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MORE COMPLEXITIES IN SUBTRACTION

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Four ways in which subtraction is more difficult than addition are discussed: Verbal solutions are not always parallel to object solutions. Two correct methods exist for counting down a certain number of words, and these methods may interfere with each other. Special problems exist with subtraction on the number line. And subtraction is not just take away but has multiple situational interpretations. These points, along with some recent research results, suggest the research question: Should children be taught to solve subtraction statements such as 8 − 5 = ? by counting up from 5 to 8?

Baroody (1984) has discussed ways in which the counting down solution procedure for subtraction is considerably more difficult than the counting-on solution procedure for addition and has proposed some ways to improve counting down performance. This paper complements the Baroody paper. Four additional complexities in subtraction (three concerning counting down) will be discussed, and a proposal for future research on teaching subtraction will be made.

This paper involves a detailed discussion of children's addition and subtraction solution procedures. An example below describes each counting procedure discussed. More detailed presentations can be found in Carpenter and Moser (1984), whose terminology is followed here.

- *Counting-on* (a given number): For 5 + 3 = ?, "5, 6, 7, 8. The answer is 8." [I shall not distinguish between counting-on from first and counting-on from larger because no point here depends on that distinction.]
- *Counting up to* (a given number): For 5 + ? = 8 or 8 − 5 = ?, "5, 6, 7, 8. The answer is 3."
- *Counting down* (a given number): For 8 − 5 = ?, "8, 7, 6, 5, 4, 3. The answer is 3." [Carpenter & Moser, 1982, 1984, call this "counting down from."]
- *Counting down to* (a given number): For 8 − 5 = ? or 5 + ? = 8, "8, 7, 6, 5. The answer is 3."

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In most of the paper I shall be discussing only one of the counting down procedures: counting down a given number (counting down from). For simplicity, I shall use only the words "counting down" in discussing this procedure.

In each of these verbal solution procedures, children must keep track of the number words that are said. Different ways in which children keep track are described by Steffe, von Glasersfeld, Richards, and Cobb (1983) and by Steinberg (1983). These keeping-track methods vary in their complexity; levels of difficulty in such methods are described by Fuson (1982). For simplicity, only the most common keeping-track method—the successive extension of fingers as number words are said—will be referred to in this paper. However, the discussion also applies to children's other keeping-track methods.

LACK OF PARALLELISM IN OBJECT AND VERBAL SOLUTION PROCEDURES

One special complexity of take-away subtraction is that although the verbal solution procedures for addition parallel rather well the object solution procedures that they come to replace, the take-away verbal counting down procedure does not parallel the more primitive separating-from object solution procedure (see Figure 1). For addition problems, the words said in a verbal solution are the same as those said in the final step of the corresponding object solution. For take-away subtraction, the object solution requires one to count up to the number being subtracted (while taking away an object with each count) and then to count the remaining objects: that is, all counting is forward. The verbal solution procedure entails entirely different counting—a *backward* word sequence is produced starting with the sum (the number being subtracted from), and a finger is sequentially extended with each successive word to represent the number being subtracted. This requires quite a shift in behavior from the forward counting done in the subtraction object solution.

TWO CORRECT COUNTING DOWN METHODS

In the counting down verbal solution, fingers can be matched to the number words being said by two different correct methods. These two methods are outlined in Figure 2. Most researchers describe one or the other method but not both (e.g., Carpenter & Moser, 1984, and Steffe, Spikes, & Hirstein, 1976, describe Method A, and Baroody, 1984, and Secada, 1982, describe Method B). Personal communication with these authors (except Steffe et al., because in that study children were taught the method of counting down) has established that they did not focus on children's finger keeping-track methods in a way detailed enough to discriminate between the two methods. Therefore, to date we have little hard evidence about which method is actually used by children more frequently. The distinction between

Addition

Object Solution: 5 + 3 = 8

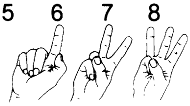
	Step 1	Step 2		Step 3	
Count All	00000	00000	000	00000	000
	12345		123	12345	678

	Step 1	Step 2	
Count On[a]	00000	00000	000
	12345		678

Verbal Solution: 5 + 3 = 8

Count All[a] 1 2 3 4 5 6 7 8

Count On 5 6 7 8



The words said in the verbal solutions are the same words that are said in the final step of the object solutions. The difference is that objects are counted and words are matched by fingers.

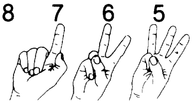
Subtraction (Take-away Solutions)

Object Solution: 8 − 3 = 5

Step 1	Step 2	Step 3
00000000	ØØØ00000	00000
12345678	123	12345

Verbal Solution: 8 − 3 = 5

Count down[b] 8 7 6 5 There are 5 left.



In the object solution the counts are all forward. In the verbal solution the sum count is backward.

Note: Numerals represent words said by the child. Objects are put out by the child. Fingers are put up by the child.

a. This method is not used very frequently.

b. This is just one of the count down methods (see Figure 2), but in both methods the sum count is backward.

Figure 1. Parallelism in object and verbal solution procedures for addition and subtraction.

the methods is important for two reasons. First, each matches a different aspect of counting-on, suggesting different ways of teaching each method. Second, components of each method are sometimes mixed by children (Stein-

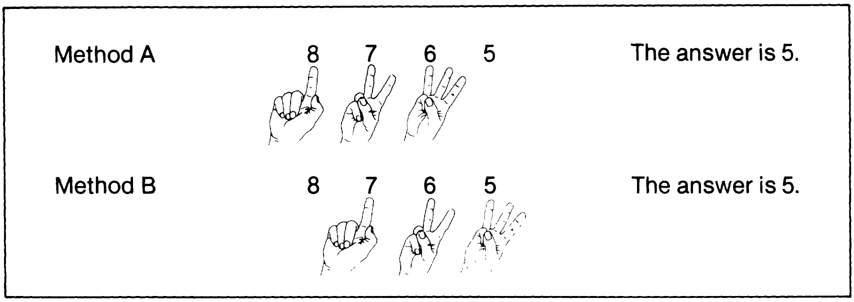


Figure 2. Two correct verbal counting down methods to solve $8 - 3 = ?$

berg, 1983; see below for details), leading to errors. Knowledge of the differences between the two methods may help lead to ways of eliminating such errors.

If done correctly, each method yields the correct answer. For the problem $8 - 3 = ?$, both methods begin with the child saying 8. In Method A, the word "eight" is counted (i.e., a finger is extended for 8), as is each word going backward until 3 words have been said (and matched by fingers): 8, 7, 6 (matched by 3 fingers). The next number word going down then tells how many words are left after the 3 words have been taken away from the 8 words: 8 (one finger), 7 (two fingers), 6 (three fingers)—the answer is 5. In Method B, the first word, "eight," is not matched by a finger. Finger extension begins with the second word said going down from 8, and words continue to be said until 3 fingers have been extended. In this method, no further word is said. The word that is said with the third finger is the answer: 8, 7 (one finger), 6 (two fingers), 5 (three fingers)—the answer is 5.

These methods vary in their relationships to the counting-on procedure (for addition). Method A is a direct *undoing* of counting-on: The *object situation* underlying Method A matches the object situation underlying counting-on, and the words said in Method A refer to the same objects as they do in counting-on (see Figure 2). Method B differs from counting-on in its object interpretations but is an exact *procedural* copy of counting-on as a verbal procedure removed from objects. (I am grateful to both Jim Moser and Ruth Steinberg for independently pointing out this similarity to me.) In counting-on, one says a starting word (the first addend) without extending a finger and then says n more words, extending a finger with each; the answer is the nth word. Method B involves the same sequence of uttering words, except that the starting word is now the sum: for $8 - 3$, say "8 (starting word), 7 (one finger), 6 (two fingers), 5 (three fingers—stop saying any more words, because the word said with the stopping finger is the answer)—5 is the answer."

Both methods of counting down obviously can be done by rote without any real understanding of the object contexts created by the number words. In such cases, it seems likely that children will make systematic errors that mix components of the two methods. They might keep track of the words counted down as in Method A (i.e., match the first 3 words to fingers) and then derive the answer as in Method B (give the word said with the last finger as the answer): 8 (one finger up), 7 (two fingers up), 6 (three fingers up)—6 is the answer. Alternatively, they might keep track as in Method B and derive the answer as in Method A: 8, 7 (one finger up), 6 (two fingers up), 5 (three fingers up)—the answer is 4. Second-grade children demonstrate both kinds of systematic errors (Steinberg, 1983).

An ingenious method of avoiding such errors by making the object meanings clear at each step was displayed by 2 of the 23 children in the Steinberg (1983) study. In both Method A and Method B, number words are said without specifying whether the meaning of the word has a sequence, counting, cardinal, ordinal, or measure meaning (see Fuson & Hall, 1983, for a detailed discussion of these meanings). The two children used ordinal words instead of sequence words and then made the unambiguous ordinal-cardinal translation: "I am taking away the eighth (one finger up), the seventh (two fingers up), and the sixth (three fingers up), so five are left" (Steinberg, 1983, p. 175).

Whereas the cardinal interpretation of Method A is quite clear, the interpretation of Method B is not so clear. At least two interpretations seem quite possible: cardinal and measure/cardinal (see Figure 3). The mixed measure/cardinal interpretation is the one often given to the number line. This interpretation presents some special problems. Because the number line is used so widely in textbooks and in schools, I shall briefly discuss these problems in the next section.

METHOD B AND THE NUMBER LINE: SPECIAL PROBLEMS

The number line, a learning aid often used in schools, is a special object context that combines some features of an object solution with some of a verbal solution. However, the meanings in the number line subtraction procedure (at least as it is usually taught) may not be clear. The number line is a measure model, not a count model. Five is represented on the number line as five unit intervals, as the length from the 0 to the 5 on the number line. Five on the number line is *not* the point labeled 5. Children are usually taught to use the number line by making unit hops forward or backward on the line. The left side of Figure 4 illustrates the usual method of teaching subtraction on the number line: The child begins at 8 and hops backward 3 hops, saying, "1, 2, 3" as the hops are made. The hopping ends at 5, so the answer is 5.

One problem concerns what children understand about the meaning of the answer they get. The real measure meaning of the hopping is indicated on the right side of Figure 4: One begins with a length of 8 units, subtracts 3 units from that, and is left with 5 units. The use of a centimeter number line with Cuisenaire rods or the use of strips of paper matched to whatever number line

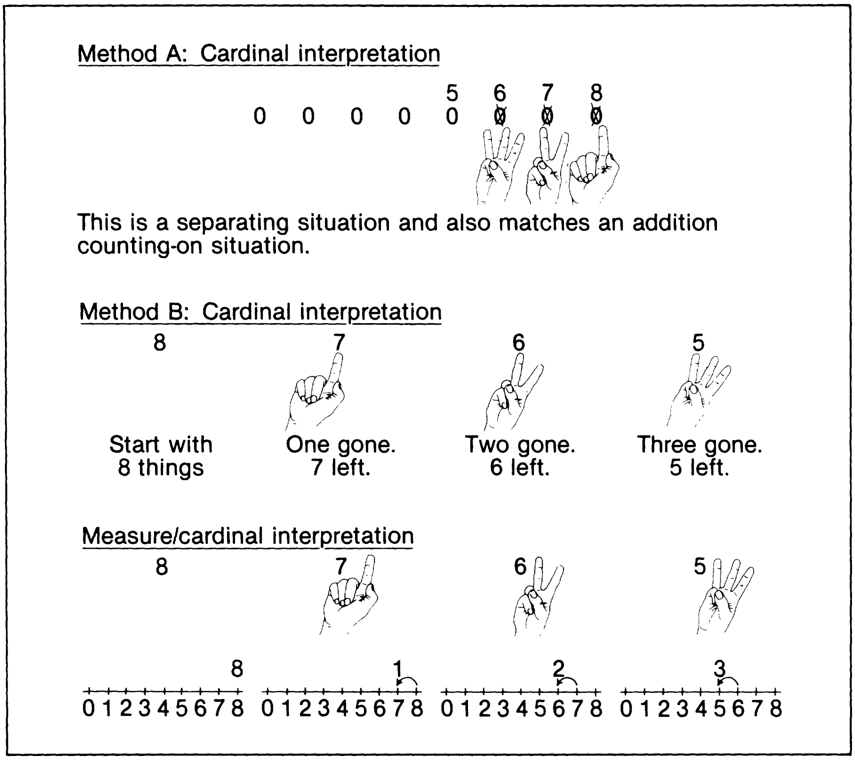


Figure 3. Object interpretations of correct verbal counting down methods for 8 − 3 = ?

is used can easily make clear the underlying measure meaning of the number line (Bell, Fuson, & Lesh, 1976, give a more detailed treatment of differences between count and measure models of number). Without such measure supports, the number line is often interpreted incorrectly. My years of observation of teachers and of children have rarely revealed anyone who understood that numbers are represented on the number line by lengths—instead, numbers are thought to be represented by the points they label. Thus the hopping solution on the number line is given the mixed measure/cardinal interpretation of Method B pictured at the bottom of Figure 3. The child is supposedly counting hops and not numbers (or numerals), but the hopping ends on a numeral. In the example pictured, there are three hops involving four numerals. Without an unambiguous interpretation of the number line as a measure model, it is not obvious why the final number should be the answer. The number line may serve adequately as an answer-getting device but may or may not help the child understand take-away subtraction. It seems preferable to use the number line in an unambiguous length manner as pictured on the right side of Figure 4 or to use a row of numbers that have a simple count/

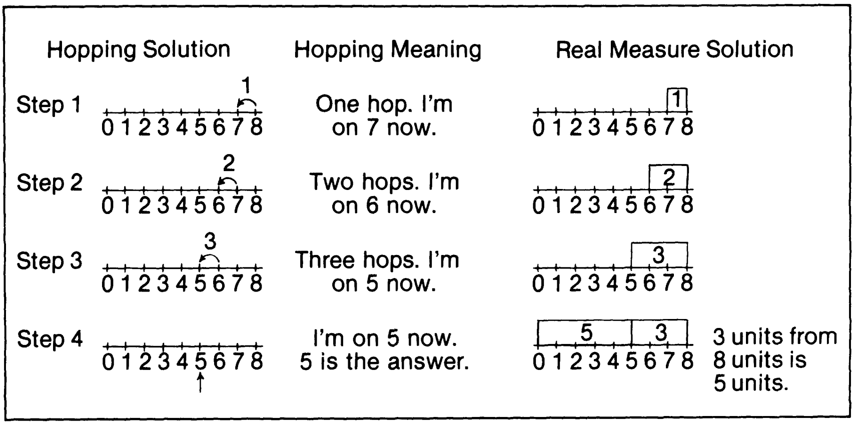


Figure 4. The number line and counting down take away for 8 − 3 = ?

cardinal interpretation not open to the confusion of endpoints and intervals by the number line as it is usually used.

These problems with subtraction on the number line are related to an unfortunate use of the term *number line* in describing the mental representations that children use to solve addition and subtraction problems. Resnick (1983) reviews much of the evidence concerning children's solution procedures in addition and subtraction and concludes that children possess an internal "mental number line" that they use to solve such problems. She is referring to verbal solution procedures like those described here in which children use an internal representation of the number-word sequence along with the cardinal, ordinal, and counting meanings of the words in the sequence. Resnick presents no evidence that children possess a *measure*-based representation such as a number *line,* nor does such evidence exist, to my knowledge. Rather, all her discussion is quite consistent with the developmental levels in the use of the number-word *sequence* described by Fuson, Richards, and Briars (1982) and the developmental changes discussed by Steffe et al. (1983) in what children take to be the counting unit in various counting solution procedures. Furthermore, Fuson and Hall (1983) review evidence concerning children's understanding of various meanings of number words, and this evidence clearly indicates that children's understanding of measure number word meanings lags considerably behind their understanding of the other meanings. The choice of the measure term "number *line*" for the mental representation of the "number *word sequence*" used by children in more sophisticated verbal addition and subtraction solutions is therefore unfortunate because it is potentially misleading: It suggests considerably more advanced notions than those indicated by the evidence and by Resnick's schematic drawing, and it falsely implies that the number line teaching aid would be readily and fully understood by children.

SUBTRACTION IS NOT JUST TAKE AWAY

A subtraction statement such as 8 − 3 = ? represents at least four different real-world situations: comparison, separate or take away, join missing addend, and combine missing addend. Examples of these situations follow:

- *Comparison:* Susan has 8 cookies. Her friend Dan has 3 cookies. How many more cookies does Susan have than Dan?
- *Separate or take away:* Mary has 8 cookies. She gives 3 cookies to her friend Scott. How many cookies does she have left?
- *Join missing addend:* Dan has 3 cookies. How many more cookies does he have to get so that he will have 8 cookies?
- *Combine missing addend:* Greg has 3 raisin cookies and some oatmeal cookies. He has 8 cookies. How many oatmeal cookies does he have?

The comparison and separate situations are inherent subtraction situations, and the join-missing-addend and combine-missing-addend problems are addition situations in which one of the addends is unknown but the sum is known. However, all four situations can be represented by the symbolic statement 8 − 3 = ? Put another way, the symbolic statement 8 − 3 = ? can be interpreted to mean any of the four types of action situations above.

Clear evidence exists that primary school children understand all four kinds of situations and, if given objects, can and do solve such situations by directly modeling the actions in the problem situation (Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982, 1984). Controversy exists concerning the age at which children can first solve each type of subtraction problem, and it is clear that differences in the wording of a problem type will affect performance (Briars & Larkin, in press; Carpenter & Moser, 1984; Hudson, 1983; Nesher, 1982; Riley, Greeno, & Heller, 1983).

There is also evidence that by the time children spontaneously use verbal solution methods, their verbal solution methods may no longer parallel the structure of the problem situation. In particular, for *separate* problems, Carpenter and Moser (1984) found in their longitudinal study that counting up to appeared before counting down from and that more children used counting up to than counting down. In the opposite direction, in a combine-missing-addend object situation (an additive subtraction situation—see Figure 5) adapted from tasks in Steffe et al. (1976), Secada found in two studies (1980, 1981) that more first-grade children counted up to than counted down to or counted down from (23 vs. 16, with 7 children counting up and down), but he did find a fairly substantial number using some count down procedure. Steinberg (1983) found that for compare and join-missing-addend problems, many more children used counting up to than either counting down method. Thus, children who do use verbal solution procedures seem

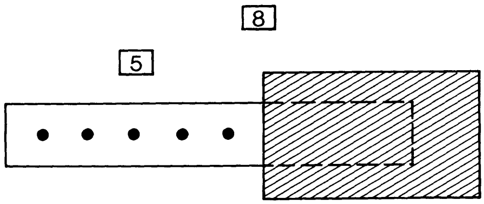


Figure 5. Combine missing-addend object situation for 8 − 5 = ? or 5 + ? = 8.

able to use them fairly flexibly. Furthermore, counting up to is used by many children in three of the four subtraction situations and is used by a sizable portion of children in their verbal solutions in the separate (take away) situation.

TEACH SOLVING 8 − 5 = ? BY COUNTING UP TO?

Steffe et al. (1976) described how difficult it was to teach children to count down, and Baroody (1984) and this paper have discussed special complexities of counting down that forward verbal procedures do not have. The considerable difficulty of counting down, combined with the evidence above concerning children's spontaneous use of the verbal counting-up-to forward counting procedure, seems to lead rather naturally to a suggestion: Why not interpret symbolic subtraction statements (such as 8 − 5 = ?) to children as one of the three subtraction situations that seem to lead naturally to a counting-up-to solution? Or, if one thinks that the take-away/separate situation is somehow the most natural and most easily understood situation, one might provide an object interpretation of take away that would support a counting-up-to procedure: take away the *first* 5 objects and count up from 5 to 8 (see Figure 6) to see how many are left. This counting-up-to take-away situation is in contrast to the taking away of the *last* 5 objects discussed earlier as the object situation for counting down 5 (see Figure 6).

Several factors indicate that if counting up to is taught as a solution procedure for subtraction, it may be wise to teach counting-on for an addition context first. First, these two procedures are very closely related (see Figure 6). The differences lie in what stops the number word production and in what provides the answer. In counting up to, number word production stops when the sum word (in 8 − 5, the 8) is said, and the answer is the number of fingers extended (here, 3). In counting-on, number word production stops when the number of fingers extended is the second addend (here, 3), and the answer is the number word said with the last finger (here, 8). Second, Secada (1982) found that the use of counting-on seemed to precede the use of any verbal procedures for subtraction: All children who used counting up to, counting down from, and counting down to also used counting-on, whereas many children used counting-on but did not use any of the verbal subtraction

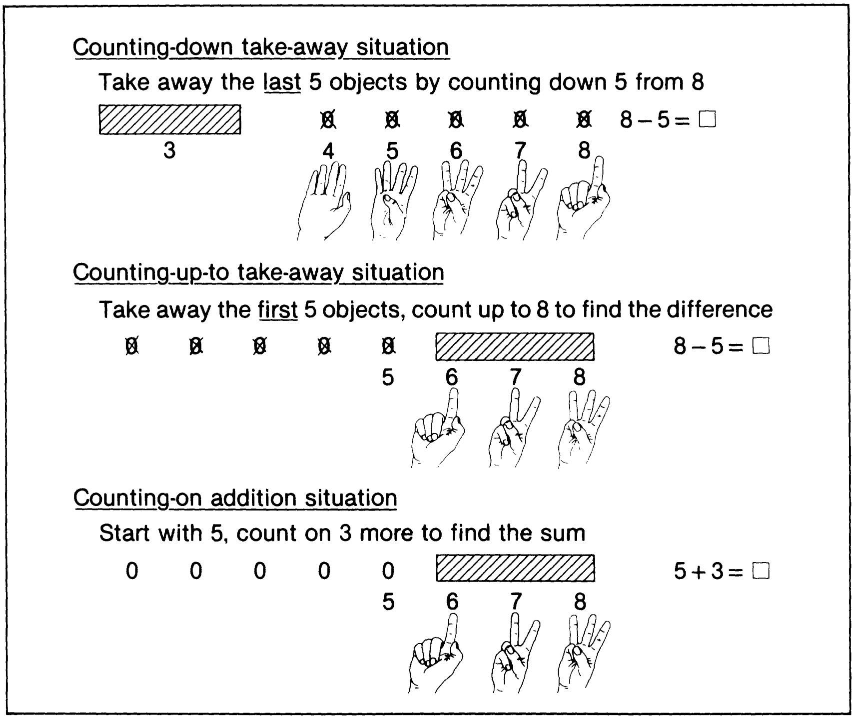


Figure 6. Counting situations for addition and for take-away subtraction.

solution procedures. Third, counting-on can be taught in an object context where objects for the second addend provide a simple means for keeping track of the words for the second addend (Secada, Fuson, & Hall, 1983). Such an object context might then be used to support the learning of the more abstract finger extension method of keeping track (i.e., dots might be shown for the second addend in the bottom picture in Figure 6 and the keeping-track fingers might be matched to the dots). When the counting-up-to procedure was then introduced within a similar object context (see the middle picture in Figure 6), the link of the extended fingers to the covered (missing) dots might then be clearer.

The answer to the questions raised in this section must, of course, await empirical results. Some children may prove to be quite resistant to a counting-up-to approach, and some may not be able to learn (at least for some period of time) any verbal solution procedure for a subtraction statement. Furthermore, counting down is an efficient solution method when the number subtracted is very small: It is not so difficult to produce one or two numbers that come before a given number; small count downs present little keeping-track difficulty (the words counted down can just be "subitized");

and counting down one or two numbers is fast. Thus, it may be quite sensible to do problems such as $9 - 1 = ?$ or $8 - 2 = ?$ by counting down from. Of course, these are also very easy problems. By the time one might teach counting up to as a general solution procedure for the subtraction of any two numbers, these simple problems might already be known as number facts.

CONCLUSION

For a long time we have known that children have considerably more difficulty with subtraction than with addition. Recent research and analyses are beginning to indicate how complex subtraction is and, more importantly, are beginning to demonstrate specific ways in which this complexity can lead to difficulties when children are learning to solve subtraction problems. The next step is to devise ways of helping children overcome these difficulties.

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