Grade Level Teaching Comments That Answer Frequently Asked Questions

These specific teaching comments address questions or issues I have frequently been asked about by teachers. In reviewing my old files of teacher questions, I have been struck again by the dedication and intelligence of so many *Math Expressions* teachers.
Thanks for caring enough to ask questions!!!

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Questions for all grade levels

Why is Quick Practice important?

As I was writing *Math Expressions*, there were crucial components of a topic for which all students need to be fluent. Some of these components needed to be built before they were required in a lesson. For example, understanding math drawings for hundreds, ten, and ones need to be fluent before using these drawings in computations. Other components need to be brought to fluency after initial teaching such as telling time to five minutes. At all grade levels I identified these special crucial components and designed short 5-minute class activities that practiced these activities. These activities **begin a class and are called Quick Practice**.

I wanted all students to have a chance to build their leadership skills and feelings of competence by leading these activities. This role is **called a Student Leader.** Some more complex Quick Practices use several student leaders doing different steps. The Quick Practices change every few days because there are many components to learn in a given year. Most of these components can become fluent enough in a few days. **Please do not omit the Quick Practices.** They will really help your students learn important content. And Quick Practices can be enjoyable for students as they engage with their classmates in these shared activities. Quick Practices also can foster confidence because growth in understanding can happen relatively quickly.

Why are Daily Routines important?

Daily Routines are used in Kindergarten, Grade 1, and Grade 2. **They focus primarily on place value concepts, symbols, and words.** They extend over months with gradual extensions because there is so much to learn about place value. Just learning the English number words takes time. Learning what number words mean and how they relate to the written numerals is complex because there are so many of each and because their mathematical structures sometimes differ. For example, we say the *four* first in *fourteen* but write the *four* second in 14. We do not say *four tens* but say *forty,* where the meaning of the *ty* is not clear. We name the place values in words, but just write numerals using relative positions to name their values: *three hundred eighty two* is written 382. Comparing numerals is complex. Why is 82 more than 28 and 346 is more than 289? **The Daily Routines bring all students to fluency with these crucial concepts.** In

Grades 1 and 2 after place value to 1000 is fluent enough, some other concepts are the focus of Daily Routines. Student Leaders place important roles leading these activities. The Daily Routines provide a familiar community activity in which choral responding is often used.

Math Talk instead of Number Talks

Number Talks are too often being done in ways that reduce what students could learn. Problems in Number Talks are summarized in the left column of the table below. How to build beyond each problem is described in the right column. If you are a *Math Expressions* teacher doing Number Talks, please **use what you have learned about student thinking to elicit and support student thinking and Math Talk every day throughout your *Math Expressions* class.** The lessons in the Teacher Edition will help you do this. Move away from doing Number Talks and instead do Math Talk in your classroom. Do not take time away from *Math Expressions* for Number Talks for all of the reasons shown in the table below.

You can read more about the problems in this article:

Fuson, K. C. & Leinwand, S. (2023). Building equitable Math Talk classrooms. *Mathematics Teacher Learning and Teaching, 116* (3), March, 164-173. 2023
DOI: https://doi.org/10.5951/MTLT.2022.0285

To download and read the article click on Publications in the menu at the top of this website and click on Download for this article in the table.

The table on the next page is from the above article. It summarizes problems with Number Talks and how to overcome them. It is used with permission.

Table 2 Building on the Foundation of Number Talks

Aspects of Number Talks	Expanding Number Talks to Math Talk Classrooms
Number Talks are often used as lesson openers, separate from, and before, the core part of the lesson to which the powerful strategies of Number Talks are not transferred.	Students need good teaching every day throughout their mathematics class. All mathematics classes should include students explaining their thinking. Number Talks can help teachers learn how to elicit and support student thinking. But then it is vital that teachers elicit and support student thinking during the rest of grade-level mathematics lessons.
Many teachers incorporate student justification and discussion into their Number Talks, but fail to adapt regular mathematics teaching to include similar and consistent justification and discussion.	A key aspect of the Common Core State Standards for Mathematics is the third of the Standards for Mathematical Practice (SMP 3): "Construct viable arguments and critique the reasoning of others." Number Talks are perfect vehicles for launching and practicing this focus on justification and argument, but it is vital that teachers use such prompts as "Why is that?", "Can you explain your reasoning?", "Can you convince us?", and "How did your brain picture that?" to expand these Number Talk elements to mainstream instruction.
As often recommended, many teachers limit students to only mental methods when conducting Number Talks and rarely expand the Number Talk to include visual representations that can support solutions and enhance instruction. Giving students access to whiteboards, markers, or pencils can reduce student thinking into just using an algorithm, but broadening Number Talks beyond mental mathematics can be a safe place for students to practice the use, demonstration, and defense of multiple representations.	Many state and district standards ask teachers and students to use visual representations because they support meaning-making and learning. Not allowing students to use visual supports violates those standards and reduces the opportunity to understand those standards. Because this is what is expected in mainstream instruction, these practices ought to extend, when appropriate, to Number Talks as part of Math Talk Classrooms. In addition, restricting students to mental solutions can reduce the difficulty of problems that can be used to a level that might not measure up in complexity to what is done in mainstream instruction. This is a particular concern when teaching English language learners because many such students are helped to solve and to explain by visual supports that make the mathematical language and symbols meaningful. The Math Talk Classroom benefits greatly from presentation, use, and discussion of such visual representations.
Teachers write and draw the methods that students describe orally so that they are visible to the entire class.	Students may learn from this that only the teacher can write or draw to show thinking. Students need to draw their own methods. Other students and the teacher can help to correct and extend such drawings.
Number Talks are often stimulated by naked computation exercises.	Teachers can open discussion of topics at any time within the regular mathematics class by providing students with diagrams, objects, exercises, tasks, or problems; by eliciting student methods; and by leading a discussion that compares and relates the different methods, insights, and approaches. All of a mathematics class needs to open problems to student thinking and discussing, as is typically done with Number Talks.
Parker and Humphreys (2018) explain that they ask teachers to do all of the recording of student thinking for three main reasons: "(1) clear communication, (2) accurate representation, and (3) precise mathematical notation" (p. 87).	Teachers can model how to represent student methods sometimes, but students need to learn how to do accurate recording of their thinking. Students also need to be able to make mistakes and then edit and improve their representations and explanations. Discussing student written solutions with mathematics drawings allows all of this to happen and supports improving written representations to take place by all students. If students are recording their own thinking in drawings related to mathematical notation, often individually or in pairs on whiteboards, the teacher is freed up to think deeply about the student method and has more cognitive time to make the difficult teaching decisions about what to elicit and how to focus discussion. The explaining student, other students, and the teacher can raise questions and extend thinking about a given drawing and method. Thus, everyone learns about these three crucial aspects of mathematical representing.

What is fluency with "math facts"?

It is important for students to be able to find answers to single-digit additions, subtractions, multiplications, and divisions confidently and fairly rapidly. But **this is at the end of a long learning progression** that involves identifying and working with patterns and strategies in ways **that support understanding**. *Math Expressions* fosters understanding and also supports a learning progression of becoming faster and more accurate with such problems—becoming fluent. These balanced goals have always been in the program, but in the 2018 program there are new Quick Practices that facilitate the fluency progression. For example, Grade 1 Unit 2 has Quick Practices called *After or the Same?* to practice problems that add a 0 or a 1. Such problems are conceptual problems, and the title *After or the Same?* is a prompt for those conceptual issues (adding 1 requires that you give the number after and adding 0 requires that you give the same number). But even when students understand this, they need practice moving back and forth between these two kinds of problems.

There are three levels of general strategies that work for all addition and subtraction problems: **counting all, counting on for addition and for subtraction, and make a ten or other derived facts. This learning path is taught in *Math Expressions*** so that all students can move at least from counting all to counting on for addition and for subtraction. Counting on can be fast enough to be used in multidigit computation.

Multiplication/division have many specific numeric patterns that need to be examined and learned by students. For example, the pattern in the threes relates well to the pattern in the sixes but neither relates well to the sevens. Grade 3 has **a systematic learning progression of understanding these patterns followed by detailed and individualized practice.**

The main problem with most practice on math facts is that much time on it is wasted.
Doing the 100 additions or multiplications practices mostly easy facts.
Visualize a table of such facts:

Top left quarter has 1 to 5 plus 1 to 5 which are **kindergarten** fluency goals.

Bottom left quarter and top right quarter has 1 to 5 plus 5 to 10 or vice versa which are **grade 1** fluency goals.

Only the bottom right quarter has the **grade 2** fluency goals 6 to 10 plus 6 to 10.

The same is true for multiplications and much time can be wasted.

Students need practice that <u>is in their learning zone</u> so that they can focus on what <u>they</u> need to learn. *Math Expressions* facilitates this in many ways at each grade level. The conceptual strategy practice cards are a very useful part of this, as is the individualized practice stream in Grade 3 for multiplications and divisions. These both enable students to find those "facts" on which they are incorrect or slow **and focus practice on them.** Years ago I did a study comparing students' use of flash cards to use of a computer program that varied the difficulty of problems. The flash cards gave superior learning, and students said that they could control their own practice better. For forty years I have examined computer practice programs, and they always have bad groupings of problems so that much time is wasted practicing problems that are too easy.

Timed tests: My experience is that many students are fine with this if they are only racing themselves and are practicing within a supportive environment. But it is also useful for students to be able to slow down and not have to race sometimes or always. Having control over racing/not racing on a given day (or for some students, every day) is helpful.

The term *math facts* is very misleading. This term has done and is still doing a lot of damage. These problems are not "facts" is the same sense as many other facts are facts (things you just have to memorize and cannot come to by reasoning). ***Math facts* are the result of reasoning.** They are answers to problems. **Students can find these answers by thinking**. The term "math facts" elicits an image of a multiplication table made from the alphabet where you just have to memorize all of the answers and have no way to find the answers. **This is WRONG.** It short-cuts the thinking that is at the heart of math. **There are many interesting math patterns and predictions that students can make about single-digit additions, subtractions, multiplications, and divisions.** Students should be identifying and discussing such patterns and then engaging in their own individual learning progression toward fluency, toward rapid solving/remembering of such answers.

The progression *concrete, representational, abstract* is not actually a progression

My research, and the research of others, have established that this supposed progression is not necessary in most cases. It is an old progression articulated by Jerome Bruner decades ago. What is **really important is relating visual quantities to math words and to math symbols so that the words and symbols take on meanings.** The important thing is the relationships. Math drawings are much more effective than concrete manipulatives for multi-digit numbers. Kindergarteners use manipulatives (concrete visual supports) to see tens in teen numbers, but even they move to seeing these as drawings on a page. First graders and older children make math drawings of tens and ones (and later hundreds, tens, and ones) to give meanings to these concepts, words, and written math notation, and to the operations they do with these. So a more accurate statement is: **Learning is facilitated by connecting visual representations to words and to symbols**.

Why are math drawings used instead of base-ten blocks for multidigit computation?

My classroom research project began with a focus on using base-ten blocks to show multiunit quantities. This choice built on research with base-ten blocks in which high-achieving second-grade students successfully invented many different multidigit addition and subtraction methods (Fuson & Burghardt, 1993, 1997, 2003; Fuson, Fraivillig, & Burghardt, 1992) and in which all second graders in a large urban school district learned the common algorithms by linking steps with the blocks to steps in the written methods for 4-digit numbers and then extending the methods to larger numbers (Fuson & Briars, 1990). Performance of these second graders greatly exceeded that for older students using the usual concatenated single-digit methods of teaching without multiunit quantity meanings.

However, pragmatic and conceptual issues of various kinds resulted in *Math Expressions* using math drawings of quantities instead of manipulatives. Sources of the change to math drawings were

a. The financial and logistical difficulty of getting even one set of base-ten blocks per classroom and then keeping them in the classroom because they would get lost during the summer or because teachers would take them when they left the building for a different job.
b. Students each needed their own multiunit quantities if they were to invent methods themselves. This then has logistical difficulties in getting materials out to all students in a classroom.
c. Behavioral difficulties arose sometimes when students were working with manipulatives in small groups or even individually.
d. Student and teacher difficulty in understanding student explanations of their base-ten methods because explainers could not easily show their base-ten block steps to their listeners.
e. Teachers could not easily get around to all groups or all students to see their methods before they ended in an answer. But math drawings left a record of quantitative thinking that could be used to understand and correct student errors and do on-going assessment about student conceptions.
f. Students working alone or in groups frequently did not stop to record each step with the blocks on their written problem. They solved the whole problem with blocks and then just wrote the answer on the written problem. Thus, they did not write any grouping or ungrouping step, **so they were not developing general written methods that could be used aside from the blocks.** Developing and understanding general written methods is the purpose of using manipulatives, but this was not happening.

These difficulties were experienced in the Children's Math Worlds project, and I and other project team members were hearing similar reports from around the country for other projects and programs.

Since then, tens of thousands of second graders have begun successfully with math drawings and without base-ten blocks. Math drawings are easier and better in many ways. The Overview of Teaching gives more advantages of math drawings over manipulatives.

What are recommendations regarding how much homework to give, grade, and record?

Homework is an important part of *Math Expressions*. **It helps students reflect on the work they did that day and get some more experience with those concepts.** The one page of homework is usually on the concepts done in class that day. The Remembering page on the back of the Homework page is also important. The Remembering page is the distributed practice over the year that helps students remember concepts they had earlier and relate various concepts as they go along. Some lessons are longer than others, so if you do not finish a given lesson, you can give the Remembering page as the homework assignment. If you do finish the lesson, you can give both the Homework page and the Remembering page as the homework assignment.

Because homework can take longer for some students than for others, **your district or school might want to set a time limit on how long a student must work on the homework**. Students who consistently need more time might need special support of some kind to catch up and work a bit more fluently.

Many parents want to be able to help with homework. Because *Math Expressions* has some approaches that may differ from those that parents know, **the program has parent letters to**

explain certain things. It also can be **helpful for parents to see a Teacher Edition** to understand things that happened in class. Some districts make these available at the school or on a secure district website. If these are posted on a secure district website, the district needs to request this through HMH Permissions. The Research and Math Background sections in the Teacher Edition for each unit includes lesson overviews and are helpful to parents.

Kindergarten and Grade 1 homework is often similar to the activity page done in class. Teachers sometime staple the completed page done in class to with the homework page to help parents see what to do**. Children also enjoy explaining things to their parents, which is an excellent learning method for both the child and the parent at any grade level.**

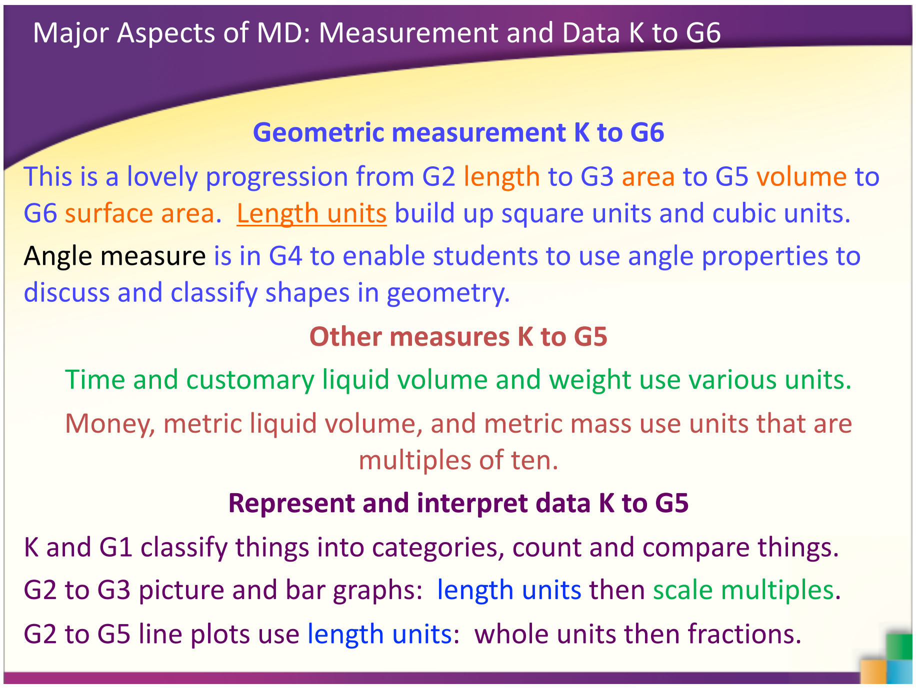
I do not think that the teacher needs to grade and record all of the homework and remembering. Many teachers just give credit for doing the homework and spot check particular problems to see how students are doing. Some teachers have students put out their homework on their desks during the Quick Practice at the beginning of math period. This makes it clear who did the homework and reinforces doing the homework. Teachers may walk around and look at certain problems to see how much homework review they need to do at the beginning of class. Some teachers give a grade for effort—for trying all of the homework, and then grade only certain key homework pages, doing so without warning so that students do not know which day will be graded. Some teachers feel that students need feedback about their homework, at least sometimes. So their students go around quickly reading their answer to a problem (the first student gives the answer to problem 1, the second to problem 2, etc.), and other students raise their hand if they got a different answer. The answers in the Teacher Edition can be used to resolve differences, or a quick discussion can often identify and solve the issue. This process gives feedback to the teacher about where students are. But "going over" homework in detail can bog down and take a lot of class time that is better spent on starting the new lesson. Sometimes the whole class might discuss a controversial problem or those with wrong answers might take home the correct answer and work to see if they understand their error.

The homework process should support students to understand and not just emphasize correct answers without understanding. Also the goal should be understanding and moving forward by all, not a competitive process in which some students feel like winners and others feel like losers. It is important for teachers to emphasize that effort (trying hard) does lead to learning and that this learning leads to more learning. And that help is always available in the classroom when anyone needs it.

What does *Math Expressions* do for geometry (G) and for measurement and data (MD)?

Math Expressions addresses all of the Common Core State Standards on measurement, data, and geometry. The Teaching Progressions on my website **karenfusonmath.com** overview the learning path in the Common Core State Standards in all math domains including measurement, data, and geometry. **The visual supports in these Teaching Progressions are all from *Math* Expressions**, so by going through these, you can see what is done within my *Math Expressions* program for each math domain. Click on Teaching Progressions at the top of the first page, and then look at any of the eight Teaching Progressions on Measurement or the five Teaching Progressions on Geometry. Click on the pale white arrow at the right end of each of these rows of progressions to get to later progressions. Slides at or near the beginning show the grade level and content of each progression.

Measurement and data standards have parts that relate to each other and to geometry. The first slide summarizes these relationships, the second slide shows the units in various parts of measurement and data, and the third slide shows major aspects of the geometry standards by grade level.



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Units in the Major Aspects of MD

	K	1	2	3	4	5
	MD Measurement and Data: K to 5					
	Geometric Measurement: K to 6 Use length to make area and volume units					
Length	(Describe attributes)	(Length)	Length	Area	Angles	Volume [G6 geometry: surface area and area of triangles, special quadrilaterals, and polygons
	Other Measures: K to 5					
Various	(Describe attributes)	(Time)	Time Money	Time Liq volume Mass	Larger to smaller units x	Convert units both ways x ÷
Base ten				Metric liq vol, mass are multiples of ten		
	Represent and interpret data: K to 5					
			Line plots	½ ¼	½ ¼ 1/8	Use fraction operations
Things	Classify into categories, count	Up to 3 categories compare	Picture & bar graphs all problems	Picture & bar graphs scale multiple unit 1- and 2-step compare		

Major aspects of the geometry standards

Geometry: In K to G6 the units are shapes.

K	1	2	3	4	5
K 1 2: Analyze, name, describe attributes and compose/decompose shapes (continues and used in higher grades)			Classify sub-categories	Classify using properties	Classify in a hierarchy

Partition a shape into equal shares/parts

Coordinate plane [G6 draw polygons on coordinate plane]

Students need considerable experience with square units and the related rectangles, right triangles, and isosceles triangles made from these units. These are units and visual subunits for MD Geometric Measurement.

Why does *Math Expressions* have some special vocabulary?

A primary goal of *Math Expressions* classrooms is sense-making by all children. I found in ten years of classroom research that there were a few words that were especially difficult for students. **Students sometimes introduced words that made more sense than the traditional math vocabulary.** I introduced some of these meaningful words in classrooms to find out if they were understandable by other children. The words that made sense and were easy to use for many children were included in the of *Math Expressions* program. These are marked in the student glossaries with an asterisk with this note at the bottom of a page: A classroom research-based term developed for *Math Expressions*. For example, the word ***partners*** is used for addends. *Partners* is a particularly powerful word because it helps children look for and pick out the two numbers in an addition or subtraction equation that **go together to make the total**. *Math Mountains* is a term that helps children remember the form of the Take Apart-Put Together drawing and that the total goes on top.

Of course students need to learn correct standard vocabulary. And we find that students have no difficulty transitioning to the formal math words at their own pace. The approach to this vocabulary is that the **teacher uses the informal meaningful word and the formal word,** but students can use only the informal word initially. Students begin to use the formal word when they want to, but certainly by second grade they can use the formal words. There is some special meaningful vocabulary at the higher grades also, and the same principles apply. But older students often move more rapidly to the formal terms than do younger students. *Math Expressions* **has high levels of expectations for all students and provides high levels of support so all students can achieve.** The informal words are one of the kinds of support.

Many 2D and 3D geometry terms have more than one definition

There is no consensus about the terms *length, width, depth,* and *height* for the sides of 2D shapes or for the edges of 3D shapes. These terms are a confusing contradictory mixture of these different sources of meaning:

common (but sometimes contradictory) usage,
defined usage in some contexts, and
math definitions and formulas.

Some meanings derive from a given point of view for the viewer, while others are independent of the viewer but are relative to the sizes of the measures involved.

On the web, for measuring boxes (or rectangular prisms), different drawings and websites vary in the meanings for terms:

From front to back: usually *depth* but sometimes *length* (even if this side is shorter than the left to right side)
From bottom to top: usually *height* but sometimes *depth*
From left to right: usually *length* but sometimes *width* or *depth*

Some of these websites use length in the relative meaning (length is always the longer side), but others have fixed terms for these dimensions.

Math Expressions* follows the CCSS usage of the terms *length, width, height, and base.
Length and *width* are used informally in Grade 2 for the sides of rectangles because these are such common terms and they will be used in later grades in the formula for volume. *Height* is used in grade 2 with 3D rectangular prisms as both the CCSS and the common meaning as a vertical measure. In grade 6 when areas of triangles, special quadrilaterals, and polygons are discussed, the 2D terms *base* and *height* are used. Because these figures can be seen from different viewpoints (by moving the figure or the viewer), any side can be the *base*. The *height* is the perpendicular distance from a *base* to a vertex not on the base. For 3D figures in both grade 5 and grade 6 *Math Expressions* the CCSS formula V = b x h (or V = bh) is written V = B x h (or V = Bh) to distinguish the base B as a surface area and not a base length. This distinction is discussed, and students can write this either way.

Because the terms for the sides of 2D or the edges of 3D figures have different meanings, and these meanings can change according to one's viewpoint of the figure, it is important to have students discuss different meanings and how they can change as the viewer or the figure moves. Here is an email from a Grade 2 *Math Expressions* teacher who did just that:
"We just had the greatest discourse in our 2nd grade math class around this topic.

One group talked about a snake
Length from tail to head
Width from side to side
Height from top to bottom

One group talked about a standing person
Height feet to top of head
Width side to side
Depth front to back

One group talked about the person lying down
Length feet to head
Width side to side
Height from the floor to the top of the body

One group talked about standing by a pool and the *depth* of the water. If there was a building right next to them, they would talk about the *height* of the building not the *depth*.

All of these child-led examples widened our understanding of what makes something 3D**. It is so important that these discussions take place not only for deeper understanding for the children, but also for the educators.** I am wiser from the examples provided by the children in my class."

A teacher could end such a discussion by saying that **these different meanings show how important definitions are in math so that everyone has the same meanings.** But for some math topics there is not yet agreement on the definitions, although there is agreement for most math topics. So a person or a class might need to agree on a definition they can share so they can discuss clearly.

Trapezoid has two definitions

The word *trapezoid* is an example of a math term that has two different meanings. There is no agreement in the United States about the definition of *trapezoid*.

A trapezoid can be defined as a quadrilateral with exactly one pair of parallel lines. OR

A trapezoid can be defined as a quadrilateral with at least one pair of parallel lines (this means that a trapezoid can have 1 pair or 2 pairs of parallel lines).

The CCSS did not define trapezoid. ***Math Expressions* uses the first definition with *exactly*.**

PARCC decided at one point to use the second definition with *at least*.

The major difference in these two definitions is in how special quadrilaterals relate to each other.

1. The *exactly one pair* definition makes trapezoids a special shape within quadrilaterals such that parallelograms, rectangles, rhombuses, and squares are NOT trapezoids.
2. The *at least one pair* definition makes parallelograms, rectangles, rhombuses, and squares BE trapezoids.

There are advantages and disadvantages for each definition. It is an excellent sense-making activity for students to discuss these two definitions, consequences of each definition, and advantages and disadvantages they see for each. Materials to support such a discussion are in the file *Trapezoid lesson.docx*. This lesson can be done with grade 3, 4, 5, and 6 students if desired.

The *exactly one pair* definition I used in *Math* Expressions is especially useful in elementary school where students are developing their spatial understanding and learning various spatial attributes of shapes and vocabulary to discuss these attributes. Trapezoids are interesting shapes that allow students to draw and examine variations in attributes while keeping one simple attribute the same **(the exactly one pair of parallel sides that are always present in a trapezoid**). Decompositions of trapezoids are related to decompositions of parallelograms but result in more spatial variations because the angles and sides of a trapezoid can vary more than the equal pairs of sides and of angles in parallelograms. One can also find the area of a trapezoid using various approaches, many of which work for parallelograms, but they often look more surprising for a trapezoid and so may develop spatial understanding more broadly.

Kindergarteners ask and debate: Is ten a teen number?

A kindergarten teacher emailed me asking if 10 was considered a "teen" number. She said her class had an amazing discussion about teen numbers, and they encountered a problem as they can't decide if 10 is a teen number or not. Some of her students said "yes" because it has exactly one ten, while others said "no" as ten is not hiding (meaning that in the secret code cards, the 0 is not covered by a ones card 1 to 9). She said both sides are very adamant—hence her reason for contacting me.

My response was: What a wonderful report from the classroom!!! This is exactly what we want our children to be doing—thinking and asking good math questions. In this case both sides are proposing sensible definitions of a teen number.

A. A teen number has exactly one ten. So 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19 are teen numbers.

B. A teen number has ten plus 1, 2, 3, 4, 5, 6, 7, 8, or 9 ones that cover the 0 in 10 to show 11, 12, 13, 14, 15, 16, 17, 18, or 19 as a teen number.

Both sides have good clear definitions and good reasons for the definitions.

Many words and symbols in math have only one definition so that we are sure about their meanings. But some words, like *teen numbers*, are not really official math words. *Teen number* is a shortcut way to say either A or B above even though *ten*, *eleven*, and *twelve* do not have the word *teen* in them. And sometimes people just do not agree about what a math word means, and they use different definitions. So in the United States, a billion means a thousand million. In England a billion used to mean and still sometimes does mean a million million.

So what can you do when you can't agree about which is the best definition? I suggested to the first grade teacher that first you all agree that both sides are good thinkers and have a good definition. Then in the next month, pay attention to when and how you all are using the words *teen number* and which definition is working the best. And have another discussion in a month to see how you are all thinking about this concept. Meanwhile, congratulations on being such good math thinkers. And I know that you will be respectful of the different ideas in your class.

Becoming confident explainers

It takes time for all children to become confident explainers. Sometimes having a friend to back them up can be helpful, especially if they are up at the board. The less-confident child explains first (which might only be pointing to a drawing or fingers and saying a word or two) and then the friend expands on the explanation, checking with the first child to see if that is what they meant. Sometimes just having a friend standing or sitting there is enough; they do not even need to say anything.

Have students write unit test items or state test items

Students can write test items for themselves and their fellow students. This is a good way for them to process ideas more deeply. Writing items for the unit test helps students reflect on the important content of the unit. Discussing formats of the test items also can help students become more aware of format issues. For all test items the writer must also provide a key with the correct reason and an explanation.

Students can also write test items for the state test. Doing this can help shift feelings about the test from threatening or unknown to "just problems" that I can work to learn how to solve. They can continue to do this next year, and everyone will be in better shape for the test next year. This also makes the test-taking process involve a learning path for everyone improving over time.

Questions for Grades PK, K, 1, and 2

Why number lines are NOT appropriate for PK, K, or G1 children

There is a great deal of confusion about what the term *number line* means. Both National Research Council reports *Adding It Up* and *Mathematics in Early Childhood: Paths Toward*

Excellence and Equity (Kilpatrick, Swafford, & Findell, 2001; Cross, Woods, & Schweingruber, 2009) recommend that number lines not be used until Grade 2 because they are conceptually too difficult for younger children. In early childhood materials, the term ***number line*** or ***mental number line*** often really means a ***number path***, such as in the common early childhood games where numbers are put on squares or circles and children move along such a numbered path. **Such number paths are count models in which things are counted**. Each square or circle is a thing that can be counted, so these number paths are appropriate for children from age 2 through grade 1. A number path is shown in Figure 1 along with the count and cardinal meanings that children must understand and relate when using these models. In contrast, **a number line is a length model** like a ruler or a bar graph in which **numbers are represented by the length from zero along a line segmented into equal lengths**. To use a number line, children need to count the length units, not the numbers (see Figure 1). Young children have difficulties with such a number line representation because they have difficulty seeing the length units. They are used to seeing things, so they focus on the numbers or the segmenting marks instead of on the lengths. Thus they may count the starting point 0 and then be off by one. Or they may focus on the spaces and are confused by the location of the numbers at the end of the spaces.

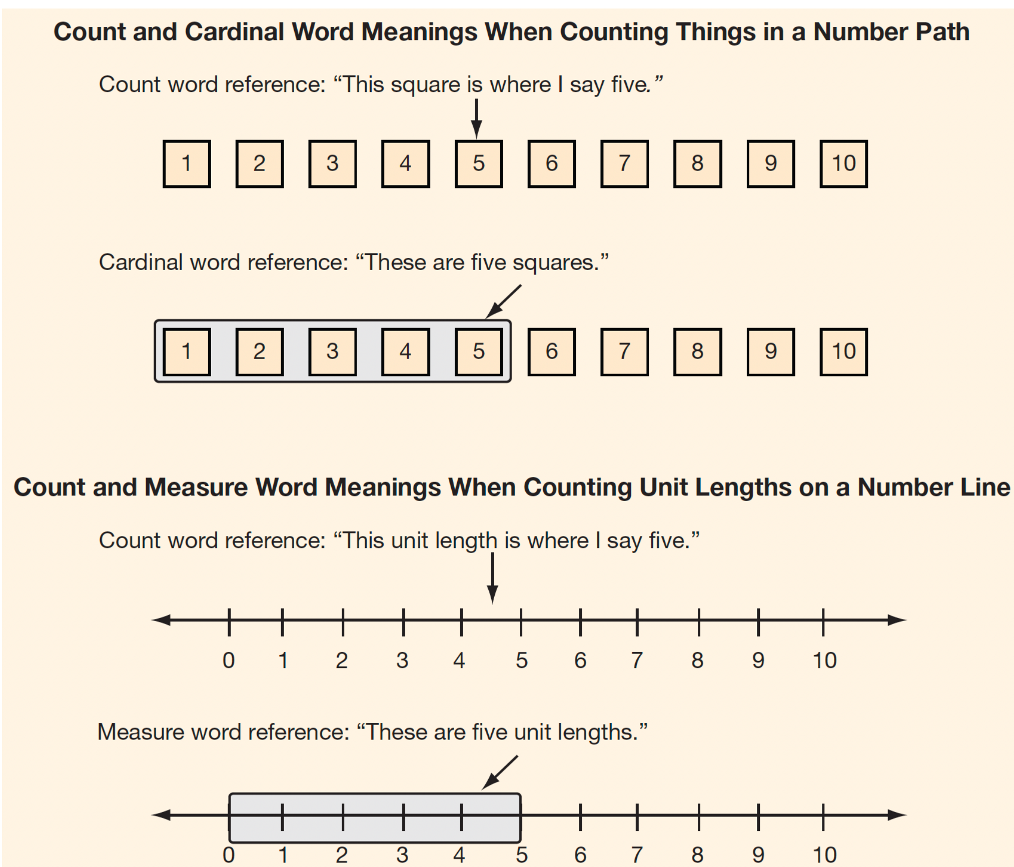

Figure 1 Seeing count units on a counting path and length units on a number line

It is important to show in classrooms for young children a **large number list** (a list of numbers in order) or a **number path** (a list of numbers inside identical objects such as the squares in

Figure 1). Or the numbers can be beside the objects as in the grade 1 *Math Expressions* Math Board. Groupings of five and ten are helpful to learn the 10-structure of English words. Children can play games and count along such number paths or number lists. Number lines should be avoided because they are confusing. Learning to count to ten and understand the written number symbols are facilitated by games such as the **number path** game in Ramani and Siegler (2008). In this game children say the numbers on which they are landing (e.g., *I got 2, so I count the next two squares: six, seven.*) instead of the usual counting of the number they rolled or spun (e.g., not *I got 2, so I count two squares: One, two.*).

Children will come to use the number word list (the number word sequence) as a mental tool for solving addition and subtraction problems. They are able in grade 1 and 2 to use increasingly abbreviated and abstract solution methods such as counting on and the make-a-ten methods. At this point the number words themselves have become unitized mental objects to be added, subtracted, and ordered. **The original separate sequence, counting, and cardinal meanings have become related and finally integrated over several years into a truly numerical mental number word sequence.** Each number can be seen as embedded within each successive number and as seriated: related to the numbers before and after it by a linear ordering created by the order relation *less than* applied to each pair of numbers. This is what Piaget (1941/1965) called *truly operational cardinal number*: any number within the sequence displays both class inclusion (the embeddedness) and seriation. But this fully Piagetian integrated sequence will not be finished for most children until grade 1 or grade 2 when they can do at least some of the Step 3 derived-fact solution methods. These methods depend on the whole teaching-learning path and are presented in more detail in the grade 1 and grade 2 Focal Point books by NCTM and in *Math Expressions* lessons.

Many researchers have noted how **the number word list (*one, two, three, four*, etc.) turns into a mental representational tool for adding and subtracting.** A few researchers have called this a *mental number line.* However, for young children this is a misnomer, because children in kindergarten and grade 1 are using **a mental number word list** (sequence) as a count model: each number word is taken as a unit to be counted, matched, added, or subtracted.

The **use of number lines such as in a ruler or a bar graph scale** is an important part of measurement. In grade 2, these all form an appropriate important part of the learning goals. But they are too complex at earlier ages. The Common Core State Standards follow the recommendations of the National Research Council reports and do not use number lines until grade 2 when length measurement is also introduced.

Why are the tens in the 120 chart vertical and not horizontal?

Look at a column of vertical tens in the 120 chart shown in Figure 2, for example 61 to 70. **The ones are above each other, and the tens are above each other.** So children can see that all of the sixties have 6 tens and that the ones increase by one. All of the thirties have 3 tens, all of the forties have 4 tens, etc. **It is easier to see the tens and ones patterns involved in the numbers to 100 if these numbers are above each other, i.e., are aligned vertically.** This may be harder for teachers to see because they are used to the horizontal form, but if they really look, they see how much easier this pattern is for students.

1	11	21	31	41	51	61	71	81	91	101	111
2	12	22	32	42	52	62	72	82	92	102	112
3	13	23	33	43	53	63	73	83	93	103	113
4	14	24	34	44	54	64	74	84	94	104	114
5	15	25	35	45	55	65	75	85	95	105	115
6	16	26	36	46	56	66	76	86	96	106	116
7	17	27	37	47	57	67	77	87	97	107	117
8	18	28	38	48	58	68	78	88	98	108	118
9	19	29	39	49	59	69	79	89	99	109	119
10	20	30	40	50	60	70	80	90	100	110	120

Figure 2 The *Math Expressions* hundreds chart

Anything you can so with the horizontal 120 chart, you can do with the *Math Expressions* vertical chart. Things just go in a different direction. You can go forward and backward by ten by moving to the right or the left one column. And this does not have to be an arbitrary rule because you can go down the rest of the ten (e.g., from 47, go down 3 to get to 50), and then go down 7 in the next column (and you get to 57).

Children draw ten-sticks (quick-tens) by drawing down a column of ten dots. So having the columns of ten in the 120 chart is consistent with this direction. **This consistent vertical direction is used for a ten-group from Kindergarten through Grade 4 in place-value work and in making math drawings for addition and subtraction.**

Why use *total*, *addend*, and *addend* for addition and subtraction problems?

The technical terms for subtraction *subtrahend* and *minuend* are difficult and are not really used. Making students use those terms interferes with another much more important relationship: the inverse relationship between addition and subtraction. Calling the three quantities for addition and for subtraction *total*, *addend*, and *addend* greatly helps students relate these operations and understand how they are alike (they involve the same 3 quantities) and different: **adding is putting together the two addends** and **subtracting is breaking apart the total to get both addends** (8 = 6 + 2) or **is taking one addend from the total to get the other addend** (8 - 6 = 2). I do use *difference* for comparison problems because this word makes sense in those contexts. I have been doing this for 30 years, and never have heard of any problem this has created for students. If *subtrahend* and *minuend* are on your state test, suggest to grade 3 teachers that they review this vocabulary so that students know it. But in grades K, 1, and 2, I would stick to *total, addend,* and *addend* for both addition and subtraction.

The **word *total* is used instead of *sum* because** *sum* has the difficulty of sounding like *some*, so in discussion it can be taken to be an *addend* instead of the *total*. Also, *total* is close to the

Spanish word *total*. By grade 3 students can use the word *sum* as well as the word *total* if it does not confuse the discussion.

Questions for Grade 2 and Grade 3

Why is a number line not used as a tool for multidigit adding and subtracting?

The number line is not used as a tool for adding or subtracting multidigit numbers because it is not an effective tool for doing so and **the *Math Expressions* student math drawings are an effective math tool.** *Math Expressions* does do adding up strategies in Grade 2 Unit 4 Lessons 16 and 17. If you look there, you can see how the drawings of ones and tens which students are already using can be used effectively for adding up. Students often invent such methods themselves, especially for unknown addend situations. So there is no need to introduce a new tool such as a number line for such problems when students already have an effective tool.

The number line is not effective as a tool for multiple reasons. First, number lines are exact and not approximate. The CCSS standard about number lines only asks students to relate addition and subtraction to length. This standard is in Measurement and Data and not in Number and Operations in Base Ten, so this is about understanding and seeing length as a measure and about using a tool for adding or subtracting place-value quantities. A number line from 0 to 100 that fits across a regular page is 6 4/4 inches long and each unit is 1/16 inches long. Students cannot draw such number lines, and they are difficult to use exactly. In *Math Expressions* such number lines are only used to meet the standard of relating addition and subtraction to length so students see that one addend is placed after the other to add and that one addend is placed within the total to subtract to find the other addend within the total.

The Dutch use a related approach called an open number line, where students label a line segment with numbers and with jumps of tens and ones. This is not really a number line because the units are not the same length. The open number line is one method of keeping track when adding on by writing numbers above and below the line segment. But research in many classrooms during the development of *Math Expressions* indicated that using the usual math drawings of ten-sticks and circles in 5-groups was easier and more accurate than using such open number line drawings. Students can do any open number line method with their *Math Expressions* ten-stick and circle drawings more accurately and flexibly, and the methods then can move to entirely numerical methods more easily (some of these are shown in Unit 4 Lesson 16 Word Problems with Unknown Addends).

The open number line is especially helpful in Dutch classrooms because their language reverses all of the words that have tens and ones, saying the ones word first. So they say the Dutch words for *four and sixty* for 64 and 364 is *three hundred four and sixty*. So counting-on methods benefit from having the written numerals in order. Fortunately English does not have this problem so there is no linguistic advantage for the open number line.

Finally, open number lines are problematic because they support adding on methods rather than methods that add like places. For adding on, the first addend is marked on the open number line. The second addend is the amount added on and must be made by jumps from the first addend along the open number line, not by making that number from zero. Puzzled Penguin makes that mistake in Unit 7 Lesson 5. For example, for 57 + 29 the mark (not the length) for 57 is circled and the mark for 29 is circled instead of showing 29 more than 57 as 29 added on. **The number line is a length model in which small line segments are counted.** Number line methods are more difficult for many children than are methods that add or subtract like places. And the number line methods do not generalize easily to larger numbers especially for 3-digit numbers.

Questions for Kindergarten

Meanings of zero

Zero is both a cardinal number that tells how many in a group (none, no things) **and a place-value holder as in 10** which means 1 ten and 0 ones. For the cardinal number meaning, children can have fun generating lists of 0 things such as zero giraffes in this room or zero children named Maria in our class.

Why don't we list zero in the lists of partners of numbers?

Partners are addends that make a certain number. When we write lists of partners, we do not write a partner for zero because we want to learn all of the partners that are not 0. It makes the lists too long if we add the partners with 0 to the lists, for example, for 5 the usual list of
1 + 4
2 + 3
3 + 2
4 + 1
would have two more rows (0 + 5) and (5 + 0) and would distract from the other more distinct partners of 5 that we want children to know. But it should be discussed that 0 is a partner with every number, you can write two partner equations with 0 (e.g., 0 + 5 and 5 + 0), and the partner of 0 is always the number itself (the total).

Why are spaces for writing numerals smaller than we used in the past?

When designing *Math* Expressions I examined international math textbooks from many counties. I noticed that **children in other countries write in much smaller spaces than in this country.** For example, in France, they write numerals in 1 square centimeter in first grade. So then I worked with teachers in Connecticut and in Chicago who identified the kindergarten and grade 1 children having the most difficulty writing numbers. **We found that it was easier for the children to write smaller rather than larger numbers.** Writing in a larger space seemed to require more coordination. The spaces in *Math Expressions* are actually larger than the spaces we used in that classroom research because I knew that teachers were so used to larger spaces that I thought they might not try even smaller spaces.

I also searched for research about the size of writing. I could not find any so I asked the head of the Learning Disability Clinic at Northwestern (an internationally known researcher who has worked with hundreds of children). She said that she knew of no evidence that writing larger was easier and they did not have students write larger in her clinic.

Why are five-frames and ten-frames not used more frequently in the program?

***Math Expressions* uses throughout the program the visual support of ten as two groups of five under each other.** Ten-frames also use two groups of five under each other. But the way that the 5-groups are shown in the *Math Expressions* materials in all grades is having **the circles in the rows closer together** than are the two rows **so that you see the group of 5**. It is important that teachers discuss and that children understand that all of the 5-group arrangements (on the Number Parade, Secret Code Cards, etc.) do have a row of 5 and then some more under that 5. This should be related to the 5 fingers on one hand. When you know that there are 5 circles and you see 1 or 2 or 3 or 4 below, you can see by the number of dots/circles above how many more are needed to make ten (4 or 3 or 2 or 1). Once children see the arrangement in this way, they do not need the support of the empty squares in the ten-frame**. They see the dots/circles above that do not have a dot/circle below.** When children are drawing tens and ones**, it is important that the ones are drawn in these 5-groups** because all children can then see at a glance how many there are and it reduces errors. And it will support children to use the mental make-a-ten method because **they can see how many more to make ten.**

The other issue about the ten-frame is that it is an array (the dots/circles in them are equally-spaced in all directions), so children can see and make any number using doubles instead of 5-groups. Of course we want children to use doubles, but our experience in *Math Expressions* is that these doubles patterns are so easy that children do not need special visual supports for them. **But the 5-groups as spaced in *Math Expressions* are a support for thinking with two groups of five that make ten, the crucial building block for place value.**

So in summary, a ten-frame puts children into conflict about whether to use 5-groups or doubles. They do not need additional experience with doubles. So it is better to focus them just on the **powerful 5-groups to create understanding of numbers between 6 and 10, to add and subtract between 6 and 10, and to gain the prerequisites for make-a-ten strategies.** Children also can easily draw circles in 5-groups for multidigit adding and subtracting, **so this gives a coherence from Kindergarten through Grade 3.** Finally, the **ten-frame also is visually more confusing** than just having dots (or circles) because children have to process or draw the frame and what is inside it and not just the circles.

Why is subtraction introduced so early in kindergarten?

I introduce subtraction early because I found that **if students only experienced addition for a long time, it was more difficult for them to learn subtraction.** They would just add two numbers without paying attention to the situation or to the sign if there was a written expression or equation because they only knew one operation.

To help students differentiate addition and subtraction, it is important for students to act out the subtraction situation so that they see and act out the taking away. Then students also need to see a math drawing (just circles) of the subtraction situation. For 4 – 1, draw 4 circles and then draw a horizontal segment (a minus sign) through the first circle. As you draw this, you say, “We are taking away this circle.” and gesture to pull it away. So this taking away motion is the minus sign. **You are connecting the taking away in the situation to the taking away action to the minus sign.** If it is 4 – 3, you will draw a long minus sign through the first three objects. We draw through (take away) the first objects because that helps children to count on to subtract in first grade.

For the + sign you can ask children for their ideas about how to understand and remember this plus sign. Children often have good ideas including "There are two parts for the plus sign because we are putting together two numbers." and "I can grab a (pretend) number in each hand and put my two arms together like a plus sign. So that means I am adding those two numbers to each other."

How do I help my students switch between addition on the top half of a page and subtraction on the bottom half of the same page from Unit 3 on?

Students should work across so that they do all adding first. It can be helpful for students to fold a blank paper in half and use this as a guide to cover all problems below the row on which a student is working.

Such pages should be introduced by eliciting a story about the first problem so that students are remembering what addition means. You can also elicit that directly without a story: What does this symbol + mean? [Putting two numbers together, finding how many in all, finding the total] And have students look at the numbers and find related problems (all of the +1 problems, etc.) and discuss how to solve these pattern problems (+1 is the next counting number.).

For switching to subtraction, make a big deal about this switch. Say something like: WOW, what is happening now? These have a different symbol. What are we doing now? Repeat the sense-making steps for addition above only using subtraction (e.g., subtracting 1 gives the counting number just before that number). Then the next time students come to such a page, ask first: What do we have to be careful about here? Where is the WOW on this page? [Yes, I see the take-away sign for subtraction on the bottom part, so we all have to be careful to switch to subtraction.] Discuss more as needed.

Kindergarten is not workbook based

Most of the important work in kindergarten in *Math Expressions* happens in activities and Math Talk discussions. Manipulatives are used initially and eventually students make drawings. I developed special manipulatives to meet the CCSS focus on situations in the real world reflected in word problems, for the ten hiding in teen numbers, and for all of the prerequisites for the make-a-ten methods that will be done in Grade 1 for totals in the teens. The CCSS Kindergarten standards are heavily based on the National Research Council Report "Mathematics Learning in Early Childhood." I was on the committee that wrote that report. I know the research literature on which it is based. The *Math Expressions* kindergarten is based on that research. It is ambitious and rigorous. The Teacher Edition for all units describes the activities to do in class. Units 3, 4, and 5 are crucial, especially the activities about the meanings of teen words, the partners of numbers (the addends hiding inside a number), and adding and subtracting with real-world situations and word problems and written equations. Much specialized work is needed in kindergarten to prepare students deeply to have a flying start in grade 1 with respect to more-advanced strategies. Few programs are written by people who know the research deeply enough to reflect all of this vital prerequisite work.

Learning from activities in kindergarten does not mean that students should not see math work on printed pages. The written work on student pages is important as follow-up work to the class activities and discussion especially for students with little literacy experience before school. So there is a Student Activity Book for kindergarten. But this work on pages is culminating and practice work, not beginning work, for concepts. This is what the NRC Early Childhood Report recommended. Many of the activities in *Math Expressions* kindergarten are discussed and explained in the NCTM book for teachers: National Council of Teachers of Mathematics (NCTM) (2010). *Focus in Kindergarten: Teaching with Curriculum Focal Points*. Reston, VA: NCTM. This book was endorsed by NAECY, the National Association for the Education of Young Children.

Questions for Grade 1

Why are there word problems where the picture is not the word to write in the answer label?

Some word problems in the lower grades have a picture with each word problem. These pictures are to help children understand the problem context. But we do not want children to learn the old solution technique of looking at the numbers in the problem and doing something without reading the problem. If every picture for every problem was the word that needed to be written on the blank to label the answer, children could begin to follow that bad approach**. Therefore some of the pictures are about things in the story but are not the answer label.** Make it a game for children to find the word problems for which the picture is not the label for the answer. Children can underline the word in the problem that is the label for the answer, write the label for the answer in the blank, and can even draw a picture of that answer if they wish. You and they can call such problems *tricky picture problems* or another term you all make up.

When do students need to be able to do the Make-a-Ten strategy?

The make-a-ten strategy is difficult for many first graders. In the make-a-ten strategy you rewrite the teen problem as one ten and some ones. For example, for 8 + 6 you first ask how many makes ten with 8, and then how many are left in the 6: 8 + 2 + 4 to get 10 + 4. Then you need to know quickly all of the ten plus some ones examples, so 10 + 4 is 14. There are three prerequisites for this strategy: K.OA.4 and K.OA.3 and K.NBT.1. These are taught in *Math Expressions* kindergarten. If children are not fluent in these, the make-a-ten method can be difficult and even then it takes practice. It is important to introduce these prerequisite strategies in kindergarten and to continue to discuss them sometimes during first grade because then more children will begin to use them. Even more children will begin to use them in Grade 2 when they are adding or subtracting multidigit numbers where make-a-ten is very useful. Make-a-ten is not for mastery in grade 1 for all children, but counting on is. So just be sure that all children are counting on to find the total and to find the unknown addend when subtracting, and nudge those who can to try make-a-ten strategies. Some children learn this strategy easily or even invent it, so it is important to introduce and discuss it.

How can I adapt the first grade Daily Routines for new 2018 users?

Kindergarten Unit 1, 2, 3 Daily Routines develop the basic understandings of numbers to 100 as tens and ones, counting to 100, and why numbers are written as they are. **Children entering grade 1 as a new 2018 *Math Expressions* user may not have this knowledge fully developed.** So in the first year, first grade teachers should do this:

a. For 16 school days, use the Kindergarten Unit 1 Daily Routine but add 3 circles/pennies per day (reaching 48).
b. For the next 13 school days, use Kindergarten Unit 2 Daily Routines but add 4 circles/pennies per day for 13 days (reaching 100).
c. For the next 10 days, do Grade 1 Daily Routines Unit 1.
d. For the next 10 days, continue Grade 1 Daily Routines Unit 1 and add Unit 2 routines (Unit 2 Daily Routines are short).
e. For the next 10 days do Grade 1 Daily Routines Unit 3.
f. Do Grade 1 Daily Routines Unit 4 during Unit 4 (or at least for 10 days).

Questions for Grade 2

How can I adapt the second grade Daily Routines for new 2018 users?

If many second graders are new to *Math Expressions* and they are not confident about tens and ones and counting in the grade 2 Daily Routines Unit 1 Lessons 1 to 6, **continue working with numbers less than 100 for a few more days until children are confident.** Then continue on to the Unit 1 Daily Routines that use numbers greater than 100. **Be sure that children are confident about these numbers to 200.** These numbers are more important than the money value Daily Routines that begin in Unit 2 and continue to Unit 3. Do these Unit 2 and Unit 3 Daily Routines about money for a shorter time if needed to give time for full confident knowledge of tens and ones in 2-digit and 3-digit numbers.

The word problems in Unit 1 seem too difficult for the beginning of the year

There are four major reasons for these word problems being here in Unit 1. **First, these are all Grade 1 problem types and should have been done in Grade 1.** Perhaps Grade 2 teachers need to check with their Grade 1 colleagues about being sure they do these problems. The problems in Big Idea 2 were for mastery in Grade 1 and the more difficult problems in Big Idea 3 were just for experiencing in Grade 1. **The grade-level goal for grade 2 students is to solve all of these types of word problems with 2-digit numbers,** which is more difficult than with the single-digit numbers here in Unit 1. So it is helpful to review these word problems at the beginning of the year so the students can move on to the 2-digit versions later in the year.

Second, this unit also gives teachers a good idea of where their students are. Many students forget things over the summer, but if they do the Math Talk explaining, many will remember and be able to solve fairly readily. Teachers should not spend a great deal of time on these lessons because students will be solving them for the rest of the year. Big Idea 3 problems are included because some to many students can solve them and they need to see more difficult problems at their level as well as easier problems. If teachers have classes that are really struggling, they could skip Big Idea 3 and do it between Lessons 20 and 21 in Unit 4. But in that case they should have students cut out and practice with the blue Make-a-ten cards from SAB pages 71 to 74. These are described on TE page 146. It is important for students to practice with these cards. But students do need to do Big Ideas 1 and 2 at the beginning of the year.

Third, students need the time to practice their strategies for symbolic single-digit addition and subtraction and move toward mastery on these. They work on this every day in the Quick Practice and in some lessons. This is needed before students begin the work in Unit 2 on Addition Within 200. But we also want to get the Math Talk Community started with students explaining their thinking. Word problems with math drawings are a perfect way to do this because students draw and solve in different ways. Also, if some students are new to *Math Expressions* and have not solved word problems in these ways, this is a good introduction for them.

The fourth reason to cover the problem types and models is so that in every subsequent unit, **students have the groundwork they need to work on those problem types all year**. These problems appear on Remembering pages, and students do progress in their understanding of these problems.

Questions for Grade 3

Multiplication and Division in *Math Expressions*

Math Expressions explicitly relates multiplication and division to each other for many reasons. The CCSS and many other standards specify that these operations are to be related because doing so increases understanding of both operations. Also, relating the strategies by which you are finding multiplication and division answers helps you find and remember the multiplications and divisions. For example, students practice and learn count-bys for each number and use these count-bys to find particular multiplications and divisions. **To find 7 x 4**, students can say the 4 counts: 4, 8, 12, 16, 20, 24, 28 and keep track with their fingers of how many groups they have counted. **They stop when they see 7 fingers.** Or they could use the 5s shortcut and start with 5x4 is 20 and count by 4 two more times: 24, 28. **Dividing 28 by 4** is the same process: count by 4s and keep track of how many you are counting. But **you stop when you hear 28** and you look at your fingers to see that there are 7 4s in 28. You could also use the 5s short-cut to divide: I know that five 4s is 20, so 24, 28 is two more 4s so that is 7 4s in all. So 28 divided by 4 is 7. When watching students use these methods, you cannot tell whether they are multiplying or dividing unless they say something extra.

Learning multiplications and divisions involves a lot of specific number knowledge about repeated numerical groups. Learning and practicing the multiplications and divisions near the same time shortens this process as well as strengthening the relations between these operations. Ask your students to tell you specific things they notice about how multiplication and division are related for a particular number. They will share with you many helpful things as they become used to doing this.

Why start third grade with multiplication/division instead of addition/subtraction?

In the Common Core State Standards and most other standards, the most important Grade 3 standards are single-digit multiplication and division. All of its components—situational meanings, equations, patterns in the different count bys, practice to fluency of all of the factors 1 through 10—have to be completed in grade 3. And students will suffer if they do not reach fluency by the end of grade 3 because they need that knowledge for grade 4 topics,. There is not time for much fluency practice in grade 4. **Third grade teachers have a crucial job**, and it is **getting everyone to understand multiplication and division meanings, find patterns for specific numbers, and then become fluent with all of the multiplications and divisions with factors 1 to 10.**

There is not time to do the traditional review of multidigit addition and subtraction for weeks or months before starting multiplication and division. Also, such an extensive review in grade 3 is no longer as necessary because there are no new addition and subtraction computational standards for grade 3. Instead, students are to become fluent with the problems they solved in grade 2, problems with totals within 1000. **Grade 2 teachers must teach addition and subtraction within 1000** so that the third grade teachers can begin the huge and hugely important task of helping grade 3 students understand and become fluent in multiplication and division.

So in grade 3 teachers need to concentrate on multiplication and division and just do enough addition and subtraction later in the year so that students remember how to add and subtract 3-digit numbers within 1000 and then learn rounding. Students will have a chance in grade 4 to generalize their multidigit adding and subtracting to larger numbers and reflect on multidigit adding and subtracting.

Some students may come to third grade without a strong understanding of all single-digit additions and subtractions, especially with totals from 11 to 18. However, students who have not yet been able to remember these do not need to "memorize" these additions and subtractions, and having students practice these rotely in Grade 3 will interfere with learning multiplication and division. Students do need to be able to count on to find the unknown total for adding and count on to find the unknown addend for subtracting. These strategies are taught in grade 1 and grade 2. You can review these at the beginning of grade 3 if necessary. But the main thing to do is to emphasize to your grade 1 and grade 2 colleagues how important these strategies are and not to give mindless timed tests and but instead practice these strategies for totals from 11 to 18 so students will be ready for grade 3.

Questions for Grade 4

Please explain the choice of Grade 4 multidigit division methods

For multidigit multiplication the place-value sections method was invented by students who could not see what to multiply by what in any symbolic expanded notation method. All *Math*

Expressions methods use an area model for understanding what multidigit multiplication is. Many students began drawing and filling in this area model quickly and then adding up the four partial products out at the side. **This method was fast and accurate for them**, and they could explain it, so it seemed sensible for them to use it.

The related Rectangle Sections model for division uses the same area model but now one knows the product and one factor and **must find the other factor** that is the length of the top of the rectangle. So one finds the product of the tens number in the unknown factor (40 x 67), writes that below the product 2881, and subtracts that product 2680 from the known total product 2881. The resulting partial product 201 is written in the next Rectangle Sections. The ones number in the unknown factor (3) is then found. This process is like finding and adding up the partial products to make the total partial product. That is the same process is in the Current Common division method except in that method the place values are ignored and numbers are treated as single digits. The Expanded Notation Method is like the current common digit-by-digit method but it uses place values.

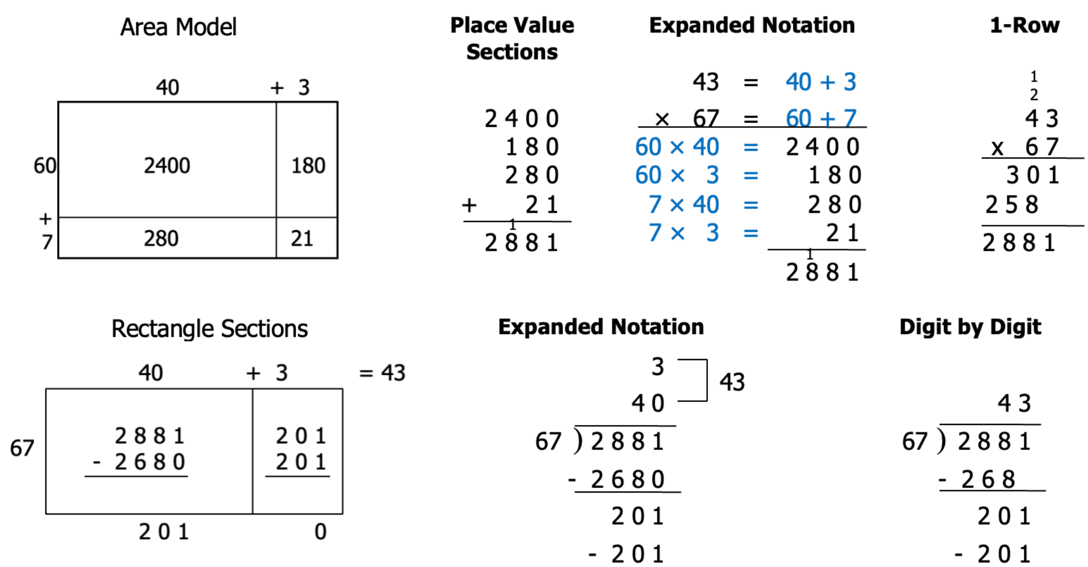

Figure 3 Drawings, good written methods, and not-so-good methods for multidigit multiplication and division

Why does measurement to the nearest fraction of an inch come before the fraction units?

Some teachers ask if they can do the fraction units (Units 6 & 7) before the measurement unit (Unit 5) because they feel the knowledge of fractions helps when measuring to the nearest fraction of an inch. They wonder what is the reasoning behind the measurement unit coming before the fraction units?

Students in grade 3 measure to the nearest quarter inch (Unit 4 Lesson 6), and this lesson follows five lessons on fractions. So it is a simple extension in grade 4 to measure to the nearest eighth of an inch in unit 5 in grade 4. Students do not need to do all of the grade 4 fraction lessons to do grade 4 Unit 5 Lesson 4 Activity 2. This is a simple activity in which they focus on the eight small lengths between the inches and count how many extra they have after the whole inches.

They do not even need to find equivalent fractions that equal those eighths (e.g., ½ = 4/8) but they can do that if they wish to do so. If teachers want students to know these equivalents, they can revisit inch lengths in Unit 7 lessons 4, 5, 6 by using fraction strips (on TRB M31 to 38) for eighths, fourths, and halves or by drawing number lines with these lengths and discussing the equivalent fractions. If teachers delay the Unit 5 measurement unit until after the fraction units, students will not get Remembering practice on measurement, and they need that.

What is the purpose of the Grade 4 introduction to array and area models in Unit 2 Lessons 1 and 2?

In some ways the lessons after Lessons 1 and 2 are easier because Lesson 1 uses such small numbers that drawings are not really needed for many students. So emphasize throughout all of the lessons that **the drawings are to help make sense of the patterns in the numerical methods** and eventually the drawings will not be needed. These first two lessons might be easy for students and might go rapidly.

Lesson 2.1 has several purposes. It is reviewing Grade 3 concepts but with the visual representations and properties of grade 4. To be discussed are:

1. Being sure that all students understand the connections among arrays and area shown at the top of student page 53 and can start to talk about these representations using easy numbers. **Especially important is to count unit lengths in area models.**
2. Relating different ways of looking at the same area and see and generalize that a x b = b x a.
3. Focusing students on unit lengths when they are making area models so that their models are accurate.
4. Practicing relating the associative property to the area models that "show" it. Students all should have experienced this property in grade 3, but many will need support to remember or see it again, so you may be leading heavily for page 54 #3. Or some students might be able to lead this discussion. **Each step in the equation relates to the drawings at the top.**

   2 x 30 is shown in the top model where 2 is broken into two units of 1.
   Then 2 x 30 is shown in the second model where 30 is broken into three units of 10.

   The second side of the equation $(2 \times 1) \times (\underline{\ 3\ } \times 10)$ shows the third drawing where both the 2 units of 2 and the 3 units of ten are shown in the drawing. Support students to see and state in their words this relationship.

   Then the next part of the equation $= (\underline{\ 2\ } \times \underline{\ 3\ }) \times (1 \times 10)$ is using commutativity to switch the 2 and the 3, and then the numbers within the parentheses are multiplied to give the final products in the equation $= \underline{\ 6\ } \times 10 = 60$. So you get 6 tens numerically and also in the third drawing.

Why include the difficult multiplication methods in Lesson 13 Activity 2?

In Unit 2 Lesson 13 the methods in Activity 1 are simpler than those in Activity 2 because those in Activity 1 do all of the multiplying first and then all of the adding. Students should discuss all of these methods, but they can choose which method they wish to use. The methods in Activity 2 alternate multiplying and adding, which is more difficult. The New Groups Above Method is discussed in detail because many parents may know it and try to teach it to students. This method is difficult because it alternates multiplying and adding and has a misleading step in which the product of the tens x tens has some hundreds but this number of hundreds is written above the tens place in the top number. Students do not need to learn this method or the Shortcut New Groups Below method. You can skip Activity 2 if you wish, but if some students write the new groups above the problem, discuss that method so students understand it.

Questions for Grade 5

Why start the year with fractions?

Math Expressions starts the year with fractions in Grade 5 because this helps students review their single-digit multiplications and divisions before they need to use them in the more complex multidigit and decimal multiplication and division in later units. Fraction concepts are not difficult with the visual supports provided in *Math Expressions*. Especially for students who have gaps in their multiplication or division facts or in other number concepts, it is much easier to do the fraction Unit 1 first. This lets you and individual students identify issues they may have with their multiplications and divisions and work on them before Unit 3. Also, there are not very advanced fraction concepts in grades 3 and 4. So if teachers are worried that their students do not understand fractions well, that is ok because grade 5 Unit 1 on fractions includes the basic concepts students need.

Teachers have asked this question at the beginning of the year, and they have been happily surprised at how well the first fraction unit went. They emailed me to tell me that.

Questions for Grade 6

Grade 6 *Math Expressions* is and always has been totally aligned with the Common Core State Standards and other high-level standards.

Grade 6 2013 in its final form was written after the Common Core State Standards were released in final form in June, 2010. But much of grade 6 consists of units that had been developed and piloted in the Children's Math Worlds Research Project that is the basis for the K to grade 5 *Math Expressions.* These units were in the original grade 5 *Math Expressions*, for example, division of fractions, ratio and proportion, and the geometry units. The grade 6 Common Core State Standards did have two new standards for which units did not exist—Units 5 and 8. I asked Dr. Sybilla Beckmann, a mathematician from the University of Georgia who had worked on the Common Core State Standards and the CCSS Learning Progressions to work on these new units with me. In working on these units, she drew on her decades of working with teachers and writing books for teaching teachers. The 2013 Grade 6 *Math Expressions* has a somewhat different format from the kindergarten to grade 5 *Math Expressions* 2013 Common Core State Standards version, but these differences are minor. **Grade 6 2013, 2015, and 2018 *Math Expressions* are all totally aligned with the Common Core State Standards.**

Grade 6 does have a feature that no other grade has. **The first activity in each lesson focuses on discussing homework.** I carefully selected a problem for this discussion which would provide useful follow up for the previous lesson and help the next lesson get off to a good start.

We did this because homework is so crucial from grade 6 on, and we wanted to encourage students to do it and discuss it. This discussion permitted teachers to emphasize points that might not have been clearly understood in the lesson the day before.

Why does Grade 6 Unit 3 include fractions as well as decimals?

I worked on the Common Core State Standards and on the learning progressions. There are some complexities in the standards concerning decimals and fractions. First, we realized after the CCSS were released in their final form that we had never specified a fluency standard for fractions. An appropriate place for this is **in grade 6 to go along with 6.NS.3, the fluency standard for multi-digit decimals.**

Second, in 6.NS.4 we mention the least common multiple of two whole numbers, but we do not mention a common use of that notion--in finding equivalent fractions in order to compare, add, or subtract fractions with different denominators. Because the CCSS are trying to build coherence and meaning-making, **it seems important to relate finding the least common multiple in 6.NS.4 to these uses, so this requires these operations with fractions.**

Third, the cluster heading for 6.NS.1 does specify that students are **to apply and extend previous understandings of multiplication and division to divide fraction by fractions.** It is important for students to relate and differentiate the meanings for and the specific steps taken in carrying out these two operations because **the results for a multiplier or divisor less than one are opposite to the very familiar results for whole numbers**: multiplying by a fraction or decimal less than one results in a product smaller than the number multiplied, not larger as with whole numbers.

Fourth, grade 6 integrates the kindergarten to grade 5 domain Number and Operations in Base Ten (NBT) that included decimals and the domain Numbers and Operations–Fractions (NF) to become **The Number System (NS). Decimals and fractions are just different ways to write the same numbers.** Therefore it is appropriate for Grade 6 students to reflect on both of these systems and discuss how the **operations with decimals and fractions are alike and different**: The operation meanings and contexts are the same, but the notation and methods for solving these operations are different, although even here there are important commonalities such as that adding and subtracting for decimals and for fractions require adding or subtracting like units.

Therefore Unit 3 builds all of the above relationships and reflections. **It is a summative experience of lower grade meaning making and computation for whole numbers, decimals, and fractions.** It has coherence, focus, depth, and rigor.