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Research Commentary

Critiques of the Common Core in Early Math: A Research-Based Response

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We respond to a call to analyze issues of curriculum standards and to present alternative storylines by addressing criticisms of the Common Core State Standards in early childhood. We describe a storyline from multiple media and evaluate this storyline's criticisms, focusing on the criticism that the standards are developmentally inappropriate. We review research and conclude that the criticism is invalid and may reflect a historical belief in the primacy of development over learning rather than the research record. Misinterpreting or ignoring relevant research has equity consequences because it may particularly harm those children most in need of learning support in learning grade-level mathematics. Fortunately, theory and research illuminate learning trajectories that help all children meet these standards.

Keywords: At-risk students; Classroom research; Common Core; Early childhood; Learning trajectories; Mathematics; Primary; Standards

In this Research Commentary, we respond to two recent *JRME* commentaries that asked researchers to (a) analyze issues of curriculum standards (Tran, Reys, Teuscher, Dingman, & Kasmer, 2016) and (b) provide alternatives to storylines in the media that are inconsistent with research syntheses (Herbel-Eisenmann et al., 2016, pp. 104–105). We do this in the context of the critiques of the *Common Core State Standards for Mathematics* (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) for early childhood (kindergarten to Grade 2 and ramifications for preschool). First, we describe our storyline about the CCSSM in the early grades as a research-based alternative to storylines that are inconsistent with research findings. We address the media's critical storyline and show why and how it is not based on the full body of relevant research. We then provide a detailed response to one criticism, that the CCSSM in the early years are not developmentally appro-

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priate, because this criticism is persistent (Clements, Fuson, & Sarama, 2017a) and is purportedly supported by a large body of research. We find that many critiques reflect a long-standing debate about the primacy of development versus learning and may not fairly represent the relevant research corpus.

The CCSSM Calls for Research-Based Teaching Through Learning Trajectories

Children in some communities are provided with more opportunities to learn mathematics than children in others (e.g., Baroody & Purpura, 2017; Clements et al., 2017a). This gap hurts children who live in poverty and who are members of linguistic and ethnic minority groups (Clements et al., 2017a; Cross, Woods, & Schweingruber, 2009). These differences in opportunity to learn are especially problematic because of evidence of the importance of early learning in math (Baroody & Purpura, 2017; Clements & Sarama, 2009; Sarama & Clements, 2009). As an example, preschool mathematics knowledge predicts achievement even into high school (National Mathematics Advisory Panel, 2008). For these reasons, all children deserve opportunities to develop a full range of mathematics competencies in the early years.

Awareness of these issues led to the appointment of a National Research Council (NRC) Committee that issued a report in 2009 entitled *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (Cross et al., 2009). This report summarized research-based, foundational, challenging-but-achievable goals for children in preschool through Grade 2 based on learning trajectories. Because the Common Core standards were based on the NRC report and research on learning trajectories, the standards describe and thus can guide paths of learning across the grades. Thus, the CCSSM can be a tool for equity because it specifies what children need to learn to catch up to others with more learning opportunities. Preschool programs based on such learning and teaching trajectories can enable children to enter kindergarten ready to engage with the learning involved in the Common Core standards (e.g., Clements & Sarama, 2011, 2014; Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Frye et al., 2013; Hachey, 2013; Klein, Starkey, Clements, Sarama, & Iyer, 2008; Lewis Presser, Clements, Ginsburg, & Ertle, 2015).

To summarize, the Common Core standards are consistent with teaching in the early years that supports children through research-based learning trajectories, which can serve as a helpful equity tool to focus efforts to provide opportunity to learn for children who have less opportunity than others.

The Media Storyline About Early Childhood Math Is Contradicted by Research

To identify the media storyline about early childhood math, we searched the research literature and news and social media platforms and consulted with the presidents of the National Association for the Education of Young Children (NAEYC), the National Council of Teachers of Mathematics (NCTM), and the National Council of Supervisors of Mathematics (NCSM; see Clements et al., 2017a). The main categories of criticisms included that the Common Core standards are not developmentally appropriate; that they dictate scripted curricula and didactic instruction rigidly applied to all children at the same pace; and that they emphasize academic skills and leave no time for play, exploratory approaches, or social–emotional development. Many of the criticisms are grounded historically in the debate, famously argued by Piaget and Vygotsky, as to what comes first, development or learning (Fuson, 2009). The criticisms have their foundation in the Piagetian developmental primacy hypothesis (i.e., the ontological precedence of biologically driven cognitive development, viewed as a prerequisite of learning). We discuss these criticisms and the evidence that strongly refutes them in detail elsewhere (Clements et al., 2017a).

In this commentary, we compare this critical storyline to other more productive storylines that have been identified in mathematics education. The critical storyline is similar to the first storyline described in Herbel-Eisenmann et al.'s (2016) commentary "There Are Two Dichotomous Ways of Teaching Mathematics": basic and discovery learning (see pp. 104–105). In the United States, these have often been characterized as skills and memorization versus concepts and meaning making. This dichotomous "math wars" storyline has been replaced in U.S. research-based national reports since 2001 by a multifaceted (Herbel-Eisenmann et al., 2016), balanced approach that includes both understanding and fluency and generally moves in each math domain from meaning making and supporting understanding of concepts to a focus on practice to gain fluency to prepare for the next level of conceptual learning (Cross et al., 2009; Kilpatrick, Swafford, & Findell, 2001; National Mathematics Advisory Panel, 2008; NRC, 2000). This perspective has been extended to the cycles of understanding and fluency in math talk communities that support all children through their own learning trajectories (Clements & Sarama, 2011, 2012, 2014; Fuson, 2012; Fuson & Murata, 2007; Fuson, Murata, & Abrahamson, 2015). These learning trajectories synthesize myriad research programs (Fuson, 2009; Sarama & Clements, 2009); here, we emphasize that they include the static descriptions of knowledge states and their progressions that Piagetian research engendered as well as the dynamics of learning and development that Vygotsky and others spearheaded, with subsequent studies showing that, although they interact (Fuson, 2009; Sarama & Clements, 2009), generally, learning produces development (NRC, 2000).

The major criticism of the CCSSM for young children is that the standards are not developmentally appropriate. These assertions are often general and do not discuss specific standards (Clements et al., 2017a). However, one example of this criticism of the CCSSM, *Selected Standards from the Common Core State Standards for Mathematics, Grades K–3: My Reasons for Not Supporting Them* by Kamii (2015), focuses on many different standards, discusses them in detail, and cites research ostensibly supporting this position. For this reason, we respond to this report in detail and explicate problems with the research used in it.

Research and Specific Common Core State Standards

This report (Kamii, 2015) is found on the website of Defending the Early Years (DEY). The foreword introduces the report as a research-based "report showing how selected Common Core mathematics standards for Kindergarten-Grade 3 cannot be supported by research" (Carlsson-Paige, 2015, p. 3). It goes on to say that "Dr. Kamii shows that selected Common Core math standards for Kindergarten-Grade 3 are not grounded in the large body of research on how children learn mathematics" (p. 3). In this commentary, we argue that there is in fact a large body of research that supports these standards, and we describe several problems with the critique presented in the report.

The remainder of this section discusses the nature of the problems in the evidence in the Kamii (2015) report: (a) relying only on Piagetian structures and results that have since been modified by a substantial research corpus and ignoring relevant non-Piagetian research about specific standards, (b) using complex tasks not matched to standards, (c) confusing teaching by rote and teaching meaningfully, (d) not considering opportunity to learn or the quality of instruction, (e) ignoring related standards (arithmetic problems), and (f) just waiting for children to develop ideas. As we shall see, many of these problems stem from a commitment to the earlier phases of Piaget's work, which emphasized the developmental primacy hypothesis.

Exclusion of Relevant Non-Piagetian Research

Kamii's report (2015) defines children's learning and understanding of number and counting only from Piaget's original position as requiring a synthesis of hierarchical inclusion and seriation operations. From this perspective, counting is ineffectual with "no connection between the acquired ability to count and the actual operations of which the child is capable" (Piaget [& Szeminska], 1941/1952, p. 61; see also Piaget, Inhelder, & Szeminska, 1948/1960). However, research for many years, including 17 chapters of research by Piagetians (Bideaud, Meljac, & Fischer, 1992), has substantially changed this position, showing that preschoolers and kindergartners understand the relationship between cardinality of the collection and the individual items in the collection and can use counting meaningfully to quantify, judge equivalence, and solve arithmetic problems (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Clements, 1984; Cross et al., 2009; Fuson, 1988). None of this research was referenced in Kamii's (2015) report.

The references in the Kamii report consist of 27 Piagetian papers, two references to the National Assessment of Educational Progress (NAEP) used to show how students cannot do area tasks, one reference to Ravitch, and one reference to the CCSSM. Of the 27 Piagetian papers, 14 have Kamii as an author, one is a dissertation from Kamii's university, 11 papers have Piaget as an author, and one is by Morf (1962). This constitutes a very narrow review of the literature that does not sufficiently represent the full breadth of relevant work in the field over many decades.

As a first example, counting objects accurately is a kindergarten standard (K. CC.4a, CCSSM): "When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object" (NGA & CCSSO, 2010, p. 11). Kamii's (2015) report critiques this standard as inappropriate by appealing to Piagetian theory:

> Children become able to "pair each object with one and only one number name" when they have constructed these logico-mathematical relationships. Hierarchical inclusion and order cannot be taught *directly*, but they can be taught *indirectly* by encouraging

> children to *think*. . . . children can be encouraged to *think* in daily living while cleaning up spilled milk and in activities like Pick-Up Sticks. (p. 9)

However, many research studies indicate that children can learn to count accurately if given opportunities to learn to do so (e.g., Clements & Sarama, 2014; Cross et al., 2009; Fuson, 1988). Children as young as 4 years of age can learn to count (Clements & Sarama, 2014). For example, low-SES 4-year-old children who lacked these and other core competencies not only learned them but spontaneously demonstrated performance characteristic of middle-class 6-year-olds when given many opportunities to count (Griffin, Case, & Capodilupo, 1995).

Kamii's (2015) report critiques kindergarten (Standard K.NBT.1) and Grade 1 (Standards 1.NBT.2a and 1.NBT.2b) place-value standards as inappropriate for their grades. She states that two levels of understanding of ten are involved in place value. We agree, as do the CCSSM. Kindergartners are to think of the ten ones within any teen number, and in Grade 1, children can think of those ten ones also as one group of ten. This distinction was suggested to the authors of the CCSSM by early childhood educators who thought that the original draft with the notion of "one ten" was too ambitious for kindergarten; therefore, this concept was placed at Grade 1 in a revision of the draft.

After making this distinction, Kamii (2015) rejects the kindergarten level of ten ones by saying

> To gain "a foundation for place value," however, children need to become able to think about a *ten* logico-mathematically . . . . Note that when children think about "one *ten*" logico-mathematically, they abstract the *ten* out of the *ones* that are in their heads and think about one *ten* and ten *ones* simultaneously. (p. 10)

But this "one ten" is not the ten ones in the kindergarten standard.

Kamii's (2015) critique of the Grade 1 standards that do involve this one ten is as follows:

> A "ten" . . . is logico-mathematical knowledge, which is not observable, but a bundle of ten ones is observable. It is not possible to use this physical knowledge with the social-conventional knowledge of words like "ten" to teach the logico-mathematical knowledge of "ten." (p. 11)

So, Kamii rejects these Grade 1 standards because of their use of descriptions of the conceptual "one ten" that use observable things rather than her language of logico-mathematical knowledge of "ten," even though the Grade 1 standard is the same as her picture of one ten that shows observable things.

Considerable research indicates that children can construct mathematical meanings for symbols and words from seeing quantities or situations for those symbols and words, acting on them, and talking about them. This is the dominant approach used in most of the studies reported in this section and in the research about early childhood math in the NRC report (Cross et al., 2009). Specifically, about "one ten," researchers working with predominantly Latino low-SES first graders promoted their thinking about two-digit quantities as tens and ones by having them use and discuss objects and then drawings of tens and ones. By the end of the year, most of the children could accurately add and subtract two-digit numbers that require trading (regrouping) by using drawings or objects, performing well above higher SES U.S. first graders and equaling the performance of East Asian children (Fuson, Smith, & Lo Cicero, 1997). Children also demonstrated their Grade 1-level understanding of the higher level of a ten in teen numbers using a digit-meaning task that Kamii used in several of her studies (i.e., Kamii, 1985, 1989). Kamii (2015) reported that many U.S. students, even up through Grade 5, pointed to one object instead of to ten objects as the meaning of the 1 in 16. However, the Latino low-SES first graders given opportunities to understand tens and ones using objects and their own drawings performed at the same level as Kamii's affluent sixth graders in identifying the 1 in 16 as ten objects.

We conclude that Kamii's summary is misleading because it is based solely on Piagetian theoretical structures and omits a large body of subsequent research. More recent research strongly supports children's ability to learn Common Core standards if provided with research-based opportunities to do so.

Using Complex Tasks Not Matched to Standards

Standard K.CC.4c, "Understand that each successive number name refers to a quantity that is one larger" (NGA & CCSSO, 2010, p. 11), is rejected by Kamii's report in this way: Morf's (1962) research "has shown that it is not until third grade that children become able to relate each subsequent number with the +1 operation. Morf came to this conclusion with experiments about 'connectedness' that I replicated" (Kamii, 2015, p. 9) by dropping 30 cubes one at a time into a glass with 2 cubes sitting beside a glass with 15 cubes. Children were asked, "'When I was dropping one cube after another into this glass, was there a time when the two glasses had *exactly the same number*?'" (Kamii, 2015, p. 9). Only by third grade did children give "clear, logico-mathematical justifications" (p. 9). This task introduces complexities far beyond the standard that it ostensibly invalidates. The proficiency that the standard describes is achievable by kindergartners. Indeed, a common preschool activity is to add one to a group of five and ask "How many now?" (Clements & Sarama, 2007/2013; Greenes, Ginsburg, & Balfanz, 2004; Klein et al., 2008). Both preschoolers and children with special needs can use the generalization by giving the next larger number name if they are given educational experiences that include such tasks (Baroody, 1999; Clements et al., 2011; Klein et al., 2008).

A second example is the third-grade Standards 3.OA.1 and 3.OA.2 on multiplication in situations involving equal groups. Kamii's (2015) critique is that such "hierarchical thinking was not possible for about a third of the middle-class third graders interviewed by Clark and Kamii (1996)" (p. 14). But the task that they used is a multiplicative comparison task (CCSSM Grade 4 Standards 4.OA.1 and 4.OA.2), which is more complicated than the Grade 3 standards involving equal groups.

Confusing Teaching by Rote and Teaching Meaningfully

Kamii's report states that "two-digit subtraction is still too hard for some fourth graders" (p. 12), and criticizes Standard 2.NBT.7 about using concrete models or drawings to add and subtract within 1,000 because "only a minority of second graders can add and subtract three-digit numbers (Kamii, 2004)" (Kamii, 2015, p. 12). A further critique is that the CCSSM require students to use the algorithms (rules) of carrying and borrowing in Grade 2 and that this is too early because "too many third graders unlearn place value because of using these algorithms (rules)" (p. 16). Her evidence is from interviewing children taught traditionally with a focus on the steps in the algorithm.

People often confuse any use of algorithms with teaching algorithms by rote because that is the predominant history of such teaching in the United States. However, such methods can be taught meaningfully so that students can understand them and explain their reasoning, as specified in the CCSSM (e.g., in East Asia and some math programs in this country, Fuson & Li, 2014). Fuson and Briars (1990) enabled second-grade children to add and subtract four-digit numbers in a large urban district, and these children could explain their thinking. Fuson and Beckmann (2012), who worked on the learning progressions explicating the Common Core standards, discussed the mathematical meaning of the standard algorithm as using understanding of adding and subtracting single-digit numbers and then using understanding of place value when it became necessary to compose ten of one kind of unit or decompose a unit to get enough to subtract. They identified several different ways to write methods that met this meaning of the standard algorithm, and they identified criteria for which methods are better than others. The methods described by Kamii as those that children naturally use are included in these methods. The Common Core standards on multidigit addition and subtraction emphasize understanding, use of place-value meanings for quantities, and explaining your method. They are not consistent with rote teaching of algorithms and thus are not subject to Kamii's criticism.

Not Considering Opportunity to Learn or Quality of Instruction

A related issue is that for several critiqued standards, Kamii's (2015) report includes statements about what children cannot do, but the research used gives no information about children's opportunity to learn that topic or the quality of that opportunity, suggesting, again, an implicit grounding in the developmental primacy hypothesis. Three examples are (a) it is "not possible for about a third of the middle-class third graders interviewed by Clark and Kamii (1996)" (p. 14) to do multiplication, (b) "two-digit subtraction is still too hard for some fourth graders" (p. 12), and (c) NAEP "has been showing repeatedly that even 7th graders or 13-year-olds cannot use the formula of length × width" (Kamii, 2015, p. 18). We already discussed the first two examples above. Regarding the area topic, Barrett, Clements, and Sarama (2017) found that U.S. second graders who are provided research-based experiences learned a wide range of area concepts and skills, including meaningful understanding of area as length times width.

Ignoring Related Standards: Arithmetic Problems

Kamii critiques the standard about first graders working with addition and subtraction equations that have the unknown in various positions (Standard 1.OA.8) such as 3 + _ = 5 as cognitively out of range until second grade when children have reversible thinking (Kamii, 2015, p. 11). However, most 4- to 5-year-olds in high-quality environments, when asked, "Give me 5 cubes. OK, now watch, I'm going to hide some! [Hides 2 in one hand, then shows the 3 in the other hand.] How many am I hiding?" will eagerly answer, "*Two!*" Opportunities to learn conceptual subitizing—quickly recognizing parts and wholes of small numbers—are important and developmentally appropriate (Clements et al., 2011).

Furthermore, the position that first graders cannot solve such problems ignores the fact that this standard is the eighth standard in the Operations and Algebraic Thinking (OA) domain. Earlier standards support meaning making for equations. There is a large international research base that outlines the learning trajectory in children's ability to do all eight of the Grade 1 OA standards (e.g., see the research summary in Fuson, 1992). Most children, some even in kindergarten, can use equations to represent various unknowns including the unknown addend, and they do so spontaneously (Fuson, 1992). Cognitively Guided Instruction (CGI; Carpenter & Franke, 2004; Carpenter et al., 1998) is an extensive program of training for teachers about this research so that they can use it in the classroom. CGI has been implemented successfully in districts in many states, indicating that the OA standards are developmentally appropriate and are accessible by using objects or drawings to model situations.

Just Waiting for Children to Develop Ideas: Cognitive Development and Learning Trajectories

In several passages (e.g., missing-addend problems and measurement), Kamii's (2015) summary states that children simply develop mathematical ideas if we wait, the most explicit sign of her commitment to the developmental primacy hypothesis and rejection of more recent research that shows that learning does not depend on development, as defined by Piaget (Fuson, 2009; NRC, 2000; Sarama & Clements, 2009). For example, "children become able to iterate a unit in fourth grade, without any instruction," and thus "it is a waste of first graders' time" to teach such iteration in first grade (Kamii, 2015, p. 12), as specified in Common Core Standard 1.MD.2. This contradicts evidence that children can learn these tasks with understanding before fourth grade and that many do learn them earlier (e.g., Barrett, Clements, & Sarama, 2017; Sarama & Clements, 2009). Such separation of development from teaching and learning is contradicted by theory and research (e.g., Fuson, 2009; NRC, 2000; Sarama & Clements, 2009) and invites a passive approach of "waiting" that is especially deleterious to children with fewer opportunities in their homes and communities to learn mathematics. Denying children the opportunity to learn important mathematical knowledge when others have such opportunities is not equitable or sensible. For some children with limited previous opportunities to engage with mathematics, denying such instruction relegates them to trajectories of academic failure.

Summary

In the discussion above, we have shown that the report is based on a narrow review of relevant research that fails to incorporate findings from large bodies of Piagetian and non-Piagetian research that have built on, challenged, and modified Piaget's early work. Therefore, the report does not show "that selected Common Core math standards for Kindergarten-Grade 3 are not grounded in the large body of research on how children learn mathematics" (Carlsson-Paige, 2015, p. 3). In fact, we argue that the Common Core mathematics standards for young children are solidly based on decades of research. It is important that we do not deny children, especially those most in need, the opportunity to learn the developmentally appropriate foundational and achievable research-based goals outlined in the NRC report (Cross et al., 2009) and contained in the CCSSM for Grades K–3. Mathematics educators at all levels can help correct problematic uses of research such as those we identified above so that discussions about the CCSSM are based on adequate research-based knowledge.

Final Words

We suggest that one powerful way for mathematics educators at all grade levels to provide alternatives to the misleading dichotomous storylines is to share examples of balanced research-based teaching based on learning trajectories. We shared such examples in our two papers addressing the criticisms of the CCSSM in the early grades (Clements et al., 2017a; Clements, Fuson, & Sarama, 2017b), and the websites karenfusonmath.com, LearningTrajectories.org, and http://www.umich.edu/~devteam/ include resources for the CCSSM domains at other grade levels. The NCTM and NAEYC books give examples for number and geometry learning trajectories in pre-K through Grade 2 (see, e.g., www.nctm.org/store/Products/Focus-in-Grades-PreK-2--Teaching-with-Curriculum-Focal-Points), and other curricula and curricular approaches are available (e.g., Clements et al., 2017a; Clements & Sarama, 2007/2013; Frye et al., 2013; Fuson, 2013/2018; Greenes et al., 2004; Hachey, 2013; Lewis Presser et al., 2015).

It is time for everyone involved in mathematics education to abandon harmful storylines that pit against each other either two simplistic views of teaching and learning (skills and memorization versus concepts and meaning making) or the equally pernicious views of development versus learning. These need to be replaced by active support of the research-based storyline that we identified at the beginning of the paper: The CCSSM Grades K–2 call for balanced teaching of developmentally appropriate, research-based learning trajectories. We especially need to address equity issues in the early grades so that children enter later grades ready and eager for the mathematical challenges and opportunities of the CCSSM.

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