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Relating Math Words, Visual Images, and Math Symbols for Understanding and Competence

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ABSTRACT

This paper briefly overviews my research in supporting children to learn number concepts by relating number words, research-based visual supports, and math symbols. I first outline my approach to helping children build relationships between the use of concrete materials and the building of abstract concepts. I then focus on two crucial early aspects of building meanings for numbers: (1) understanding break-apart partners such as 5=3+2 that support addition and subtraction with small numbers and children's moving on to Level 2 counting on and algebraic problem representations, and (2) the use of visual five-groups in understanding numbers 1–1000 and in drawings to support multi-digit computations. The research-based learning path of visual-spatio supports is shown and discussed for each topic, including examples of children's math drawings for representing word problems algebraically and for multi-digit computations. I have found math drawings to be a key visual support that helps children transition to working meaningfully with symbols and words alone. I close with a brief discussion of the difficulties children have with the number line. This overview can provide a framework within which future research on number learning by individuals with trisomy 21/Down syndrome can proceed.

KEYWORDS

addition/subtraction; concrete materials; counting learning; Math; number; trisomy 21; visual supports

This paper briefly overviews my research in supporting children to learn number concepts by relating number words, research-based visual supports, and math symbols. My research has focused on finding powerful visual images (concrete objects or math drawings) that work in classrooms conceptually and practically. I develop teaching/learning materials using these visuals, research their effects in detailed ways, and then articulate the learning web and learning path for a math domain so that others may understand and use it. Learning paths involve levels of increasing abbreviation, abstraction, and internalisation but the visual images remain as an accessible base. I have drawn on research and teaching approaches from around the world and woven them into a coherent stream of learning supports from age 4 to 13. I offer this overview with the hope that it will be useful to those who work with people with trisomy 21.

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An earlier and longer version of this paper was presented at Trisomy 21, Mathematics, and Thought, September 7 to 9, 2017, Department of Mathematics, Zaragoza, Spain. A narrated video of that presentation is available at karenfusonmath.com in the presentations section. Relating Math Words, Visual Images, and Math Symbols for Understanding and Competence
Please also see karenfusonmath.com for many other papers and resources concerning this paper.

Relationships between the Use of Concrete Materials and the Building of Abstract Concepts

I use a Piagetian theory of learning and a Vygotskiian theory of teaching with a focus on the sociocultural tools for learning. This is not Bruner's concrete to iconic to abstract but a web of connections among math words, math symbols, and visual images (concrete or math drawings). This web becomes connected through modelling and explaining by a more knowledgeable other and by the learner using the visual tools and explaining her/his thinking. A nurturing Math Talk community in which everyone is a teacher and a learner is crucial for children constructing the conceptual web in a given math domain.

Young children do need concrete things they can move to build math concepts, but math drawings they can produce themselves are engaging and often easier to manage. The visual images in math drawings also support Math Talk so that listeners can understand better. Kindergarteners (age 5–6) can make math drawings for some topics, and first graders can do so for many topics.

A strength of the conceptual web connected by Math Talk is that most children have at least one modality in which they are strong enough and that can help support the other modalities that need to become related. So explaining may be difficult for one child, but she perhaps can make drawings of her thinking or use concrete objects to show her thinking. Then with enough modelling of explaining by others, her own explaining will also improve. Another child might have difficulties with drawing, but can explain or enact with concrete materials. And gradually drawing can improve. These conceptual webs use the power of geometric and spatial images and concepts to support learning. This may be particularly important for people with trisomy 21 because math talk often is not a strength.

This approach has been powerful for children from a range of economic and cultural backgrounds, including students learning English as a second language and children categorised as having various intellectual disabilities including having visual impairments or being on the autism spectrum. Decades of research have gone into the design and fine tuning of the visual supports. These are specifically detailed in a mathematics programme for classrooms that moves from PK to Grade 6 (*Math Expressions*, 2018). Various versions of this programme are in use in every state of the United States, indicating that the approaches outlined here are viable for a wide range of students and teachers. Because children can build the conceptual web in various ways, it seems worthwhile to try some of the visual supports and at least the theoretical approach of a connected web of visual-spatial-sensorimotor/verbal/written symbolic math forms with children or adults with trisomy 21/Down syndrome. For more about this conceptual web approach to building mathematical ideas find this paper in *Publications* on my website: karenfusonmath.com (Fuson, 2009). Also on that website you can find many publications about different math domains, classroom videos that show children making math drawings and explaining their thinking, and Teaching Progressions that detail for different math domains the learning paths and visual supports in those domains. During the years of classroom research and interviewing of children and teachers about this approach, all university informed consent procedures were followed.

The Learning Webs for Specific Concepts

In this paper, I overview the learning webs for two crucial aspects of early number development:

(1) break-apart partners such as 5=3+2 that support early addition/subtraction and moving onto Level 2 counting on methods for addition and subtraction, and
(2) the use of visual five-groups in early number and in multi-digit computations.

Partners are two addends that make a total. We use the word *partners* initially with children because we found that it was powerful in helping them to see each group of objects as making one number and also to see those two numbers 'hiding inside a bigger number' to make that bigger total number.

Break-Apart Partners Hiding Inside a Number

Figure 1 shows visual learning materials and outlines steps in building the connected web of early count cardinal-numeral meanings. The numbers with circles above them are called a 5-Parade and are printed on an 8½ inch by 11 inch piece of paper. The movable darker numbers

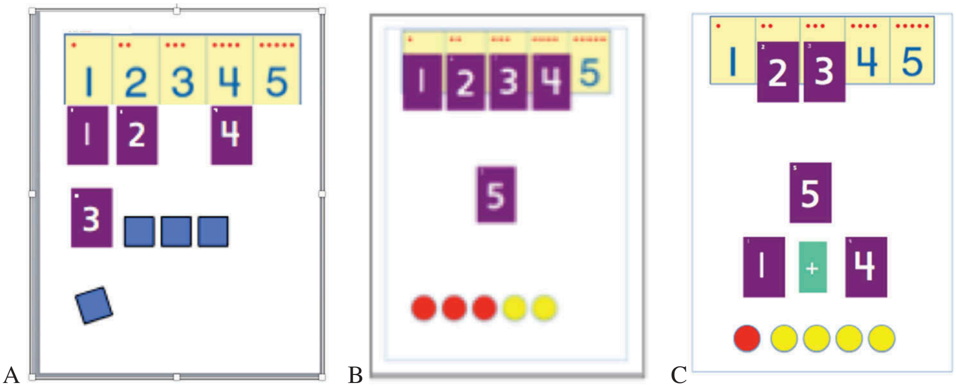


Figure 1. Building count-cardinal-numeral and break-apart partner knowledge.
(a) Count-cardinal-numeral relationships: This is a long period of building these relationships for the numbers 1, 2, 3, 4 and then for 5.
Use a page with the 5-Parade of numbers 1, 2, 3, 4, 5 each with their number of circles above.
Children count on this 5-Parade 1, 2, 3, 4, 5 pointing to each number as they count.
Later children can raise 1, 2, 3, 4, 5 fingers as they count.
Children then put the purple foam number tiles under the numbers at the top.
Children can use the 1 to 5 circles on the back of the tiles to match by the number of circles.
A number is said, and children pull down that number tile (here, 3) and say the number.
Children make that number using circles or squares or cubes.
Children can show that number with fingers (in various ways).
Children relate that quantity to sounds and body movements.
Children practice visual imagery (Close your eyes. Visualize.)
Children describe different arrangements that make that number.
Children copy the arrangement of another child.
(b) Showing break-apart partners of a number using colour or spacing: This uses the knowledge from A as children move from perceptual subitising of 1, 2, 3 to conceptual subitising to make and take apart totals up to 5.
A number is said, and children pull down that number tile (here, 5) and say the number.
Any two partners (addends) that compose that number are made using different colours or spacing, here 3 and 2.
Children look at and discuss the different partners that are made.
(c) Showing a numerical expression for the partners of a number.
Children make a numerical expression using the purple number tiles and a+tile (here, 1+4). Then they place the total number broken apart to make those partners (addends) above the + sign (here 5).
Eventually children will see and write equations of the form 5=1+4 to show a total breaking apart to make partners.

are 3 by 5 cm purple foam number tiles that have small circles on the back in the same pattern as the circles above numbers on the printed 5-Number Parade. The squares are 1 square-inch foam squares in various colours, and the circles are 3 cm foam red on one side and yellow on the other side.

Children carry out over time all of the steps outlined in A. They can go as slowly or rapidly to all of the steps as their performance warrants. Children may vary in the steps or parts of the web that are easy or difficult for them. We start with the number tiles 1, 2, 3 and later add 4 and later still, 5. The focus here is building subitising capacity (rapidly recognising a small quantity) and connecting that quantity to number words and to numerals.

Once children can subitise 1, 2, 3 and relate each quantity to number words and numerals, they can use those quantities to build up other numbers by using colour or spacing (5 as o o o o o) or shapes (see B and the tasks in Table 1). The tasks in B use objects and those in Table 1 are written tasks used with kindergarten children (age 5) after they have done tasks with objects. In Table 1 a break-apart stick (a vertical line segment between two objects) is used to make the partners rather than spacing apart the partners. Children can use physical break-apart sticks with objects before they use the drawn versions.

Figure 1(c) shows the learning web lay-out of materials for connecting numeral expressions to the partner objects: Children put the number tiles for the partners above the objects for each partner and connect these tiles with a + tile. They place the tile that shows the total above that. All of these steps are discussed and modelled. Children can start with 1+1 or 1+2 or 2+1, the really small subitisable numbers.

Table 1 shows in the third and sixth row another visual support introduced after the materials in Figure 1. This is called the Math Mountain and is introduced with a story about the Tiny Tumblers who live at the top of a mountain. Every day the Tiny Tumblers go out to play and some of them play on one side of the mountain and the rest play on the other side of the mountain. This visual support is useful to show the Put Together-Take Apart word problem type where the addends are contained within the total.

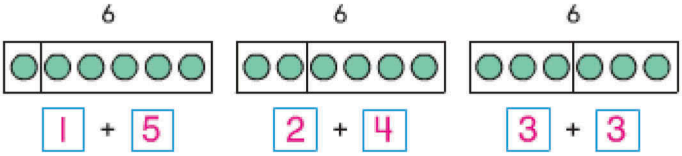
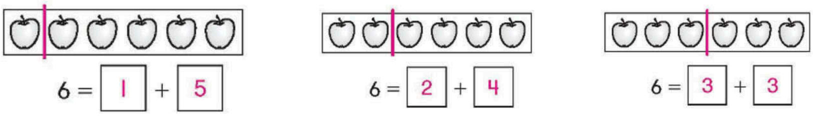
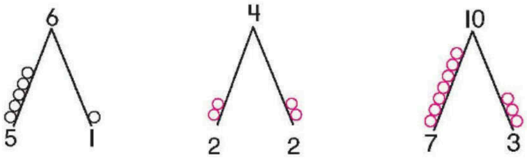
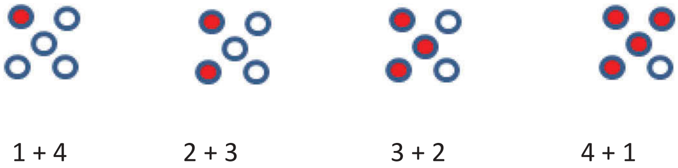
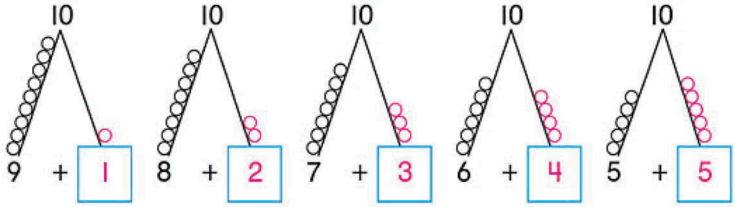
Three red apples and two green apples are on the table. How many apples are on the table? Or Five apples are on the table. Three are red and the rest are green. How many apples are green?

The line segments connecting the total to each the partner show the partners as breaking apart the total to make the partners. We found that children could use this action model to move rapidly between the total and the partners and to see the total as composed of the two partners. The non-action forms used in the rest of the table sometimes were more difficult to relate the total and the partners because the same objects show both of these. Children use this Math Mountain form to show partners and to represent word problems with unknown addends or totals.

Children can gradually work with larger totals to see the partners hiding inside and so support adding and subtracting with visual images. This work also helps them relate adding and subtracting. Gradually children can embed both addends within the total and thus count on to find the total: 6+2 is *six* (here in one partner), *seven, eight* (two more in the other partner). They can think of subtracting as counting on from the known addend to the total: 8–6 is 6+?=8. So I can count *six* (I have taken away), *seven, eight* (I stop when I hear 8); I counted on two more, so the partner I do not know is 2.

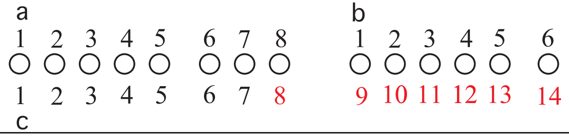
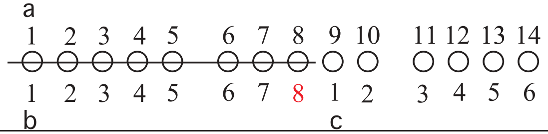
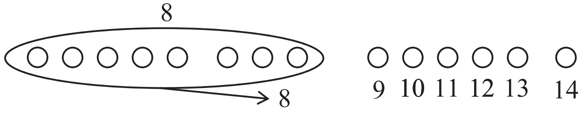
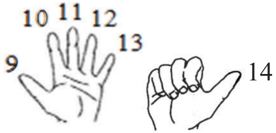
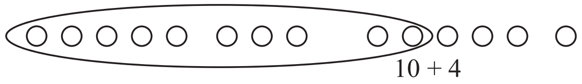
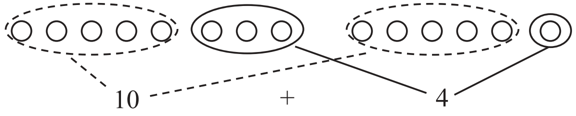
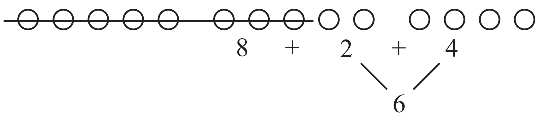
Counting on is shown in the second row of Table 2. The three levels in Table 1 are the worldwide levels of children's addition and subtraction methods (see Fuson, 1992, 2003).

Table 1. Percentage Correct on Partner (Addend) Tasks for Kindergarten Children

Unit	%	Task
3	90	1. Write the Partners 
4	92	2. Draw a line to show the partners. Write the partners. 
4	92	3. Draw Tiny Tumblers on the Math Mountain 
4	85	4. Write the partner equation. 
5	88	5. Shade to show all the 5-partners in order. Write the 5-partners. 
5	83	6. Draw Tiny Tumblers on the Math Mountain and write the partner. 

Note. Items are from unit tests. Units 3, 4, 5 are taught in January through May. Kindergarten children range from age 5 to 6 at this time.

Table 2. Levels of Children's Addition and Subtraction Methods

Unit	%	Task
Levels	**8 + 6 = 14**	**14 – 8 = 6**
Level 1: Count all	Count All 	Take Away 
Level 2: Count on	Count On 	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten  	14 – 8: I make a ten for 8 + ? = 14  8 + 6 = 14
Doubles ± n	6 + 7 = 6 + 6 + 1 =12 + 1 = 13	6 + ? = 13 = 6 + 6 + 1 = 12 + 1= 13

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

These levels are shown with totals in the teens because that is when many children really need counting on and the Level 3 recompose methods because they can not easily show numbers above ten with their fingers.

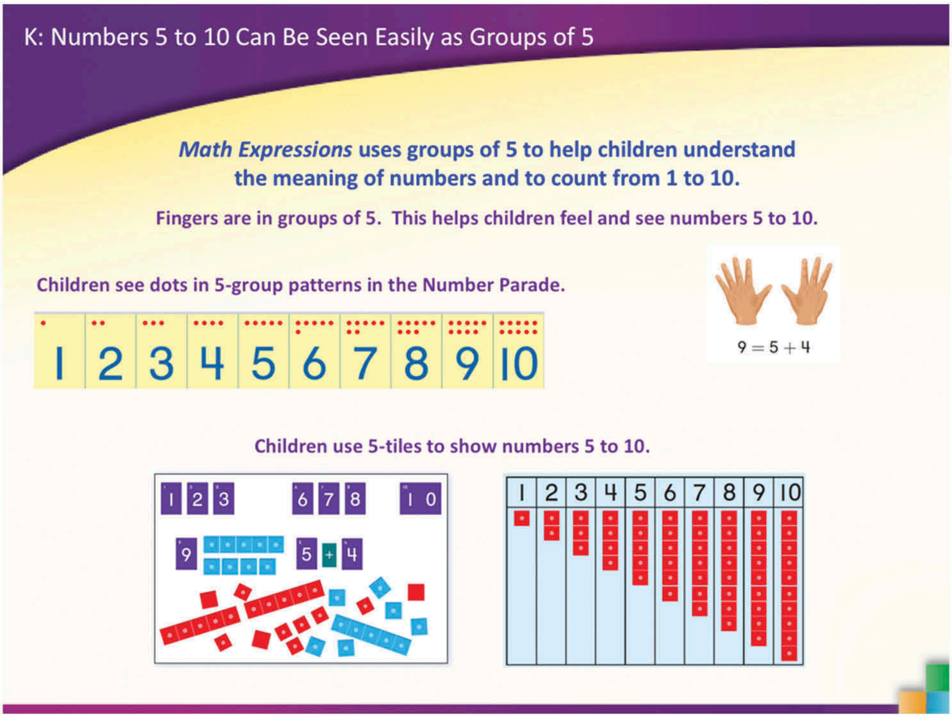
You can find more about partners and levels of adding and subtracting on my website karenfusonmath.com in the *Teaching Progression* OA Part 2, the *Classroom Videos* E G1 Single-Digit Addition and Subtraction Part 1, and in *Visual Presentations* (Fuson K.C.& Smith, S., 2016, April, Children living in poverty can solve CCSS OA word problems). You can find more about drawings for word problem solving and levels of adding and subtracting in word problems in *Teaching Progression* OA Part 1 and Part 2 and in *Classroom Videos* E G1 Single-Digit Addition and Subtraction Parts 2, 3, 4, 5. The words in italics are menu titles and can be reached by clicking on the menu.

Visual Five Groups in Single-Digit Numbers, Teen Numbers, and Multi-digit Computations

Arranging quantities in five groups is widely used in East Asian classrooms and in other places around the world. Fingers also are arranged in five groups, so visual five-groups support sensorimotor links with the fingers. The top of Figure 2 shows materials that help children learn and work with five groups. The top row of the Number Parade poster (on the top left of Figure 2) shows five-groups making 6, 7, 8, 9, 10 as 5+1, 5+2, 5+3, 5+4 and 5+5, respectively. Children also can see these patterns in their fingers. In such displays it is important that there is more space between the rows than between the circles. If those distances are the same, the circles make an array in which children see pairs of circles rather than the five groups. The five groups help children know how many more to make ten by looking at how many circles in the top row of five do not match a circle below: 6+4 makes 10, 7+3 makes 10, 8+2 makes 10 and 9+1 makes 10. This is the first step in the Level 3 recomposing make-a-ten method that is taught in East Asian and many other classrooms around the world. The Number Parade poster is a large poster permanently on the wall of the classroom. The other materials are used by children to show numbers between five and ten using five groups.

The bottom of Figure 2 shows how money is also used to show five groups. On the left are nickel strips that fold to show five pennies on one side and one nickel on the other side. These help children learn the value of one nickel and use that value of five when counting money. The top left of Figure 3 shows nickel strips and also dime strips that fold to show ten pennies on one side and one dime on the other side. These strips also show that two nickels make one dime, a difficult equivalence in the United States where the dime (10ç) is smaller than the penny (1ç) and the nickel (5ç). The length of the strips on the nickel and dime sides and the exact quantity of the pennies on the back are very helpful in learning the values of dimes and nickels. Strips like this that show the unit quantities for major money amounts could be made for the many different systems of money around the world.

European systems of number words above ten have many irregularities that make them difficult for children to learn and use (Fuson & Kwon, 1991; Fuson & Li, 2009; Ho & Fuson, 1998; Menninger, 1958/1969). The teen numbers in English are particularly difficult because the tens and ones order is reversed in the words and written numerals (sixteen but 16). Fortunately the number words from 20 to 100 do not have this reversal



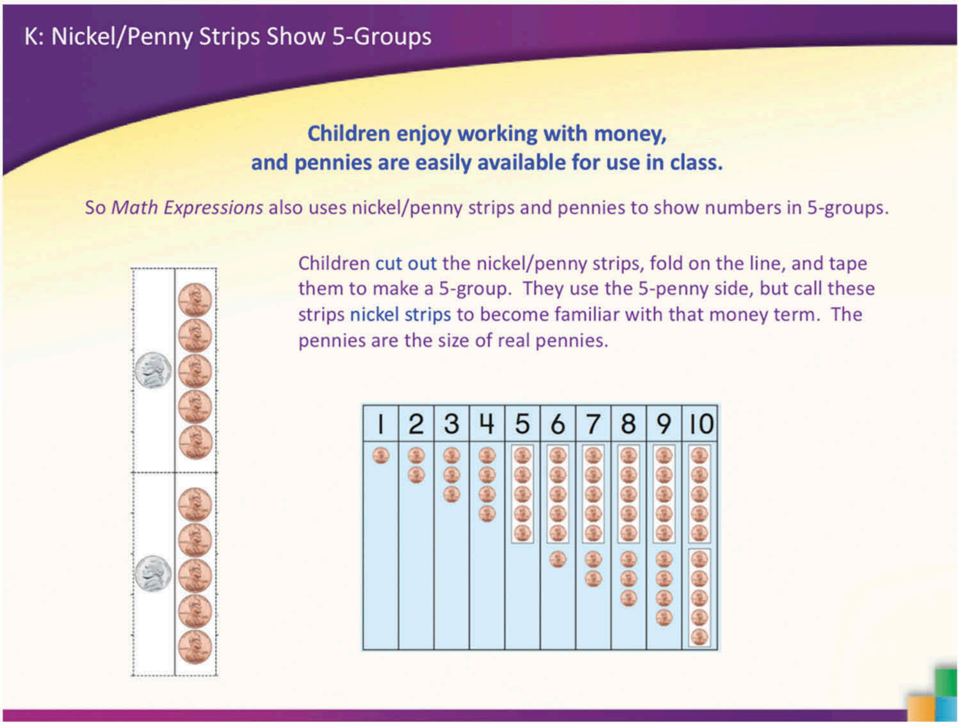
K: Numbers 5 to 10 Can Be Seen Easily as Groups of 5

***Math Expressions* uses groups of 5 to help children understand the meaning of numbers and to count from 1 to 10.**

Fingers are in groups of 5. This helps children feel and see numbers 5 to 10.

Children see dots in 5-group patterns in the Number Parade.

Children use 5-tiles to show numbers 5 to 10.



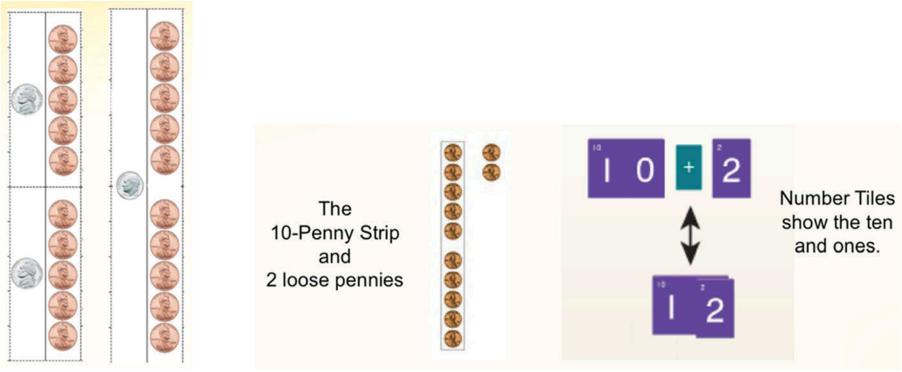
K: Nickel/Penny Strips Show 5-Groups

Children enjoy working with money, and pennies are easily available for use in class.

So *Math Expressions* also uses nickel/penny strips and pennies to show numbers in 5-groups.

Children cut out the nickel/penny strips, fold on the line, and tape them to make a 5-group. They use the 5-penny side, but call these strips nickel strips to become familiar with that money term. The pennies are the size of real pennies.

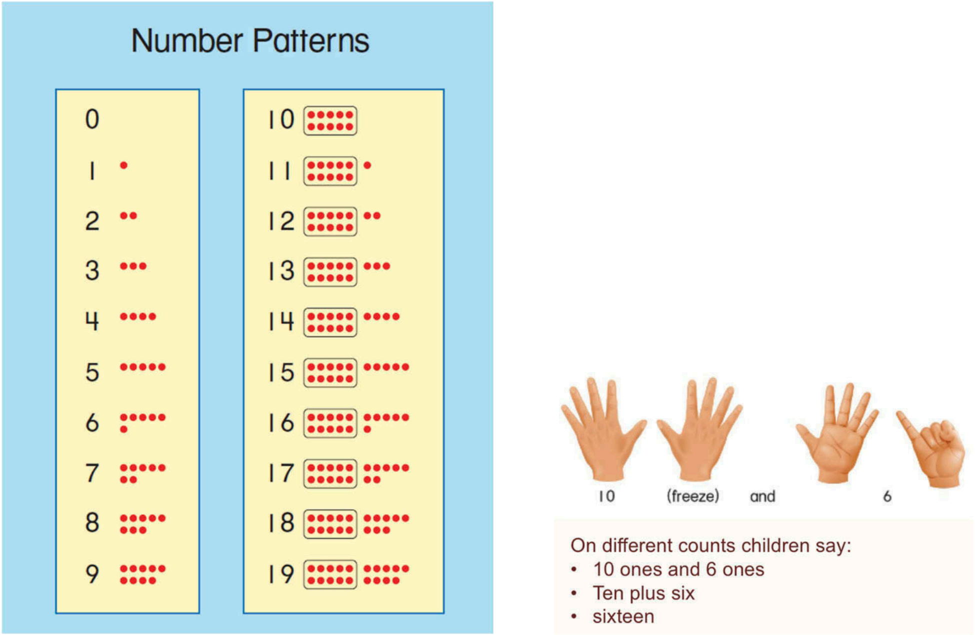
Figure 2. Using five groups to understand numbers 6 to 10 and money.



Nickel Strips and Dime Strips

Children make quantities with Dime Strips and pennies.

Children make 11 to 19 with Secret Code Cards that show the 0 In 10 hiding under the ones digit.



Children discuss patterns they see in numbers 10 to 19.

Children show 10 to 19 with fingers.

Figure 3. Using 5-groups and 10-groups to understand teen numbers and money.

as German and Dutch do. English and some other European languages also do not clearly say the *ten* in teen or decade numbers (16 is sixteen, not ten six or even six ten, and 60 is sixty, not six tens). So children need visual supports to see and understand the tens in teen numbers and in numbers from 20 to 100.

Figure 3 shows on the bottom left a Number Patterns poster the right column of which shows the teen numbers as ten and some ones. Children can feel these numbers on their fingers as shown on the bottom right of Figure 3: They flash ten fingers to the left (to match the 1 in the teen numbers) and then flash fingers to the right to show how many ones are in the teen number. They can say *ten and six make sixteen* to connect the number words for each part of the teen number to the English number word made by those quantities. But still there is the puzzle of the written numeral that shows 16 made by 1 and 6, not by ten and six. Layered cards such as are at the top right of Figure 3 can help children visualise a teen number as having a 0 in the 10 (ten) hiding under the ones number. Work with the cards related to quantities such as the 10-penny (dime) strip and loose pennies (top right of Figure 3) and to finger flashes can help children see teen numbers as 10 and the ones in the teen number (12 is 10 and 2; 15 is 10 and 5; etc.).

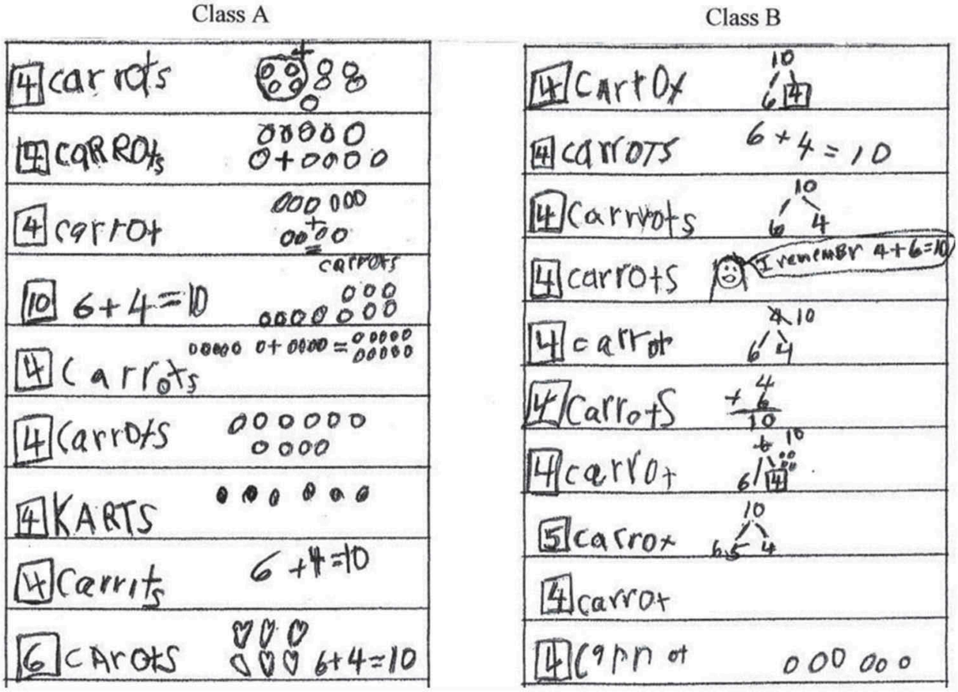
For the numbers 1–100, children count by ones and by tens using a 100 poster that has the numbers arranged in groups of 10 vertically. This allows children to see that the tens number is the same in each column and they can relate that tens number to the decade word (e.g. relate the 5 in 53 to the *fifty* in *fifty three*). Children also flash fingers to show numbers: flash ten to the left five times and then flash three fingers to show 53. They use layered cards like the teen number cards in Figure 3 to show 53 as putting the 3 on top of the 0 in the 50 card. And they also learn to show these numbers by drawings of tens and ones as shown in Figure 4. They draw through a column of ten circles to make a ten that is called a *ten-stick* or a *quick-ten*. Such ten-sticks take on quantity meanings rapidly because children have been seeing columns of ten in 10-penny strips and in the columns of ten numbers in the 100 poster and they have been making such columns of ten with various objects.

So as shown in Figure 4, 58 is five ten-sticks and eight ones (shown as circles drawn using five groups) and 36 is three ten-sticks and six ones (shown with five groups as 5 and 1). The Grade 1 (age 6–7) girl solving this problem says that ‘eight needs two more to make ten so I get two from the six (see the loop in Figure 4) and that makes 1 ten that I write down over here at the bottom of the tens column and 4 ones left that I write in the ones column.’ Writing the new one ten at the bottom of the tens column is easier than writing it at the top: You can see the 14 made by adding 8 and 6, you can write 14 in the usual order (write 1 then 4) rather than write 4 and ‘carry the 1’, you can add the two larger tens numbers first and then add the one ten, and you don’t forget to add in the new one ten or add it to the top number and then ignore the number that is there to add 6 and 3. The five-groups support accuracy in drawing, help listeners to explanations see the quantities immediately, and support the make-a-ten methods for adding to a teen number, such as 8+6=8+2+4=10+4=14.

Children can also use visual five groups and Math Mountains using partners in drawings to show word problem situations. Figure 5 shows the range of such drawings and written equations that children might use to represent a word problem situation. Drawings of circles and/or fingers can also be used to represent and solve simpler problems.

You can find more about five groups on karenfusonmath.com in *Publications* and in *Teaching Progression OA Part 2* for numbers within 20; in *Teaching Progression NBT Part 2* for adding and subtracting with numbers through 2000; in *Classroom Videos: D Kindergarten* (most videos) to see classroom videos of kindergarten children using materials with 5-groups; *Classroom Videos, A Classroom Components*, then *Math Talk segments 3, 4, 5, 6* to see classroom

Figure 4. Using five groups to add 2-digit numbers making a new ten (a Level 3 method).



Rosa picked 6 carrots. Her sister picked some too. Together they picked 10 carrots.
How many carrots did Rosa's sister pick?

Figure 5. Drawings of some first graders to represent and solve an algebraic word problem including five groups and Math Mountains in some drawings.

videos of children explaining math drawings in adding and subtracting for 2- and 3-digit numbers (segment 4 is the problem 58+36 shown in Figure 4).

The Difficulty of the Number Line

In the United States the number line has been used historically in some programmes beginning in kindergarten. But number lines are quite difficult for children and lead them

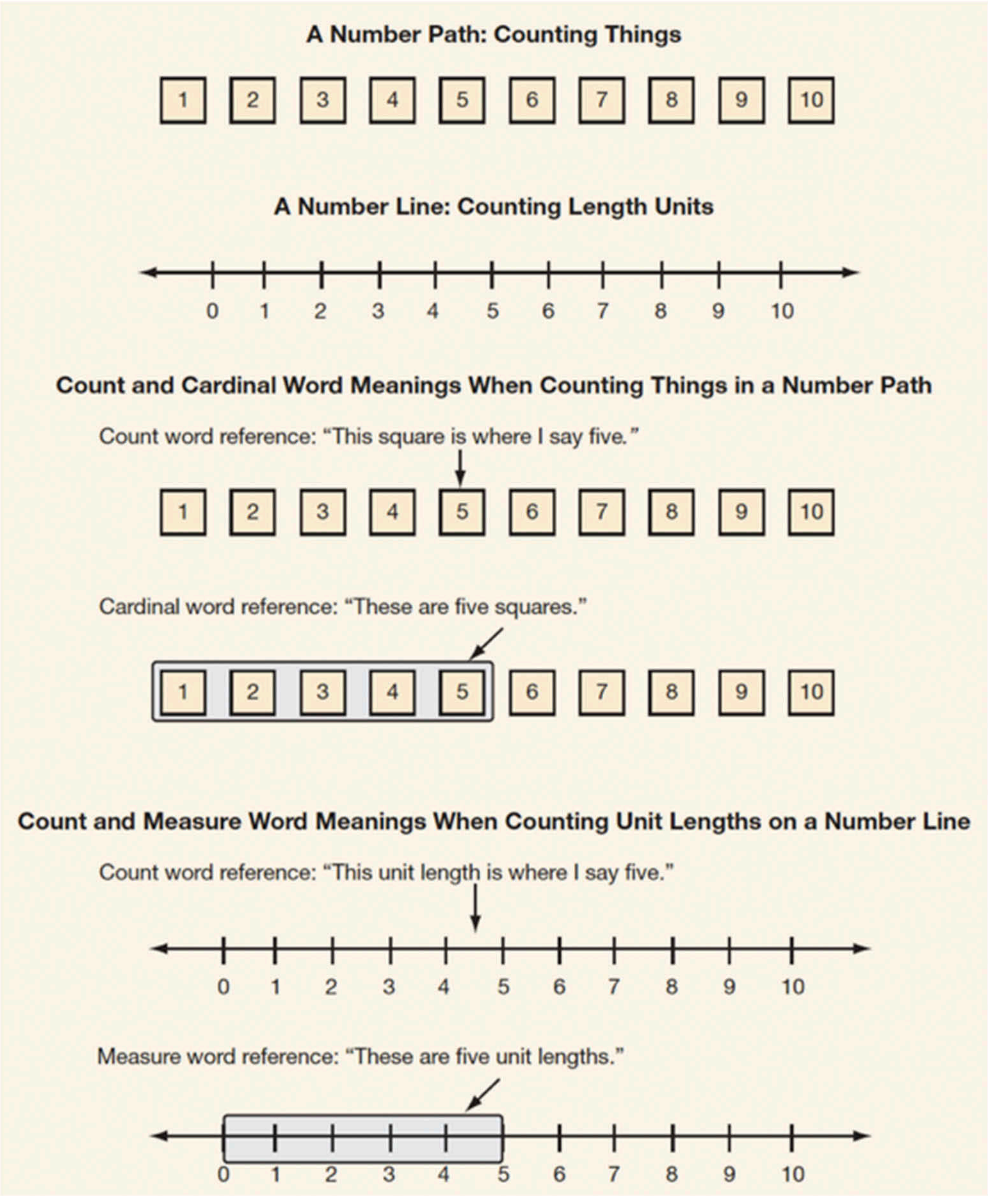


Figure 6. Number Path versus Number Line: Relationships between counting number, cardinal number and measure number.

to make errors. In Figure 6 you can see the count and cardinal word relationships for counting objects or numbers written on objects in a number path: These are the same as for counting objects. In contrast, a big problem with the number line is that the things counted are lengths between the numbers written below the intervals on the number line. Children are wired to see and count objects, so they tend to count the interval *markers* (the short vertical markers) instead of the interval *lengths*. This gives them answers that are off by one and interferes with learning accurate number relationships. All major national research and policy committees in the United States in the past 17 years have recommended that the number line not be used until Grade 2 (age 7/8) when length measure is introduced and one actually needs quantities between the whole numbers. Even then it is a difficult tool that takes a lot of time to learn accurately and that continues to stimulate errors. So the number line does not seem to be a productive learning support for children with intellectual disabilities including trisomy 21.

Looking Forward

In my 45 years of teaching mathematics to learners of many ages, backgrounds and intellectual abilities, I have been repeatedly reminded that learners are always surprising us by learning more than we may be thinking they can learn. Those of you who do research with learners with trisomy 21/Down syndrome of course know this, but I want to end this contribution to the special issue by celebrating the possibilities of increasing learning by pursuing new paths for supporting such learning.

Acknowledgments

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