Balanced Learning-Path Teaching in the Classroom and Remotely

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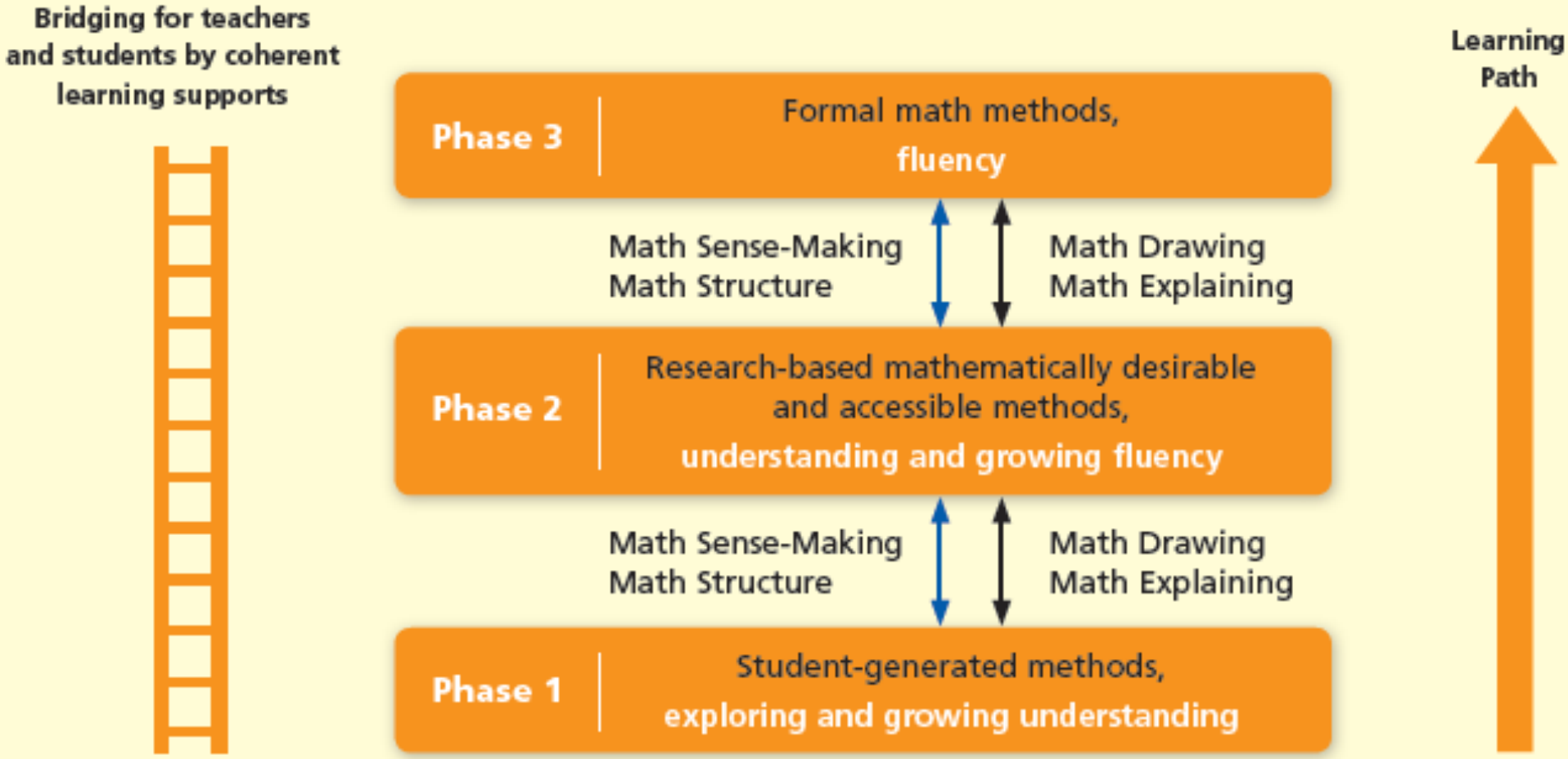
Please see my website karenfusonmath.com for 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, and presentations.

Inquiry Learning Path

in the Math Talk Community



Mathematical Practices

Math Sense-Making	Math Structure	Math Drawings	Math Explaining
Make sense and use of appropriate precision.	**See structure and generalize.**	**Model and use tools.**	**Reason, explain, and question.**
MP1 Make sense of problems and persevere in solving them. **MP6** Attend to precision.	**MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	**MP4** Model with mathematics. **MP5** Use appropriate tools strategically.	**MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others.
Teachers continually assist students to do math sense-making about math structure using math drawings to support math explaining.			

Teachers continually assist students to do **math sense-making** about **math structure** using **math drawings** to support **math explaining**.

Create a Nurturing Sense-Making Math Talk Community

The teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of students, using responsive means of assistance that facilitate learning and teaching by all.

- Engaging and involving
- Managing
- Coaching*

*modeling, cognitive restructuring/clarifying, instructing/explaining, questioning, feedback

The teacher supports the sense-making of all classroom members by using and assisting students to use and relate:

- Coherent mathematical situations
- Pedagogical supports
- Cultural mathematical symbols and labels

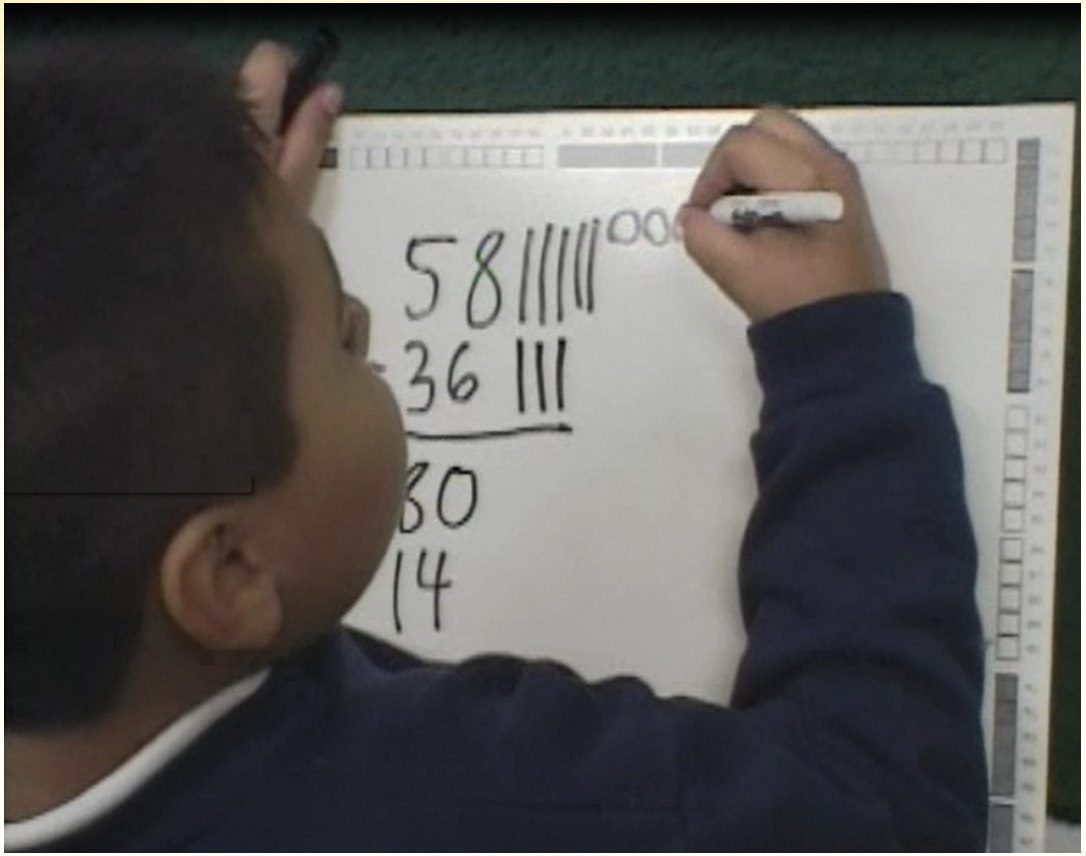
Solve and Discuss Classroom Structure

Solve	Explain	Question	Justify
All students solve. Some solve at the board, and the rest at their seats.	**One student at the board** explains and then asks, "Are there any questions?"	**Other students** ask questions to clarify or extend.	The original explainer **responds to the questions** by explaining more (justifying the original explanation).

Any student at any time can ask for help from anyone.

For more practice, Solve and Discuss can take place in pairs or small groups.

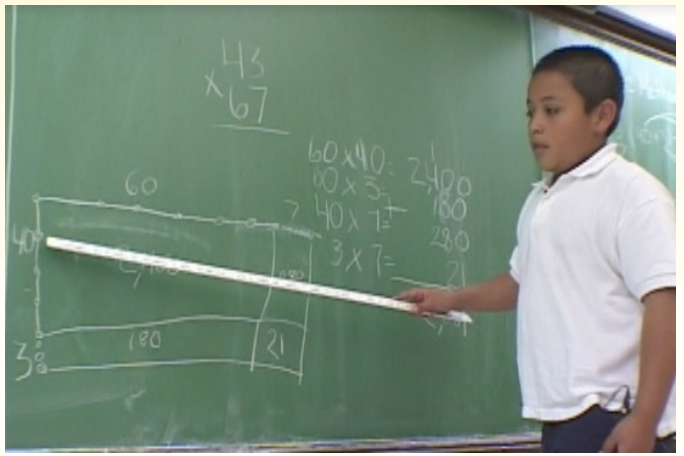
Make the math thinking visible



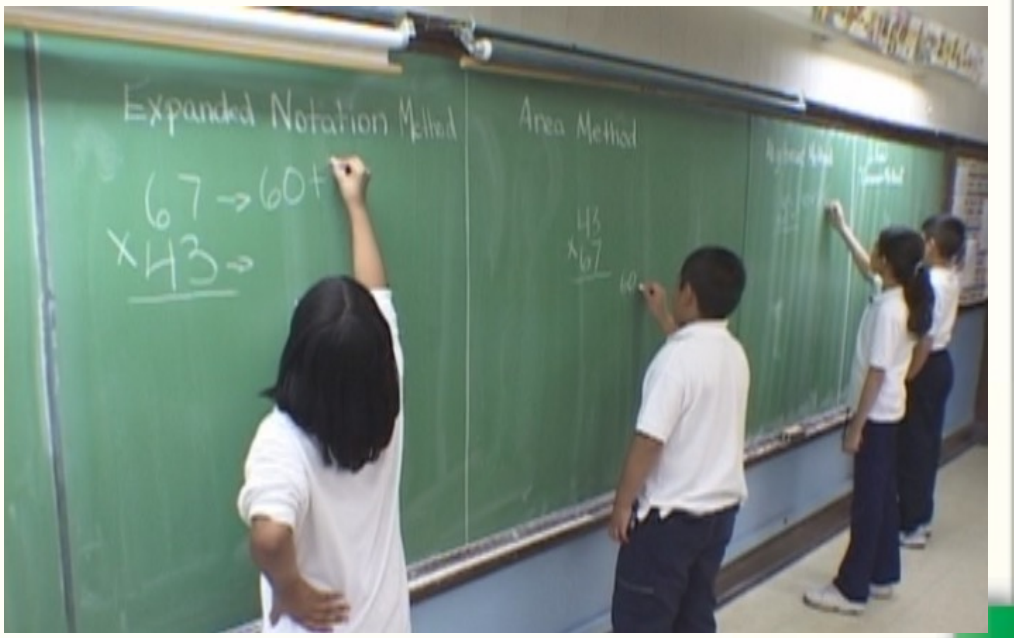
- Students must make some kind of math drawing related to the math symbols to show their thinking.
- This supports understanding by the listeners and promotes meaning.

Make the math thinking visible

- This is important for equity: less advanced students and English Learners are helped by the math drawing linked to the explanation by pointing.



- Be sure that important methods remain on the board or can be made visible again (e.g. on a Math Board) so they can be compared with other methods.



Students must speak and not just listen

1. Structure opportunities to explain to a partner and repeat what the partner says, if needed. Students eventually find their own words, but may need the security of saying an explanation they know is correct.

2. "Bite your tongue" to provide wait time. Students will explain, ask questions, or add a comment if you wait.

3. Help students speak to classmates by moving to the side or back of the room. Later remind students with a silent gesture to address each other.

A nurturing meaning-making visual Math Talk Community:

is an inquiry-based teaching/learning environment, and has continual focus on sense-making by all participants.

Students are expected:

- to understand what they are doing,
- come to be able to explain their thinking,
- understand the thinking of other students,
- learn to seek help when they need it, and
- help others who need it.

Balanced Inquiry Learning Path teaching requires

mathematically desirable and accessible situational diagrams

mathematically desirable and accessible drawings of numbers

mathematically desirable and accessible computational methods

My many years of classroom research focused on finding, developing, and testing these in varied classrooms of students.

They all involve learning paths to bring students from where they start to fluency with advanced enough methods.

You can find details of all of these on my website in the Teaching Progressions and in the classroom videos. karenfusonmath.com

The Common Core State Standards support schools to teach using learning paths of students because the math standards progress and build sensibly.

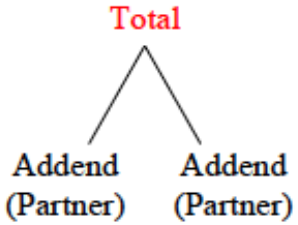
The 6 Situations

K — Add To / Take From

Start + Change = Result
Start – Change = Result

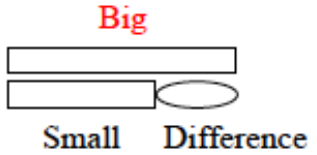
⟶ (becomes)

K — Put Together / Take Apart



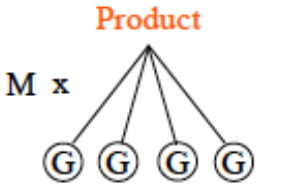
T = A + A
$\overset{i}{=}$ (identical)

Gr1 — Additive Comparison



Small + Difference = Big
Big – Difference = Small
Big – Small = Difference
$\overset{n}{=}$ (same number)

Gr3 — Equal Groups



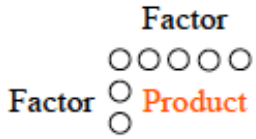
G + G + G + G = P
M times

M x G = P

⟶ (becomes)

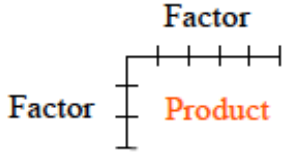
Gr3 — Rectangular Everything Times Everything

Array



(Long Division Format)

Area



Table

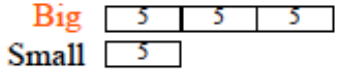
F x F = P
P ÷ F = F

(table)

$\overset{i}{=}$ (identical)

Gr4 — Multiplicative Comparison

Big = 3 x Small



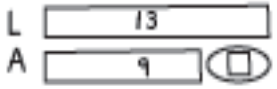
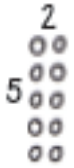
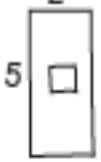
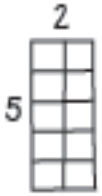
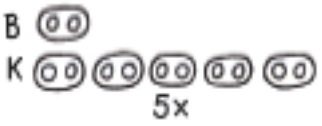
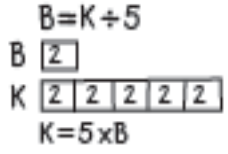
Small = $\frac{1}{3}$ x Big

Big ÷ 3 = Small

$\overset{n}{=}$ (same number)

CCSS Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each

Math Drawings Then Diagrams

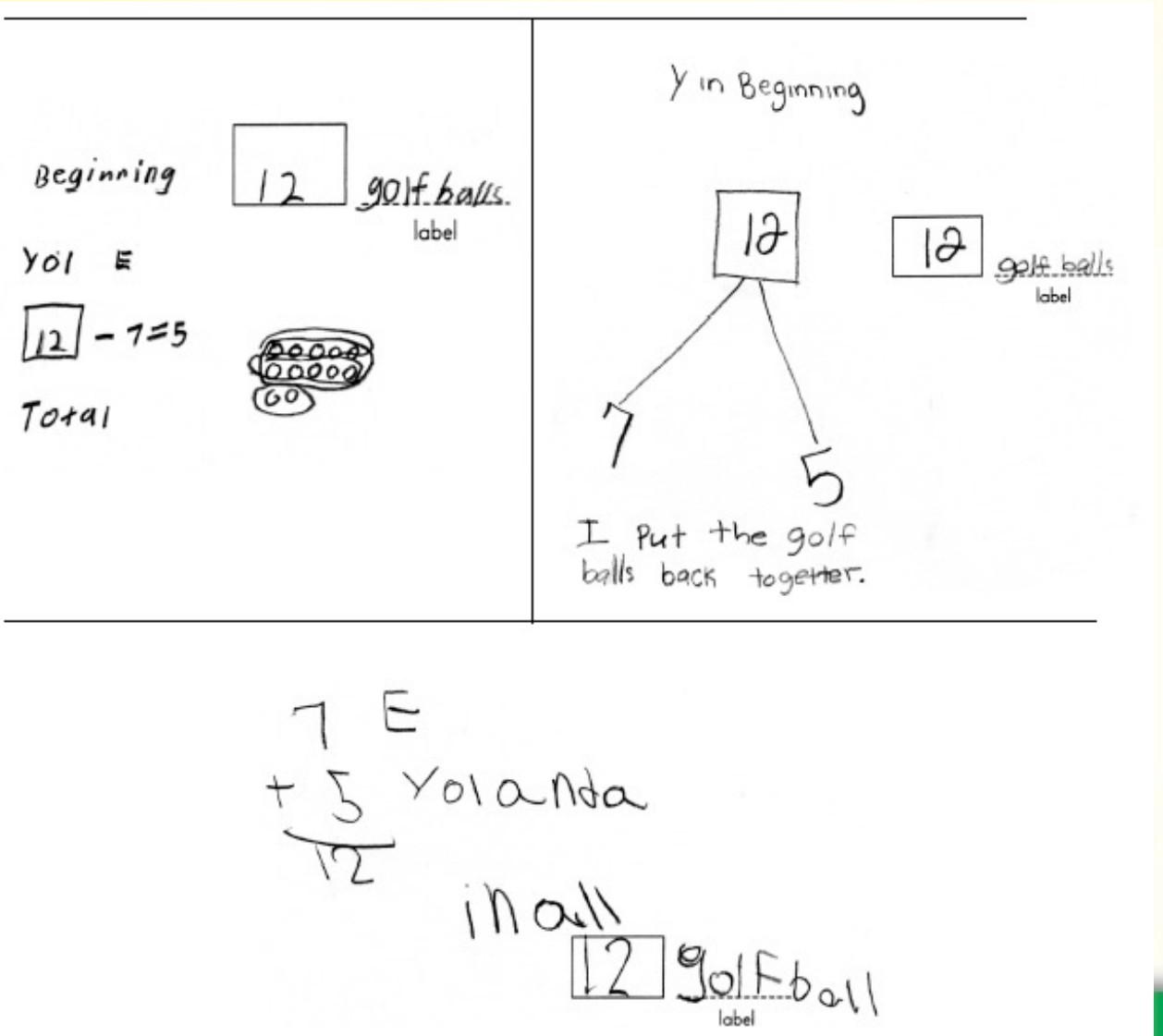
Problem Type	Word Problem	Representation	
		Math Drawing	Diagram
Add To	Dan had 9 cherries. Then he picked 4 more. How many does he have now?		9+4=□ (situation/solution equation)
Take From	Dan had 13 cherries. Then he ate 9 of them. How many does he have now?		13-9=□ (situation/solution equation)
Put Together/ Take Apart	Ana has 9 dimes and 4 nickels. How many coins does she have in all?		Math Mountain Diagram 
Additive Comparison	Ali has 9 balloons. Lisa has 13 balloons. How many more balloons does Lisa have than Ali?	Matching Drawing 	Comparison Bars 
Equal Groups	Amy has 5 cousins. She is making 2 puppets for each cousin. How many puppets will Amy need to make?	Grouping Model 	Equal Shares Diagram 
Array	A garden has 5 rows and 2 columns of bean plants. How many plants are there in all?	Array Model 	Fast Array Diagram 
Area	The garden is 5 yards on one side and 2 yards on the side touching this. What is its area?	Area Model 	Fast Area Diagram 
Multiplicative Comparison	Bill has 2 apples. Kim has 5 times as many apples as Bill. How many apples does Kim have?	Grouping Model 	Comparison Bars 

Grade 2 Labeled Math Drawings for a Start Unknown Problem

Yolanda has a box of golf balls. Eddie took 7 of them. Now Yolanda has 5 left. How many golf balls did Yolanda have in the beginning?

The key to solving story problems is **understanding the situation**. Students' equations often show the **situation** rather than the solution.

Students drawings should be **labeled** to show which numbers or objects show which parts of the story situation.



Grade 2 Labeled Math Drawings for a Start Unknown Problem

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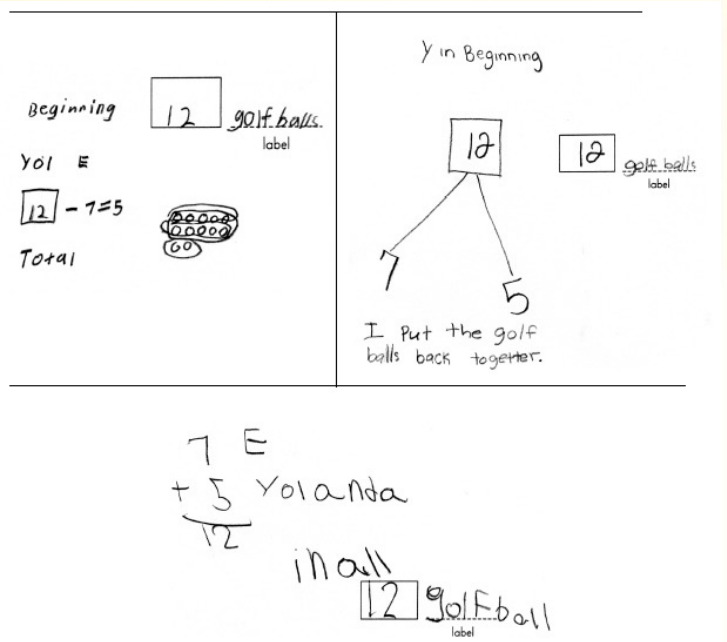


Student drawings vary in their representations. There are levels of solutions.

Students can use any numeric diagram for any problem if they can explain it. (Individuals vary in their choices.)

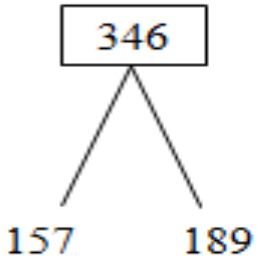
Students should work on writing a situation equation even if they have a solution representation because with larger numbers they cannot draw a solution: They will have to write a solution equation ? – 7 = 5 and then a situation computation or equation 7 + 5 = ?.

The location in addition and subtraction equations of the total and the addends is the key to problem solving, now and later in algebra.

Grades 3 and 4: Represent a Start Unknown Situation

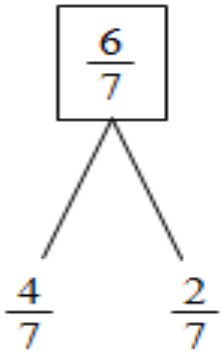
Numerical Relationships in Math Mountain

Y beginning



$$\begin{array}{r} 157 \\ +\ 189 \\ \hline 346 \\ {\scriptstyle 1\ 1} \end{array}$$

Y at first



Situation Equation

Y	E	end
346	- 157	= 189

$$\begin{array}{r} 157 \\ +\ 189 \\ \hline 346 \\ {\scriptstyle 1\ 1} \end{array}$$

Y	E	left
$\frac{6}{7}$	$-\ \frac{4}{7}$	$= \frac{2}{7}$

total

$$\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}=\frac{6}{7}$$

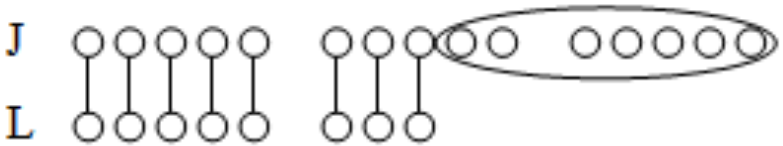
or

$$\frac{4}{7}+\frac{2}{7}=\frac{6}{7}$$

***Figure 4.* Grade 2 Solution Approaches to an Additive Comparison Problem**

In March Jana read 15 books. Lisa read 8 books. How many fewer books did Lisa read than Jana?

Matching Drawing of Quantities



7 books

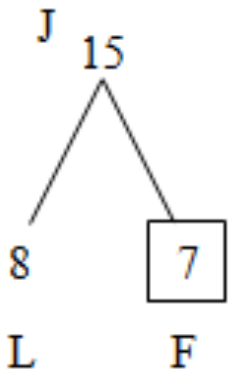
Situation Equation

8 + 7 = 15

Lisa more Jana

7 books

Numerical Relationships Shown in Math Mountain



7 books

Solution Equation

15 - 8 = 7

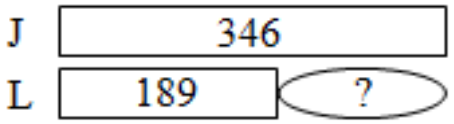
Jana Lisa fewer

7 books

Grade 3 Solution Approaches to an Additive Comparison Problem

In the summer Jana trimmed 346 bushes. Lisa trimmed 189 bushes. How many fewer bushes did Lisa trim than Jana?

Comparison Bar Drawing of Quantities



$$\begin{array}{r} {\scriptstyle 13} \\ 2\,\not{1}\,\not{4}\,{\scriptstyle 16} \\ \not{3}\,\not{4}\,\not{6} \\ -\,1\,8\,9 \\ \hline 1\,5\,7 \end{array}$$

157 bushes

Situation Equation

L	*d*	J
189	+ **157**	= 346

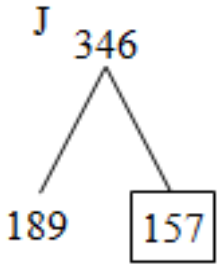
189 + 11 = 200

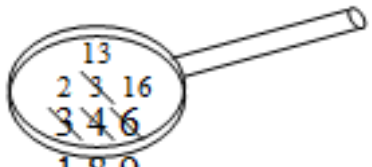
200 + 146 = 346

157

157 bushes

Numerical Relationships Shown in Math Mountain





157 bushes

Solution Equation

Jana	Lisa	fewer
346	- 189	= **157**

$$\begin{array}{r} {\scriptstyle 13} \\ 2\,\not{1}\,\not{4}\,{\scriptstyle 16} \\ \not{3}\,\not{4}\,\not{6} \\ -\,1\,8\,9 \\ \hline 1\,5\,7 \end{array}$$

157 bushes

Making situations meaningful:

The same diagrams for all kinds of numbers

From drawings for small numbers to

diagrams containing numbers to solve for numbers of any size

Situation equations to solution equations

Major steps in making computation meaningful:

Relate drawings to numbers and

do not use drawings to find answers

Later students can go from numbers to drawings

sometimes to retain or recall meanings

Making computation meaningful:

First, there is no one "standard algorithm."

There are only ways to write computations that people erroneously take to be the standard algorithm.

The 2021 proposed California framework is confused about standard algorithms.

There is no one "standard algorithm." There are variations in ways to record efficient, accurate, and generalizable methods that form the collection of standard algorithms.

There are better methods; my research is about these. These are in classroom videos, papers, and Teaching Progressions on my website. These are the mathematically desirable and accessible methods that are standard algorithms.

Most taken to be "standard algorithms" are difficult or misleading.

The CCSS say in the critical area for the first year of a given computation: "Students develop, discuss, and use efficient, accurate, and generalizable methods." They do not say to wait until Grade 4 to do "standard algorithms."

More details are in the paper below (it is on my website under publications).

Fuson, K. C. & Beckmann, S. (Fall/Winter, 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14 (2)*, 14-30.

What Is the Standard Algorithm?

The NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach

with minor variations in how the algorithm is written:

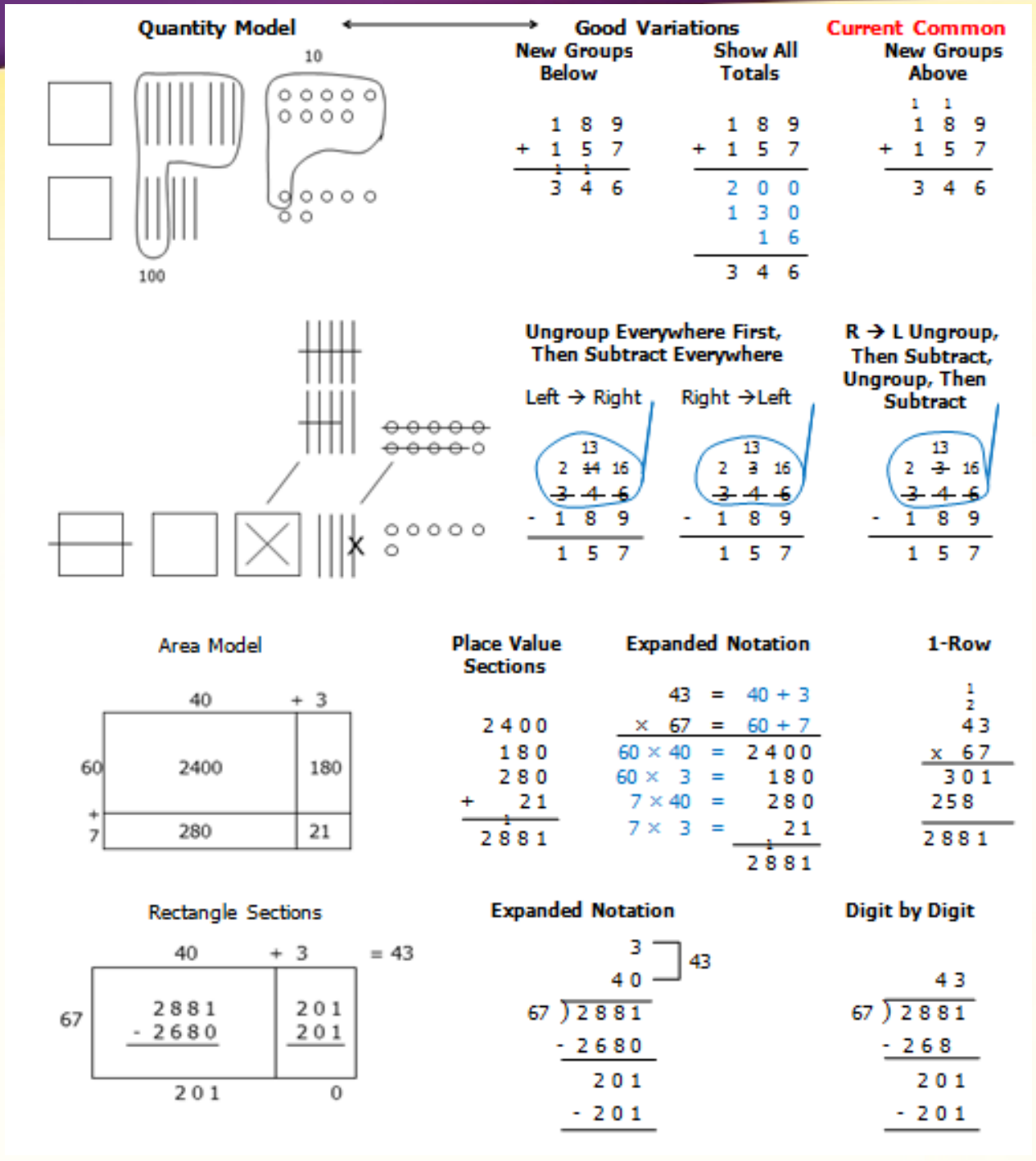
- **decompose numbers into base-ten units and then carry out single-digit computations with those units using the place values to direct the place value of the resulting number; and**
- **use the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.**

To implement a standard algorithm one uses a systematic written method for recording the steps of the algorithm.

There are variations in these written methods

- **within a country**
- **across countries**
- **at different times.**

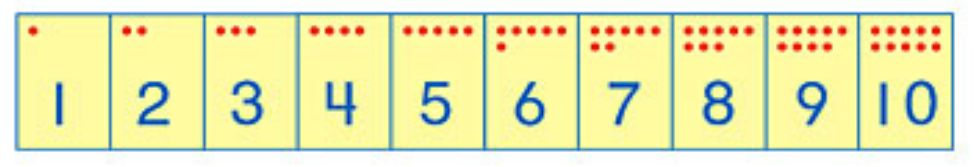
Drawings and Written Variations of Standard Algorithms



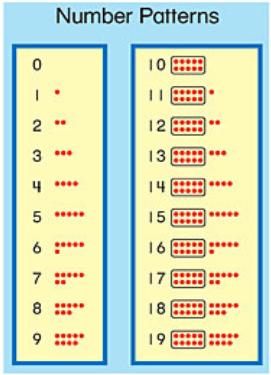
K Number Patterns in Order

Visual Supports for Patterns in Numbers and Quantities in Order

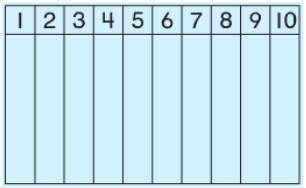
Number Parade



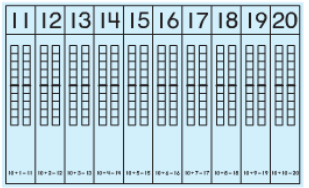
Number Patterns to 19



1-20 board

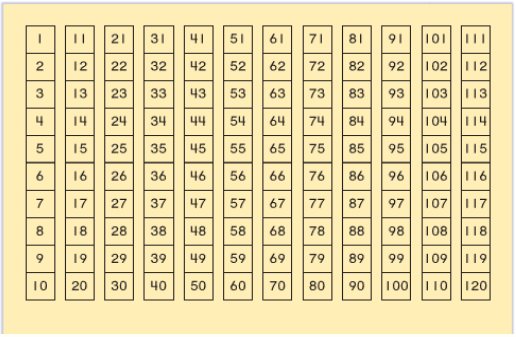


1–20 Board (front)



1–20 Board (back)

120 poster

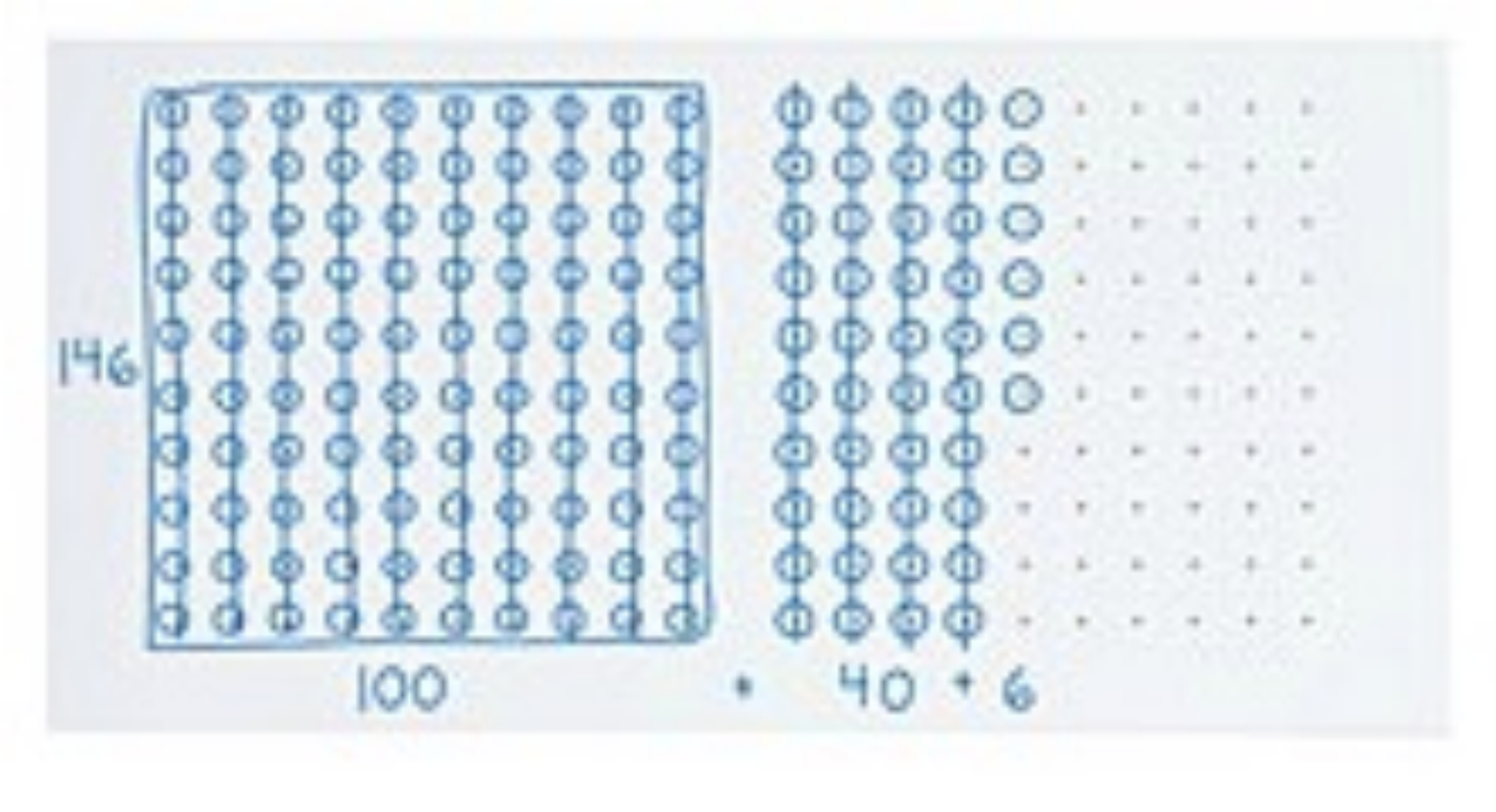


The Vertical 120 Poster

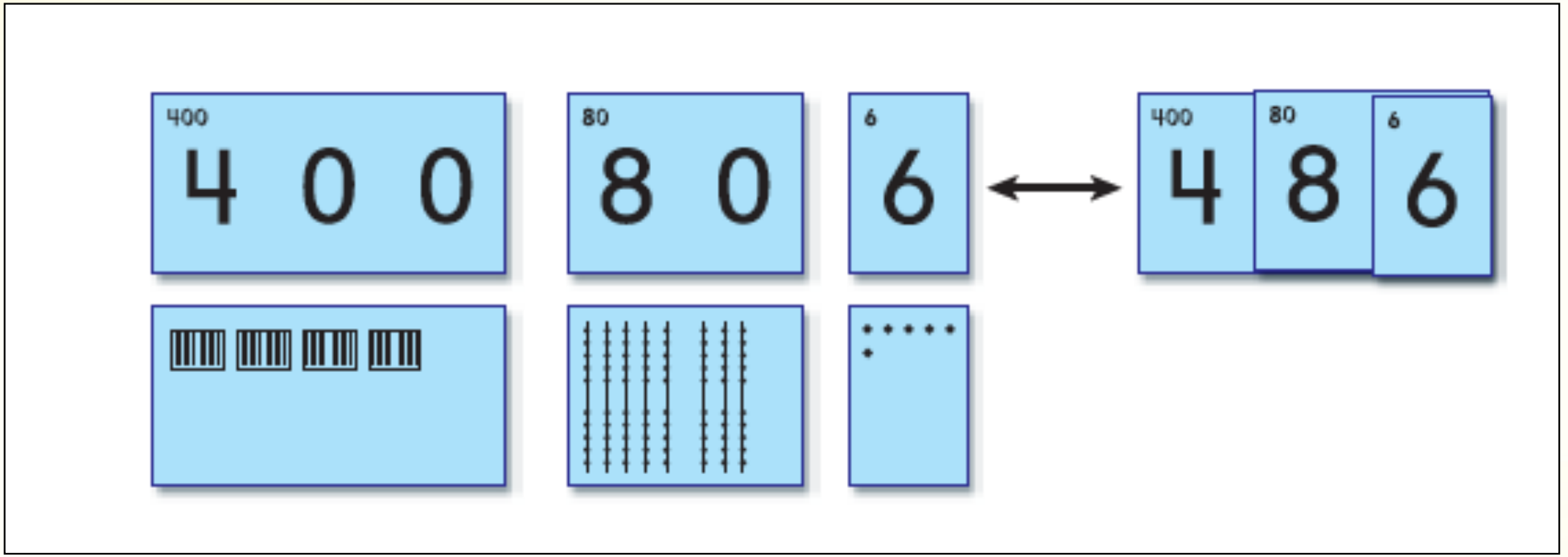
1	11	21	31	41	51	61	71	81	91	101	111
2	12	22	32	42	52	62	72	82	92	102	112
3	13	23	33	43	53	63	73	83	93	103	113
4	14	24	34	44	54	64	74	84	94	104	114
5	15	25	35	45	55	65	75	85	95	105	115
6	16	26	36	46	56	66	76	86	96	106	116
7	17	27	37	47	57	67	77	87	97	107	117
8	18	28	38	48	58	68	78	88	98	108	118
9	19	29	39	49	59	69	79	89	99	109	119
10	20	30	40	50	60	70	80	90	100	110	120

G2 Place Value Drawings 2.NBT.1 and 2.NBT.3

Hundreds **Tens** **Ones**



G2 Secret Code Cards for 486 2.NBT.1 and 3



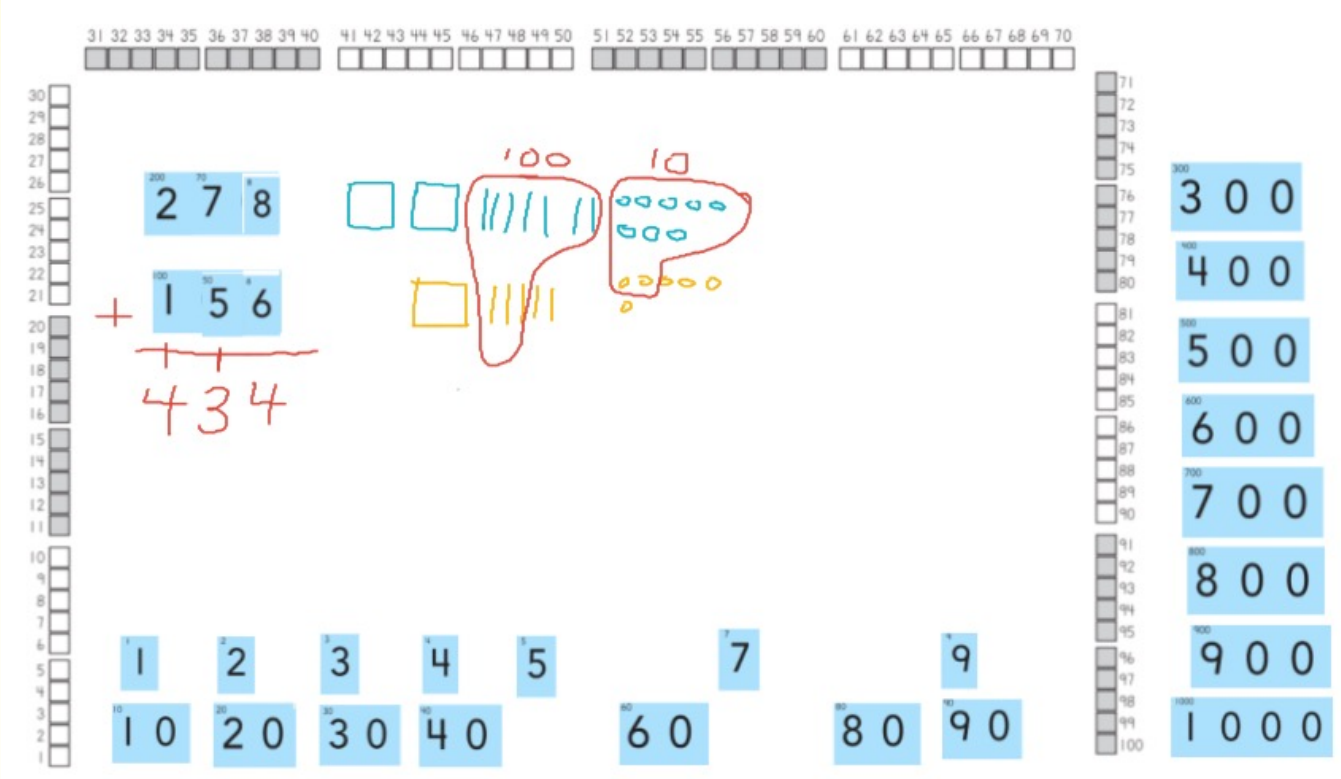
Grade 2 Standard Algorithms

A mathematically-desirable and accessible method

New Groups Above

```
  1 1
  2 7 8
+ 1 5 6
  4 3 4
```

New Groups Below



Now we will watch videos from public school classrooms with children from backgrounds of poverty and many children who are not native English speakers.

The first 5 videos are on my website karenfusonmath.com under Classroom Videos and

A. Classroom Components and Part 3 Math Talk at the beginning.

Math Talk Introduction

Grade 2 bar graph

Grade 1 addition with regrouping invented method and

New Groups Below method

Grade 2 subtraction with ungrouping

This video is on my website under Classroom Videos and A. Classroom Components and Part 4 Learning Math Talk starting at 4:34

Grade 5 fraction multiplication

Many schools around the country were successfully using Balanced Learning Path Teaching with a range of students.

And then covid struck and required remote learning.

How to use Balanced Learning Path Teaching remotely?

especially

How can students use manipulatives and make drawings and teachers can see the work?

This stimulated a now almost year-long collaboration with

Shannon Kiebler: www.empowerlearngrow.com

Robyn Decker: ultramathpd@gmail.com

We built google slide decks with manipulatives and visual representations that students can move to show their thinking.

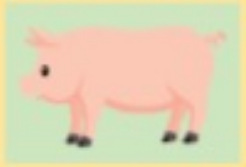
We used jamboards (a google app) to allow students to write and also move manipulatives around to show thinking.

Each student gets a copy of a particular slide so that the teacher can see what the student is doing in real time. These can be saved at the end for the teacher to see them.

Next are 5 examples of Digital Learning Environments that are in the Digital Learning Environment document on my website.

These are physical manipulatives usually used in the classroom.

Find different partners for each kind of animal.
Then play Animals Hiding in the Barn.





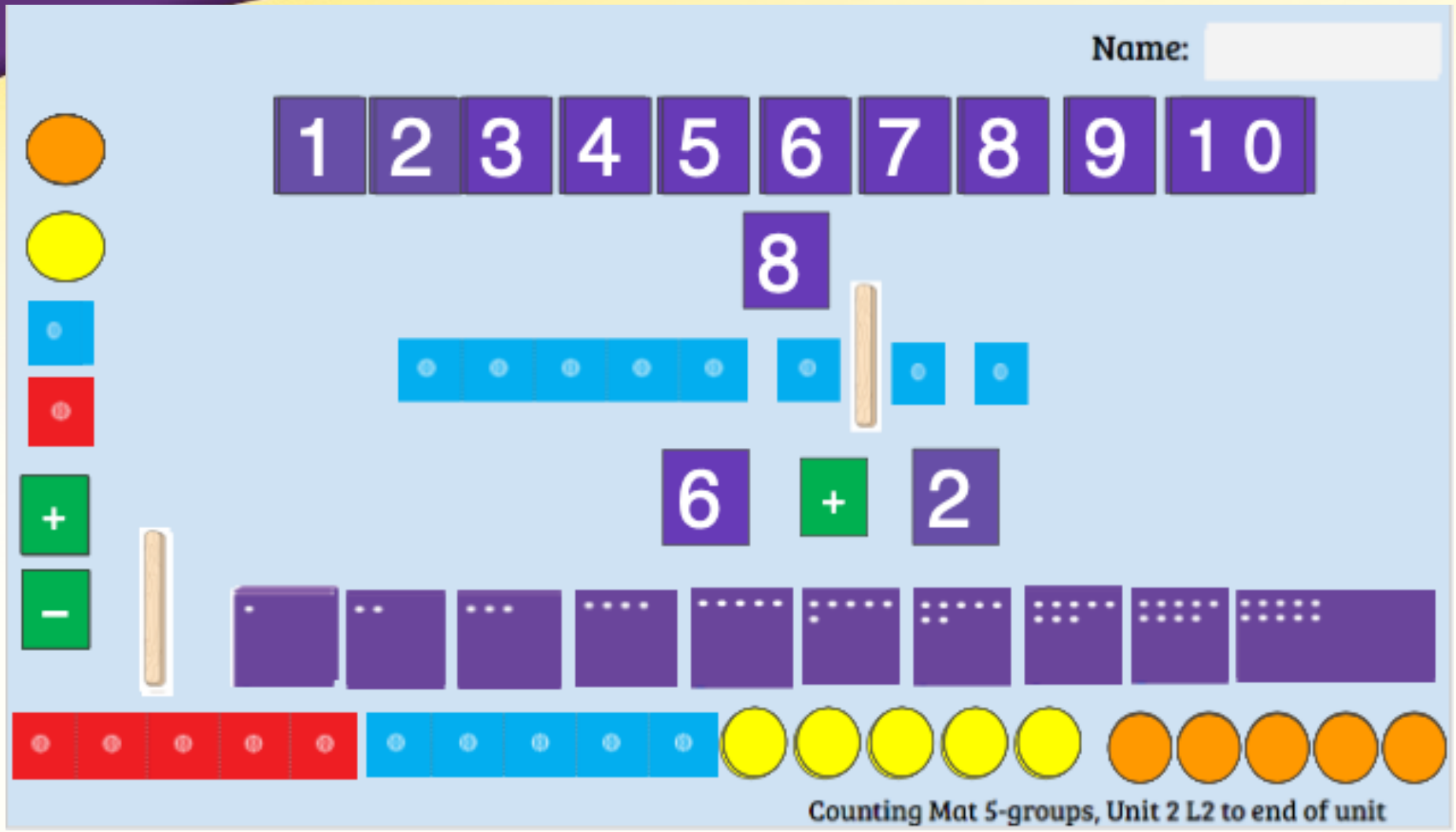








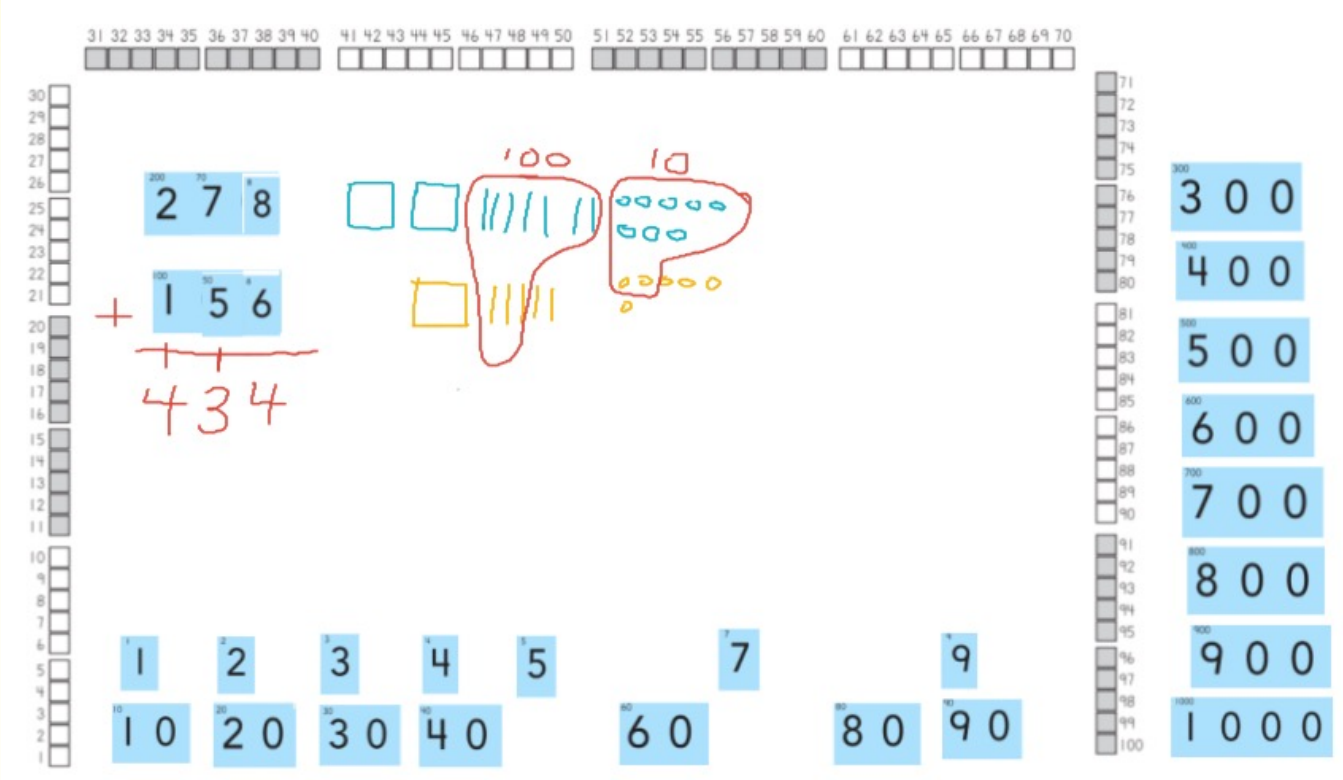


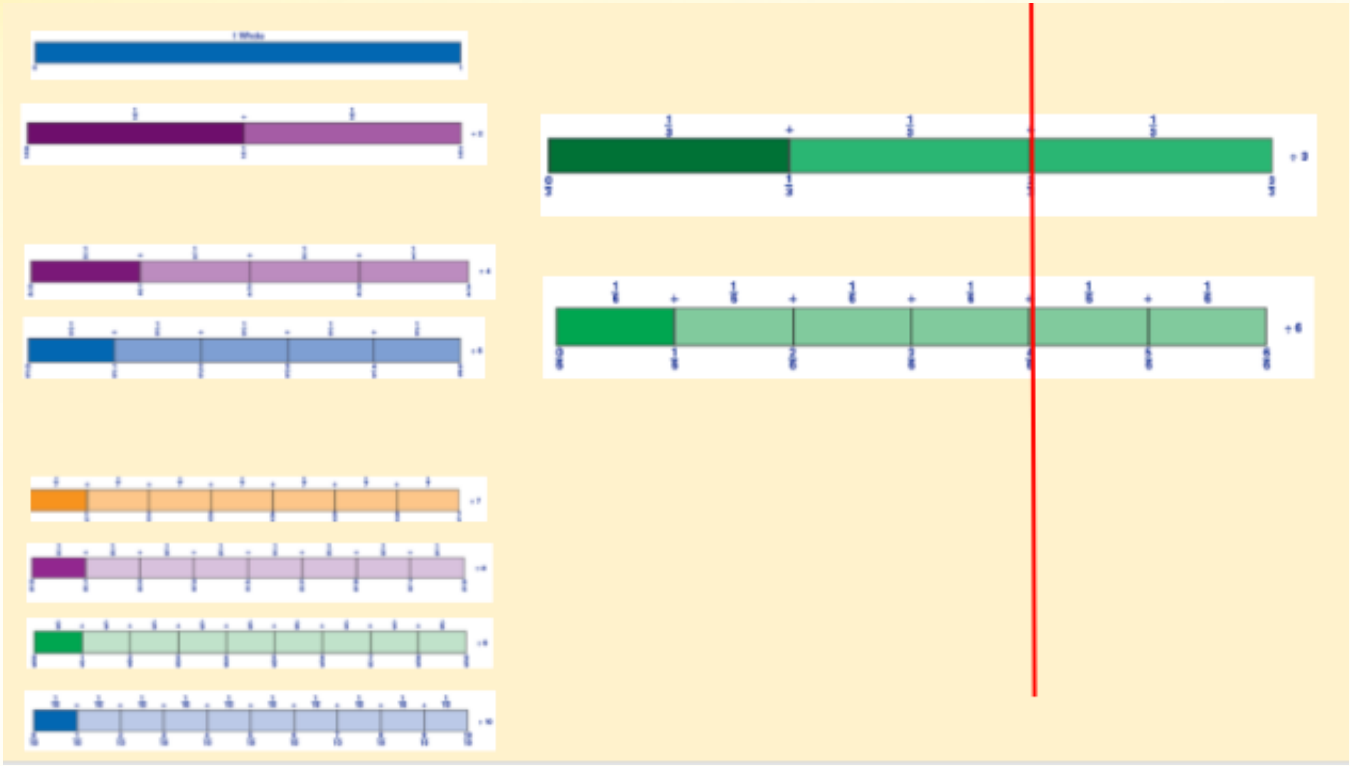


Name:
1 2 3 4 5 6 7 8 9 10
8
6 + 2
Counting Mat 5-groups, Unit 2 L2 to end of unit

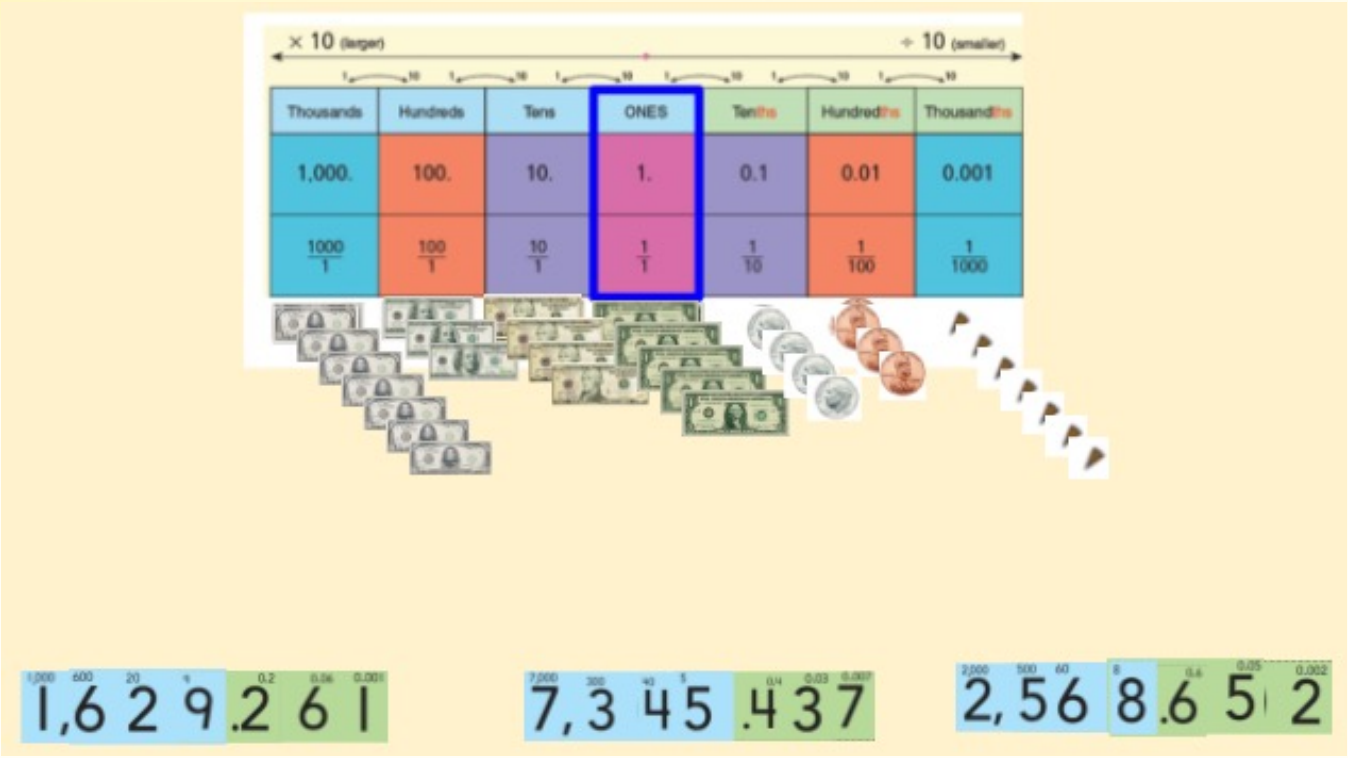
New Groups Above

```
  1  1
  2 7 8
+ 1 5 6
  4 3 4
```

New Groups Below






× 10 (larger)
÷ 10 (smaller)
Thousands
Hundreds
Tens
ONES
Tenths
Hundredths
Thousandths
1,000.
100.
10.
1.
0.1
0.01
0.001
1000/1
100/1
10/1
1/1
1/10
1/100
1/1000
1,629.261
7,345.437
2,568.652

How to see student drawings and written work?

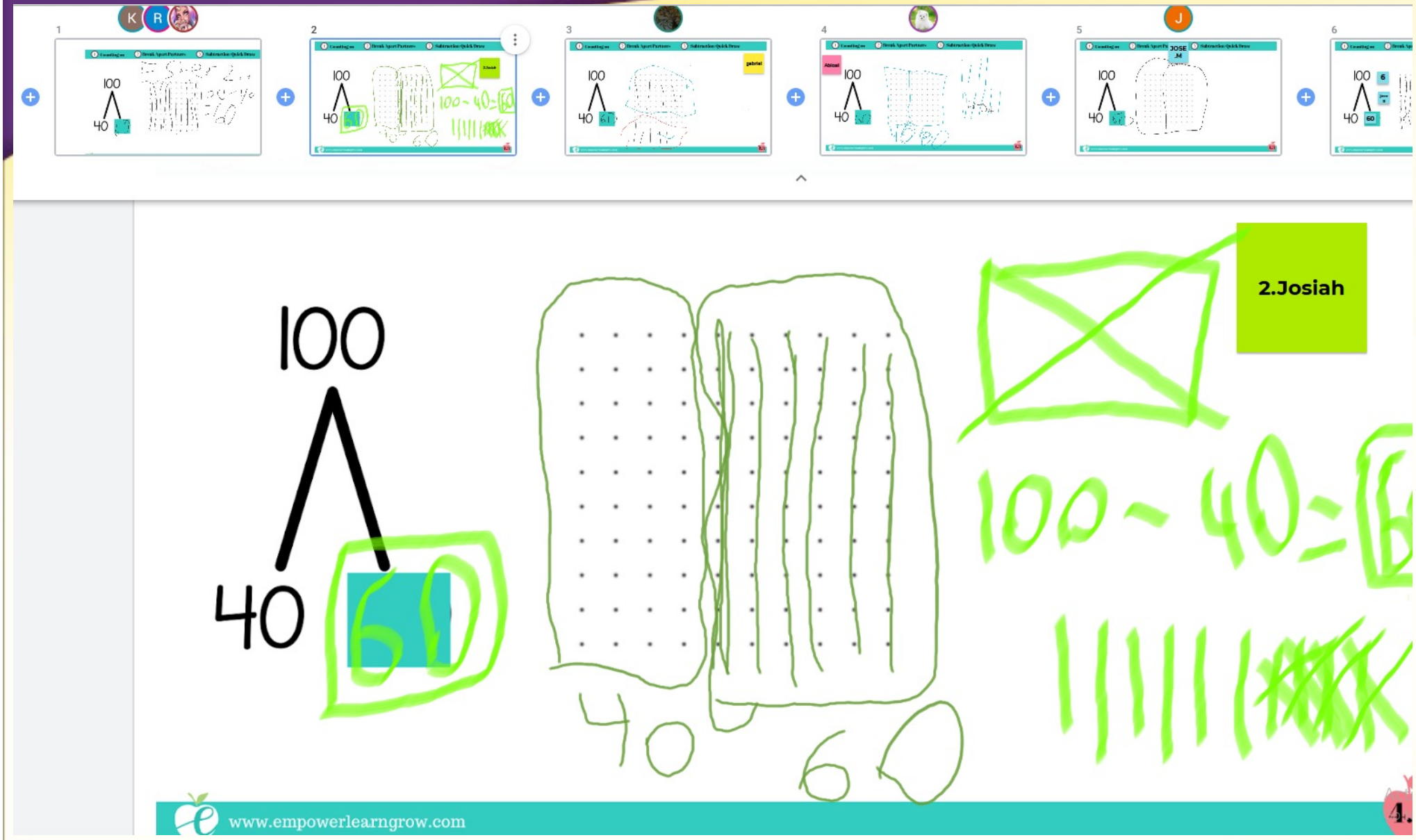
Use our digital learning environments on google slides and jamboards. Get free access from my website karenfusonmath.com

Use Zoom annotating tools and write on teacher screen (the writing tool on google slides is bad).

Use other programs such as whiteboard.fi or Nearpod

Monitor student work using district security programs to view student screens while using itools:

use programs such as Securly, Aristotle, DyKnow, Classkick


1
2
3
4
5
6
100
40
2.Josiah
100 - 40 = 60
40
60
www.empowerlearngrow.com

How to see student drawings and written work?

Send student mathboards and manipulatives and books home and students write on these and show in the camera (it is difficult to show manipulatives). Teachers can screenshot camera view to save student work to evaluate or share at a later time.

Students write on paper or small whiteboard and show the screen (can be wavy because it is hard to hold still).

Use the hack by Professor Michael Peshkin (Northwestern University) mirror and laptop webcam as a document camera

https://www.mccormick.northwestern.edu/news/articles/2020/08/back-to-school-hack-shares-students-handwritten-work-and-teacher-response-in-real-time.html

These Digital Learning Environments are free and can be accessed on my website karenfusonmath.com by clicking in the top right menu choice Remote Teaching Materials.

There also are Quick Practices for grades K to 6 and Daily Routines for grades K to 2 that help to build and practice vital grade level knowledge. You can use these with any math program.

Teachers have used our tools in various ways for assessment:

Breakout rooms, small group sessions: Teacher asks questions, students show thinking on physical or digital tools.

Insert pages of assessment onto a jamboard:

Students write directly onto the assessment.

Thoughts from almost a year of supporting teachers to teach remotely in varied situations:

Teach the technology to students in small steps at the end of a math lesson so that if students get lost in tech space, they do not miss a whole lesson.

Practice, practice, practice each small step such as copying a shape, using chat, moving or reshaping manipulatives, inserting text, using annotation tools, navigating from zoom.

Believe that students can learn these steps. Even kindergarteners can learn to navigate to other windows, share screens, collaborate in breakout rooms, and move manipulatives.

After teachers are functional with the technology, you will be teaching teachers what coaches or colleagues usually are doing: how to elicit and lead math talk with math drawings and relate, build on, and extend student thinking.

Many teachers forget good practices they had used in the classroom and revert to teacher show and tell everything.

But they can learn to recreate those practices remotely.

Best practices in the classroom can be modified to be best practices online, but this often takes support.

Daily Routine and Quick Practice

In classroom

- Student leader with choral response
- Student centers used for extra practice

Virtual

- Teacher leader uses icons for small group choral response.
- Students record themselves teaching their stuffed animals and send to teacher.

Digital Learning Environment

In classroom

- Manipulatives for hands-on practice
- Whiteboards to show drawings
- Small group/pairs to learn from peers and see other student work.

Virtual

- Google slides to move, slide, flip manipulatives
- Jamboards to show drawings
- Jamboard/Google slides students share slides/boards and work together.

Assessments

In classroom

- Unit test paper pencil spaced out in the room
- Small group facilitated by teacher.

Virtual

- Jamboard or alternative
- Online test, but show work on Jamboard
- Small group facilitated by teacher, student uses jamboard or white board and holds up response