Journal for Research in Mathematics Education
1986, Vol. 17, No. 3, 172–189

TEACHING CHILDREN TO SUBTRACT BY COUNTING UP

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Symbolic subtraction problems such as 14 − 8 = ? were interpreted to children as story problems with dot pictures that supported count-up solutions. Children in first-grade mathematics classes were taught with considerable success to solve such symbolic subtraction problems by counting up from the smaller number to the larger ("8, 9, 10, 11, 12, 13, 14; 6 more make 14") while keeping track of the number counted up by using one-handed finger patterns. The children improved quite considerably on a timed test of the more difficult subtraction combinations, and this improvement held up over a month. Interviews indicated that almost all children could count up to solve subtraction combinations they did not know. Many used counting up to solve subtraction story problems with different semantic structures: performance on compare, separate (take-away), and equalize (how many more to make the same?) story problems was similar and good.

Many studies (see Carpenter & Moser, 1984, and De Corte & Verschaffel, 1985, for recent reports and Carpenter & Moser, 1983, and Riley, Greeno, & Heller, 1983, for recent reviews) indicate that for a considerable period of time children solve subtraction word problems by using solution procedures that model the semantic structure of the problem. The earliest solution procedures use objects to do the modeling; later procedures use the number-word sequence (e.g., counting down for 8 − 5 = ? is "8, 7, 6, 5, 4, 3" and counting up to for 8 − 5 = ? is "5, 6, 7, 8"). The different object solution procedures seem to be about equally difficult. However, counting down presents children with considerable difficulty (Baroody, 1984; Carpenter & Moser, 1984; Fuson, 1984; Steffe, Spikes, & Hirstein, 1976; Steinberg, 1985a), which may be one reason subtraction is so much more difficult for children than addition is. In a previous article (Fuson, 1984), I suggested that to avoid the difficult counting-down procedure subtraction might be introduced to children in ways that would lead naturally to the use of the simpler counting-up-to procedure. Two ways to do this were proposed. The first was to interpret symbolic subtraction statements such as 13 − 5 = ? as one of the subtraction story situations that are easily modeled by a counting-up-to solution procedure. The second was to use a take-away interpretation of subtraction in which the objects taken away are the *first n* objects rather than the *last n* objects (see Figure 1). I also suggested that counting on for addition be taught before subtraction was taught as counting up because these procedures are so closely related conceptually. Furthermore, Secada (1982) re-

Many thanks go to the teachers, who were willing to try something new and who made it work; to Maureen Hanrahan, for efficiency and sensitivity in watching over all the important details; and to Tom Carpenter and Paul Cobb, who gave helpful comments on an earlier version of the paper. This research was funded by the AMOCO Foundation.

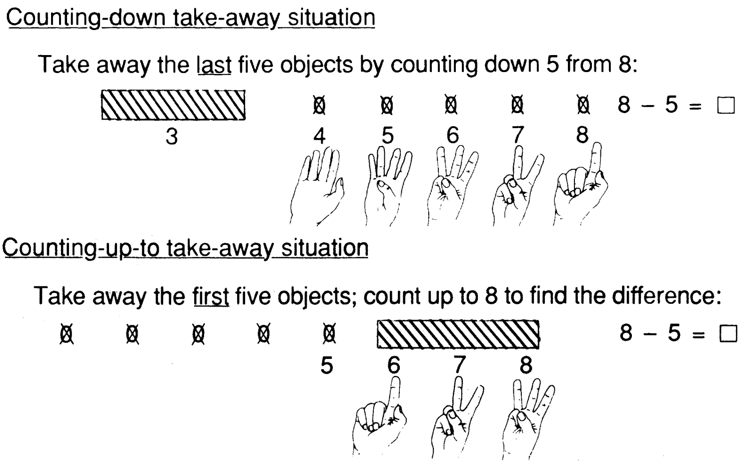


Figure 1. Counting-down and counting-up take-away object situations.

ported that counting on developmentally precedes all subtraction number-word-sequence solution procedures, and the model of counting types (Steffe, von Glasersfeld, Richards, & Cobb, 1983) makes a similar proposal.

In the present study, we explored whether first-grade children could learn to subtract by counting up. Because many children of this age are still using objects to solve subtraction problems, it was not clear how many could learn the more sophisticated number-word-sequence solution procedure. To try to maximize the number of children who would succeed, all the recommendations concerning how counting up might be taught were followed. The children were first taught to count on to solve addition problems. Subtraction was then introduced as the operation used to solve three different kinds of story problems (see Table 1). Two of these problems were modeled easily by counting up; the third was a take-away problem. Dot pictures were drawn for all three kinds of stories, and the counting-up dot picture in Figure 1 was used for the take-away story.

Several other features of the study were dictated by educational concerns about whether teaching subtraction as counting up would be practical for schools. First, regular classroom teachers (rather than special research staff) did all the teaching. Second, to ensure that the method could be used easily with all single-digit subtraction problems (those with sums up to 18), the teaching focused on the more difficult single-digit problems. This was also done because Baroody (1984) suggested that teaching subtraction first with small numbers may reinforce a tendency to solve subtraction problems by counting down. Third, a particularly efficient method of keeping track of the number of words counted up was taught. Children ordinarily keep track of the number of words counted on or counted up in various ways (Fuson, 1982; Steffe et al., 1983; Steinberg, 1985a, 1985b). The most common is to extend one finger with each word said. However, observations of children using this

Table 1
Subtraction Story Situations

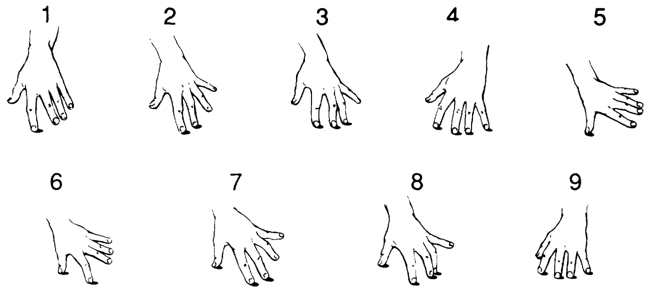
Compare		
Joan has 8 stickers. Suzanne has 5 stickers. How many more stickers does Joan have?		
Joan	○ ○ ○ ○ ○ ○ ○ ○	8
Suzanne	○ ○ ○ ○ ○	5
Take-Away		
Joan has 8 stickers. She gives 5 to Suzanne. How many stickers does Joan have now?		
Joan	⊗ ⊗ ⊗ ⊗ ⊗ ○ ○ ○	8
Equalize		
Joan has 8 stickers. Suzanne has 5 stickers. How many more stickers does Suzanne have to get to have the same as Joan?		
Joan	○ ○ ○ ○ ○ ○ ○ ○	8
Suzanne	○ ○ ○ ○ ○	5

Note. Small numbers were used for the introduction of subtraction; however, all subtraction practice and all interview stories had larger numbers.

method indicate that it is fairly slow because the children put down their pencil either on problems with both digits over five in order to use the fingers on their second hand or on all problems in order to use their writing hand for keeping track (Fuson & Secada, in press). A method that can be used with the nonwriting hand alone is one-handed finger patterns (see Figures 2 and 3). This method has been used to teach counting on (Fuson, 1985; Fuson & Secada, in press) and was used in the present study (see Figure 3) for counting up. The method was chosen for trial because it could be done very rapidly, it was always available, and it was close to the method ordinarily used by children.

Effect on Solution of Subtraction Story Problems

Teaching subtraction as counting up is a fairly radical educational departure. Although this study was designed to address the question "Can first-grade children learn to subtract by counting up?" rather than "Should they so learn?" two kinds of data were gathered to begin to address this second question. First, the effect on the solution of subtraction story problems was examined. Two aspects of these solutions were of interest. The first was whether children would use counting up to solve problems that are not ordinarily directly modeled by counting up or whether they would persist in modeling the structure of the problem. Baroody (1984) had observed children who counted down or tried to count down for such problems even though they had been taught to count up using the Wynroth (1980) curriculum. By third grade many children free themselves from the semantic structure of subtraction problems and use a solution procedure of their own choosing, often counting up (Carpenter & Moser, 1984). However, it was not clear



The finger patterns for 1 through 9 are made by touching certain fingers or the thumb or both to some surface such as a table. Thus there is kinesthetic as well as visual feedback for the finger patterns. The finger patterns use a subbase of 5. The thumb is 5, and 6 is the thumb plus the 1 finger (6 = 5 + 1), 7 is the thumb plus the two fingers (7 = 5 + 2), and so forth. The motion from 4 to 5 is a very strong and definite motion — the fingers all go up and the thumb goes down, all in one sharp motion with the wrist twisting.

Figure 2. One-handed finger patterns used in counting up.

Addition	Subtraction
8 + 5 = ?	13 − 8 = ?
The counting-on procedure:	The counting-up procedure:
1. Count on five more words from 8.	1. Count up from 8 to 13.
2. Stop when finger pattern for 5 is made.	2. Stop when 13 is said.
3. Answer is last word said (13).	3. Answer is what the hand says — the finger pattern for 5.

Words said: "8" "9" "10" "11" "12" "13"

Finger patterns:

Hand up in air ready to begin the finger pattern for 1

Figure 3. Counting on and counting up to with finger patterns.

whether first-grade children could do so, especially because the solution procedure was a more advanced number-word-sequence procedure. A second aspect of interest was the comparable difficulty of different problems. Riley et al. (1983) reviewed several studies that found very large performance differences for first-grade children across the kinds of problems used in our initial introduction of subtraction as counting up. These differences were hypothesized to occur because first graders do not have a representation of the more difficult problem types that relates the problem to an available solution procedure. We were considerably interested in the extent to which the instruction in the present study could provide children with representations of the different problem types that they could and would easily relate to counting up.

Relation to Mental Operational Capacity

The second kind of data pertained to mental operational capacity (M-space or M-power), as discussed by Case (1978) and Case, Kurland, and Daneman (1979). M-space refers to the amount of information that a person is able to keep in short-term memory and thus has immediately available for use in some task. M-space is measured by having a person do a task while remembering some new bit of information; the M-space is the number of new bits of information that can be remembered while doing the task. Counting up requires that a child do at least three things at once: count up starting at some number, make each successive finger pattern, and monitor the counting so as to stop when the larger number has been reached. It therefore seems possible that counting up may exceed the operational capacity of some first graders; they simply may not be able to do all these things at once. If so, it would be useful to have a simple test (such as an M-space test) that would identify such children so that counting-up instruction might be delayed for them.

METHOD

Subjects

A brief description of the study was presented to the elementary school principals of a small city on the northern border of Chicago. Two principals volunteered their schools for the study. The subjects were the 110 members of five first-grade mathematics classes in the two schools. Two classes contained children identified as average and below average in first-grade mathematics achievement, two contained above-average first graders, and one contained second graders considerably below average in mathematics (i.e., functioning at the first-grade level in mathematics). The student population was racially and economically heterogeneous. Seven children moved during the study or missed several days of the teaching unit; they were dropped from the study, leaving a final sample of 103.

Instruction

We prepared lesson plans and student worksheets for use by the five teachers. Two months previously, all the children had been taught by the teachers to count on to solve symbolic addition problems with sums up to 18 (e.g., 8 + 6) and to keep track with one-handed finger patterns of how many words they were counting on so that the counting on would be accurate (Fuson, 1985; Fuson & Secada, in press). Counting on for addition was taught with the dot-and-symbol setting (see the top part of Figure 4) used by Secada, Fuson, & Hall (1983).

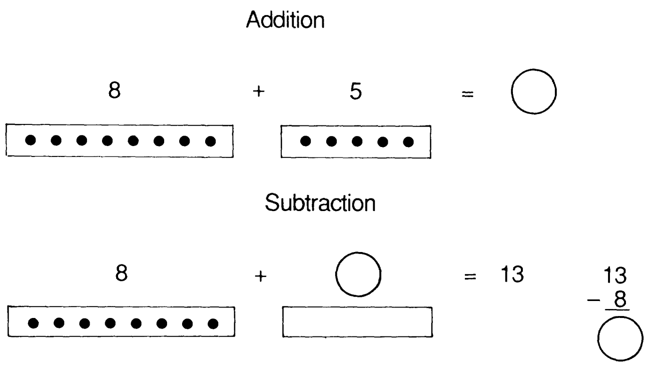


Figure 4. Dot-and-symbol layout for addition and subtraction.

Before the instruction on subtraction was begun, counting on to solve symbolic addition problems was reviewed. Then the teachers wrote on the board and drew pictures for three kinds of subtraction story situations (see Table 1). Subtraction was identified as the solution procedure for each situation. The symbol for subtraction (−) was always read as "minus" and never as "take-away" so that the latter word would not interfere with the children's understanding of the compare and equalize problems as subtraction situations. Symbolic problems (e.g., 13 − 8 = ?) were given meaning as the story situations written and pictured on the board and were also linked to the subtraction counterpart (see the bottom part of Figure 4) of the dot-and-symbol context in which the children had learned to count on. The children were taught to count up from the known part to the known whole while using finger patterns with their nonwriting hand to keep track of the number of words they had counted (see Figure 3).

The first symbolic subtraction problems were written in column form because that form supports so well the count-up-to approach: the numerically lower number is also spatially lower, so one can count *up* from the lower (in both senses) to the higher number. While the children were counting up, they were asked to put their pencil on the larger number—the number up to which they were counting—to help them focus their attention on the number at which they had to stop counting. After practicing problems in column form, the children were introduced to, and then practiced, subtraction problems in row (horizontal) form. At the end of the unit, the children had some further practice with the three kinds of subtraction story problems.

The class sets of student worksheets given to the teachers contained two optional worksheets to review addition as counting on, two worksheets with subtraction problems in column form, two worksheets with subtraction problems in row form, and one worksheet with both column and row form problems. Two sheets of story problems, each with five subtraction problems and one addition problem, were available at the end of the unit.

Two subtraction skills test (one column test and one row test) were given during the unit. These tests as well as the daily worksheets were graded by the project staff, and a list of low-scoring students was given to the teacher at the end of each day. This information permitted the teachers to work further with low-achieving children either individually or in small groups while the other children did their practice worksheets.

Each practice sheet and each skills test sheet contained 20 problems. Half the problems had both parts of the whole (i.e., the known part and the unknown part) between 6 and 9, and half the problems had one part between 6 and 9 and the other between 3 and 5. No problems with both parts the same ("doubles") were given as these problems are easier than other problems.

Before the unit began, a meeting was held with the teachers in each school. The counting-up-to procedure with finger patterns was demonstrated, and the important points of the teaching method were emphasized (e.g., the − symbol was always to be read as "minus," never as "take-away"). Two meetings had been necessary to prepare the teachers to teach counting on with finger patterns. However, counting up was so similar to counting on that learning the procedure presented little difficulty to the teachers. Most of the session was spent discussing the types of story problems and emphasizing that subtraction should not be thought of as just take-away.

At the teachers' request, the teacher plans were made sequential but without tight time lines so that the teachers could feel free to adapt the lesson progress to the wide range of mathematical ability across the five classrooms. The teachers were to teach counting up until they believed that most children had learned it. They were to make up more worksheets or provide practice in other ways if the worksheets provided were not sufficient. The 12 story problems were provided at the end as another way to practice counting up, not as an instructional treatment for teaching story problems. No directions were given to the teachers about teaching these story problems. Some teachers worked through the problems with their students; others just read the stories and let their students work them.

Observations made during the teaching indicated that the teachers were following the lesson plans and were concentrating on teaching counting up to solve symbolic subtraction problems. The observations were not detailed enough to indicate the extent to which the teachers continued to use the story problem contexts to give meaning to subtraction after the initial introduction. The teaching required from 8 to 14 periods of 40 minutes each.

Tests

Before the unit began, the students were given a 2-minute subtraction pretest of 20 problems in column form and two 2-minute addition tests of 20 problems each (one having the problems in column form given in counterbalanced order with one having the problems in row form). Two 2-minute subtraction immediate posttests were given at the completion of the unit. One consisted of 20 problems in row form; the other (given in counterbalanced order) contained 20 problems in column form. The posttests were repeated again 1 month later. The number combinations used on all tests were like those used in the instruction (see above).

Interview

Individual interviews were held with 50 children—about half of the sample—within 4 school days following the immediate posttest. From each classroom, the five children having the highest gains from the pretest to the immediate posttests and the five children having the lowest gains were selected for the interview sample with the condition that children with pretest means of 16 or more items correct (out of 20) were excluded because they already were subtracting very well by a method likely to be different from counting up. (Four children were so excluded; our teacher-observer knew that three of them already knew most of their subtraction facts.) The interviews were conducted individually in a room outside the classroom. They took from 20 to 35 minutes each.

Two story problems for each of the three kinds of story situations used in the counting-up instruction (compare, take-away, and equalize—see Table 1) were given. The numbers for these problems were 16 − 9, 12 − 5, 15 − 6, 13 − 6, 17 − 9, and 14 − 8. Two addition story problems (one join problem and one combine problem) were also given. The addition problems had numbers similar to the numbers in the subtraction problems (13 and 6, 12 and 5) so that the addition problems could not be identified as having two single-digit numbers. Because the children had been taught to add by counting on with finger patterns, these problems were not more difficult to solve than problems with both parts below 10 if one started counting on with the larger number. The story problems were given in four different orders such that each kind of problem was given in each of the eight possible positions. Each problem was read as often as the child asked to hear it. If a child just sat and did not do anything for some time after a problem was read, the child was told, "You can use this piece of paper and pencil (pointing), or your finger patterns, or any way you want to find the answer." Five children received such a prompt.

Three symbolic subtraction problems (13 − 7, 17 − 9, 14 − 8), each written on a card, were presented next.

Two measures of M-space were given last. They were selected because of their similarity to counting up for subtraction. Both measures required that one number be kept in mind during another count. In one task the count was of objects; in the other the count was verbal only. The first task was an adaptation of the original Case et al. (1979) task; it was used by Romberg and Collis (1980). In this task a child must count the darker stickers on a card having both dark and light stickers and report the count result, then count a different card and report first the count result for the first card and then that for the second card, then count a third card and report the count results for the first, second, and third cards, and so on. The second task was similar except that no objects were counted (Keranto, 1983). Before each count the child was told what number to count up to ("Now count up to 8"); after counting, the child had to report the last counting word said on the preceding counts in order. The numbers used on all trials were between 2 and 9.

The M-space tests were given in the following way. Each child began on the task that required a final repetition of 2 count words (Level 2 task). If the child succeeded on that task, the child was given the task that required a final repetition of 4 words (a Level 4 task). A success at Level 4 resulted in a Level 5 task. A failure at Level 4 resulted in a Level 3 task. At any level, a task failure with no success or two task failures with only one success resulted in a drop of one level. The second success at a level resulted in a rise of one level. The tasks were given until the child had five successes at a level.The child's score for each M-space test was the level of five successes (1 through 5) plus .2 for any success one level higher and .4 for any success two levels higher. The scores thus ranged from 1 to 5.

RESULTS

Performance on Symbolic Problems

Overall, the children learned to count up to solve symbolic subtraction problems. They improved significantly over the 2- to 3-week instruction, $t(102) = 16.21, p < .01$, and maintained this level of learning over a period of 1 month (see Table 2). The level of performance for addition problems and for subtraction problems was similar (about 73% correct). However, a somewhat different group of children evidently did well on these two types of problems, as the correlation between the two scores was only .53.

Table 2
Test Means (and Standard Deviations)

Form	Addition test	Pretest	Immediate posttest	1-month posttest
Column	14.3 (5.0)	6.3 (4.6)	14.4 (4.7)	14.7 (5.5)
Row	14.3 (5.1)		14.3 (4.7)	14.9 (5.7)

Note. These tests were 2-minute 20-item tests of the most difficult addition and subtraction combinations (sums to 18).

The children learned to subtract equally well for the column and the row forms of the symbolic subtraction problems. The mean scores on the two forms did not differ significantly ($p < .01$) on the immediate posttest or on the 1-month delayed posttest. The children showed somewhat more variability across the two forms on the immediate posttest than on the delayed posttest (the correlations between the two problem forms at these times were .74 and .82).

The direct observations in the interviews of children solving symbolic subtraction problems enabled us to check whether they were in fact counting up for such problems. Our first question was whether the children *could* use finger patterns to count up correctly. Of the 50 children interviewed, 44 used finger patterns to count up from the smaller to the larger number for the symbolic subtraction problems (e.g., 17 − 9) without having had any help with counting up earlier in the interview. Of these, 43 counted up correctly on all three problems. Two of the 44 initially used other methods to solve the problems (one drew sticks on the paper and one knew some facts) but could use finger patterns when they were asked to do so later. Of the remaining 6 children, 4 demonstrated learning (or probably relearning) of counting up during the interview: One had had help with counting up earlier in the interview and did all the symbolic subtraction problems correctly, 2 had some help on the first symbolic subtraction problem and then did the next two correctly, and 1 had help on all three symbolic problems but seemed to learn with this help and got 16 correct on a 2-minute 20-item posttest after the interview. Thus, 48 of the 50 children interviewed demonstrated the capability of learning to count up for subtraction.

Classification by Count-Up Performance

How representative of the rest of the sample was the interview sample? By choosing the high-gain and the low-gain children, we intended to select children for whom the instruction had been successful and those for whom it had not. However, there turned out to be two problems with this approach. First, many of the low-gain children later scored quite high on the 1-month delayed posttest, demonstrating that the instruction had in fact been successful for them. Second, the high-gain children initially had known very little about subtraction of the larger numbers (twenty had gotten pretest scores of 0 to 4, and the remaining five had scored between 5 and 7 out of the 20 pretest items), whereas the low-gain children had showed more initial knowledge about these problems (six had scored between 0 and 4, twelve between 5 and 8, and seven between 9 and 15). These two factors together suggested that some children who had entered instruction with a fairly well-worked-out method for solving subtraction problems might initially have made slower progress with learning counting up or might have showed lower immediate posttest scores because of interference between their "old" method and the new counting-up method.

A reclassification of the interview sample was made in order to address both this possibility and the question of the representativeness of the interview sample. The children were classified on the basis of their immediate posttest scores, their 1-month posttest scores, and, for six children, their interview results, into the following groups:

A: *Initial fast count-up*. Scored 16 or above on the row or column initial posttest (n = 22).

B: *Delayed fast count-up.* Scored less than 16 on the initial posttest but 16 or above on the row or column 1-month posttest ($n = 9$).

C: *Slow count-up.* Did not score above 15 on any posttest but did count up in the interview ($n = 13$).

D: *Interview learner.* Learned or relearned to count up in the interview ($n = 4$).

E: *Non-count-up.* Did not count up in the interview ($n = 2$).

The children in Groups D and E scored 15 or less on the immediate posttest; 5 of the 6 also scored 15 or less on the 1-month posttest.

Of the interview sample, 44% were in Group A, 18% in B, and 38% in Groups C, D, and E. Of the rest of the sample, 65% were in A, 24% were in B, and 11% were in C, D, or E (interviews would be needed to discriminate among these three groups). Therefore the interview sample contained most of the children whose test scores indicated they might not have learned to count up. Only six of the children not interviewed did not at some time score quite well (at least 15) on a 2-minute subtraction test. The test scores of these six children were similar to those of the interviewed children in Groups C, D, and E. Because the interview results indicated that most of the interviewed sample could count up with finger patterns for subtraction, it seems clear that almost all the children taught to count up with finger patterns learned to do so. The teacher reports also supported this conclusion.

Effect of original knowledge. To examine the possibility that original subtraction knowledge might have interfered with learning counting up, or with a rapid use of counting up, each interview group was partitioned into two parts based on their pretest scores: those children showing very little initial knowledge of subtraction with larger numbers (pretest score from 0 to 4) and those children showing some such knowledge (pretest score from 5 to 15). The resulting partitions were A, 15 and 7; B, 0 and 9; C, 6 and 7; D, 3 and 1; and E, 2 and 0. Group B (consisting of those children who did not subtract rapidly just after instruction but did so 1 month later) was the only group in which all the children had some initial knowledge of subtraction. The interview indicated that they all could count up, so the initial counting-up instruction was successful. It seems possible that on the immediate posttest they were a bit slower because they were using different procedures (their old ones plus counting up) on different problems and that a month later they had gotten faster at making such decisions or perhaps had shifted entirely to counting up. The partition results also indicate that not all children with some initial knowledge of subtraction showed such a pattern of delayed but ultimately fast performance. The children with some initial knowledge of subtraction were spread across all four groups: A, B, C, and D. The relationship between original subtraction knowledge and learning to count up might be examined further in future research.

Effect of mental operational capacity. The M-space scores of the interview children were examined to see whether they could predict membership in Groups A through E. Two children were dropped from the sample for this analysis because they were not native speakers of English, and M-space measures assume developmental levels of automaticity associated with one's native language. Mean M-space scores are given by group and by M-space task in Table 3. The mean M-space scores of Groups A and B are higher than those of the three other groups on the object-counting task but not on the word task. Note that one would expect scores to be about one point higher on the object task than on the word task because each word task requires that you remember one more number than the object task (on which you need not remember the number up to which you are counting). There was a difference across tasks for all groups except Group C, in which the object and word scores were about the same. Group C looks like the higher performing groups on the word task and like the lower ones on the object task. Thus their number-word sequence may be as automatic as that of children in Groups A and B but their counting of objects (including fingers) may not be as automatic. This possibility might be examined in future research. Although the M-space means were somewhat lower for the lower scoring count-up groups, an examination of all scores below 3.0 indicated that they were distributed across all five groups. Thus there did not seem to be a minimum M-space score necessary for fast counting up.

Table 3
Mean M-Space Scores by Count-Up Group

	Group				
Task	A Initial fast count-up (n = 22)	B Delayed fast count-up (n = 9)	C Slow count-up (n = 13)	D Interview learner (n = 4)	E Non-count-up (n = 2)
Object	4.0	3.9	3.2	3.0	3.3
Word	3.1	3.2	3.1	2.3	2.7

Performance on Story Problems

Comparison across problem types. The mean percent of story problems of each type that was solved correctly is given in the last row of Table 4. In each case, all but 4% to 10% of these problems were solved by the use of count-up finger patterns. Other methods that led to correct solutions were using known or derived facts, drawing sticks on paper, counting up with fingers on both hands, counting down, and doing the problem "in one's head." At least one compare, take-away, equalize, and add problem was solved by 86%, 83%, 88%, and 79% of the sample. Thus, unlike most other samples of first-grade children, the performance was as good for compare as for take-away and add problems. The level of performance was also quite good in general considering that the numbers in the problems were larger than those usually used in

Table 4
Percent of Story Problems Solved Correctly by Type and by Count-Up Group

Group	Story problem type: Compare	Take-away	Equalize	Add	Mean
A: Initial fast count-up	100	86	90	86	91
B: Delayed fast count-up	78	83	89	94	86
C: Slow count-up	77	77	68	59	70
D: Interview learner	13	38	38	25	29
E: Non-count-up	0	0	0	0	0
Mean	79	76	77	72	

research on word problems with first graders (sums are usually 10 or less) and considering that the performance consisted almost entirely of solutions that were more advanced than object solutions.

There was evidence that the children were not just mindlessly applying counting up on all problems, that is, that the relatively equivalent performance across all four types of problems did not result from a rote application of subtraction counting up. Only one of the interviewed children subtracted on all eight problems; two other children subtracted or tried to subtract on one of the two addition problems. All other children added or tried to add for one or both addition problems. Many children (40% of the interview sample, 70% of those not answering all story problems correctly) showed variability in their ability to solve the three types of subtraction problems, that is, their scores varied across the types of problems.

Most of the incorrect solutions to the addition problems (71%) resulted from difficulty the children had in counting on with finger patterns. Some of the difficulty seemed to stem from the double-digit number in the problem; the children sometimes started counting on from the single-digit number and then did not have enough finger patterns to finish the adding. For other children, learning to count up for subtraction clearly had interfered with their memory of counting on for addition ("I don't remember how to add with finger patterns"). These children had just completed the subtraction unit. Some work clearly needs to be done after the subtraction teaching to help children straighten out the two counting procedures.

Comparison across groups. The percentage of correct solutions of the story problems by each count-up group (shown in Table 4) suggests the extent to which an ability to count up quickly and accurately affected the use of counting up to solve story problems. The three groups who demonstrated counting up (A, B,C) showed across the four types of story problems roughly similar patterns of correct performance except that Group A was better at compare problems than Groups B and C were, and Group C was worse than A and B on equalize problems and considerably worse on add problems. All but one of the addition errors made by the Group C children were mechanical rather than semantic—they could not figure out how to count on to add the two numbers. Thus children who were slow on both the immediate count-up posttest and the delayed posttest seemed to experience particular interference with the counting solution procedure for addition. For the other groups on the addition problems and for all groups on the subtraction problems, the kinds of errors were roughly equally split between semantic errors (trying to add for subtraction problems and vice versa or just not knowing what to do) and mechanical errors (difficulty with the counting-on or counting-up solution procedures). Thus successful word problem instruction apparently needs to include both further practice on, and discussion of, different addition and subtraction word problems and some work on differentiating counting on from counting up.

Effect of instruction. The amount of instruction on story problems actually given by the teachers is unknown. The lesson plans did not suggest that any instruction precede the 1-day practice on story problems at the end of the unit, but such instruction might well have been given by some teachers. In any case, the amount of practice on story problems was quite low: Only 12 problems were given on the worksheets.

Only two of the solutions for the 300 subtraction problems given in the interviews modeled taking away: One child counted down on a take-away problem, and one child drew sticks and crossed them off for a compare problem. All the other solutions involved counting up (with finger patterns or with fingers on both hands) or using known or derived facts. Thus the setting of initial counting-up instruction within the take-away story situation, as well as in the compare and equalize situations, seems to have enabled the children to see take-away situations as solvable by counting up. That the level of success on both the compare problems and the take-away problems was roughly equal and quite high seems to indicate that teaching subtraction as counting up provides a solution procedure that can be, and is, used for solving compare problems without interfering with solution procedures for take-away problems.

DISCUSSION

The subtraction teaching in this study followed a recommendation by Carpenter and Moser (1984) that instruction in addition and subtraction might well capitalize on the rich knowledge of addition and subtraction story situations that young children possess. The counting-up instruction was set within various subtraction story situations from the very beginning as well as being linked to the dot-and-symbol layout found by Secada et al. (1983) to facilitate teaching counting on for addition. This approach proved to be quite effective: First-grade and slower second-grade children were able to learn counting up to with finger patterns fairly easily and rapidly. Interviews indicated a widespread use of counting up to with finger patterns on symbolic problems and on story problems. The performance on symbolic subtraction problems was equal to that on symbolic addition problems, and the subtraction performance on all three kinds of subtraction story problems was equal to that on addition story problems. The children used counting up on all three types of subtraction story problems. Thus the instruction seems to have been successful in teaching the counting-up procedure and in helping children learn representations of compare, take-away, and equalize subtraction problems that easily related to the counting-up procedure.

These initial results are encouraging and seem to warrant further research on teaching subtraction as counting up. There was some evidence that for many children the use of counting up was not just a rote procedure. Hardly any of the children counted up for an addition story problem, even though the numbers in those problems were the same as the numbers they had used in subtraction problems. Individual children showed variability in their ability to solve the three kinds of subtraction story problems, even though the group means turned out to be roughly the same.

Some questions obviously remain, however, that need to be examined in future research. First, how the instruction helped the children use counting up for the various subtraction stories needs to be explored. The instruction may have helped the children construct a representation of each story problem type in which counting up modeled the required solution action. Or, the children may have learned to think of each problem as a subtraction problem (rather than as a compare, take-away, or equalize problem) and then to use counting up as the method. Of course, different children may have learned each approach, and some children may have used a mixture of approaches. Second, research should explore how learning to subtract by counting up affects performance on other types of story problems. Future counting-up instruction might even use addition story problems with an unknown part, as these are natural counting-up situations (Carpenter, Hiebert, & Moser, 1981). Third, the present sample of chronological first graders contained as many above-average as average and below-average children. Future studies might examine more average and below-average first graders. Such studies might also examine what experiences with subtraction should precede teaching subtraction with larger numbers. Some first graders may be able to move fairly directly to counting up, but many may first need experiences with objects. Fourth, the method children who are taught counting up use to solve problems such as $9 - 1$ or $8 - 2$ should also be examined, as it is very inefficient to solve such problems by counting up.

One reason for the ease with which the children learned to subtract may be that counting up for subtraction is actually easier than counting on for addition. In counting on, the feedback cycle that controls when the counting ends is visual and kinesthetic and focuses on the completion of a given finger pattern (e.g., when doing $8 + 6$, the finger pattern for 6 must be completed). In counting up, the feedback cycle is auditory, and counting up ends when the bigger number is said (e.g., for $14 - 8$, counting up ends when 14 is said). Thus in subtraction the finger patterns do not have to be as well known as in addition. No monitoring of them is required because only the final finger pattern needs to be interpreted—as the answer. This difference between counting on and counting up—which makes subtraction actually easier—seems likely to be responsible for the finding of a relatively low correlation between performance on symbolic addition and subtraction problems and the untypical finding that performance on symbolic subtraction problems was as good as that on symbolic addition problems. The teachers expressed surprise that the subtraction teaching went so quickly and easily. Teaching the children to count on with finger patterns had required considerable effort and short periods of individual help with many children. The teachers had always found before that teaching subtraction was more difficult and time consuming than teaching addition. The ease with which the children learned to count up for subtraction seems to indicate that the conceptual structures necessary for counting up were already present or were readily constructed. The most likely source of such structures would seem to be counting on, which these children had learned earlier. Thus future research might examine the relationship between the conceptual structures required for counting on and those required for counting up.

Teaching children to count up for subtraction is of course not the final step in subtraction instruction. The next steps seems to be to teach thinking strategies (e.g., Rathmell, 1978; Steinberg, 1985a; Thornton, 1978) and then to try to move on to known subtraction facts. During this time, counting up would increasingly become only a "fallback" method—to be used when one cannot remember a fact. In the meantime, counting up with finger patterns gives children a general, powerful, and rapid means of finding subtraction answers. One reason for undertaking this study was that I thought that if children learned at an earlier age the efficient methods of solution that most children spontaneously construct later (counting on and counting up to), they might be freer to concentrate on learning other mathematical concepts and procedures. This possibility received some modest support in a report from one of the teachers who, after the delayed posttest, taught her above-average first-grade children the subtraction algorithm with and without regrouping. She reported that the teaching went much more easily than ever before and that the children really understood regrouping that year. Her explanation was that usually the children struggled with the subtraction combinations, which ordinarily got in the way of their learning the new algorithmic procedure. That year, however, the children had spontaneously used counting up with finger patterns to do any subtraction combinations they did not already know. Counting up freed them to concentrate on learning and understanding regrouping and the subtraction algorithm. Thus counting up seems to have been automatic for these children.

The approach of teaching subtraction as counting up rather than counting down has a generalization that might be tested in future research: The subtraction solution procedures that are taught should go forward, *up* the counting word sequence, rather than backward, *down* the sequence. The forward counting word sequence is much more automatized (Fuson, Richards, & Briars, 1982) than the backward sequence, and all forward procedures ought to be easier and faster than backward ones. The generalization implies that thinking strategies should be forward rather than backward. For example, the "over 10" strategy for 14 − 8 would be thought of as "8 plus how much to 10 plus 4 more to 14" rather than as "4 from 14 down to 10 plus 2 more down to 8." The exception to the generalization is that giving the number word just before a given number word is relatively simple, so that a backward "doubles" strategy for 13 − 7 (7 plus 7 is 14, so the answer is 1 less, 6) might be almost as easy as a forward "doubles" strategy for 13 − 6 (6 plus 6 is 12, so the answer is 1 more, 7). One possible benefit of such forward solutions is that they may be mechanically easier for children to do, just as counting up is mechanically easier than counting on. A possible problem with all forward subtraction procedures, including counting up, is that they may create additional interference between addition and subtraction solution procedures because the procedures are more similar.

A final important issue for future research is the effect on children's conceptual structures of their learning to think of subtraction as a forward "how many more" procedure. An exploration of the effects of this conception on the solution of various kinds of verbal problems was begun in the present study and should be extended in future work. Another possible beneficial contribution of a counting up or a general forward approach that should be examined is the possible construction of stronger links between addition and subtraction. In particular, the approach seems likely to facilitate the learning of subtraction facts in relation to addition facts, so that a problem such as 13 − 8 = ? is thought of from the very beginning as "8 and how many more to make 13?" Such links might also make it possible for children to construct earlier a unified conception of addition and subtraction as related operations.

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[Received September 1984; revised June 1985]

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