Learning Paths and Learning Supports for Conceptual Addition and Subtraction in the US Common Core State Standards and in the Chinese Standards

Karen C. Fuson and Yeping Li

Abstract The results of the Fuson and Li (ZDM Math. Educ. 41:793–808, 2009) analysis of the major early numerical aspects and learning supports for single-digit and multi-digit adding and subtracting in a representative Chinese textbook series and a US textbook series (*Math Expressions*) are related to the Chinese standards and to the US Common Core State Standards for these topics. Similar learning paths and visual-quantitative supports for mathematical thinking were identified in the textbooks from both countries, the US standards, and the experimental Chinese standards (2001). The new Chinese standards (2011) were less specific about learning paths and supports, though these appeared in examples. Criteria for judging the best variations of the multi-digit adding and subtracting variations were proposed and used. This analysis identified the best variations as the "New Groups Below" for adding and the "Ungroup First" for subtracting. The somewhat different levels in the adding and subtracting learning paths for East Asia and the US are summarized.

Keywords Addition · Subtraction · Language effects · Learning supports · Cross-cultural textbook analysis · Standards

Introduction

Several studies document ways in which East Asian mathematics textbooks support conceptual understanding by students (Cai 2008; Li 2007, 2008; Li et al. 2009a,b,c; Murata 2004, 2008; Murata and Fuson 2006; Watanabe 2006). Some of these studies also describe ways in which mathematics textbooks from the US fail to support conceptual understanding in these ways. One exception (Fuson and Li 2009) is the support for single-digit and multi-digit addition and subtraction in the most-used Chinese text and in a US second-generation NSF math textbook *Math Expressions*

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(Fuson 2006/2009). In that study, Fuson and Li reported extensive linguistic, visual-quantitative, and written-numeric supports for mathematical thinking in the Chinese textbooks and for the US textbook for the grades analyzed (Grades 1, 2, and 3 in both countries and Kindergarten for *Math Expressions*).

The diverse results from textbook analyses across countries suggest the need to examine and understand possible influences from curriculum standards that help guide the development of textbooks and possible learning paths embedded in textbooks. In many countries textbooks are a commonly used teaching resource that embodies the mandated curriculum. Therefore, it is also important to analyze the curriculum documents themselves as they are the primary source on which teachers and textbook writers draw in creating learning resources and planning learning experiences. Thus a major objective of this chapter is to analyze the extent to which curriculum standards in China and the US provide a coherent learning path that supports conceptual understanding of addition and subtraction.

China has a centralized education system, where curriculum standards are developed and used to provide overall guidelines for school education across the nation (Liu and Li 2010). While textbooks are developed as aligned with the curriculum standards, instructional activities and planning also follow information provided and highlighted in the curriculum standards including curricular goals, content, and their specifications at different grade levels. Before 2001, mathematics curriculum standards were called national mathematics teaching syllabus. The current version of the Chinese Mathematics Curriculum Standards (CMCS, Ministry of Education 2011) was revised from an experimental version that was published in 2001 (Ministry of Education 2001).

The United States has an education system with goals set by each of the 50 states, and districts within each state may also set goals. In the past, this has led to standards that vary extremely across states, resulting in a "mile wide and inch deep" curriculum (Schmidt et al. 1997). In the last decade, however, governors of many states organized a process to produce a set of standards that would be the same across all states that chose to participate. Standards were written by an appointed committee with advisors. The standards were then revised to accommodate the many comments elicited from across the nation. The final standards are now called the Common Core State Standards (CCSS) for the United States (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010). Almost all states agreed to use these standards; states that did not agree have standards that are similar to the CCSS.

If our analysis indicates that the new US CCSS standards require conceptual supports, then all mathematics books in the United States will need to include such supports, as the US textbook *Math Expressions* did in the Fuson and Li analysis (2009). This would be a significant step forward given that other analyses have not found such supports in many US textbooks. For China, the related question is whether the conceptual supports in textbooks exist because they are mandated in the mathematics curriculum standards or for other reasons. If they are not mandated, it seems that there is widespread cultural agreement that such supports facilitate learning and thus there is no need to mandate these in the mathematics standards.

In the following sections, the major results of the Fuson and Li (2009) analysis will first be summarized. Then features of the CCSS and Mathematics Curriculum Standards for China for the target mathematics domains will be presented and compared to the analysis of the textbooks. Then criteria for mathematically-desirable and accessible multi-digit methods that support conceptual learning and fluency will be summarized and applied to the variations of methods found in Fuson and Li (2009). Finally, the somewhat different learning paths for multi-digit adding and subtracting in the US and East Asia will be summarized.

A Coherent Learning Path of Meaning-Making Supports in Textbooks

An important framework for the Fuson and Li (2009) analysis was the Fuson and Murata (2007) Class Learning Path Model that integrated teaching principles from two US National Research Council reports (Donovan and Bransford 2005; Fuson et al. 2005; Kilpatrick et al. 2001), the NCTM process standards (National Council of Teachers of Mathematics 2000), and from teaching in Japanese classrooms. This model discusses the importance of a coherent learning path that supports student movement from primitive to more advanced methods that are mathematically desirable. Mathematically-desirable methods show important mathematical features, generalize across numbers and situations, and are efficient in computation. Fuson and Murata also discuss how it is possible to teach mathematically-desirable methods so that students can understand them. These accessible but mathematically-desirable methods have the numerals organized and written to support place value. These methods also do not have misleading written-numeric features that interfere with understanding or stimulate errors.

Consistent with the previous study (Fuson and Li 2009), written-numeric supports in this paper refer to the extent to which the written numerical method is presented with notated intermediate steps that make the written method more accessible to students and easier to carry out. Visual-quantitative supports refer to those clarifying text illustrations to show visually important quantitative aspects of the concepts involved. Linguistic supports mean how clearly a language expresses mathematical ideas. For example, Chinese language often expresses ideas more clearly than English, especially in the initial learning of numerical concepts.

Single-Digit Adding and Subtracting

For single-digit adding and subtracting, Fuson and Li (2009) found that both the East Asian and US textbooks took a coherent learning path consisting of methods that moved through the three levels identified in extensive international research (e.g., see research in Fuson 1992):

- Level 1: Count all and take-away
- Level 2: Count on keeping track of the second addend
- Level 3: Recompose the addends to make a new problem.

The mathematically-desirable recomposing method is make-a-ten, because it can be used for all single-digit sums over ten. For example, one can recompose the 6 in $8 + 6$ to become $10 + 4 : 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$. In both textbooks, children's methods are elicited and discussed, and then support is provided to move through the entire learning path to an accessible and mathematically-desirable method.

There are three conceptual prerequisites for the make-a-ten methods (see also Murata 2004, and Murata and Fuson 2006, for a discussion of how these are taught in a Japanese classroom). The prerequisites are easier to discuss if we introduce terminology used in the US *Math Expressions* program. Two addends that compose a number are called *partners* (e.g., in the make-a-ten method above for $8 + 6$, 8 and 2 are partners of 10, and 2 and 4 are partners of 6). To carry out the addition or subtraction make-a-ten methods, children must

(a) know the partners to ten for the numbers 9, 8, 7, and 6 to do the first step,
(b) know all of the partners of a given number to find the second step, and
(c) know the total $10 + n$ composed to be written as $1n$ (or know that $1n$ decomposes to be $10 + n$).

Step (c) is the step that is very easy if you say the written teen number using ten as in many East Asian languages (e.g., 12 is said as *ten two*). This step is more difficult in English where there are no verbal cues that 12 is equal to ten plus two. Both textbooks helped children learn the prerequisites for the make-a-ten method. The extensive work involved in finding two partners of a number for prerequisites (a) and (b) also supports children during the embedded addend thinking required for Level 2 of "counting on to add or to subtract" (e.g., $14 - 8$ is thought of as $8 + ? = 14$).

Details of how the Chinese books provide a coherent sequence of visual-quantitative supports (illustrations and drawings in the book) are given in Fuson and Li (2009) including seeing numbers 6 to 10 as 5-groups, which facilitates adding and subtracting and the make-a-ten method. Various linguistic supports in the math words in Chinese are then summarized including the easier teen numbers with the ten said explicitly and the more meaningful words for the parts of an addition and of a subtraction equation. Written-numeric supports that show the make-a-ten method are also displayed beginning with break-apart addend (partner) drawings (an upside down V with the total at the top and addends at each leg).

Details of how the *Math Expressions* books provide a coherent sequence of visual-quantity supports (illustrations and drawings in the book) are also given in Fuson and Li (2009). Because the Level 3 make-a-ten method is more difficult in English than in Chinese, mastery of Level 2 counting on methods was facilitated and emphasized early in Grade 1. These counting on methods are the accessible and mathematically-desirable methods for adding and subtracting that can become very fluent and then merge for many children into the make-a-ten methods or directly into known sums. Emphasizing subtracting as finding an unknown addend and counting on to find an unknown addend eliminated the many common errors made by US children counting down to subtract. The visual-quantitative supports for moving to these Level 2 methods and for the prerequisites for the Level 3 methods were described, e.g., 5-group patterns as in East Asia shown on a Number Parade and on student pages and penny strips showing ten pennies on one side and a dime on the back used to make teen numbers with an obvious group of ten. Written-numeric supports such as the addend drawings (developed independently and called Math Mountains) were also described. Various linguistic and visual-quantitative supports needed to compensate for various difficulties in English were also described, e.g., using tens-in-teens words as well as English words: one ten four ones for 14; secret-code cards in which a unit card, for example a 4, is placed on top of the 0 in the 10 card to make 14 so children could imagine the 0 hiding under the 4 and be supported to think of 14 as ten and 4 (10 and 4) even though their word for fourteen obfuscates this composition from a ten and 4 ones.

Multi-digit Adding and Subtracting

For multi-digit adding and subtracting, Fuson and Li (2009) found in both textbooks a coherent learning path of methods that moved rapidly to accessible and mathematically-desirable methods. Again, students' methods are initially elicited and discussed, but then support was provided to learn one or more accessible and mathematically-desirable methods. The irregularities in the English words for 11 through 99 required special learning supports relating drawings of tens and ones (later: hundreds, tens, and ones and larger quantities) to secret-code cards that showed the numbers in expanded notation with all zeroes but could be layered to show just the place value numbers (e.g., 379). Students used English number words and place-value number words to describe all of the quantities *three hundreds seven tens nine ones*.

East Asian books showed the meanings of multi-digit adding and subtracting with pictures of quantities such as bundled sticks or base-ten blocks. The feature that differs from many US texts that show similar pictures is that each step is shown so that students can relate what happens with the quantities to what happens in the written numeric method. In *Math Expressions* math drawings that showed hundreds, tens, and ones were built up and used so that students could make math drawings that would relate to their written numeric method as they explained and related steps in the drawing and the written method.

For multi-digit adding, all methods found in both textbooks, and in the Japanese and Korean books examined, used the same core approaches based on place-value: Two multi-digit numbers were added by adding like multi-units (numbers in like places) and making new larger unit(s) as needed (grouping/carrying). There were variations in the written-numeric supports for this core approach (i.e., how much a method showed the place value meanings) and in the order in which steps were carried out (e.g., adding hundreds or ones first). Numbers for the steps were written in different places and in different ways. Different variations existed in the books for the same country. These variations also varied in how clear and easy the method was to carry out.

For multi-digit subtracting, all methods used the same core approaches: Like multi-units were subtracted and ten more next-smaller units were made from one larger multi-unit as needed (ungrouping/borrowing) in order to do these subtractions. There were variations in the written methods including methods that showed only part of the step of getting ten next-smaller units when needed. Most methods involved alternating between the two aspects of the core approach (ungrouping then subtracting one place), but one method did all of the ungrouping first and then all of the subtracting.

Are There Coherent Learning Paths and Meaning-Making Supports in the US CCSS and Mathematics Curriculum Standards for China?

Core attributes of the US Common Core State Standards (CCSS) for single-digit and multi-digit adding and subtracting are given in Table 1. Table 1 indicates that most of the central aspects of the examined textbooks are specified in the CCSS. Coherent learning paths that begin with understanding and move to fluency are identified for single-digit and multi-digit addition and subtraction. These learning paths initially use visual-quantitative supports and then move to fluency without such supports. The answer for Chinese Mathematics Curriculum Standards (CMCS) standards is more complex and is given separately below for single-digit and multi-digit numbers where the contents of the US standards are also discussed in more detail.

Single-Digit Adding and Subtracting in Curriculum Standards

US The three levels of the CCSS single-digit learning path begins in Kindergarten with Level 1 and moves to Levels 2 and 3 in Grade 1 (see Table 1). Students continue with these levels in Grade 2. Students are to use visual-quantitative supports such as objects or drawings for this learning path. Fluency is specified for particular numbers at each grade level. Inclusion of these levels of student thinking is an important step forward because Fuson and Li (2009) explained that the learning path in many US textbooks at the time of the development of the *Math Expressions* program was to move from Level 1 directly to recall of memorized facts. In the US, there was, and still is, the idea that these "facts" can and should be learned as separate rote bits of information rather than as a set of interrelated triads that have many relationships accessible to reasoning and that go through the well-established research-based learning path of three levels.

The CCSS Operations and Algebraic Thinking (OA) Progression (The Common Core Writing Team 2011a) that explicates the OA standards makes it clear that math

Table 1 Core attributes of the US common core standards for single-digit and multi-digit addition and subtraction

Overall attributes

1. Research-based learning paths and the use of learning supports are specified.
2. Understanding and fluency are both crucial foci and mentioned specifically, and the standards are focused and coherent across grades so there is time to focus on understanding and on fluency.

Single-digit addition and subtraction

The standards specify a learning path of three levels of single-digit addition/subtraction strategies from Kindergarten through grade 2: (1) direct model counting all, (2) count on, (3) make a ten and other methods that recompose the addends.

Kindergarten children add and subtract within 10 using Level 1 methods (K.OA.1, 2), and they learn prerequisites for Level 2 and Level 3 methods (K.OA.3, 4; NBT.1). Children use visual-quantitative supports such as objects or drawings.
Grade 1 children add and subtract within 20 using Level 2 and 3 strategies (1.OA.6). Visual-quantitative supports are not mentioned explicitly, but are implicit because they are described in the OA Progression (2011a) and are used in the many studies about these strategies.
Grade 2 children fluently add and subtract within 20 using Level 2 and 3 strategies (2.OA.2) as outlined in 1.OA.6.

The standards specify fluency for adding and subtracting to specific totals by grade level: K.OA.5: totals $\leq$5; 1.OA.6: totals $\leq$10; 2.OA.2: totals $\leq$20 including by end of Grade 2, know from memory all sums of two one-digit numbers.

Multi-digit addition and subtraction

For multi-digit computation, the standards specify a learning path in which students first develop, discuss, and use concrete models or drawings and strategies based on place value and properties of operations, and they relate the strategy to a written method and explain the reasoning used (explanations may be supported by drawings or objects). They use the visual-quantitative supports for adding within 100 in Grade 1 (1.NBT.4) and for adding and subtracting within 1,000 in Grade 2 (2.NBT.7, 9). Grade 1 specifies "understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten."
Students then move to fluency to specific totals by grade level:

In Grade 2 they fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction (2.NBT.5).
In Grade 3 they fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction (3.NBT.2).
In Grade 4 they fluently add and subtract using the standard algorithm for totals through 1,000,000 (4.NBT.4).

Note: There are place value standards that support these computation standards

textbooks and teachers are to support this learning path of student reasoning and use of strategies and not just move immediately to fact fluency: "The word *fluent* is used in the Standards to mean "fast and accurate." Fluency in each grade involves a mixture of just knowing some answers, knowing some answers from patterns (e.g., "adding 0 yields the same number"), and knowing some answers from the use of strategies. It is important to push sensitively and encouragingly toward fluency of the designated numbers at each grade level, recognizing that fluency will be a mixture of these kinds of thinking that may differ among students. The extensive work relating addition and subtraction means that subtraction can frequently be solved by thinking of the related addition, especially for smaller numbers. It is also important that these patterns, strategies and decompositions still be available in Grade 3 for use in multiplying and dividing and in distinguishing adding and subtracting from multiplying and dividing. So the important press toward fluency should also allow students to fall back on earlier strategies when needed. By the end of the K-2 grade span, students have sufficient experience with addition and subtraction to know single-digit sums from memory (2.OA.2). As should be clear from the foregoing this is not a matter of instilling facts divorced from their meanings, but rather as an outcome of a multi-year process that heavily involves the interplay of practice and reasoning (pp. 18–19)."

The extended East Asian experience in supporting students through this meaningful learning path suggests a further clarification of what "known from memory" means. Murata (2004) reported that many Japanese students interviewed about their use of the make-a-ten method did not clearly distinguish between using the strategy and "just knowing" the total. These seemed to merge into such a rapid use of the strategy that it was not externally, and even sometimes internally, distinguishable from "just knowing." Therefore, "known from memory" might be a strategy that is done so quickly that others cannot tell whether the answer is obtained by just knowing or by rapid use of a strategy. These "known from memory" sums for the US CCSS can be seen as the culmination of a three-year process in which patterns (e.g., for adding 0 or adding 1), strategies, and remembered results merge to become sums "known from memory" but in ways that might differ across students. What fluency actually entails may differ between East Asian and at least some US students because the general make-a-ten method is more difficult in English, and thus fewer students use this Level 3 method.

No specific linguistic supports are described in the US CCSS, but the need for additional supports for the irregular English number words from 11 to 99 is identified and discussed in the Numbers Base Ten (NBT) Progression (2011b). In addition to these English-Chinese number word differences, Fuson and Li (2009) found that Chinese words for the parts of addition and subtraction equations were more meaningful than those in English. The auditory confusion between *sum* and *some* led to the use in *Math Expressions* of *total* instead of *sum* in the early grades. The difficult words for subtraction (*subtrahend, minuend*) led to the use of *addend* in subtraction as well as addition, providing the further benefit of relating addition and subtraction in the context of equation forms. Further analyses of other linguistic differences reported in Fuson and Li (2009) for these or other topics might be helpful (see also Song and Ginsburg 1987).

China An examination of the Chinese Mathematics Curriculum Standards (CMCS, Ministry of Education 2011) indicates that these standards emphasize the development of students' understanding of numerical concepts (up to 10,000 from grades 1–3) and computations (adding/subtracting of two-digit and three-digit numbers, and multiplying 1-digit times 3-digit and 2-digit times 2-digit numbers, dividing two-digit or three-digit numbers by a 1-digit number). In classroom instruction, CMCS requires teachers to use real world contexts to develop students' number concepts through observations, hands-on work, and problem solving. It emphasizes oral computations, estimations, and students' sharing of different computation methods. Further suggestions are also provided for teaching, assessment, and textbook development. For instance, CMCS in its experimental version (Ministry of Education 2001) included six general suggestions for textbook development, including (1) selecting and using problems and tasks that are engaging and closely related to students' daily lives; (2) providing students with opportunities for active thinking, collaborations, and discussions; (3) using multiple forms and representations to present materials; (4) introducing important mathematics concepts and ideas step by step; (5) leaving some flexibilities in content design; and (6) introducing related background information of selected mathematics concepts. These suggestions provide general guidelines for textbook writers to develop and include written-numeric, visual-quantitative, and linguistic supports in textbooks. Its current version (Ministry of Education 2011) still contains six general suggestions for textbook development, with similar but more comprehensive intentions. These suggestions include (1) textbook writing should emphasize scientific quality, (2) textbook writing should emphasize coherence and structure, (3) textbook content should show its development process, (4) textbook content should connect with students' reality, (5) textbook content design should leave some flexibilities, and (6) textbook should be readable and user friendly. In contrast to the US CCSS, the Chinese CMCS do not provide many specifics except some examples. Fluency is not specified for particular grade levels. The Chinese CMCS uses sample problems and solutions to illustrate its general suggestions rather than the detailed descriptions in the US CCSS. And the CMCS does not specify the single-digit learning progression from Levels 1–3 in Kindergarten to Grade 1 even though Fuson and Li (2009) found these levels in the textbook. However, this learning progression has been used for many years in China, so perhaps it is less necessary that it be specified.

Multi-digit Adding and Subtracting in Curriculum Standards

US The CCSS coherent learning path that begins with understanding and moves to fluency for multi-digit addition and subtraction starts with methods that use place value and properties of operations. These methods are initially grounded in visual-quantitative supports (concrete models or drawings) and related to a written method and explained (see Table 1). This is the approach reported by Fuson and Li (2009) for the Chinese (and other East Asian) and *Math Expressions* textbooks. In the US CCSS, fluency without the use of visual-quantitative supports is at Grade 2 within 100 and at Grade 3 within 1000. In the US CCSS and in

Math Expressions students generalize the standard algorithm through 1,000,000 in Grade 4.

China An examination of the Chinese Mathematics Curriculum Standards (CMCS) and textbooks indicates that sharing and discussing different ways of doing multi-digit computations is part of the approach. Computations with numbers larger than three digits are not emphasized in Grades 1–3, but estimation may involve simple 4-digit numbers. Numbering in Chinese uses different names for larger numbers through the first 5 places that students need to learn by Grade 3 (i.e., 个—ones, 十—tens, 百—hundreds, 千—thousands, 万—ten thousands). The use of distinct names with the meaning of different place values likely helps students to extend their computation skills to larger numbers when needed.

Many Methods or One in Curriculum Standards?

Fuson and Li (2009) reported variation in how multi-digit adding and subtracting methods were written even within the same country. The standards of both countries explicitly or implicitly allow different methods to be used. But the US CCSS calls the early methods that are used with drawings *strategies*, the later methods *algorithms*, and specifies that students are eventually to use the standard algorithm. The differences in these terms are discussed in Fuson and Beckmann (2012). These terms are not differentiated or even used in China, so the differences are not important for this paper. We will call all of these *methods*. But it is important for this paper to convey that "the standard algorithm" does not mean one single algorithm, but rather a collection of variations of written methods that use the same mathematical approach. The term "standard algorithm" is not defined in the CCSS, but in the NBT progression (The Common Core Writing Team 2011b) and in Fuson and Beckmann (2012), standard algorithms implement and are characterized by this mathematical approach:

- they decompose numbers into base-ten units and then carry out single-digit computations with those units using the place values to direct the place value of the resulting number; and
- they clearly use the one-to-ten uniformity of the base ten structure as they generalize to large whole numbers and to decimals.

All of the methods identified in Fuson and Li (2009) use this mathematical approach. This approach was classified by Fuson et al. (1997) as *decompose-tens-and-ones* methods rather than as *begin-with-one-number* methods that add or count on from one addend. Fuson and Beckmann analyze both of these kinds of methods and conclude that only the *decompose-tens-and-ones* methods (more generally, *decompose-place-values* methods) qualify as a standard algorithm.

Criteria for Mathematically-Desirable and Accessible Algorithms for Multi-digit Adding and Subtracting

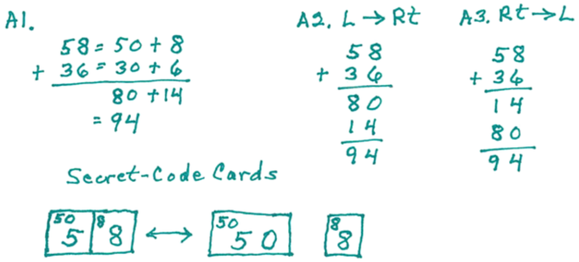
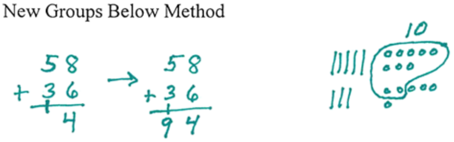
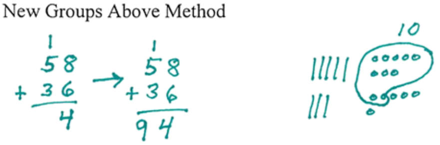
Given the variation between methods reported in Fuson and Li (2009) and supported in the standards for both countries explicitly or implicitly, an important educational question is: Are there variations that are more supportive of understanding and explaining and also easier to carry out than other variations? To address this question, we first identify aspects of multi-digit adding and subtracting in Table 2 that vary in how methods are written. These issues arise from mathematical aspects of multi-digit adding/subtracting or from the research literature about typical errors or preferences of children.

In Table 3 we show the major methods from Fuson and Li (2009) and then answer the questions from Table 2 to identify which methods clarified more conceptual issues or were easier to carry out. Because, as discussed above, all of the methods used in Fuson and Li (2009) benefited from visual-quantitative supports, these supports are shown in Table 3. The Secret-Code Cards show the 0 in the tens numbers hiding under the ones, and thus enable students to move from the first Expanded

Table 2 Conceptual issues for multi-digit adding and subtracting

Adding
(a) Add like quantities Is adding like quantities made easier by using vertical form and aligning like places (units)?
(b) Group (carry) 10 units to make 1 new next-larger unit (ten, hundred, etc.) if needed
(B1) Is it easy to see the total that includes the new grouped unit (e.g., 14 ones or 14 tens)? (B2) Do you write that teen total in the usual order (1 ten then the ones)? (B3) Is it easy to see where to write the new unit? (B4) Is it easy to add units that include that new unit?
(c) MDN + MDN = MDN Are the two addends and the total kept separate? If not, the problem is changed.
(d) Can you go from left to right? Many English-speaking students prefer to do math in the same direction as they read. So this preference would vary by language.
Answers of yes in Table 3 mean that the written method addresses that conceptual issue.
Subtracting
Issues a (Subtracting like quantities), c, and d are the same as those described above for Adding.
(b) Ungroup (borrow) 1 to make 10 new next-smaller units if needed
(B1) Is it easy to see the total that includes 10 of the new ungrouped unit (e.g., 14 ones or 14 tens)? (B2) Do you write that teen total in the usual order (1 ten then the ones)? (B3) Is it easy to see and/or to see where to write the new ungrouped unit? (B4) Is it easy to subtract from the units that include that new ungrouped unit?
(e) Does the method avoid the common subtracting error of subtracting top from bottom? (f) Does the method do all of one kind of step first and then all of the other kind of step?

Table 3 The support of multidigital addition methods for the conceptual issues in Table 2

A. Expanded Notation Methods		a) Yes all b1) Yes all b2) Yes all b3) Yes all b4) Yes 2 & 3 c) Yes all d) Yes 2
A. New Groups Below Method		a) Yes b1) Yes b2) Yes b3) Yes b4) Yes c) Yes d) No
A. New Groups Above Method		a) Yes b1) No b2) No b3) No b4) No c) No d) No

Notation Method (A1) to the two forms that do not write out the expanded forms for the addends but do for the sums (A2 and A3). The quantity drawings that show tens and ones help students see how to align the numbers vertically (especially for cases where the number of digits differs such as 8 and 65) and why one adds the numbers in like places (because they are like quantities tens and ten or ones and ones that can be readily distinguished in the math drawings). Each of these visual-quantity supports is helpful because they address different conceptual issues. The drawings for tens and ones in the Chinese books serve the same functions as the drawings in Table 3. The quantity drawings can support any method.

One of the Expanded Notation methods (the right to left A3 method) and the New Groups Below method (method B) address all of the mathematical issues a, b, and c from Table 2. The first and second Expanded Notation methods read from left to right (and are yes for d), so they are easier for some students. The New Groups Above method C does not support any conceptual issues except that it aligns like places (a). The problems with this method are:

- b1: The total is separated in space so that it is difficult to see as a total 14.

- b2: Usually students are to write down the 4 and carry/group the 1, so they must write 14 opposite to their usual order of writing 1 then 4.
- b3: Writing the new 1 ten above the left-most place instead of the next-left place is a well-documented error; it arises more with problems of 3-digits or more.
- b4: To add the column with the new unit above, students must add the 1 to one of the numbers in that place, remember that number and ignore the number they just used, and add the mental number to the other number they see. Or they add the two numbers there originally but then often forget to add the 1 on the top.

The Expanded Notation methods are useful for initial understanding, and the second and third such methods (A2 and A3) easily generalize to 3 or even 4 places. But these methods get complex for numbers as large as 1,000,000, so they cannot be considered totally general methods. Therefore, two approaches seem sensible. In the initial Chinese Mathematics Curriculum Standards (Ministry of Education 2001), the sharing of different computation methods is specifically emphasized, including the Expanded Notation method, for example, $58 + 36 = (50 + 30) + (8 + 6) = 80 + 14 = 94$. Methods adding on from the first number were also described (e.g., $58 + 36 = 58 + 30 + 6 = 88 + 6 = 94$; $58 + 36 = 58 + 2 + 34 = 60 + 34 = 94$) as was the method of New Groups Below. However, in the current new version of the Chinese Mathematics Curriculum Standards (Ministry of Education 2011) these examples do not appear, implying that students are to generate different methods but that teachers do not necessarily teach all different methods. In *Math Expressions* the Expanded Notation methods are used through Grade 3 because some English-speaking children benefit from working with the expanded forms for a longer time because of the irregular decade words in English. However, New Groups Below is introduced in Grade 1 and continued in all subsequent grades because of its conceptual advantages and because it generalizes. New Groups Above is introduced by the teacher in *Math Expressions* as a method used by some people if a student does not bring it into the class because it is considered by some people to be "the standard algorithm." All methods in Table 3 can be related to each other.

Results of the related analysis of Table 2 issues for the multi-digit subtracting methods reported in Fuson and Li (2009) are shown in Table 4. Conceptual issues a, b, c, and d for subtraction were similar to those for addition. Two new issues arise for subtracting. The use of the vertical form for adding like places suggests an extremely common subtraction error: Subtract the top from bottom number instead of ungrouping to get more (e.g., for 94–36, get 62). This error is increased in the US partly by the common practice of introducing problems with no ungrouping (e.g., 78–43) in Grade 1 and only moving to ungrouping problems a year later in Grade 2. The US CCSS will hopefully prevent this common textbook practice because no general 2-digit subtractions are in the US Grade 1 standards. In the East Asian textbooks examined, problems with no ungrouping are introduced first but the textbooks move immediately to problems requiring ungrouping, so Chinese students do not experience this difficulty.

Visual-quantitative supports of models or drawings that show hundreds, tens, and ones were used in all textbooks in the Fuson and Li (2009) analysis for subtracting. These might not have been needed so much for knowing issue (a) *subtracting like*

Table 4 The support of multidigital substraction methods for the conceptual issues in Table 2

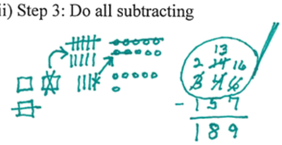
D. Do All Ungrouping First

i) Step 1: Ungroup 1 hundred

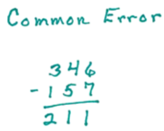
Step 2: Ungroup 1 ten

a) Yes c) Yes
b1) Yes d) Yes
b2) Yes e) Yes
b3) Yes f) Yes
b4) Yes

ii) Step 3: Do all subtracting

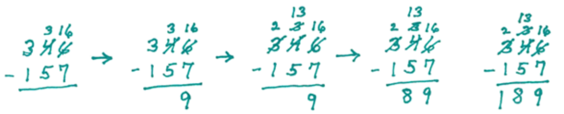


The major parts of this method (i ungroup and ii subtract) can be done Left to Right or Right to Left



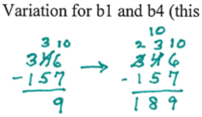
E. Alternate Ungrouping and Subtracting

Ungroup Subtract Ungroup Subtract Subtract



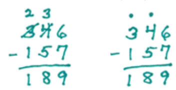
a) Yes c) Yes
b1) Yes d) No
b2) Yes e) No
b3) Yes f) No
b4) Yes

F. Variation for b1 and b4 (this alternates but does not have to)



a) Yes c) Yes
b1) No d) No
b2) Yes e) No
b3) Yes f) No
b4) Yes

Partial c1 and c4 (they alternate but do not have to)



a) Yes c) Yes
b1) No d) No
b2) No e) No
b3) No f) No
b4) No

quantities because of the previous work on multi-digit adding. These models illustrate directly the ungrouping (borrowing) needed to get more units in a top number in order to be able to subtract from it. *Math Expressions* also used a special visual-quantitative support—the magnifying glass (see Table 4)—to interfere with the common subtracting error and to help with conceptual issue c: seeing the three multi-digit numbers involved in the subtracting rather than only seeing the vertical frame for (a) subtracting like quantities. Students draw a *magnifying glass* (an ellipse) around the top number that is big enough to hold all of the ungrouping, with a little stick at the top right for the handle. The *magnifying glass* is introduced as something that reminds us to *look inside the top number* to check in each column to see if there is enough to subtract. This support serves to inhibit the subtract-smaller-from-larger number error that is often made before students even think about ungrouping. The *magnifying glass* also makes a visual grouping that emphasizes the top multi-digit number as a whole and thus facilitates a discussion about whether the value of the top number is changed when it is ungrouped. Because many US students view multi-digit subtraction as successive vertical operations on columns of single digits, many think that ungrouping does change the value of the top number (e.g., see literature reviewed in Fuson 1990). Students enjoy the metaphor of the looking glass, but they drop this step when they no longer need it.

Table 4 shows the two major subtracting methods Fuson and Li (2009) found in the textbooks. These methods are the same for 2-digit numbers but differ for numbers of 3 or more digits. The common top from bottom error is shown at the right of the second row. In the third row is an alternating method of subtracting in which a student can ungroup to make more ones if needed and then subtracts the ones, ungroups to make more tens if needed and then subtracts tens, etc. This method increases the common error even for students who know they should ungroup. For example, in the second step students see a 3 on the top and a 5 below and they have just subtracted the ones, so they are in subtracting mode. *Two* pops into their mind as the difference of 3 and 5, and they write 2 in the tens column instead of ungrouping to get more tens on the top. In the bottom row are variations of this method found in East Asian books. In Method F the ungrouped 10 of the new unit may be written above that place to make it easier to subtract by using make-a-ten. In the first method G only the new, reduced, larger unit resulting from the ungrouping is written. This method was also invented in *Math Expressions* classes by Grade 4 students who said they did not need to write the 1 for the tens because they knew it was there. The second method G is an even more abbreviated ungrouping recording: Neither the reduced larger unit or the increased smaller unit is shown; a dot shows a column that has been decreased by 1 to ungroup.

The top two rows show the *Math Expressions* Do All Ungrouping First method (D). Doing all ungrouping first eliminates top from bottom errors. Answers to the conceptual issues for subtraction in Table 3 show that this Do All Ungrouping First method (D) is the most conceptually supportive. It has the further advantage that either major step (ungrouping or subtracting) can be done left-to-right, which many students like. Both major steps can also be done right-to-left, for students who prefer this. Deep mathematical discussions ensue when students explain why they can go in either direction and why they get the same answer both ways. Any of the East Asian methods shown in Table 4 could be done as a non-alternating method by doing all ungrouping first.

Somewhat Different Learning Paths in China and the US

Ma (1999) identified three steps in multi-digit adding and subtracting that Fuson and Li (2009) verified in other East Asian books. Ma called these levels, but we use the term "steps" to differentiate these from the three levels used in single-digit adding/subtracting. In Ma's Step 1, the make-a-ten methods for teen addition and subtraction were developed, each in a separate unit. In Step 2, multiple methods were given for 2-digit problems. In Step 3 for problems with 3-digit and larger numbers, the books focused on one generalizable mathematically-desirable method, with the variations as discussed above.

Our analysis of the US CCSS indicates a related but somewhat different sequence for students in the United States, partly because of the limitations of English number words that cause difficulty with the make-a-ten methods for children speaking English (or other European languages with irregular tens). The East Asian Steps 1 and 2 become mixed in the US. Make-a-ten methods can be introduced and discussed in Grades 1 and 2. But because these methods are more difficult in English than in Chinese, many students stick to Level 2 counting on methods. The Level 3 make-a-ten methods may begin to be used by more students when they are grouping and ungrouping in multi-digit adding and subtracting because the group of ten is salient and important then. Limited use of the make-a-ten methods is also due to limited teaching of the make-a-ten methods and their prerequisites in the US. Multi-digit expanded notation methods as well as the New Groups Below method are conceptually important to introduce early on. Students may invent other methods especially for totals within 100. The US Step 1 is formed by this mixture of single-digit and multi-digit methods and occurs in Grades 1 and 2. Step 2 occurs with numbers greater than 100 but less than 1000, where it is important for more children to move to New Groups Below although some may continue to use Expanded Notation methods or adding on methods especially in Grade 2. The Do All Ungrouping First method is now important for subtraction involving 3 digits (this method D and the alternating variation E are not distinguishable for 2-digit subtraction). In Step 3 all students focus as in East Asia on a generalizable mathematically-desirable method in Grade 3 within 1000 and go on in Grade 4 to larger problems within 1,000,000.

Conclusions

The analysis of the US and Chinese standards indicated differences between these countries' standards. The US CCSS have explicit learning paths and supports as can be seen in Chinese and the US *Math Expressions* textbooks in Fuson and Li (2009). Therefore, all US programs in the future should have such supports as mandated in the standards even though such supports have not appeared consistently before. This is an important step forward for students and teachers in the US. The Chinese standards were less explicit about learning paths and supports, but the presence of these learning paths and supports in textbooks suggest that there is cultural knowledge that may make it less necessary to be explicit in the standards. Hopefully this cultural knowledge will be sufficient to maintain such learning paths and supports in future textbooks.

Our analysis identified methods that show variations of written steps that can support conceptual issues in multi-digit adding and subtracting, reduce errors, and make steps easier to carry out. These New Groups Below and Do All Ungrouping First methods are as efficient and succinct as related variations that are viewed by some in the US as the standard algorithm (New Groups Above C and the Alternating Subtraction Method E), and these superior methods do not have the disadvantages of these inferior variations. Thus, the best variations are the New Groups Below method (B) for addition and the Do All Ungrouping First method (D) for subtraction. Hopefully the use of these superior variations will become widespread in both the US and China.

Finally, we summarized the somewhat different organization of steps in Ma's learning path for China and the US. These differences were primarily the results of linguistic limitations of English teen (11 to 19) words compared to Chinese words, but they also are due to limited teaching of the make-a-ten methods and their prerequisites in the US. The final Step 3 in Ma's book (1999) is similar in China and the US and could become even more similar if both countries moved to using the multi-digit variations we identified as superior.

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