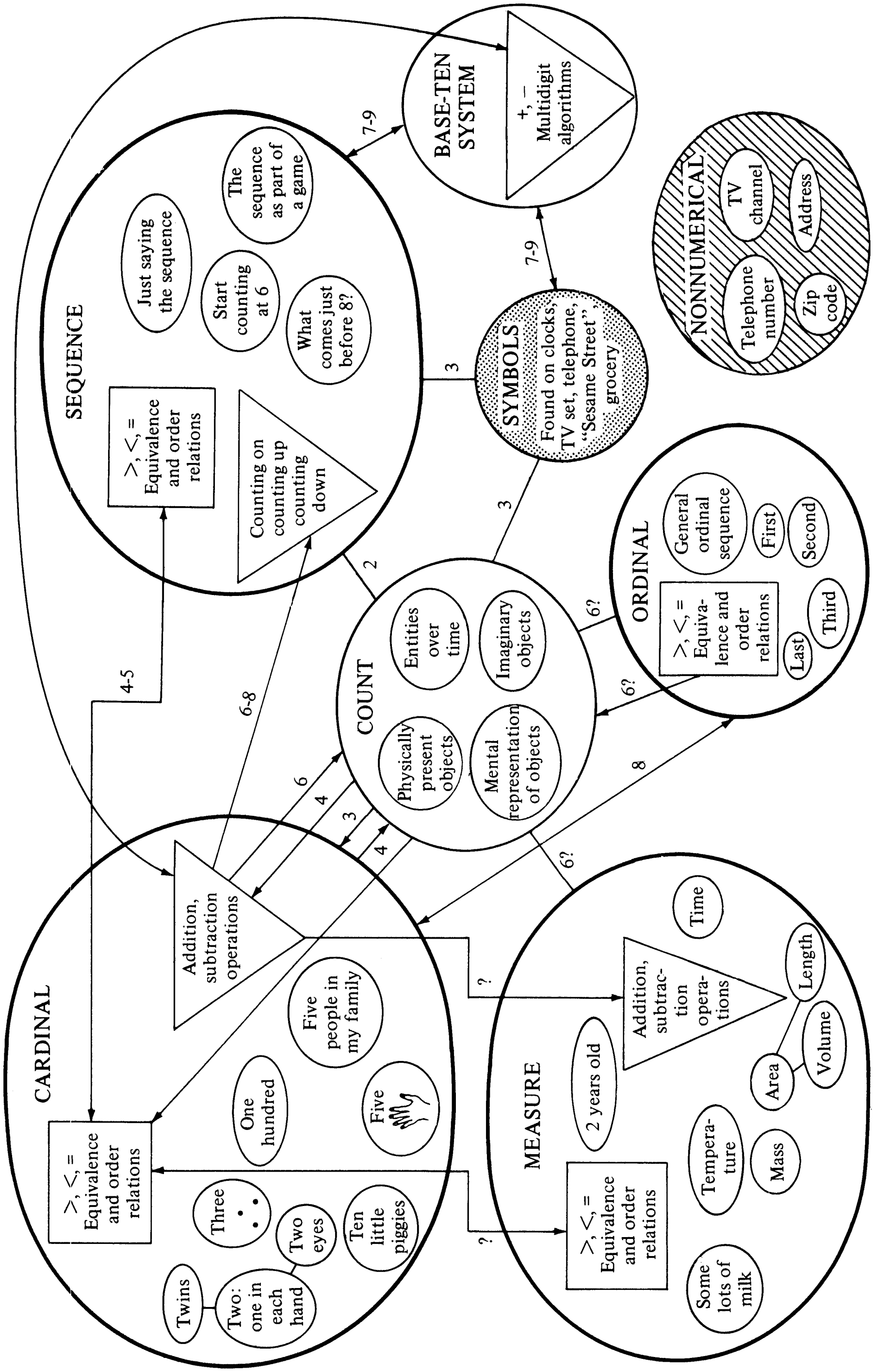

Figure 11-1. A partial mental map of a young child's early experiences with number words.

within each situation. Arrows connect various situations. Approximate ages in years have been entered for these relationships, but these ages of course depend very heavily on the size of the numbers involved. The ages entered are for the more general cases of number words larger than five (but not necessarily larger than ten). There is considerable variability in the amount of data to support each entry. Those for which there are practically no data have a question mark placed after them. The relationships between counting and cardinal situations were discussed in chapters 7 through 9, between cardinal and sequence situations in chapters 8 and 9, and those between cardinal and ordinal situations are briefly discussed in this chapter.

Figure 11-1 captures one major aspect of learning in the period from age 2 through 8: the increasing number of relationships between different kinds of situations. Another major aspect of learning during this period that is not reflected in the figure is the very considerable increase in knowledge about different specific situations within each kind of situation. Portraying this would require successive versions of Figure 11-1 over different ages. These versions would show increasing numbers of entries within each kind of situation and changing networks of relationships within each kind of situation. These versions of Figure 11-1 would also seem to vary quite considerably for different children. The entries would be determined by the specific experiences of individual children, which could be quite different. For example, some children might live in an environment containing many different number symbols that would be labeled for the child. An older sibling might teach a given child addition at a young age. One preschool might have the children doing a lot of counting and saying the number-word sequence, while another might do practically nothing in these areas. Thus, it does not seem likely that one could develop a single series of figures like Figure 11-1 that would apply to all children.

A second major aspect of learning over the age span 2 through 8 that is not reflected in general in Figure 11-1 is that larger and larger number words are learned over this age span. This learning is discussed in chapters 2 and 10. One ultimate outcome of this learning that is reflected in Figure 11-1 is a new later mathematical situation: the base-ten system of numeration. This system has obvious links to symbol, sequence, and cardinal situations and is discussed in chapter 8. Some research indicates that addition and subtraction of multidigit numbers and some concepts of place-value and of base-ten numeration are within the understanding of second graders of all ability levels if they are taught the multidigit algorithms with materials that physically embody the base-ten system (Fuson, 1986a, 1987). Other more advanced concepts within this base-ten domain are the English names for very large numbers, exponents, and scientific notation.

Figure 11-1 includes only uses of whole numbers. Other numerical domains—decimals less than one, integers, rational numbers—also come to be learned in the middle and later elementary school years. However, learning about these different kinds of numbers may suffer from a problem that is the oppo-

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site of the young child's problem of nonexistent or inadequate connections among different number-word situations. These domains seem to suffer from being too closely connected to the cardinal numbers, that is, from inadequate differentiation from the cardinal numbers. Thus, for example, many children think that $\frac{1}{6}$ is smaller than $\frac{1}{7}$ because 6 is smaller than 7 or that $.2 \times 3$ should be bigger than 3 because 2×3 is bigger than 3. This problem probably stems from at least two sources: simple interference from the very well-learned relations and operations on cardinal numbers and an inadequate experiential basis for the decimal, integer, and rational number domains. Certain aspects of the measure number domain may also suffer from the latter deficiency. School mathematics instruction in this country for these domains typically involves verbal instruction about the correct manipulation of symbols for the domain. This poverty of concrete experiences that could provide meaning to the basic numerical constructs and to the order/equivalence relations and operations on these numbers is in sharp contrast to the richness of the experiences young children have with cardinal numbers.

A third major aspect of learning over the age span from 2 through 8 years that is not reflected adequately in Figure 11-1 is the nature of the changes in children's conceptual structures over this period. These changes underlie many of the relationships given by arrows in Figure 11-1. This book concentrates on such changes in three situations: the cardinal, counting and sequence situations. These conceptual changes are summarized in Table 11-1. The changes in conceptual structures for the sequence are discussed in chapters 2 and 8 and those for counting and cardinal situations in chapters 7 through 9 (see those chapters for any clarification required for this necessarily brief overview).

The sequence moves from an initial *string* state, in which number words may not even be differentiated into separate words (no word units may exist), to an *unbreakable list* of number words, in which words are differentiated but the sequence must always be said beginning with *one*. The counting situation[1] moves from some number words and points being directed vaguely at entities to one in which the counter "sees" a counting situation as consisting of equivalent "countable" entities, of perceptual unit items (Steffe et al., 1983). At this time it is possible to establish a one-to-one correspondence between the perceptual unit items and the sequence words. This is done by the use of an indicating act (such as pointing) that simultaneously creates a temporal word-point correspondence and a spatial point-entity correspondence (see chapters 3 through 6).

At first, counting situations are only counting situations; counting is done only for itself. However, sometime during age 3 (for most children) children

[1] Because there are so few data concerning children's counting of entities distributed over time, this book concerns only counting of entities distributed in space and all existing at the same time.

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answer the question "How many entities are there?" with the last counted word (see chapter 7 for a discussion of how and why this occurs). Some children at this time also seem to be able to make a cardinal integration of the counted entities and make a count-to-cardinal transition in which the last counted word takes on a cardinal meaning as the manyness of the cardinal set of entities. For others, this step seems to take some further time. Once a child has reached this level, however, counting can be used for cardinal addition and subtraction operations.

The next conceptual level within the sequence is that of a *breakable chain* in which a child can begin saying the sequence at any word. In counting and cardinal situations, perceptual unit items become capable of simultaneously representing a sum and an addend embedded within that sum. The cardinal conceptual operation of embedded integration allows a child to integrate these simultaneous perceptual unit items into the embedded sum situations shown in Table 11-1. This sequence ability combined with the simultaneous perceptual unit items allows children to carry out new, more efficient solution procedures in addition and subtraction situations: counting on, counting up, and counting down with entities. To count on for 8 + 6, for example, one would represent the 6 by some entities, begin the counting with 8, and continue counting the representation of 6 (thus counting on six more words: "eight, nine, ten, eleven, twelve, thirteen, fourteen").

At the next level the sequence, counting, and cardinal situations become more closely related. The sequence becomes a *numerable chain* in which the number words themselves form the cardinal addition and subtraction situations. The number words can be used as sequence unit items in counting and can simultaneously represent the sum and the addends embedded within the sum. Sequence versions of counting on, counting up, and counting down become possible in which some method of keeping track of the second addend is used instead of the second addend being represented by entities. These keeping-track methods usually involve matching or double counting the sequence units items for the second addend. This requires a paired integration of the sequence second addend words and the representation of the second addend by the keeping-track method. The sequence unit items provide a very flexible means of representing and solving many different kinds of addition and subtraction situations. Children first learn forward-sequence counting solution procedures and then learn backward-counting situation procedures (the counting down procedures: see chapter 8).

Finally, the sequence becomes a *bidirectional chain* in which the forward- and backward-counting solution procedures are related to each other. Over the previous levels the unit items required to represent a cardinal situation have become increasingly abstract and decreasingly have required physically present entities. At this new level no entities are required for a cardinal situation. A given specific cardinal number has now itself become "concrete" to a child. The child can conceptually operate on and relate specific cardinal numbers. When necessary, cardinal situations can be represented by ideal

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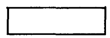
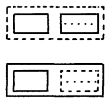
Table 11-1. Developmental Levels in Sequence, Counting, and Cardinal Conceptual Structures

Sequence structure	Counting and cardinal conceptual units	Cardinal conceptual structures	Cardinal conceptual operation	Cardinal relations
String No units				
Unbreakable list Separate words; start from one	Perceptual unit items Single representation addend *or* sum			
Unbreakable list Separate words; start from one	Perceptual unit items Single representation addend *or* sum		Cardinal integration	Count equivalence ⇒ Cardinal equivalence
Breakable chain Separate words; start anywhere	Perceptual unit items Simultaneous representation addend *within* sum		Embedded integration	Count >, < ⇒ Cardinal >, <

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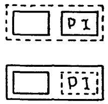
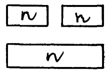
				Inductive generalization about displacement transformations
Numerable chain Sequence unit items	Sequence unit items Simultaneous representation addend *within* sum		Embedded integration	Deductive generalization about displacement transformations
Bidirectional chain	Cardinal numbers Can be decomposed into ideal unit items[a]		Numerical equivalence	Truly numerical counting

[a] At this level a child may still solve a problem using an object or sequence procedure, but the procedure will not necessarily directly model the problem representation.

Note. In the cardinal conceptual structures, a dotted line represents the unknown set and a solid line represents a known set. PI is paired integrations: the pairing of the embedded integration of the second addend with the integration of the keeping-track method for that addend.

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unit items, that is, by abstract identical iterable *ones*. However, cardinal addition and subtraction situations can now be represented by a triplet of three cardinal numbers (e.g., 8, 6, 14) related to each other by cardinal equivalence (Table 11-1). This triplet relationship entails both an addition (addend + addend = sum) and a subtraction (sum − addend = addend). Thus, at this level a child first has a representation of, a meaning for, a cardinal number without having to represent it by a set of perceptually apprehendable entities.

Children's understanding of equivalence and order relations on cardinal situations also goes through a developmental sequence that is described in chapter 9. Evidence is fairly clear about the first part of this sequence, but the later parts of it are much more tentative. This developmental sequence is given in the right-hand column of Table 11-1. Children come to understand a "count equivalence imples cardinal equivalence" rule, namely: if two cardinal situations yield the same count word, then they have the same number of entities. A bit later children become able to use counting to establish an order relation on two cardinal situations.[2] With numbers below ten, children may know order relations directly for pairs of cardinal words. For larger number words, the order relations on the cardinal situation are derived from the sequence order relations (if a word is later in the sequence, it represents the larger cardinal situation). Children also use matching in cardinal equivalence situations. Considerable experience with counting or matching in cardinal equivalence situations may then lead to the inductive generalization that a displacement transformation (one in which entities are merely moved around in space) on one cardinal situation does not affect any cardinal equivalence or order relation on that situation. Children in the Piagetian conservation of number task may then immediately respond correctly and give justifications that refer to the irrelevance of the transformation or to the type of transformation (one not involving addition or subtraction). Later, considerable experience with simultaneous perceptual unit items in addition and subtraction situations may allow a child to deduce (rather than induce) conservation by mentally "following" matched or counted perceptual unit items through the transformation and "seeing" that the end state is composed of the identical unit items that constituted the initial state; this deduction would be justified by a logically necessary operational reversibility. A related ability to represent two aspects of the situation would lead to the use of the compensation justification. Finally, a fourth postconservation stage of truly numerical counting occurs in which a child's number-word sequence is unitized, embedded, seriated, and cardinalized (see following).

The developmental levels within the sequence are quite clear and fairly well supported by data (see chapter 2). Most of the levels for the count and

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[2] Knowledge of these equivalence and order rules is displayed considerably later in situations in which length of an array is misleading than in those in which length cues are consistent with number.

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cardinal situations and for the relationships between the count and cardinal situations are also fairly clear and well supported (see chapters 7 and 8). However, the relationship between the two sequences of levels is not so clear. In general the sequence conceptual levels are required by the counting and cardinal levels, and so they occur earlier than the counting or cardinal levels in which they are used. The relationships between children's understanding of cardinal equivalence and order relations (the right-most column in Table 11-1) and their understanding of cardinal addition and subtraction operations are not at all clear at present. One possibility is the way in which these levels relate to each other in Table 11-1. However, the deduction using simultaneous perceptual unit items may occur earlier than the sequence unit item level, and some aspects of truly numerical counting may occur with sequence unit items (see the following discussion).

Table 11-1 provides some sense of the changes that occur in children's conceptualizations of sequence, counting, and cardinal situations. However, it does not demonstrate one major aspect of these changes: that the representation of the sequence changes over this period so that sequence, counting, and cardinal meanings become integrated within the sequence itself. Figure 11-2 displays major aspects of this increasing integration. At the string level the sequence is not related to any other situation. At the unbreakable list level, the sequence is used in counting, and one-to-one correspondences can be established with perceptual unit items "seen" by the counter in the counting situation. Later in the unbreakable list level, the conceptual operation of cardinal integration enables a counter to make a count-to-cardinal transition from the count meaning of the last counted word as paired with the last counted entity to the cardinal meaning of the word as describing the manyness of all the entities.[3] At the breakable chain level a cardinal-to-count transition for the first addend enables a child to count on by moving from the cardinal meaning of the first addend (*eight* in Figure 11-2) to the count meaning as paired with the last entity in the first addend and then continuing to count the entities representing the second addend. A count-to-cardinal transition at the end of the sum count up to thirteen enables the thirteen to refer to the cardinality of all of the entities (that is, to the sum). At the numerable chain level no entities need to be present. The same transitions occur, but now the cardinal situations are represented not by sets of entities but by sets of number words.

The evidence supports the developmental progression up to this level. We propose in chapter 9 a fourth postconservation stage of truly numerical counting (see the discussion there with respect to Piaget's statements) and also

[3] Resnick's characterization of the preschooler's representation of number as a mental number line (1983, p. 110) is roughly similar to the unbreakable list level. However, what is described by the term mental number *line* is actually a mental number-word *sequence* such as discussed here. A number *line* is a measure model in which numbers are represented by *lengths* on the line, such as by Cuisenaire rods. Number words are discrete entities, not lengths.

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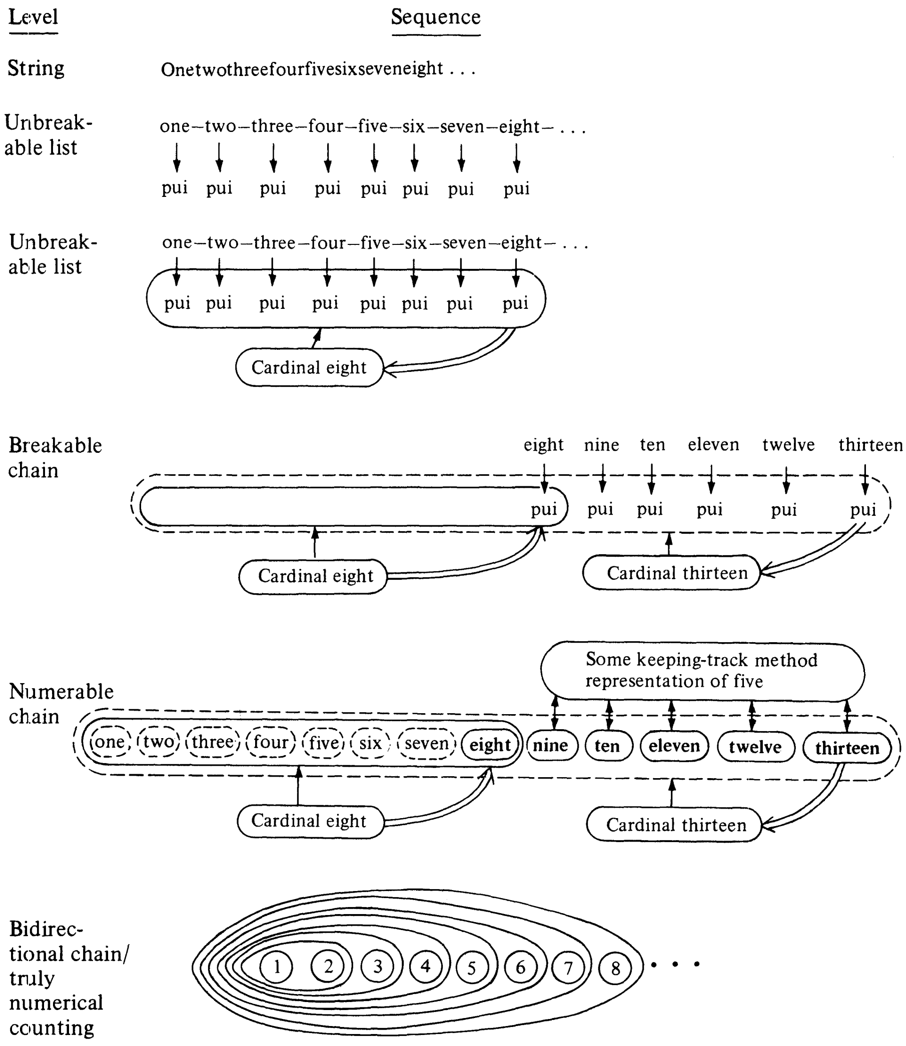

Figure 11-2. Major aspects of the increasing integration of sequence, count, and cardinal meanings.

suggest that there is a bidirectional chain level in the sequence (chapter 2; also Fuson et al., 1982). There are several aspects of relationships among sequence, counting, and cardinal situations and of relationships between cardinal and ordinal situations that do occur relatively late. We place these all within Figure 11-2 at a single *bidirectional chain-truly numerical counting* level. Research is clearly needed at this level, and such research may find sublevels in which certain aspects in the table occur before others. Ways in which various thinking strategies children use for addition and subtraction

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Bidirectional chain/truly numerical counting (*cont.*)

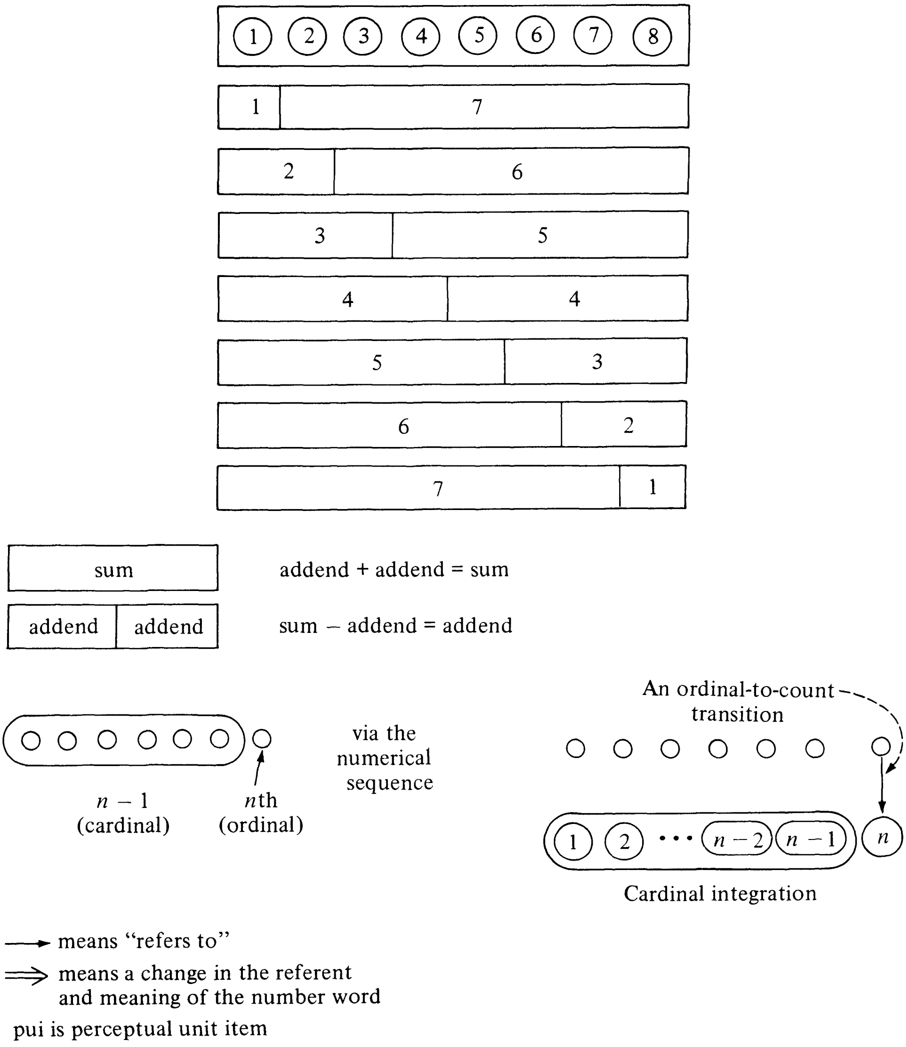

Figure 11-2. *Continued*

(see chapter 8) relate to aspects of this level are also of interest and might well be a focus of future research.

One aspect of this truly numerical counting level is that the sequence is both seriated and embedded. Because each word of the sequence is now an ideal identical iterable *one*, and because each word is now both a sequence word and a cardinal word (that is, it can refer to all of the words up to and including itself), each next word represents a cardinal number that is one larger than (using the cardinal as well as the sequence meaning) the earlier

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word. This cardinalized sequence thus displays both class inclusion (the embeddedness of each cardinal number within the next) and seriation. These were the requirements of a truly operational cardinal number for Piaget (1941/1965). The child at this level then is capable of the progressive summation of sequence-counting-cardinal words.[4] The conceptual operation of numerical equivalence also enables any given cardinal number to be seen as composed of all possible combinations of smaller numbers and as decomposable into all such possible combinations. The sum of two numbers is now also related to its two addends within a triplet addend-addend-sum structure such that knowing any two of the numbers will yield the third, either via addition (addend + addend = sum) or via subtraction (sum − addend = addend). Finally, cardinal and ordinal situations are related via the embedded numerical sequence so that a child knows that there are $n - 1$ entities in the cardinal set preceding the nth ordinal entity. This is first accomplished by an ordinal-to-count transition from the nth ordinal entity to the sequence n, which is now a cardinal as well as a sequence n for the child and is known to have $n - 1$ entities before the n. For example in Piaget's staircase problem, for a doll standing on the eighth stair a child could make a cardinal integration of the stairs preceding the nth stair and use the backward chain to know that the cardinality of those stairs would be the sequence word before n, that is, would be $n - 1$.

Brainerd (1973, 1979b) presented a theory of children's understanding of number in which ordination preceded cardination. Ordination was the internal representation of transitive asymmetric relations, while cardination required the use of correspondences to establish numerical equivalence. Tasks measuring ordination were found to be much easier than those measuring cardination, and an argument was made that axiomatizations of set theory in which cardinal number was primary were problematic, whereas those in which ordinal number was primary were not problematic. It was concluded that ordinal number rather than cardinal number was the child's earliest notion of number. This theory did provide a useful focus on the importance of linear orderings (transitive asymmetric relations), but both bases of its conclusions are actually problematic. Evidence from Baroody (1982) indicates that the cardination task was so difficult that even some adults have trouble with it; Kingma and Roelinga (1984) demonstrated that one can change the order among seriation, ordinal correspondence, and cardination by varying the processing load of the different tasks; and Michie (1985) used tasks equated on other dimensions to measure understanding of cardinal number and ordinal number situations and found that children

[4] The term progressive summation is taken from Saxe (1982). He defined counting as *the progressive summation of correspondences*. However, a progressive summation that involves representing the correspondences involved in the activity of counting seems to us to be an even more abstract notion than the progressive summation of sequence-count-cardinal words and therefore would occur even later than this.

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understood cardinal number first. With respect to set theory, the various axiomatizations of set theory actually all permit the construction of the natural numbers and all permit cardinal and ordinal numbers to be derived.[5] Many aspects of Brainerd's arguments actually apply to sequence and counting situations, as we have discussed them here. In that sense understanding of certain aspects of these situations does seem to precede understanding of related aspects of cardinal situations. Sequence and counting play especially key roles in addition and subtraction cardinal operations.

We have sketched developmental sequences within sequence, counting, and cardinal situations and indicated how these become increasingly integrated over the age span 2 through 8. Many specific questions remain unanswered about these developmental sequences, and especially about their relationships to each other. One of the major unanswered kinds of questions concerns the extent to which progression along these developmental sequences is due to experience and practice with sequence, counting, and cardinal situations versus simple maturation. As in most areas, the answer is undoubtedly that a mixture of these is involved. However, the work of Case indicates that experience and practice are crucially important and can change children's level of performing quite considerably. The basic argument is that practice leads to increased automatization, which leaves more processing space for noncounting aspects of a task. Results of Case et al. (1979) indicate that the automaticity of the number-word sequence can affect the difficulty of the counting task in which it is used, and that considerable practice in counting can increase the performance of 4-year-olds so that it is like that of the 6-year-olds. Likewise, developmentally appropriate training in quantification seems to increase the performance of lower-class 4½-year-olds on tests of scientific and practical reasoning tasks to the level of normal 6-year-olds (Case & Sandieson, 1986, 1987).

The kind of conceptual analysis carried out in chapters 7 through 9 may seem quite abstract and have clearer implications for the cognitive development than for the education of young children. However, this approach has already yielded a highly effective instructional method for the addition and subtraction of single-digit cardinal numbers. An initial conceptual analysis of what young children must come to understand in order to count on (Fuson, 1982a, 1982b) was followed by assessing whether these understandings differentiated children who could count on from those who still counted all and whether teaching counting-all children these understandings would lead to counting on (Secada, et al., 1983). This was followed by classroom level

[5] In particular even in Russell's system (Russell, 1903; 1919; Whitehead & Russell, 1910–13) the Peano axioms are derivable, and thus in Brainerd's (1979b) terms the ordinals can be derived from the cardinals. Furthermore, Peano's axioms do not define the ordinal numbers; they define a structure that can be satisfied by the ordinal numbers or by the cardinals. Finally, the paradoxes discussed by Brainerd apply to any naive set theoretical development of the ordinals or cardinals but not to formal set theories.

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studies in which teachers did the teaching based on an instructional unit (Fuson & Secada, 1986) and by an extension of counting on for addition to counting up for subtraction (Fuson, 1984; Fuson, 1986b, Fuson & Willis, in press). The instructional units followed the normal developmental sequence of understandings except that a more efficient one-handed method of keeping track was taught for sequence counting on and counting up. This method of adding and subtracting single-digit numbers was then efficient enough to be used in the multidigit addition and subtraction algorithms, which could be taught to second graders of all ability levels and to high ability first graders because one did not have to wait until they had memorized their facts (Fuson, 1986a; 1987). These instructional units were demonstrated to be effective when used by classroom teachers with a range of student ability and to accelerate the learning of addition and subtraction by 2 to 3 years (cf. Fuson, Stigler, & Bartsch, in press). Thus, the conceptual analysis of children's understandings can have important educational results. A second obvious result of the developmental sequences in children's representational abilities presented here is that young children who can represent cardinal situations only with perceptual unit items will need to be provided with entities for cardinal addition and subtraction. They simply cannot represent or understand cardinal addition and subtraction without such representations.

We have been struck in all of our number work by how wide the age span is for correct performance on any task. There is frequently a span of as much as 1½ or 2 years in the age at which children respond correctly to some number task. This means that there is a very considerable span at school entry in the experiences and abilities of children. More use of "Sesame Street" by parents in all homes might help to close this gap. The program provides experiences of direct teaching of the number-word sequence, the symbol–number-word connection, and the meaning of addition and subtraction. Learning the first two requires considerable and repeated input; television is ideal for this, especially for parents who are very busy or under stress or for other reasons are less likely to provide such experience to their young child. Meanings of addition and subtraction can also be conveyed well on television, both by visual experiences over time and in the interactions among people on the ("Sesame") street. "Sesame Street" also provides certain gamelike formats, which children then carry into their own activities (see the diary entries in Tables 1-2 and 1-3 concerning "Sesame Street" activities). Such child-initiated activities then may elicit interaction from parents who ordinarily would not provide number-word experiences for their children. Progression through the levels in Table 11-1 takes several years and requires a very considerable amount of practice counting in cardinal situations, experiences that must be provided in the home and in schools and preschools.

In summary, over the age span from age 2 through 8, children come to understand increasingly complex relationships among the mathematically different situations in which number words are used. They gain a considerable amount of knowledge concerning different specific situations within each kind

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of situation. Larger and larger number words are learned. Important changes occur in children's conceptualizations of the sequence, counting, and cardinal situations. Increasingly abstract and complex conceptual units are used in these situations. The relationships among sequence, counting, and cardinal situations become closer and more automatic until finally these become integrated within the number-word sequence itself. At this level the number-word sequence is a seriated, embedded, unitized cardinalized, truly numerical sequence.

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