Research-Based Learning Progressions in the K-6 CCSS and the Mathematical Practices

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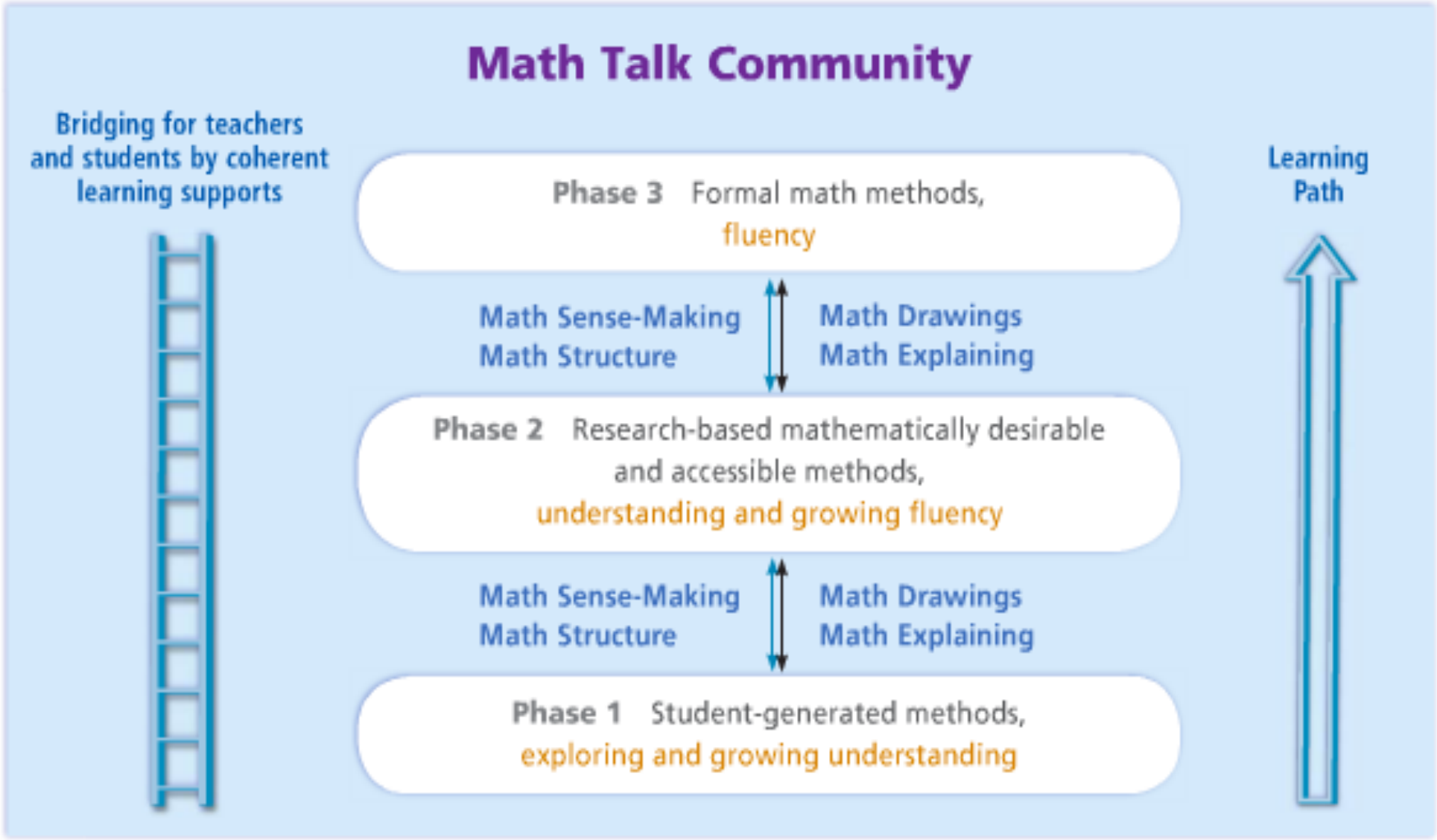
For more details about the CCSS-M and visual supports, please see the series of visual with audio Teaching Progressions I have made for various math domains. These can be found at karenfusonmath.com


Math Talk Community
Bridging for teachers and students by coherent learning supports
Phase 3 Formal math methods, fluency
Math Sense-Making
Math Structure
Math Drawings
Math Explaining
Phase 2 Research-based mathematically desirable and accessible methods, understanding and growing fluency
Math Sense-Making
Math Structure
Math Drawings
Math Explaining
Phase 1 Student-generated methods, exploring and growing understanding
Learning Path

Common Core Mathematical Practices Used in a Math Talk Community

Math Sense-Making: Make sense and use appropriate precision 1 Make sense of problems and persevere in solving them. 6 Attend to precision.	**Math Drawings: Model and use tools** 4 Model with mathematics. 5 Use appropriate tools strategically.
Math Structure: See structure and generalize 7 Look for and make use of structure. 8 Look for and express regularity in repeated reasoning.	**Math Explaining: Reason, explain, and question** 2 Reason abstractly and quantitatively. 3 Construct viable arguments and critique the reasoning of others.

Figure 2

The Math Practices in action

A teacher asks every day:

Did I do math sense-making about math structure using math drawings to support math explaining?

Can I do some part of this better tomorrow?

CCSS Domain Learning Progressions

K 1 2 3 4 5 6 7 8

CC: K only
Counting &
Cardinality

OA Operations and Algebraic Thinking: K to 5
- Single-digit numerical calculations K to 3
- Real world situation meanings of the operations K to 5

Expressions and Equations 6, 7, 8

NBT Numbers Base Ten: K to 5
- Place Value: K to 5
- Multidigit Computation: K to 6
- Decimal Computation: 5 to 6

The Number System 6, 7, 8

G Geometry: K to 8

G Geometry K to 8

MD Measurement and Data: K to 5
- Geometric Measurement: K to 6 (6 is in Geometry)
- Other Measures: K to 5
- Represent and Interpret Data: K to 5

Statistics and Probability 6, 7, 8

NF Number and Operations–Fractions: 3 to 5

Ratios and Proportional Relationships

Functions

Units for Quantities in the CCSS

K	1	2	3	4	5
CC: Counting & Cardinality ONES					
OA Operations and Algebraic Thinking: K to 5					
Single-digit numerical calculations K to 3					
Add, Sub	**Add, Sub**	**Add, Sub**	**Mult, Div**		
		TENS ONES	**A GROUP**		
Real world situation meanings of the operations K to 5 UNITS VARY					
NBT Numbers Base Ten: K to 5 Place Value & Multidigit + Decimal Computation					
Place Value: G to 5					
Teens as ten and some ones	**≤ 120 H T O**	**≤ 1000 Th H T O**		**≤ 1,000,000 M HTh TTh Th H T O**	**Tenths Hundredths Thousandths**
MD Measurement and Data: K to 5					
Geometric Measurement: K to 6 Uses LENGTH UNITS to make area and volume units					
(Describe attributes)	**(Length)**	**Length**	**Area**	**Angles**	**Volume G6: Surface area**
Other Measures: K to 5 UNITS VARY AND USE MULTIPLICATION/DIVISION					
(Describe attributes)	**(Time)**	**Time Money**	**Time Liq volume Mass**		
NF Number and Operations–Fractions: 3 to 5 Unit fractions 1/d (one whole ÷ into d equal parts)					

K	1	2	3	4	5

OA Operations and Algebraic Thinking: K to 5

Single-digit numerical calculations K to 3

Add, Sub	Add, Sub	Add, Sub	Mult, Div
Level 1	Level 2 & 3 maybe	Level 3 fluency	Level 1, 2, 3 fluency

Real world situation meanings of the operations K to 5

Add, Sub	Add, Sub	Add, Sub	Mult, Div
	more difficult unknowns		
		multistep + -	multistep + - x ÷ get more difficult

some work in G4 and 5 on equations and expressions

Learning Paths for Single-Digit Calculation

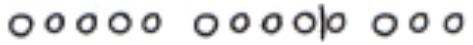
OA specifies learning paths for single-digit calculation.

- Addition/Subtraction (K, 1, 2)
- Multiplication/Division (3)

Each operation has 3 levels of increasing abbreviation, abstraction, and internalization.

Students need to be supported to move through these levels.

Figure 4. Levels of Addition and Subtraction Solution Methods

Addition Level 1 **Count All**  $9 + 4 = \boxed{13}$	**Subtraction Level 1** **Take Away**  $13 - 9 = \boxed{4}$
Addition Level 2 **Count On to Find a Total**  Stop when counted on 4 more, 13 $9 + 4 = \boxed{13}$ 	**Unknown Addend and Subtraction Level 2** **Count On to Find an Unknown Addend**  Stop when counted to 13, 4 more, so 4 $9 + \boxed{4} = 13$  $13 - 9 = \boxed{4}$ $9 + \boxed{4} = 13$
Addition Level 3 **Derived Facts** **Make a Ten** $9 + 4$ is $9 + 1 + 3 = 10 + 3$  $10 + 3 = 13$	**Unknown Addend and Subtraction Level 3** **Derived Facts Make a Ten** $13 - 9 = \square$ $9 + \square = 13$ $9 + 1 + 3$ 4 more to make 13 

Note: Children can use their fingers to keep track of how many they counted on.

Figure 5. Levels of Multiplication and Division Solution Methods

Multiplication	Division
Level 1 Count All  A) 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 3 x 5 B) (1) (2) (3) C) 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 =15	**Level 1 Count All**  A) 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 15 ÷ 5 B) (1 2 3 4 5) (1 2 3 4 5) (1 2 3 4 5) C) (1) (2) (3) =3
Level 2 Count by *n* A) 5 10 15 3 x 5 B) (1) (2) (3) =15 Stop when you count 3 fives. Answer is the last count by.	**Level 2 Count by *n*** A) 5 10 15 15 ÷ 5 B) (1) (2) (3) =3 Stop when you hear 15. Answer is the number of groups.
Level 3 Recompose (Decompose and Compose) 10 + 5 3 x 5 = (2 + 1)5 (2) + (1) = 10 + 5 = 15	**Level 3 Recompose (Decompose and Compose)** 10 + 5 15 ÷ 5 = 10 + 5 (2) + (1) = (2) + (1) = 3

Note: *(n)* is counting groups not individual objects. The answer is in red.

K	1	2	3	4	5

OA Operations and Algebraic Thinking: K to 5

Single-digit numerical calculations K to 3

Add, Sub	Add, Sub	Add, Sub	Mult, Div
Level 1	Level 2 & 3 maybe	Level 3 fluency	Level 1, 2, 3 fluency

Real world situation meanings of the operations K to 5

Add, Sub	Add, Sub	Add, Sub	Mult, Div
	more difficult unknowns		
		multistep + -	multistep + - x ÷ get more difficult

some work in G4 and 5 on equations and expressions

The 6 Situations

K — Add To / Take From

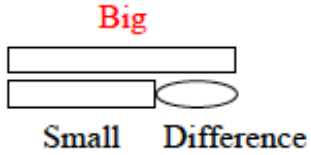
Start + Change = Result
Start – Change = Result

(becomes)

K — Put Together / Take Apart

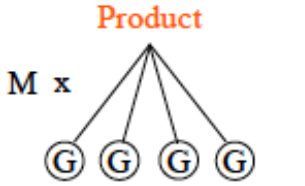
T = A + A
$\overset{i}{=}$ (identical)

Gr1 — Additive Comparison



Small + Difference = Big
Big – Difference = Small
Big – Small = Difference
$\overset{n}{=}$ (same number)

Gr3 — Equal Groups



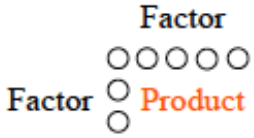
G + G + G + G = P
M times

M x G = P

(becomes)

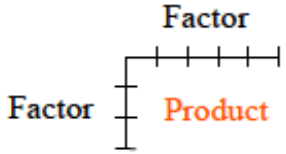
Gr3 — Rectangular Everything Times Everything

Array



(Long Division Format)

Area



Table

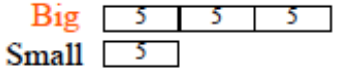
F x F = P
P ÷ F = F

(table)

$\overset{i}{=}$ (identical)

Gr4 — Multiplicative Comparison

Big = 3 x Small



Small = $\frac{1}{3}$ x Big

Big ÷ 3 = Small

$\overset{n}{=}$ (same number)

CCSS Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each

What is new in OA?

a) Solve problems with all 3 unknowns.

Each situation can have 3 unknowns.

This creates a learning path of difficulty from Kindergarten to Grade 1 to Grade 2.

b) Show the situation with a math drawing or diagram.

Problem Difficulty Learning Path:

K is dark grey.
G1 is grey.
G2 is white.

Table 2. Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? A + B = ☐	*A* bunnies were sitting on the grass. Some more bunnies hopped there. Then there were *C* bunnies. How many bunnies hopped over to the first two? A + ☐ = C	Some bunnies were sitting on the grass. *B* more bunnies hopped there. Then there were *C* bunnies. How many bunnies were on the grass before? ☐ + B = C
Take From	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? C – B = ☐	*C* apples were on the table. I ate some apples. Then there were *A* apples. How many apples did I eat? C – ☐ = A	Some apples were on the table. I ate *B* apples. Then there were *A* apples. How many apples were on the table before? ☐ – B = A

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	*A* red apples and *B* green apples are on the table. How many apples are on the table? A + B = ☐	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? C = ☐ + ☐	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? A + ☐ = C C – A = ☐

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has *A* apples. Julie has *C* apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has *A* apples. Julie has *C* apples. How many fewer apples does Lucy have than Julie? A + ☐ = C C – A = ☐	*"More" version suggests operation.* Julie has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have? A + B = ☐	*"Fewer" version suggests operation.* Lucy has *B* fewer apples than Julie. Julie has *C* apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have? C – B = ☐ ☐ + B = C

Represent the Situation
OA: Operations and Algebraic Thinking

Grade 1 and Grade 2 subtypes involve algebraic thinking:

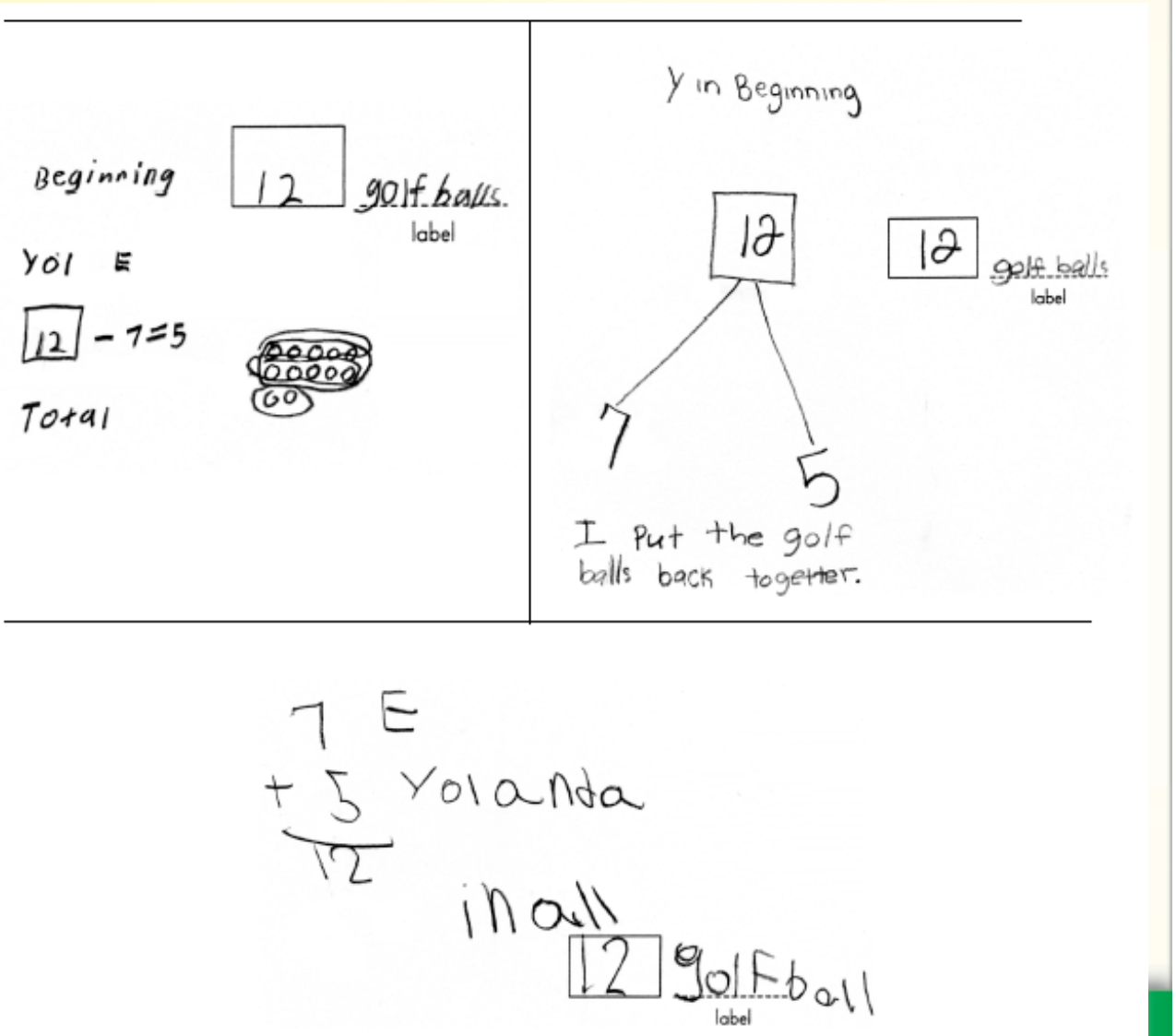
Represent the situation with a drawing, diagram, and/or an equation.

Then decide how to solve for the answer.

Grade 2 Labeled Math Drawings for a Start Unknown Problem

Yolanda has a box of golf balls. Eddie took 7 of them. Now Yolanda has 5 left. How many golf balls did Yolanda have in the beginning?

The key to solving story problems is understanding the situation. Students' equations often show the situation rather than the solution. Students drawings should be labeled to show which numbers or objects show which parts of the story situation.



K	1	2	3	4	5
NBT Numbers Base Ten: K to 5					
Place Value: K to 5					
Teens as ten and some ones	≤ 120 H T O	≤ 1000 Th H T O		≤ 1,000,000 M HTh TTh Th H T O	Tenths Hundredths Thousandths
Multidigit Computation: K to 6					
	2D + 2D compose a ten	2D + - 2D fluently Add, sub ≤ 1000	Add, sub ≤ 1000 fluently	Add, sub fluently ≤ 1,000,000 Mult 1Dx4D 2Dx2D Divide 4D÷1D remainders	Mult fluently Divide 4D÷2D G6 fluently
Decimal Computation: 5 to 6					
					Decimal fractions + - x ÷ G6 fluently

Crucial Aspects of the NBT Standards

For grade levels at which a new multidigit computation standard is introduced, the critical areas say:

Students develop, discuss, and use efficient, accurate, and generalizable methods to …

So it is not true that you start slowly with inefficient methods and only go to generalizable methods at later grades.

You must include generalizable methods that are a variation of the standard algorithm from the beginning.

Fuson, K. C. and Beckmann, S. (2012/2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, Fall/Winter, 14 (2),* 14-30.

Crucial Aspects of the NBT Standards

For grade levels at which a new multidigit computation standard is introduced, the standards emphasize:

Addition and subtraction: Using concrete models or drawings

Adding/subtracting like units and composing/decomposing when needed

Multiplication and division: Illustrate and explain by using equations, rectangular arrays, and/or area models.

All: Relate the strategy to a written method and explain the reasoning used

Crucial Aspects of the NBT Standards

Fluency

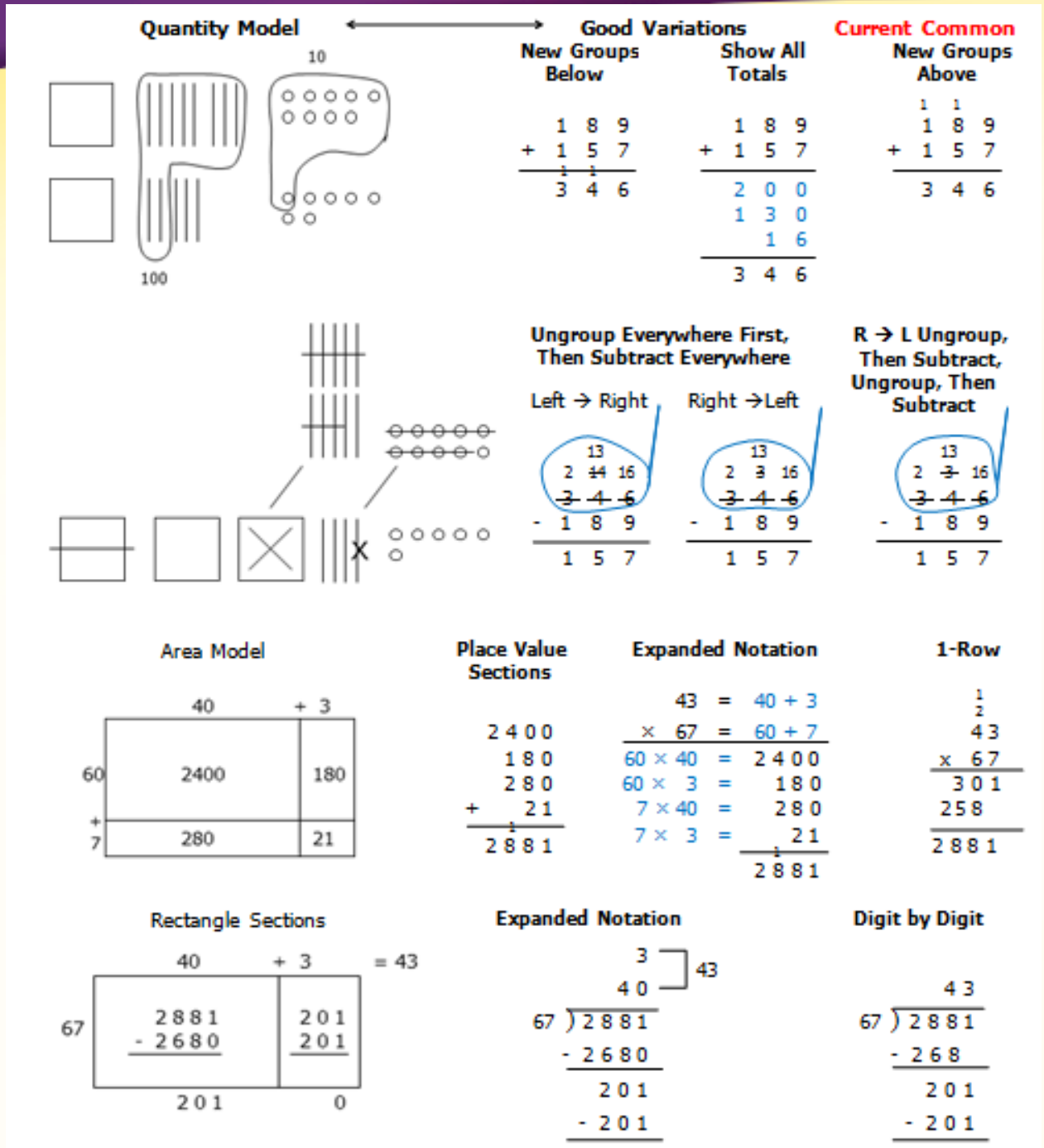
Fluency for a given operation and number size follows by 1 or 2 years the conceptual approach above.

Fluency for NBT means solving without a visual model and using a variation of the standard algorithm.

Algorithms are not bad. They use place value concepts and are general. Only rotely taught algorithms are bad.

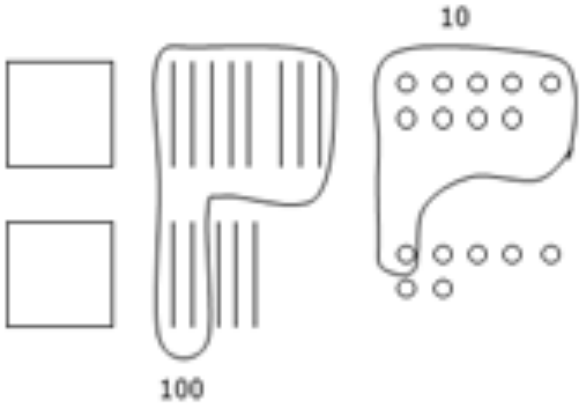
Method: strategy or procedure

Drawings and Written Variations of Standard Algorithms



Drawings and Written Variations to Work Separately with Like Place Value Units

a



Quantity Model ⟷ **Good Variations**

New Groups Below

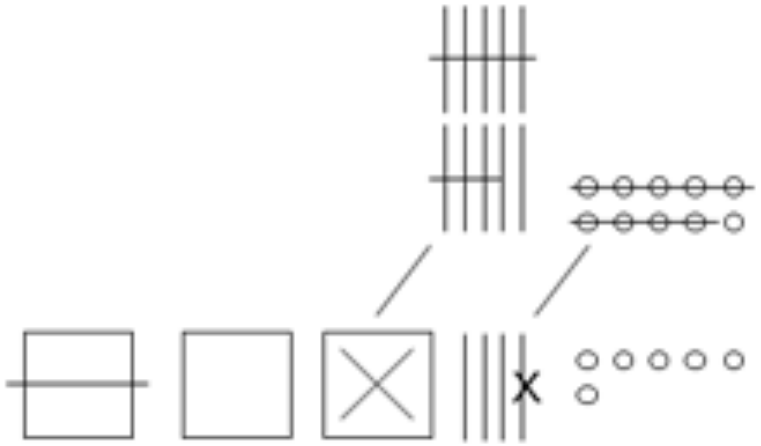
$$\begin{array}{r} 189 \\ +\ 157 \\ \hline 346 \end{array}$$

Show All Totals

$$\begin{array}{r} 189 \\ +\ 157 \\ \hline 200 \\ 130 \\ 16 \\ \hline 346 \end{array}$$

Current Common New Groups Above

$$\begin{array}{r} {\scriptstyle 1\ 1\ \ } \\ 189 \\ +\ 157 \\ \hline 346 \end{array}$$



Ungroup Everywhere First, Then Subtract Everywhere

Left → Right

$$\begin{array}{r} {\scriptstyle 13\ \ \ } \\ 2\ \not{1}4\ 16 \\ \not{3}\ \not{4}\ \not{6} \\ -\ 189 \\ \hline 157 \end{array}$$

Right → Left

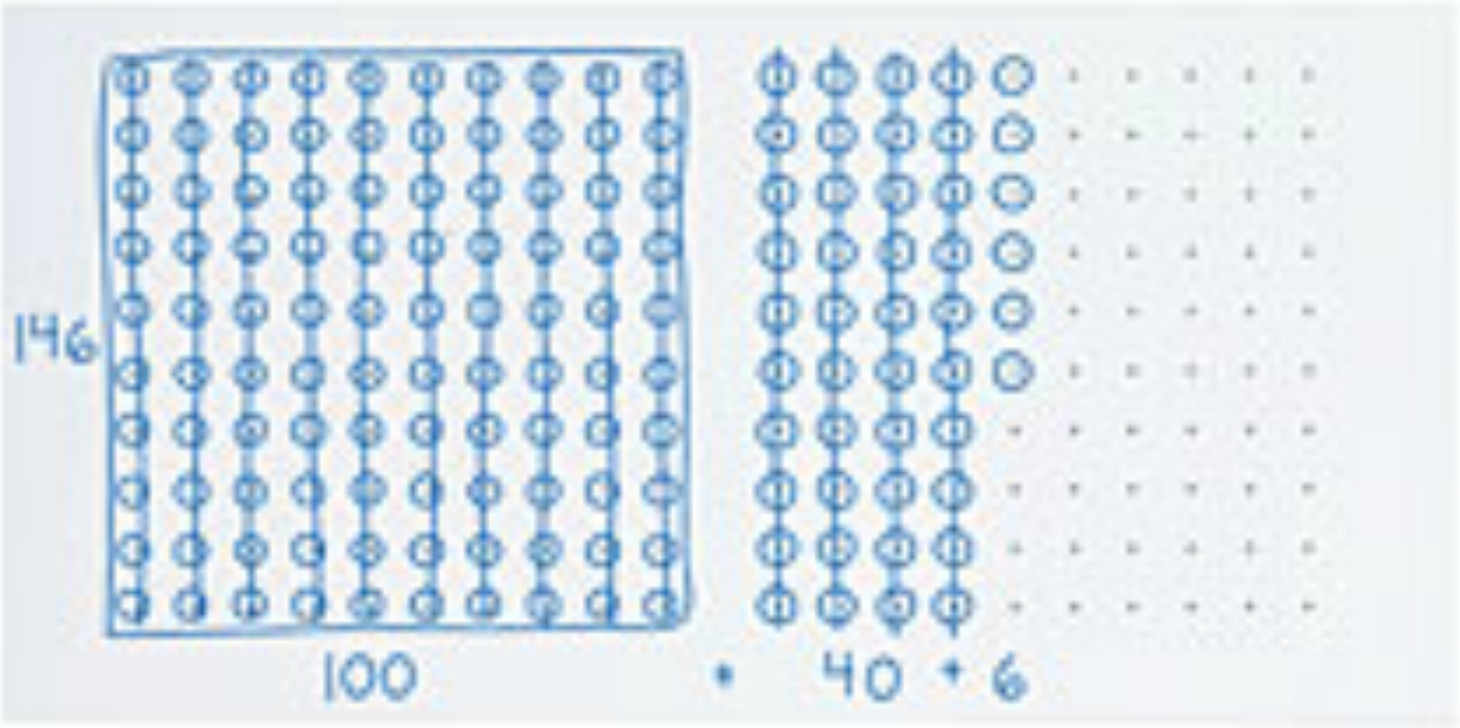
$$\begin{array}{r} {\scriptstyle 13\ \ \ } \\ 2\ \not{3}\ 16 \\ \not{3}\ \not{4}\ \not{6} \\ -\ 189 \\ \hline 157 \end{array}$$

R → L Ungroup, Then Subtract, Ungroup, Then Subtract

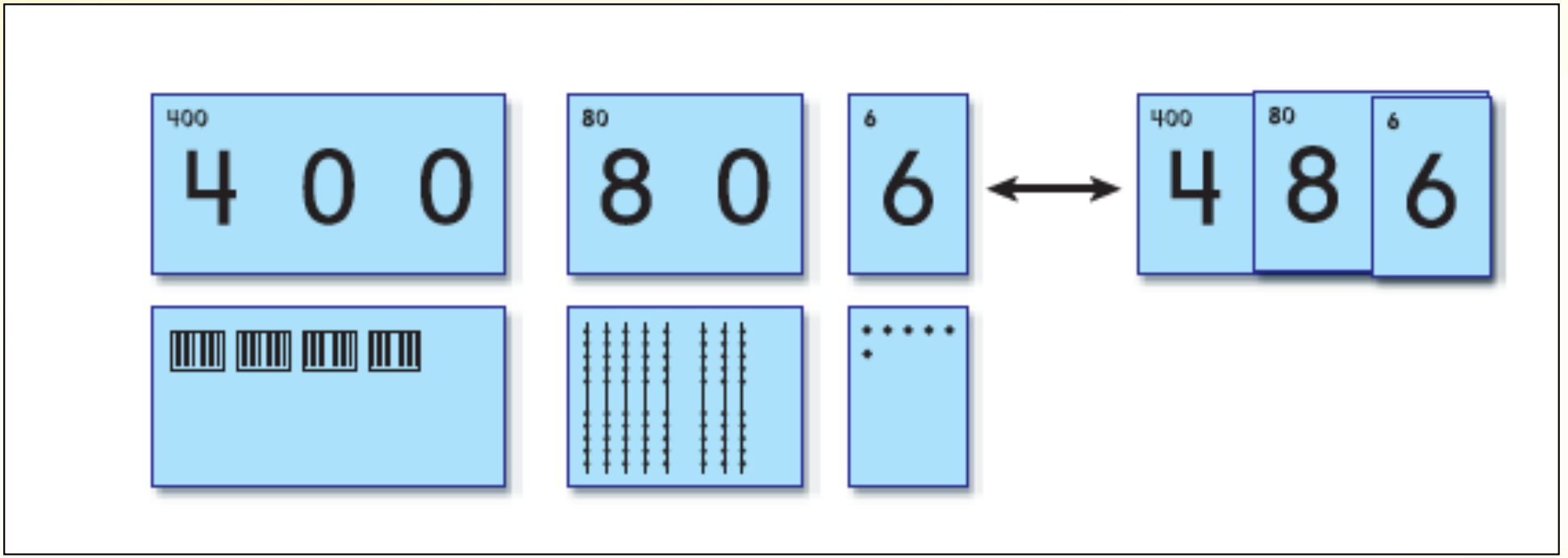
$$\begin{array}{r} {\scriptstyle 13\ \ \ } \\ 2\ \not{3}\ 16 \\ \not{3}\ \not{4}\ \not{6} \\ -\ 189 \\ \hline 157 \end{array}$$

G2 Place Value Drawings 2.NBT.1 and 3

Hundreds **Tens** **Ones**

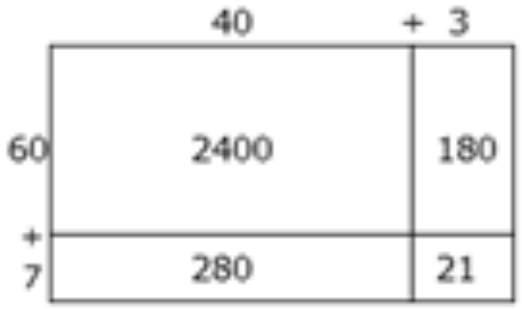


G2 Secret-Code Cards for 486 2.NBT.1 and 3



Drawings and Written Variations for Multiplication and Division

Area Model



Place Value Sections

2400
180
280
+ 21
2881

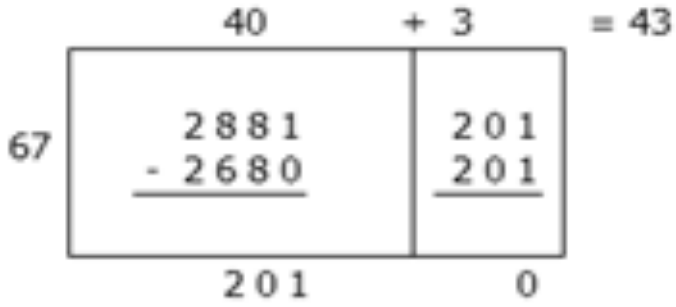
Expanded Notation

43 = 40 + 3
× 67 = 60 + 7
60 × 40 = 2400
60 × 3 = 180
7 × 40 = 280
7 × 3 = 21
2881

1-Row

$\frac{1}{2}$
43
x 67
301
258
2881

Rectangle Sections



Expanded Notation

3 ⌉ 43
40 ⌋
67) 2881
- 2680
201
- 201

Digit by Digit

43
67) 2881
- 268
201
- 201

K	1	2	3	4	5
NF Number and Operations–Fractions: 3 to 5					
		General:	Unit fractions	Compare any frs find eq frs	+- any fr eq frs fr x WN, fr [G6 fr ÷ fr]
		Special cases:	Compare like n or d	+- like denom WN x fr	n ÷ d = fr WN÷fr fr÷WN

Build Fractions From Unit Fractions

▶ Build Fractions from Unit Fractions

Write the unit fractions for each whole. Next, shade the correct number of parts. Then show each shaded fraction as a sum of unit fractions.

9.  Shade 2 parts.

Divide the whole into 5 equal parts.

$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$ $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

10.  Shade 2 parts.

Divide the whole into 3 equal parts.

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

11.  Shade 5 parts.

Divide the whole into 7 equal parts.

$\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$ $\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$

12.  Shade 7 parts.

Divide the whole into 8 equal parts.

$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{7}{8}$

13.  Shade 3 parts.

Divide the whole into 6 equal parts.

$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$

14.  Shade 8 parts.

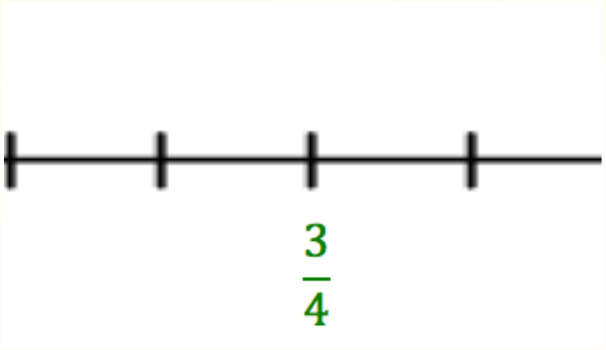
Divide the whole into 8 equal parts.

$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{8}{8}$

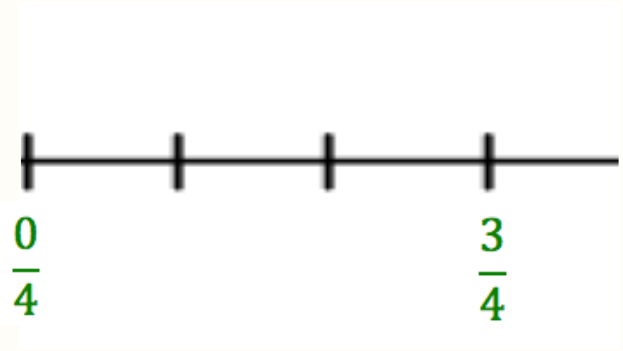
Not Enough Unit Lengths

Errors when drawing or using number lines

Error: Not enough unit lengths-
student counts marks rather than lengths.

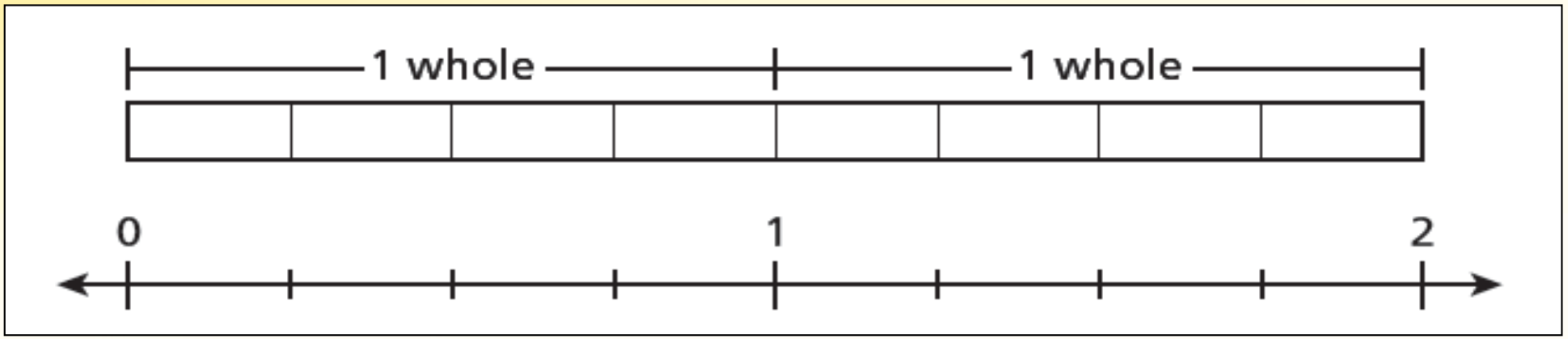


Correct: Count 3 unit lengths

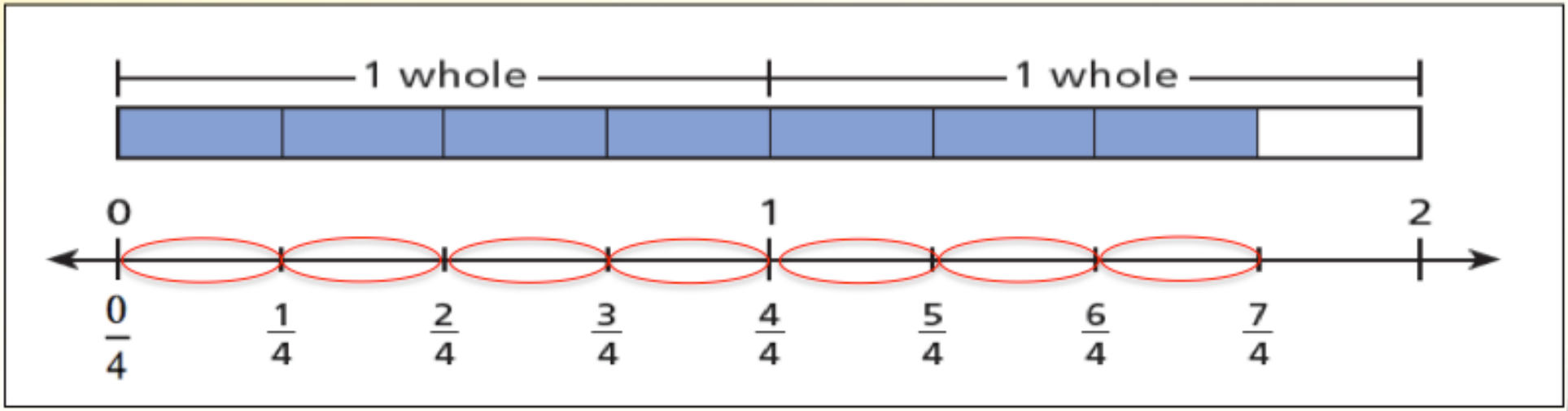


Seeing the Fraction Lengths

Step 1: Make the 4 unit fractions ¼ within each 1 whole.



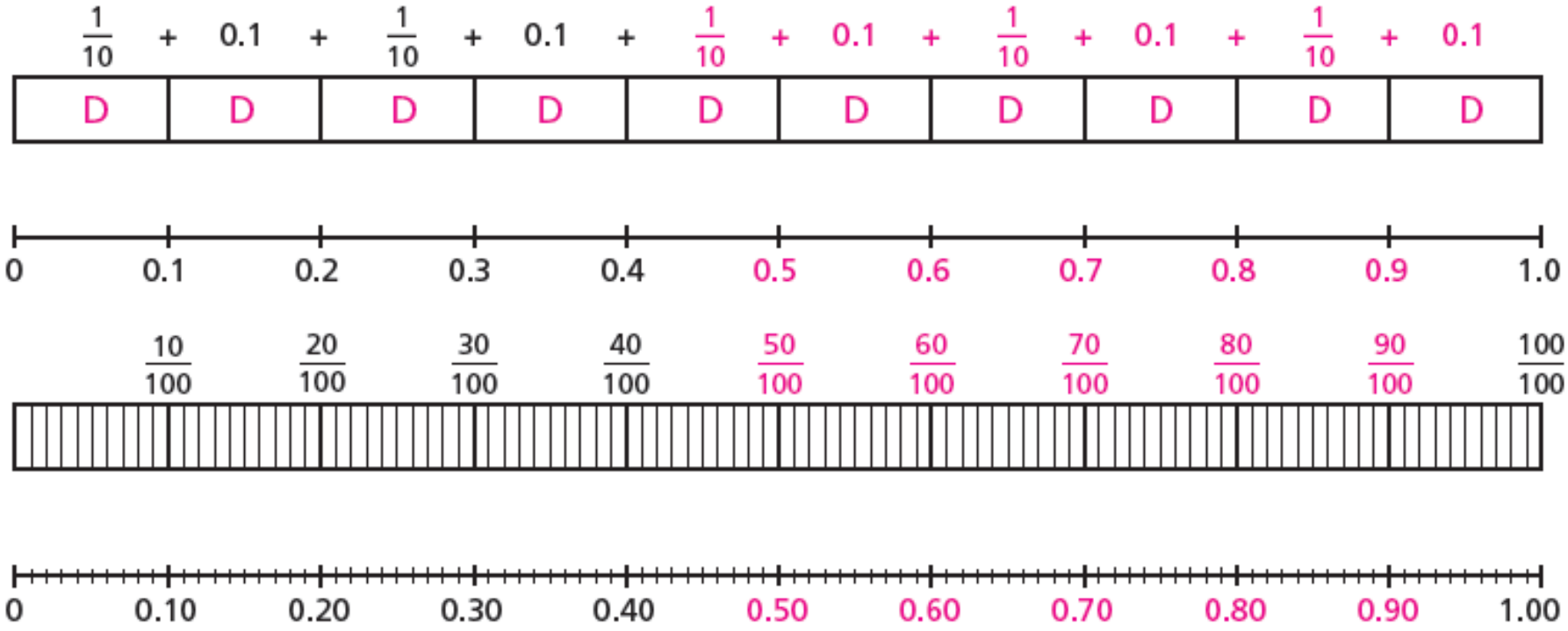
Step 2: Shade or encircle 7 unit fractions and label the number line.



The Number Line Numbers Tell the Number of Units So Far

▶ Understand Tenths and Hundredths

Answer the questions about the bars and number lines below.



Visual models are central core ideas and practices in the CCSS and support reasoning and explaining.

The models can be simple math drawings
that students can make and use in their own ways
in problem solving and explaining of thinking.

We want classrooms to be using the mathematical practices:
Students focus on math sense-making about math structure
using math drawings (visual models) to support math explaining.

K	1	2	3	4	5	6
G Geometry: K to 8						
K to 2 Analyze, name, compose/decompose shapes (this continues in higher grades)			Classification: Use sub-categories	Classify by properties	Hierarchy Coordinate plane	Surface area Area of triangles, special quadrilaterals and polygons

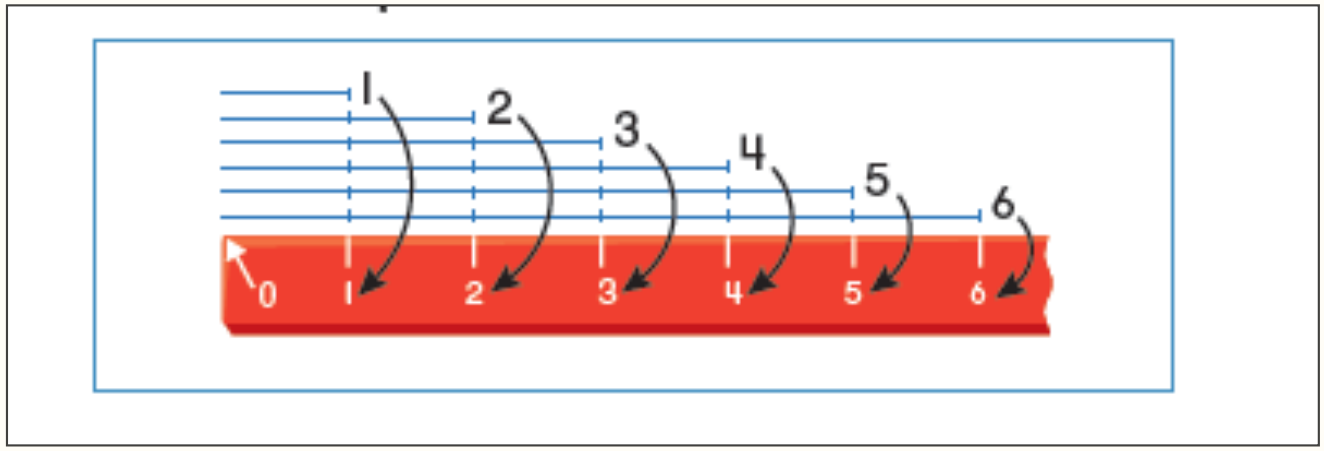
Shapes as units. Students need more experience with square units and the related rectangles, right triangles, and isosceles triangles.

K	1	2	3	4	5
MD Measurement and Data: K to 5					
Geometric Measurement: K to 6 Uses length to make area and volume units					
(Describe attributes)	(Length)	Length	Area	**Angles**	Volume [G6: Surface area Area of triangles, special quadrilaterals and polygons is in geometry]
Other Measures: K to 5					
(Describe attributes)	(Time)	Time Money	Time Liq volume Mass	Larger to smaller units x	Convert units both ways x ÷
Represent and interpret data: K to 5					
		Line plots	½ ¼	½ ¼ 1/8	Use fraction operations
Classify into categories, count	Up to 3 categories compare	Picture & bar graphs all problems	Picture & bar graphs scale multiple unit 1- and 2-step compare		

Units of Measure

- In-depth work with units of length and how a ruler is built up from units sets the stage for working with other kinds of units.

In Grade 2, students see the meaning of length units and how units are composed to make a ruler



in Grades 1, 2, and 3 representing and interpreting data is integrated with problem solving as students pose and solve word problems using data presented in picture and bar graphs. Comparison problems are a special focus.

In Grades 4 and 5 representing and interpreting data focuses on data presented in line graphs.

At Grade 6 students do significant in-depth work with statistics and probability.

CCSS Domain Learning Progressions

K 1 2 3 4 5 6 7 8

CC: K only
Counting &
Cardinality

OA Operations and Algebraic Thinking: K to 5
- Single-digit numerical calculations K to 3
- Real world situation meanings of the operations K to 5

Expressions and Equations 6, 7, 8

NBT Numbers Base Ten: K to 5
- Place Value: K to 5
- Multidigit Computation: K to 6
- Decimal Computation: 5 to 6

The Number System 6, 7, 8

G Geometry: K to 8

G Geometry K to 8

MD Measurement and Data: K to 5
- Geometric Measurement: K to 6 (6 is in Geometry)
- Other Measures: K to 5
- Represent and Interpret Data: K to 5

Statistics and Probability 6, 7, 8

NF Number and Operations–Fractions: 3 to 5

Ratios and Proportional Relationships

Functions

Visual models are central core ideas and practices in the CCSS and support reasoning and explaining.

The models can be simple math drawings
that students can make and use in their own ways
in problem solving and explaining of thinking.

We want classrooms to be using the mathematical practices:
Students focus on math sense-making about math structure
using math drawings (visual models) to support math explaining.

Research-Based Learning Progressions in the K-6 CCSS and the Mathematical Practices

Professor Karen C. Fuson

Northwestern University

For more details about the CCSS-M and visual supports, please see the series of visual with audio Teaching Progressions I have made for various math domains. These can be found at karenfusonmath.com