U.S. English-speaking Kindergarten and First-grade Children from Backgrounds of Poverty Can Understand Place Value

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The PPT used for this presentation is in the section on this website karenfusonmath.com called Visual Presentations. This paper is a written extended version of the data and concepts presented in that presentation. Please also see that PPT and the Teaching Progressions on this website karenfusonmath.com for Numbers Base Ten, NBT Parts 1 and 2, for more details of the teaching approaches used for this paper. You can also see on this website karenfusonmath.com classroom videos of some of these learning supports and how they are used in the Classroom Videos (select MENU and Classroom Videos and then A Classroom Components Parts 1, 2, 3, 6, 7; D Kindergarten all parts; and G Place Value and Multidigit Addition and Subtraction Parts 1, 2, 4).

Abstract

Children speaking European languages with irregularities in the number words do less well on place-value tasks than do East Asian children speaking regular number words such as *ten six* or *seven ten nine.* Also, children from backgrounds of poverty do less well than do children from middle or upper SES backgrounds on many numerical tasks (see research reviewed in Cross, Woods, & Schweingruber, 2009). Can these linguistic and experiential disadvantages be overcome with effective teaching approaches? The ambitious place-value understandings in the U.S. Common Core State Standards (CCSS) that include understanding tens in teens in kindergarten and adding 2-digit numbers with regrouping in grade 1 provide a good test. Year-long teaching experiments with kindergarten and grade 1 children from backgrounds of poverty found that on place-value tasks, these children outperformed U.S. children from higher grades and/or from higher SES backgrounds and were equivalent to the performance of East Asian children, who usually outperform U.S. students on place-value tasks. Conceptual place-value structures children need to learn are outlined and related to the visual, linguistic, and sensory-motor learning supports in the teaching experiment. Children from all backgrounds can understand place-value concepts. It is time for a worldwide conversation about how to support such learning for everyone.

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Cross-cultural work on East Asian children's numerical thinking indicates that they build highly effective ten-structured conceptions of numbers, use these very well in their addition and subtraction, and outperform U.S. children on single-digit and multi-digit addition, subtraction, and place-value tasks (e.g., Geary et al., 1996; Fuson & Kwon, 1992a, 1992b; Miller & Stigler, 1987; Miller & Zhu, 1991; Miura et al., 1988; Song & Ginsburg, 1987; Stigler et al., 1990).

Building conceptual structures that connect multiunit quantities, English count words, and place-value written numerals is more difficult in English than in Chinese-based languages because the latter languages say the ten and the ones in the numbers explicitly and in the order of the digits in written numerals. For example, 14 is said as *ten four* and 53 is said as *five ten three*. English has reversals in the teen words and does not say the ten clearly: In English 14 is said as *fourteen*, and *teen* does not sound like *ten*. Also in English 11, 12, 13, and 15 do not clearly say the ones numbers 1, 2, 3, 5. So it is difficult to hear the regular ten and some ones pattern for 11, 12, 13, 14, etc. as East Asian children hear the pattern: *ten one, ten two, ten three, ten four*, etc. These difficulties continue for numbers 20 to 99 in English. In East Asian languages the pattern is regular after 20: 21 is *two ten one,* 22 is *two ten two,* 23 is *two ten three*, etc. This regularity continues to 99, so that 68 is *six ten eight* and 95 is *nine ten five*. The English words from 21 to 99 do say the tens first and then the ones, consistent with the order in the written numerals. But they never say ten, instead adding *–ty* to decade words and including several irregularities for the first part of the word: *Twenty* (not *twoty* or *two ten), thirty* (not *threety* or *three tens), fifty* (not *fivety* or *five tens).* There is also auditory confusion between teen and decade words that can be confusing in the classroom: *fourteen and forty, sixteen and sixty, seventeen and seventy*, etc.

Fuson, Smith, and Lo Cicero (1997) pointed out that children speaking languages with the regular and explicit tens and ones structure as in Chinese-based languages have a simpler learning task than do English-speaking children or children with other irregularities in their number words. Research has identified a developmental sequence of place-value conceptual structures for two-digit numbers that English-speaking children use (Fuson, 1990, 1998; Fuson, Smith, and Lo Cicero, 1997; Fuson, Wearne, et al., 1997; research summarized in Kilpatrick, Swafford, & Findell, 2001). These conceptual structures vary in how quantities are conceptualized, and these concepts can arise from the visual arrangements of quantities children experience. All conceptions relate number words to written number symbols to varied conceptions of the quantities indicated by the words and written symbols. Somewhat different terms were used earlier, so for the present study we simplified the earlier frameworks of conceptual structures and renamed some structures to clarify them. These conceptual structures are:

The incorrect **Concatenated Single Digits** conception: 16 is just a *one* and a *six* and children will give one object instead of ten objects as the meaning for the digit 1 in the tens place.

Unitary conception: Count by ones a collection of single units and the last English counting word tells how many are in the collection, e.g., *sixteen*.

Tens-Total-and-Ones conception: Count the collection by ones but can see the tens-total as separate from the ones: *sixteen is ten and six* 16 = 10 + 6 and *seventy nine is seventy and nine* 79 = 70 + 9.

Sequence-Count-by-Tens-and-then-by-Ones conception: Count each group of ten ones by English tens words *ten, twenty, thirty, forty, fifty, sixty, seventy* and then shift the unit of counting to ones: *seventy-one, seventy-two, seventy-three, …, seventy-nine.*

Tens-and-Ones conception**:** Count or see each tens quantity as 1 ten and use tens and ones words to give the multiunit value: *one, two, three, four, five, six, seven tens* and *one, two, three, four, five, six, seven, eight, nine ones* to make *seven tens nine ones.*

The incorrect Concatenated Single Digits conception and the Unitary conception use only units of one for all objects in the collection. The Tens-Total-and-Ones conception still uses units of one but the total collection named by the tens digit is differentiated from the collection named by the ones digit. The final two conceptions use units of ten and units of ones but count them in different ways, using English sequence tens words or regular count words to count units of ten.

East Asian children predominantly need to use only the Unitary and the Tens-and-Ones conceptual structures. The other three conceptions may be used briefly but the regular named tens in the East Asian counting words direct attention to groups of tens and ones so that these readily become viewed by the final Tens-and-Ones conceptual structure. East Asian children do not have to learn the complex English sequence of tens words with their irregularities but would count groups of ten ones as *one ten, two ten, three ten, four ten, five ten, six ten, seven ten* and then count by ones: *seven ten one, seven ten two, seven ten three, seven ten four, etc.* Such counting readily collapses to the final Tens-and-Ones conception. Learning this Tens-and-Ones conceptual structure is also supported by the presence in East Asian children's lives of ten-structured cultural artifacts and experiences (e.g., objects packaged in fives, the metric system), the teaching of ten-structured methods for adding and subtracting single-digit and multi-digit numbers (e.g., Fuson & Kwon, 1992a, 1992b; Fuson, Stigler, & Bartsch, 1988; Miura et al., 1988; Murata, 2004), and by the support of parents and teachers in demonstrating ten-structured quantities (e.g., Huntsinger et al., 2000; Miller & Stigler, 1987; Yang & Cobb, 1995).

The U.S. Common Core State Standards (CCSS) specify for kindergarten ambitious understandings of teen numbers as ten and some ones and for grade 1 understanding of 2-digit numbers as tens and ones and of adding 2-digit numbers including adding with regrouping to compose a new ten (CCSSO/NGA, 2010). These standards drew from the foundational and achievable goals in the National Research Council report *Mathematics learning in early childhood: Paths toward excellence and equity* (Cross, Woods, & Schweingruber, 2009) that drew from world-wide research including that on East Asian children. But U.S. children from low SES backgrounds ordinarily do more poorly on number problems than do children from middle or upper SES backgrounds (e.g., see the research summarized in Cross, Woods, & Schweingruber, 2009). Therefore such children are a special concern for these ambitious U.S. Common Core State Standards tens and ones standards. Whether kindergarten and grade 1 children from backgrounds of poverty can understand place value tens and ones as specified in the ambitious Common Core State Standards is the central focus for this study. Because most of the children in the Western Hemisphere speak English, Spanish, Portuguese, or French, which have irregular place-value words, the answer to this question is important even if the place-value goals are in different grade levels in other countries.

An important aspect of such understanding is in standard 1.NBT.4, which asks first graders to relate math drawings to a written method for adding 2-digit numbers with regrouping. Fuson, Smith, Lo Cicero (1997) found that first graders from low SES backgrounds could use math drawings to add 2-digit numbers with regrouping. Children made drawings and wrote the answer in the problem. That approach was extended in the present study to support first graders to use math drawings to develop understanding of tens and ones but also to develop written methods in which the new one ten was recorded on the written problem as well as shown in the drawing so that eventually the written method could be used meaningfully without drawings.. We also wished to focus on methods that would meet the CCSS Grade 1 Critical Area (2) requirement for "efficient, accurate, and generalizable methods" that could be extended to three-digit and larger numbers but also be accessible to children.

Materials and Methods

We report here performance of low SES kindergarten and/or grade 1 children on nine kinds of tasks involving place-value performance and understandings of tens and ones. Several tasks have been used in earlier research to assess place-value understanding, especially in comparing East Asian students to students who speak European languages. We used in this study three such tasks and a fourth task requiring understanding of a ten as a dime. Where a reference is included in the list, comparison data from that study will be given in the results. These comparisons involve for tasks 4, 6, and 8 East Asian children and for tasks 4, 5, 6, 7, and 8 U.S. children who might be expected to perform better than our low SES sample (these samples were from a range of backgrounds or from high-achieving schools or from higher grades). We use these comparison data because interview data about mathematical understandings are robust and gathering such data is time consuming and difficult. These interview data can indicate aspects of place-value understanding that may not be obvious and so are especially useful to teachers.

Setting and participants

All participants came from three urban schools with high levels of children receiving free lunch, a common criterion for low socio-economic status (SES) (100%, 96%, and 68% for Schools A, B, and C). All children in School A were native Spanish-speakers; the children participating in the study were from half-day bilingual classrooms primarily conducted in English. Children in the full-day classroom in School B were from a range of ethnic and linguistic backgrounds including native English speakers, but they spoke English well enough to be in an English-speaking classroom. Most children in the full-day classrooms in School C were native English-speakers, but some were native speakers of Spanish or other languages. English was the classroom language, but teachers used Spanish occasionally to clarify issues. Schools A and B were from a large urban school district; school C was in a small heterogeneous city bordering on that district. The number and school of participants varied by task and are given below in the tables or description of results. These data were collected in year-long teaching experiments in kindergarten and grade 1 classrooms using the research-based program *Math*

Expressions (Fuson, 2006) published by Houghton Mifflin. After the data are presented, the learning approaches in the year-long Teaching Experiment will be described and related to the interview and other tasks and to the identified place-value conceptual structures.

Tasks and procedures

The nine tasks are each described in a separate section below along with the results of that task. Sources of comparison samples are listed here in parentheses and their results are described in the results section for that task.

1. Kindergarten tasks learning counting words, written numerals, and quantities;
2. Kindergarten children drawing tens and ones;
3. Kindergarten children writing numbers to 30 and to 100;
4. Kindergarten children showing understanding of Embedded-Ten Cardinality for teen numbers (Ho & Fuson, 1998);
5. Kindergarten and grade 1 children showing ten chips as the meaning for the 1 in 16 (Kamii, 1989);
6. Kindergarten and grade 1 children using tens and ones on the Miura (1988) 2-digit representation task;
7. Grade 1 children correctly giving change from a dime (Chandler & Kamii, 2009);
8. Grade 1 children adding 2-digit numbers in a word problem (Stigler, Lee, & Stevenson, 1990);
9. Grade 1 children adding 2-digit numbers in word problems and in vertical number problems.

Results

Task 1: Kindergarten tasks connecting counting words, written numerals, and quantities

Table 1 (see below) summarizes tasks done in each kindergarten unit to help children learn the English counting words to 100 and form for these number words the basic conceptual triad of a *number word* related to its *written numeral* related to *that quantity of things*. These connections for earlier units involve CCSS K.CC.1, 3, 4, and 5 and are necessary prerequisites for the K.NBT.1 standard for teen numbers that is a major focus of this paper. Children learned the order of the counting words within triad tasks in activities in which they were always focusing on at least two of the connections among number words, written number symbols, and quantities.. It takes a long time to learn all of the counting words and the written numerals and make all of the connections, especially for some children who could not count at all or recognize numerals at the beginning of the year. Teaching/learning activities focused on these tasks occurred in three settings: (a) short Quick Practice activities that began the math period and were continued for half of a unit (about 13 days), (b) lesson activities with objects and later with student activity pages that had drawings of things and of circles (or other simple shapes) and asked for numerals or showed numerals and asked for drawings, and (c) homework pages like the student activity pages done in class or at home. Children also did throughout the year adding and subtracting activities that used and provided further practice in the triad activities with totals ≤ 10. More details are given in the final section of the paper.

There were five major kindergarten units. The half-day kindergarten classrooms in this study got through Unit 4 and part of Unit 5. Full-day classes completed Unit 5 activities, which went more deeply into tens and ones and partners for (addends that totaled) 7, 8, 9.

The triad tasks summarized in Table 1 required children to show or say something, so teachers could observe what children were understanding as part of their on-going assessment for teaching. Unit observational assessments by the teacher were also done. These found 20 to 30% of the children not completely mastering a task in the specified unit, but all children whose attendance was reasonable did master those tasks within 2 months of the end of the unit. These triad tasks all build over units, and children continued to learn and become more fluent with practice. On the digit-meaning task 5 discussed below, 84 out of the 85 children interviewed at the end of the year counted out 16 cubes correctly or made 16 with a 10-block and 6 cubes, so children did learn to count out *n* objects accurately at least to 16. Thus, these Common Core State Standards K.CC.1, 3, 4, and 5 are accessible to children from backgrounds of poverty.

Table 1

Kindergarten Instructional Triad Tasks Learning and Connecting Counting Words, Written Numerals, and Multiunit Quantities

Tasks	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
Say the number-word sequence while looking at numerals in sequence, or at objects in sequence grouped in 5-groups and/or 10-group, or both. May be done with fingers to show the numbers including using flashes of all fingers repeatedly to show a group of ten.	1 to 5 then 1 to 10	1 to 20 also 11 to 20	20 to 60	60 to 100 1 to 100 by tens	100 to 120
Connect number words, written numerals, and grouped quantities: Given two of these, show or say the third.	"	"	"	"	"
Put numeral tiles in order In Unit 2 using layered tiles as in Figure 2.	1 to 10	1 to 20			
Count objects and choose (later write) the numeral to tell how many objects or pictures for things in any arrangement, in 5-groups, and in ten-groups and some ones. Later count out or draw *n* things.	1 to 5 1 to 10	1 to 10	6 to 10 in 5-groups	1 to 20 in 1 ten and some ones	1 to 20 in 1 ten and some ones
Show fingers slowly is ok	1-10	11-19 as 10&ones	20-60 as tens	10, 20, …, 90 as 10s	
rapidly		1-5	6-10		
Write numerals	1-5	6-10	11-20	1-30	1-120

Task 2: Kindergarten children drawing tens and ones on three tasks on the Unit 5 test

The Unit 5 test included three tasks that required children to show their understanding of a teen number as ten and some ones. Almost all of the children in School B correctly drew 16 things and circled ten of them. Two-thirds of these children drew 10 in a pattern (column of 10, 2 columns of 5); the rest drew a unitary 16 and circled 10 of these. In School C 66% of the children correctly drew 16 things and circled 10 of these and another 10% were partially correct. Children had done similar activities with objects and on activity sheets during the unit.

On the second task, most children in both schools (85% and 83%) correctly drew shapes in the squares of two 10-columns of connected squares to show 12 and 18, e.g., they drew ten shapes in the first column for 18 and then drew 8 shapes in the second column. Another 4% and 13% were partially correct on one or both of the columns.

On the third task most children in both schools (75% and 67%) colored (or shaded) squares in two horizontal rows of ten connected squares and also correctly filled in equations 13 = 10 + __ and 17 = 10 + __ for both numbers. Another 18% and 17% were partially correct on one or both of these parts of the tasks. So most children could show a decomposition of a teen number into ten and some ones in various ways including filling in how many ones into an equation, as stated in Common Core State Standard K.NBT.1.

Task 3: Kindergarten children writing numbers to 30 and to 100

Standard K.CC.3 only requires children to write numbers from 1 to 20. We were interested in whether kindergarteners could go farther than this. We also felt that they could appreciate the patterns involved in the tens and ones more readily if they went beyond 20 because the teen numbers are reversed and so are particularly difficult to write in the correct order. Many full-day children wrote numbers on the back of a page when they had finished the work or wrote in practice pages of horizontal or vertical boxes. They enjoyed doing this and seeing the patterns grow.

For this data collection, children wrote the numerals in connected rows of ten boxes. All scoring counted as correct the reversal of a digit (e.g., making the curves of the 3 open to the right) because this is a visual/developmental error some children make that will disappear. However, reversing the order of the ten and the one digit was not counted as correct because children need to learn to overcome the misleading order in the teen words in which the ones digit is said first but written second.

Most (91%) of the children in School A wrote all numbers to 30 correctly, and 75% of the children in School B did so. Most of the errors within 30 were systematic reversals of some segment of tens and ones: 4 children reversed 20 and 30 as 02 and 03 and 1 child reversed the teens and 20 (wrote 11, 21, 31, 41, …, 91, 02). Over half (58%) of the School B children wrote all numbers from 1 to 100 correctly. The errors all indicated partial understandings: 13% omitted 1 numeral once, 25% had most tens and ones correct but reversed some or all teens or one decade or most tens words, and one child wrote 1 to 22 and then shifted to writing teen numbers (21, 22, 13, 14, 15, etc). So most children had learned and could produce correctly most of the patterns in place-value numerals to 100, and the errors they were making seem likely to be overcome with discussion and practice.

Task 4: the embedded-ten cardinality task (Ho & Fuson, 1998)

In Ho & Fuson, a child watched the interviewer add some blocks into a closed box and counted aloud with the interviewer as each block was dropped. The interviewer stopped counting and paused, and then the interviewer and child together counted some more blocks dropped into the box. The interviewer then described what had been done, "First I put x blocks into the box, and then I put y more blocks in it. How many blocks altogether are in the box now?" Strategies and response time were recorded. An introductory problem of 2 + 1 was given first. Children received 10 + 2, 10 + 9, 4 + 2, 4 + 9 (the 10 + and 4 + problems were counterbalanced across children) or 10 + 5, 10 + 7, 4 + 5, 4 + 7 (again the 10 + and 4 + problems were counterbalanced across children). Because the interview time was shorter in the half-day *Math Expressions* classrooms and there were multiple interview tasks, the physical demonstration of putting objects into the box was not used in these classes. Children were just asked the addition problems, "What is ten plus two?" etc. So children were doing a more abstract version of the task, but the target knowledge was similar (How do children find the total of ten and some ones?). In pilot work, we found no differences in responses to these two methods. Children either knew the total of ten plus some ones or did not. As in Ho and Fuson (1998), children were classified as Understanders of Embedded-10 Cardinality if they gave rapid and accurate answers to all 10 + n problems and did not give rapid and accurate answers to all 4 + n problems.

All of the Ho and Fuson (1998) samples were middle-class kindergarten children, n = 36 Chinese Hong Kong children, n = 18 English children, and n = 12 U.S. children. The *Math Expressions* sample of 82 children was randomly drawn from four half-day and four full-day kindergarten classrooms in three schools (A, B, and C) with many English-language learners and almost all children from homes qualifying for free lunch and thus from backgrounds of poverty.

On the embedded-ten cardinality task of Ho and Fuson (1998), the *Math Expressions* kindergarten children performed at the same level as the Chinese kindergarten children: 35% versus 39% were Understanders of Embedded-10 Cardinality. Both of these groups were considerably above the English and U.S. children not receiving the *Math Expressions* tens in teens experiences (0% Understanders). This task is a high-level use of the knowledge of a teen number as ten and some ones because children could not see any collections of objects, and they had to remember the two numbers in the task and know the total rapidly without counting. Understanding a teen number as involving ten ones and some more ones is Common Core State Standard K.NBT.1 and uses the Tens-Total-and-Ones conception.

When asked how they got their answer, most of the *Math Expressions* Understanders said some version of "I just know that." but some referred to various learning tools: *We did ten and ones with our fingers. It's 1 and a 9 and it's hidden (*the 0 hiding under the 9 on the 10 tile, see later discussion of these Secret Code Cards*). I know all of these because I have a chart with all the numbers (the Number Pattern Chart)*. One of the 4 + problems given to each child in the *Math Expressions* sample had a total over 10 (4 + 7 or 4 + 9), so was challenging. But 34% of the sample counted with their fingers to solve such a problem correctly. They had not done such problems before.

Task 5: the digit meaning task: showing ten chips as the meaning for the 1 in 16 (Kamii, 1989). Students were shown a card with 16 written on it and asked to count out cubes to show that many. The interviewer then pointed to the 6 and said, *What does this part mean? Show me with the cubes what this part means.* The interviewer then pointed to the 1 and repeated the question. In the Kamii study many children showed 1 cube and not its cardinal meaning as ten ones or as one unit of ten. After this question, we gave two levels of prompts because we were interested in whether children had available a tens and ones meaning as well as whether this would be their first meaning. The first prompt was: *That is one thing it means. Can you think of something else that this might mean? Can you show me with the cubes?* The second prompt was: *This is a teen number. What does this* (point to the 1) *mean in the teen number? Can you show me with the cubes?*

The Kamii (1989) samples were from an upper-middle class school with high test scores, n = 32 and 40 for grade 2 and grade 3, respectively. The *Math Expressions* kindergarten children were from three schools (A, B, C) with high levels of students with free lunch and thus from backgrounds of poverty and many English language learners. Children were randomly sampled from four classrooms in School A, n = 39, and from 3 classroom in School C, n = 22. The School B sample was all 24 children in the class. The Grade 1 children were random samples from two classes in School A, n = 25. Understanding 16 as involving ten ones is Common Core State Standard K.NBT.1 and uses the Tens-Total-and-Ones conception. Understanding 16 as involving one unit of ten is Common Core State Standard 1.NBT.2 and uses the Tens-and-Ones conception.

More *Math Expressions* grade 1 children immediately showed ten than the number of Kamii (1989) grade 2 and the grade 3 children doing so, 64% versus 16% and 30%, respectively. Chi-square analyses found that these differences were significant at $p<.001$ for grade 2 and $p<.01$ for grade 3. Of the *Math Expressions* grade 1 children, 96% showed ten when the prompts were included. The kindergarten percentage immediately giving ten chips varied across classes from 75% for the full-day School B children to 23% for the full-day School C and 5% for the half-day children who had had less time on the ten in teens activities. An additional 21%, 23%, and 31% did show ten chips after one of the prompts.

Task 6: the Miura et al. (1988) 2-digit representation task

For this task children were given tens blocks (1x1x10 cm) and single centimeter cubes and asked to say and then make the number written in digits on a card (13 and 28). Kindergarten *Math Expressions* children had made teen numbers with such blocks but not numbers over 20. Grade 1 *Math Expressions* children had not used the blocks. The first trial assessed children's most accessible representation as some form of tens and ones (canonical or noncanonical) or as unitary (a collection of ones). Miura (1988) used five numbers: 11, 13, 28, 30, and 42 and reported little variation across these numbers in the use of canonical or non-canonical representations. We were especially interested in teen numbers for the kindergarten children, but also wanted to assess a larger number, so we used 13 and 28.

Results are given in Table 2. As with the Miura sample, there was little variation across the two numbers 13 and 28, so these data are collapsed in the table. The *Math Expressions* half-day kindergarten children looked like the Korean kindergarten children in the percentage making a canonical base-10 representation (40% and 34%), and even more of the *Math Expressions* kindergarten children made a noncanonical base-10 representation than did the Korean

Table 2

Percentage Use of Correct Representations on the Miura 2-Digit Task by Cognitive Representation Category

Trial 1 Representation	Miura (1988) upper-middle-class Grade 1				Miura Kinder-garten	*Math Expressions* low-SES Kindergarten		Grade 1
	U. S. $n = 24$	PRC $n = 25$	Japan $n = 24$	Korea $n = 40$	Korea $n = 20$	Half-day $n = 39$	Full-day $n = 46$	$n = 23$
Canonical base-10 1 ten 3 ones 2 tens 8 ones	8	81	72	83	34	40	79	94
Noncanonical base-10 10 + 3 ones 1 ten 18 ones	1	9	10	11	7	25	6	2
Unitary collection: 13 ones 28 ones	91	10	18	6	59	35	15	4

kindergarten children (25% vs 7%). *Math Expressions* full-day kindergarten and grade 1 children looked like the East Asian grade 1 children in their use of the canonical base-10 representation: 79% and 94% compared to 81%, 72%, and 83%. This representation could use the Sequence-Count-by-Tens-and-then-by-Ones or the Tens-and-Ones conceptual structure,

Task 7: giving change for a dime for a 6ç purchase (Chandler and Kamii, 2009)

This task was a store situation in which the child was the storekeeper and the adult interviewer bought something for 6 cents. The child had dimes and pennies in a box that functioned as the "cash register" and was initially asked to state the name and value of each kind of coin. The interviewer asked to buy an item that cost 6 cents and paid with a dime. Even though each child said the dime was ten cents, the issue was how the child would cope with the task of giving 4 pennies in change.

The Chandler and Kamii (2009) sample involved one class at each grade level from Kindergarten through Grade 4. In this sample 65% of the children qualified for free or reduced lunch and were attending an urban magnet school. The *Math Expressions* sample consisted of 10 children randomly sampled from each of two Grade 1 classrooms in School A, $n = 20$. All of these children qualified for free or reduced lunch, and all were native Spanish speakers but were in English-speaking grade 1 classrooms. The percentages giving 4 pennies in change for a 6c purchase in the Chandler and Kamii (2009) sample were 5%, 14%, 50%, 74%, and 84% for kindergarten, grade 1, grade 2, grade 3, and grade 4, respectively. All of the *Math Expressions* children gave four pennies, performing better than children at all grade levels kindergarten through grade 4 in the Chandler and Kamii sample.

Of the *Math Expressions* children, 90% stated the answer immediately. These all spontaneously gave some explanation:

- 15% said *because 4 and 6 are partners of 10*,
- 20% said *because 6 and 4 make 10*,
- 20% said *because if you counted 4 more you would get 10*, and
- 35% said things like *I just knew it; I have this in my head*.

The other 10% counted on from 6 to 10 to get 4 and then gave the 4 cents. So all of these first graders knew partners of ten (or could count to find an unknown partner) and understood one dime as ten pennies and could do this one-for-ten decomposing mentally (they could see one dime and think ten pennies). Knowing the partners of ten is the kindergarten Common Core State Standard K.OA.4.

Task 8: addition 2-digit word problem with regrouping (Stigler et al., 1990)

The 2-digit word problem for the Stigler et al. sample was given on a spring interview and read to the students. Paper and pencil were available. That same problem for the *Math Expressions* children was on a written test given in the spring. The teacher read the problem aloud. Chi-square analyses on the number of children answering correctly found that significantly more *Math Expressions* first graders answered correctly (79%) than did the Stigler, Lee, and Stephenson (1990) Japanese, Chinese, and U.S. first graders, 29%, 25%, and 13%, respectively, each $p<0.001$. The grade 1 *Math Expressions* children were close to the level of the grade 5 Stigler et al. (1990) sample, 79% vs. 86% correct.

Task 9: adding 2-digit numbers in word problems and in vertical number problems

To examine the consistency of the performance of and the strategies used by the *Math Expressions* children within and across word problems and vertical number problems, a 7-item follow-up test was given in one class on a later day. There were three word problems (these used 18 + 29, 26 + 45, 26 + 19) and four number problems written vertically (these used 37 + 46, 56 + 28, 29 + 63, 16 + 57). Children's responses were analyzed to see whether they just wrote an answer or recorded the new 1 ten in some written method as a step toward a generalizable method for larger numbers. Errors were examined for all problems to determine whether a child had done any of the steps in this multistep process correctly, i.e., did they exhibit partial accuracy.

All students solved at least 2 problems correctly. There was no difference in performance between the word problems and the vertical numerical problems: 78% vs 82% correct. The errors were always partial errors of one part of the multi-step addition process. Most students still made math drawings of the problem, but some did not. About a third of the students solved all 7 problems correctly, and about a third solved 6 out of 7 problems correctly. Most incorrect problems involved only one error in all of the steps required for a correct answer.

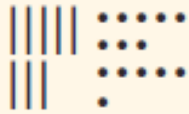
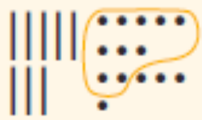
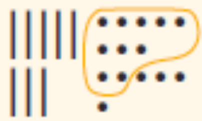
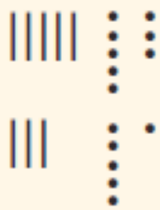
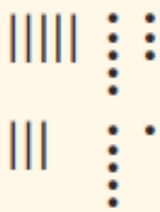
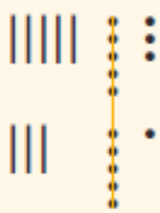
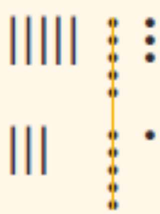
One or more children wrote the wrong numbers when solving a word problem, made a drawing in which the ones or the tens were off by one, or made a correct drawing and recorded the new 1 ten but did not add it into the final total of the tens.

All children recorded the new 1 ten somewhere in their written solution method. Across the class and problems three different methods were used. Two of these methods are shown in Figure 1. Most children used the method shown at the top: New Groups Below. Some children used the bottom method in Figure 1: Show All Totals. These methods were shown in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001, page 202) as accessible generalizable methods (see also the related discussion in NCTM, 2009). Some children used each of these methods for different problems on the test. Three children used a third method: the current common method New Groups Above, in which the new 1 ten is written above the tens in the problem instead of below as in the Figure 1 New Groups Below method. A Classroom Video showing first graders explaining these methods can be seen at karenfusonmath.com (click on MENU then Classroom Videos then C - Longer Classroom Teaching Examples then Part 1).

All of these methods had been developed in class, related to drawings of tens and ones, and discussed and explained as shown in Figure 1. Some children developed and shared other methods during the discussion but did not use them on the test. New Groups Below looks more like a Tens-and-Ones conception method, and Show All Totals looks more like a Tens-Total-and-Ones conception method or a Sequence-Count-by-Tens-and-Then-by-Ones conception method. But any of the conceptions can be used with either method in the explanation. We can see all of the conceptions in parts of the explanations in Figure 1. So repeatedly using and explaining both of these methods in class can help children to master and relate the various multiunit conceptions.

The drawings of tens and ones quantities for each step in Figure 1 show how the drawings of quantities suggest both major components in multiunit adding in ways that the written numerals do not: what to add to what (tens to tens and ones to ones) and making a ten when the ones total is ten or more (to avoid getting an incorrect 3-digit answer such as 814 for 58 + 36).

Figure 1 (next page). General and accessible methods for adding 2-digit numbers with drawings and explanations

	Problem	Math Drawing	Explanation Using Quantity Language About Tens and Ones
			New groups below
a.	58 + 36		For fifty eight I drew five tens and eight ones. For thirty six I drew three tens and six ones.
b.	58 + 36 4		I could see by looking at my eight that I needed two more to make ten. So I circled two from the six to make ten with my eight. That made fourteen. I wrote my four ones in the ones place and wrote my one ten down here in the tens place waiting until I add them.
c.	58 + 36 94		I added five tens and three tens to get eight tens and one more ten makes nine tens. I can check my answer by counting in my drawing: 10, 20, 30, 40, 50, 60, 70, 80, 90, 91, 92, 93, 94.
			Show all totals
a.	58 + 36		For fifty eight I drew five tens and eight ones. I made my 5-groups up and down. For thirty six I drew three tens and six ones.
b.	58 + 36 80		I have fifty and thirty, and that makes eighty that I write here in my problem.
c.	58 + 36 80 14		I make a ten with two fives. So ten and four is fourteen. I write fourteen in my problem.
d.	58 + 36 80 14 94		Now I need to add my eighty and fourteen to find my total. Eighty and fourteen is ninety four.

Show all totals can also be done by starting with the ones.

	a.	b.	c.
	58 + 36 14	58 + 36 14 80	58 + 36 80 14 94

Each of these methods has advantages. Major advantages of the New Groups Below method (writing the new 1 ten below instead of above the problem) are:

a. Children can see the total of the ones below the problem (see the 14 in step b in Figure 1).

b. Children can write this teen total in their usual way (write 1 then 4) instead of as the reverse as is frequently done in the New Groups Above method (write the 4 ones below and "carry" the 1 ten above).

c. Children can add the tens they see and then increase that total by the 1 new ten waiting below (5 plus 3 is 8 plus 1 is 9). With the New Groups Above method, children are often encouraged to add in the 1 new ten first so that they do not forget it (5 plus 1 is 6), resulting in their adding a number they cannot see (6) to the other number they see (3) while ignoring the number they can see (5).

d. Children do not have to change the problem (some children see writing numbers above the original problem as changing that problem, and indeed those numbers are changing that problem).

Major advantages of the Show All Totals method are:

a. Children can see the tens total and the ones total as separate totals, reflecting the two steps in multiunit adding.

b. Children can see the 0 in the Tens-Total as in the layered Secret Code Cards; many first graders are still needing to see the place-value parts that are used in the Tens-Total-and-Ones conception.

c. Children can do this method from the left, adding the tens first. Many children prefer to do math from the left when possible because it is consistent with reading in English.

There is a great deal of confusion about the use of the term *standard algorithm* in the Common Core State Standards and in general around the world. Fuson and Beckmann (2012/13) clarified these issues. First, there is no single standard algorithm. Many different algorithms have been used in this country and in other countries. *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) made this point strongly. Fuson and Beckmann (2012, page 15) also identified the crucial aspects of "the standard algorithm":

> Taken together, the NBT Progression document summarizes that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:
>
> • Decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number; and
>
> • Using the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.

Fuson and Beckmann also described variations in written methods for recording the standard algorithm for each operation, discussed what could be considered minor variations in these algorithms, and suggested criteria for evaluating which variations might be used productively in classrooms. Using their criteria, New Groups Below is conceptually clearer and easier than the other variations they showed (New Groups Above and a cross-out method where the new 1 is added into the top number and written above the problem). They identified the method called here in Figure 1 Show All Totals as a general method for 2- and 3-digit numbers that becomes unwieldy for larger methods and so is a helping-step method useful for early understanding but that then gets replaced in grade 3 or grade 4 by a variation of the standard algorithm. Fuson and Beckman emphasized that the critical area for a given multidigit computation specifies that "Students develop, discuss, and use efficient, accurate, and generalizable methods" so in Grade 1 it is important for students to see such methods as those in Figure 1.

Fuson and Li (2009) found that the two methods in Figure 1 were used in Chinese, Korean, and Japanese textbooks and that variations existed in how these were written. All of these East Asian textbooks used visual supports to show the multiunit meanings involved in these methods. Fuson and Li (2014) analyzed Chinese and U.S. math standards with respect to methods identified in the earlier paper. They concluded that the new U.S. Common Core State Standards do specify learning paths and supports as were found in the East Asian textbooks in Fuson and Li (2009). Chinese standards are less specific about such supports but perhaps this is because these supports are used widely and do not need to be specified. Fuson and Li (2014) also analyzed methods to find which methods can support conceptual issues in multidigit adding and subtracting, reduce errors, and make steps easier to carry out. They identified the New Groups Below method to be the best multidigit addition method and hoped that this superior variation would become widespread in both the U.S. and China. That low SES English-speaking first graders in the present study could be so successful with this method offers support for using this method more widely than it seems to be used at present.

Summary of Results

Our year-long teaching experiments found that the answer to the central focus of the study is yes, many children from backgrounds of poverty including those that are not native speakers of English can learn the ambitious Numbers Base Ten (NBT) Common Core State Standards for kindergarten and grade 1. Full-day kindergarten children learned more than did half-day children, which emphasizes how important it is to have 60 instead of 20 to 30 minutes for math. But even half-day kindergarteners looked like Korean kindergarten children in their use of tens and ones quantities. Full-day kindergarteners and first graders equaled the performance of East Asian children and considerably outperformed U.S. children of equal and of higher grades. Many children were able to overcome the Concatenated Single Digit meaning of the written numerals and show tens and ones quantities for a teen numeral 16 on the Kamii (1989) task and for a teen (13) and a twenties (28) numeral in the Miura et al. (1988) task. First graders learned to make drawings for 2-digit numbers and use such drawings to make sense of and to help explain a written method in which the new 1 ten was written in the method. These children used various methods, but most used the New Groups Below method found to be the best addition method in the analyses of Fuson and Beckmann (2012/13) and Fuson and Li (2014). The success of this method suggests that it be used more widely.

The Teaching-Learning Environment

So what kinds of experiences are needed for U.S. English-speaking children to understand place-value tens and ones? We now overview the classroom experiences of children in the year-long teaching experiment. We are not claiming that these are the best supports available anywhere. There are many effective supports for place-value thinking used around the world, and we read about and tried many of these in earlier teaching experiences. Some lovely and effective supports like the Montessori materials were not used because their cost was too great. We sought materials that would be relatively inexpensive and able to be used easily in the classroom. We also chose or developed materials that would form a coherent learning path to develop all of the conceptual structures identified in this paper and in earlier research.

The Math Expressions Teaching-Learning Approach

Math Expressions uses a Vygotskiian Class Learning Path Model (Fuson & Murata, 2007; Fuson, 2009; Fuson, Murata, & Abrahamson, 2015; Murata & Fuson, 2006; Vygotsky, 1978, 1986/1934) that emphasizes the use of visual, linguistic, and sensory-motor cultural tools within a nurturing Math Talk Community in which conceptual instructional conversations by the whole class relate visual quantities and steps in methods to math symbols and words. This teaching-learning approach supports and values individual student methods while helping students move along to more correct or more-advanced methods. For each new topic the teacher helps students move through four class learning zone phases:

- Phase 1: elicit, value, and discuss student methods,
- Phase 2: focus on or introduce mathematically-desirable method(s) and compare methods mathematically,
- Phase 3: gain fluency with desired method(s),
- Phase 4: ensure remembering by delayed practice.

For more information about this approach see the Teaching Progression Math Talk Community.

There is a huge amount of detailed knowledge involved in learning the related place-value conceptual structures for English-speaking children. Students must learn and relate different aspects of a given conceptual structure: visual quantities, English words, tens and ones words, and written numerals in regular and expanded form. The teacher and various students acting in a teacher role as Student Leaders lead attention to the various components (visual, verbal, and body-based) of the conceptual structures and relate these by gestures and coordination in time and/or space. Students also continually discussed relationships they saw. Learning all of this related knowledge and moving away from the incorrect Concatenated Single Digit conception and the simple Unitary conception to the multiunit Tens-Total-and-Ones, Sequence-Count-by-Tens-and-Then-by-Ones, and Tens-and-Ones conceptions was a major goal of the classrooms activities and practice.

We want to emphasize that the nurturing Math Talk Community uses a constructivist view of student thinking. When we describe a visual display such as a math drawing (a cultural tool) or an accessible solution method, we do not mean that it is sufficient for the teacher to demonstrate such tools or methods and that children will immediately see and use the relationships between quantities involved in those tools and methods. Where a new ‘level’ of complexity is involved such as the multiunit Tens-Total-and-Ones, Sequence-Count-by-Tens-and-Then-by-Ones, and Tens-and-Ones place-value conceptions, it takes a protracted collective meaning-making, relating, and explaining phase about individually varying use of the conceptions and tools to make these related conceptions accessible, meaningful, usable, and even 'seeable' to students. There was in the classrooms a continual use of the Common Core State Standards Mathematical Practices that can be summarized as everyone in the class doing *Math Sense-Making about Math Structure using Math Drawings (visual supports) to support Math Explaining* (MP1&6, MP7&8, MP4&5, MP2&3). This use of the Common Core State Standards Mathematical Practices supported what might look like practice activities to instead be building concepts.

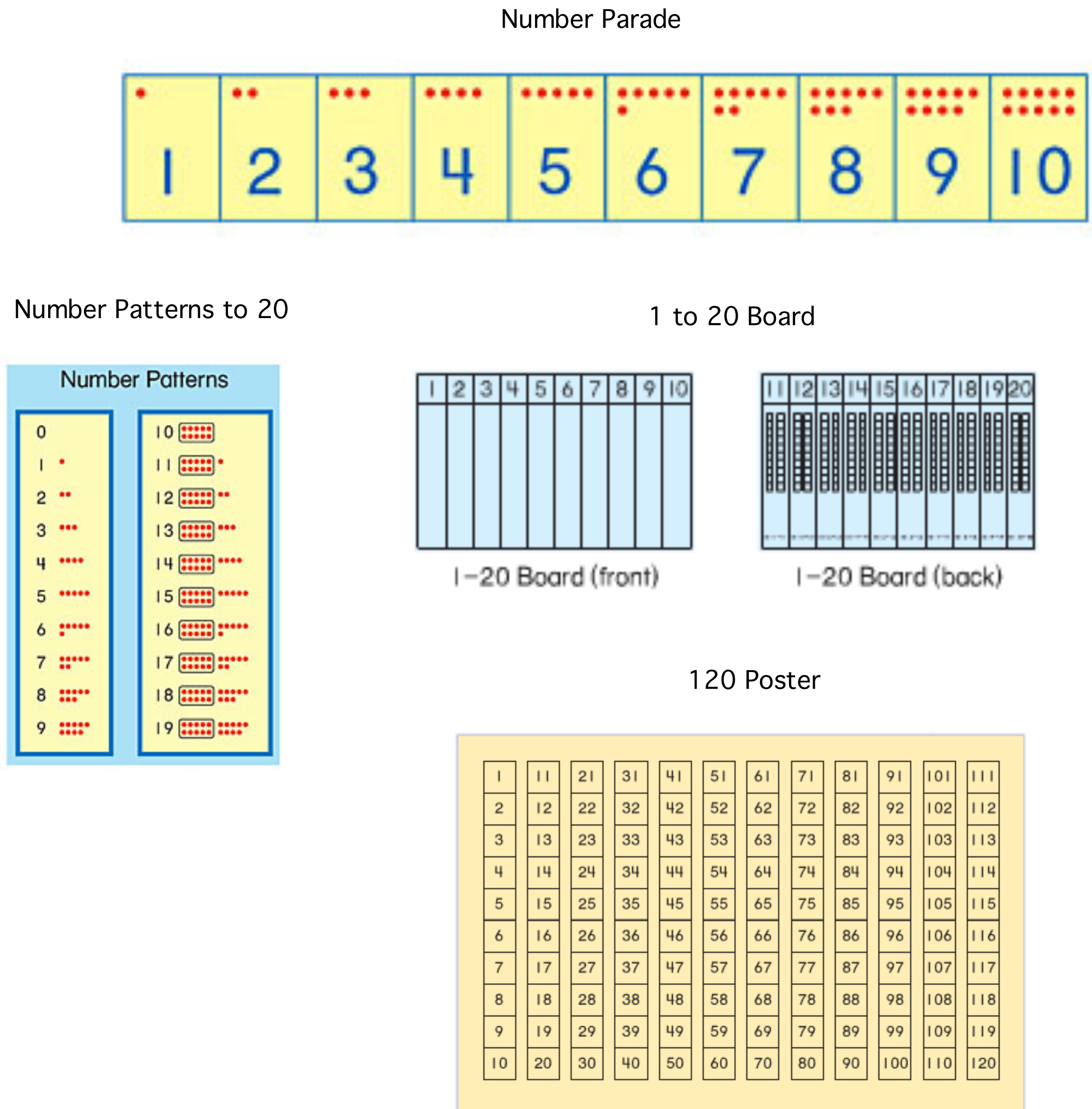
The Visual, Language, and Sensory-Motor Conceptual Supports

The visual, linguistic, and sensory-motor conceptual supports were developed over several years of research in classrooms to find powerful and easily used supports that would provide children with experiences that could help them build and connect all of the parts of the place-value conceptual structures identified by earlier research for English-speaking children. These conceptual supports were adapted from emphases in East Asia and elsewhere on quantities seen as five-groups and ten-groups. See the Teaching Progressions NBT 1 and 2 on karenfusonmath.com for more details and Classroom Videos for kindergarten to see some of these conceptual supports.

Learning to count in English was supported by the classroom posters in Figure 2 (see the next page). In kindergarten every day a Student Leader pointed in order to numbers on a poster while all children counted together. Children first counted from 1 to 10 on the Number Parade and weeks later moved to counting 1 to 20 on the Number Patterns to 20 and the 1 to 20 Board. Children also started on the 120 Poster, counting one number farther each day. These posters supported children to relate the English number words they were saying to number symbols and to quantities shown grouped by fives and by tens. Children daily at the beginning of using each poster described patterns they saw on the poster. The 120 Poster shows the numbers in vertical columns of ten because it is easier to see the patterns in the tens and ones place values than when numbers go horizontally as is common in the U.S: the tens digit stays the same as the ones digit increases by one until the next new ten is reached at the bottom of a column. The columns of ten numbers are also separated as vertical groups of ten rather than in an undifferentiated grid, as in the commonly used hundreds grid with numbers in horizontal rows.

Fingers are important cultural tools used all over the world. Fingers are used to add and subtract by counting all or taking away or by keeping track of how many are counted on. After several days of saying number words, children raised a finger with each new count word as they counted and looked at the poster. They did this in the cultural way they had learned at home: beginning with the thumb, or the little finger, or the pointing finger with the thumb last. Often children were using different fingers in the same class, and they discussed such differences. Children discussed how the 5-groups in their fingers related to the 5-groups in the Number Parade. For teen numbers children raised ten fingers to their left and the number of ones on fingers to the right to emphasize the tens digit written on the left and the ones digit written on the right: for 18, children would say, "*Eighteen is ten* (flash ten fingers to the left) *and eight* (flash 8 fingers to the right)." So children counted on the teen poster with English words (to learn these words) and also with tens and ones words to focus on the quantities: *ten and one, ten and two*, etc. For the 120 poster children would flash ten fingers as they said each English tens number *ten, twenty, thirty, etc.* The Student Leader would move the pointer down the whole column of numbers to emphasize the group of ten rectangles with numbers inside. This helped children connect the ten numbers in the column with the ten fingers each child was flashing.

Figure 2. Poster visual supports for 5-groups and 10-groups in quantities in order



Children did triad class activities to connect quantities 1 to 5 to number words and number symbols. They counted objects (circles, squares with a dot on one side, pennies, other objects), used small foam rectangles containing a numeral from 1 to 9 with dots in 5-groups on the back to show how many objects, and counted out objects to match a numeral. For 1 to 10 children used two kinds of 5-groups and single matching objects to make collections from 6 to 10: a) strips of 5 square inches on one side and a dot in each square on the other side used along with single squares with a dot on one side, and b) paper strips with 5 pennies on one side and 1 nickel on the other side with single pennies. Children used these objects to show individual numbers and placed them on a 1 to 10 board shown in Figure 2 on the left side of the 1 to 20 Board.

Figure 3 (see next page) shows visual supports used to help kindergarteners learn conceptual structures for teen numbers. At the bottom are layered cards that show numerically the Tens-Total and Ones conceptual structure of 10 and 8 making 18 when the 8 ones are put on top of the 0 ones in the ten (10). Experience in layering and unlayering the numeral and the dot sides of such cards shows numerically how 18 is really *ten and eight*. The small numbers on the upper left remind children of the 10 and 8 when the larger digits show 18 (which looks like a 1 and an 8). On the back of the cards the quantity ten is shown as an easily recognizable five and five and eight is shown as five and three. So use of the cards can help to replace the Concatenated Single Digit structure (18 *eighteen as one and eight*) with accurate multiunit meanings for the written numerals. These cards were called Secret Code Cards because they showed the secret code of written numbers, ad children enjoyed this term.

A progression of increasingly abstract organizations of quantities for 18 is shown at the top of Figure 3. Example A shows a unitary collection of 18 objects and then to the right shows ten ones gathered as a group with eight loose ones. At the top left in B is a more-advanced grouping in which the ten ones can potentially be seen as one ten. The vertical column of ten ones looks like the 1 in written teen numbers and so can support this important conceptual transition from ten ones to one ten. In C the five groupings shown on the backs of the layered cards and on the Number Parade are related to the fives in fingers and to strips that show ten ones, five ones, and single ones. In D these strips show money values with pennies on the front and a dime (ten pennies) and a nickel (five pennies) on the back. On the bottom left is shown the linguistic supports. English count words were used for all four A, B, C, D kinds of quantities, and *ten and eight* (Tens-Total-and-Ones words) and *one ten eight ones* (Tens-and-Ones words) were also used for B, C, and D quantities. In our teaching experiment kindergarten children moved through the Figure 3 experiences from A to B to C to D. During all of these extensive class experiences, they related these groupings to the English words for teen numbers, to the written numerals whose meaning is shown by the layered cards, and for B, C, D to tens and ones words like *ten and eight* and *one ten eight ones*. Children also placed ten-rods and single cubes for teen numbers on the 1 to 20 Board in Figure 2, counted on the teen board with English teen words and ten and ones words (*ten and one, ten and two, ten and three*, etc.), told the value of numbers pointed at by the teacher or another student, and read such equations such as 18 = 10 + __ on the bottom of the 1 to 20 board.

Figure 3. Relating quantities, count words, and written numerals for 10 to 19
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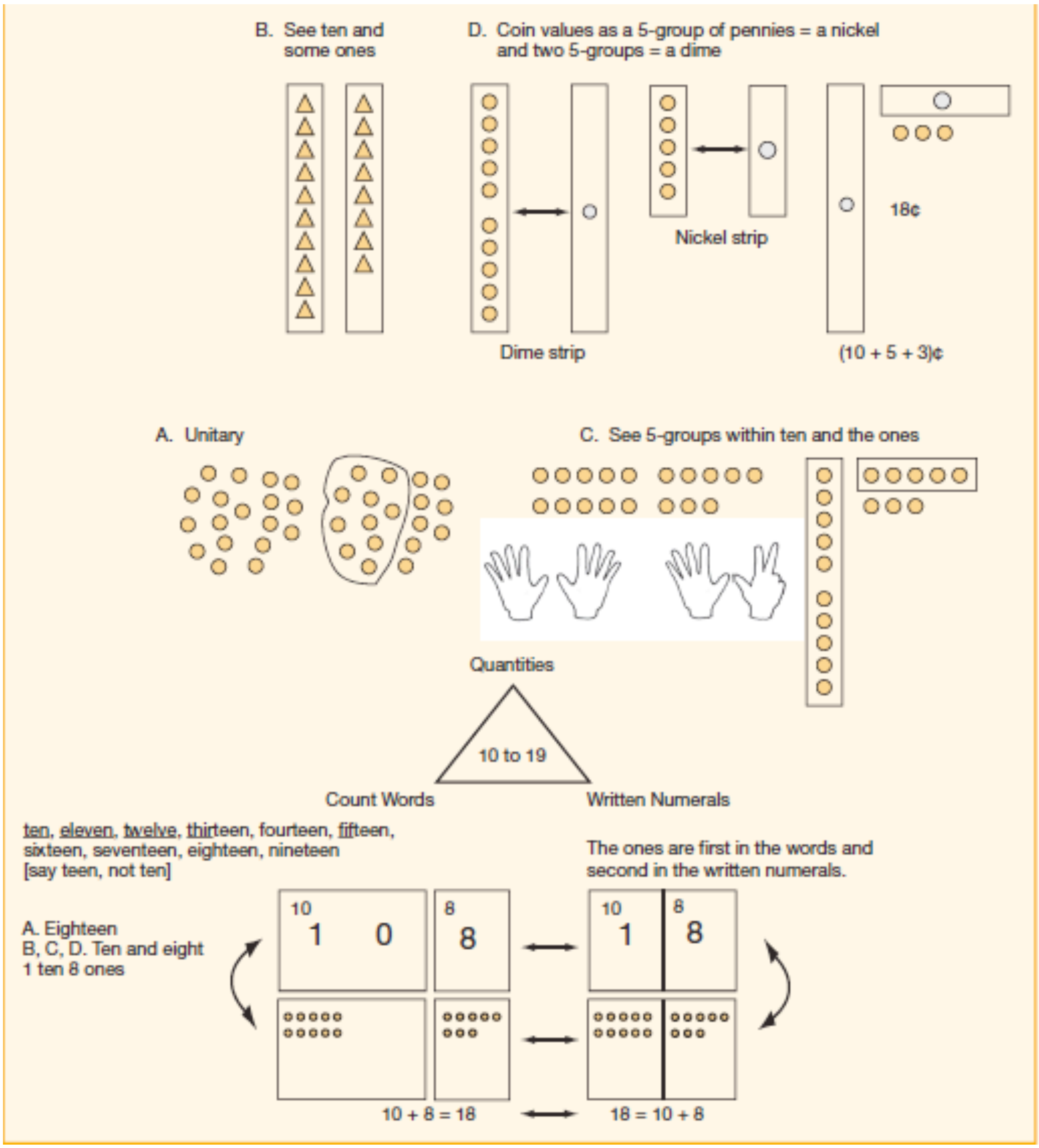


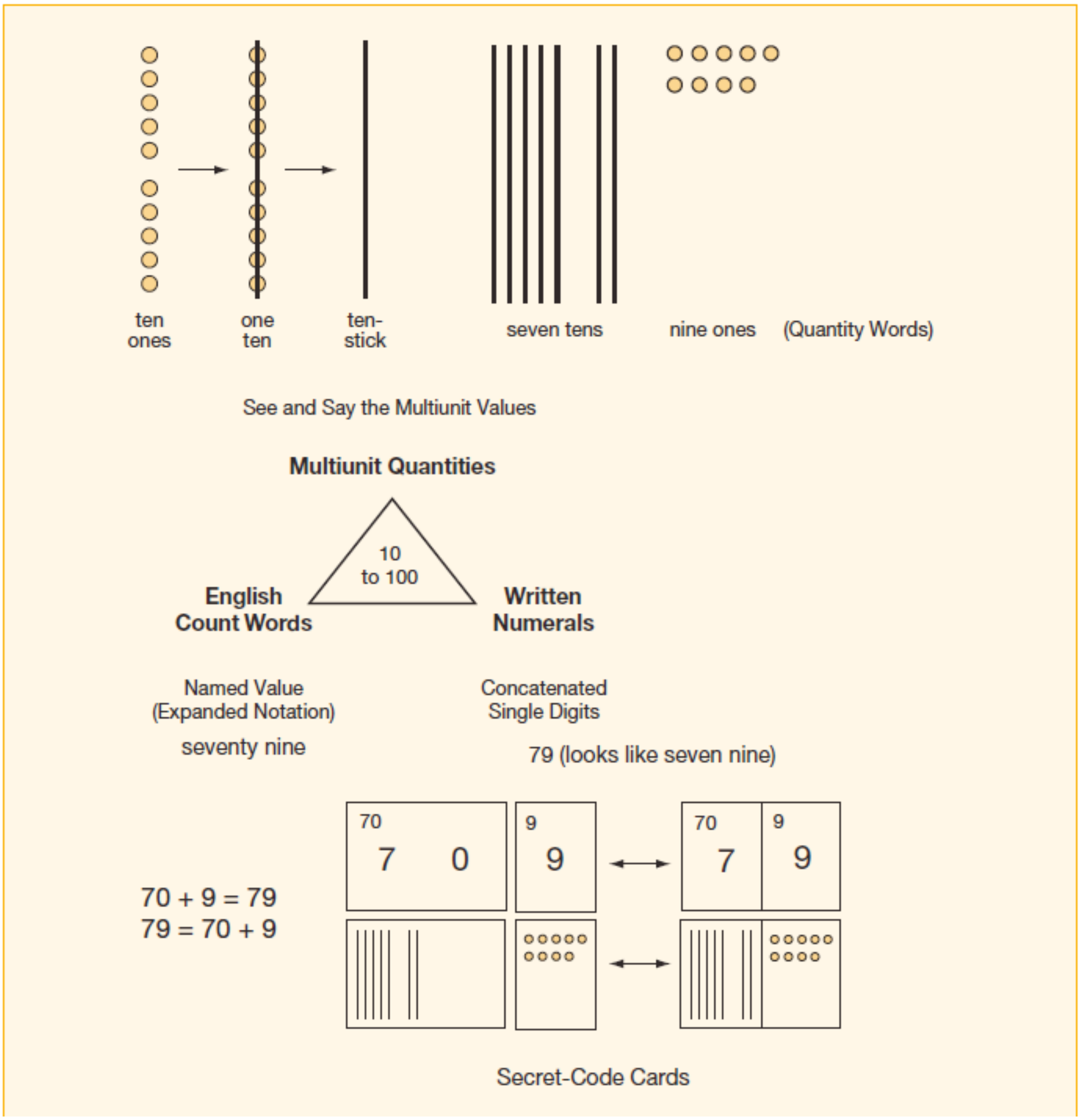
A Money Flip Chart was also used in Daily Routines with the 120 Poster so that children could see groups of ten pennies printed in columns of ten with a separation so that the five and five could be seen as in Figure 3 C and D at the top right. Each column of ten pennies was on a cardboard strip. Ten strips were attached across a board so that any number of them could be flipped to be seen. Kindergarten children made a new number each day by adding a penny to the number shown the day before. Small sticky notes covered the pennies at the bottom of a strip to make any number; for example, to show 38 three whole columns of ten pennies would be seen and sticky notes would cover the bottom two pennies on the fourth strip of ten pennies. This routine also used Giant Secret Code Cards for 2-digit numbers 1 to 100 such as are shown in Figure 4 (see the next page).

Some Grade 1 children had had none of the place value experiences in kindergarten, so it was necessary to catch them up rapidly and go on to more-advanced conceptions. All of the posters in Figure 2 except the 1 to 20 Boards were used but in more concentrated and rapid ways. For the Daily Routine using the Money Flip Chart described above, 2 or 3 pennies were added to the number of the day before. Children made predictions about when they would make a new one ten, and counted the number of that day by tens and then counted on the ones. A 10 was written below each column of ten pennies as each new ten was reached so that a number was seen as 10 + 10 + 10 + 6 and then read as "*So 3 tens and 6 ones is thirty six*." Children flashed ten fingers with each counted ten. On the 120 Poster a bracket was drawn at the bottom of each column of ten numbers as that number was reached. The extra ones were circled. Then the new number of the day was written at the bottom as 36 = 30 and 6 and read aloud: *Thirty six is thirty and six*. This routine also used Giant Secret Code Cards for 2-digit numbers 1 to 100 (see the bottom of Figure 4). The new number of the day was shown with unlayered cards 30 and 6; the 6 was then put on top of the 0 in 30 and students said aloud *Thirty and six makes thirty six.* This routine began at the beginning of the year to help all children develop robust capabilities to count by tens and then count on the ones in English and in tens and ones words before the units on place value were done near the end of the first semester.

In all of this counting by tens and then by ones children had to shift from the patterns in counting by tens (or counting the tens) to counting by a new unit, ones. Making this shift was far from trivial even when children could count by tens or count by ones; some children would continue to count on by tens. So for a while, when the last ten had been counted, the Student Leader focused attention on this shift by putting up both hands and saying *STOP. SHIFT.* and then the class would count on by ones.

Figure 4 shows extensions of the kindergarten visual supports used in grade 1 activities. Grade 1 Secret Code Cards showed all 2-digit numbers 10 through 99 (see the bottom of Figure 4). These place-value parts (e.g., 70 and 9) are the numerals used in expanded notation for a multiunit number. The drawings on the back of the card show the multiunit quantities *seven tens* and *nine ones*. These drawings can stimulate a Sequence-Count-by-Tens-and-then-by-Ones conception if the tens are counted or said as English count words (e.g., *ten, twenty, thirty, etc.*). Or they can stimulate a Tens-and-Ones conception if tens and ones quantity words (*seven tens nine ones*) are used. Each child had a set of these small cards. Putting the tens in order and counting them by tens was a useful activity, and children made various 2-digit numbers with the cards. Large versions of these cards were used as the Giant Secret Code cards in the Daily Routines described above.

Figure 4. Relating quantities, count words, and written numerals for 11 to 99
Reprinted with permission from figure 2.4 in *Teaching with Curriculum Focal Points: Focus in Grade 1*, copyright 2009 by the National Council of Teachers of Mathematics. All rights reserved.
Note. The Secret-Code Cards showed ten-sticks with dots on them as in the middle drawing at the top left (these details were too small to show in this figure).



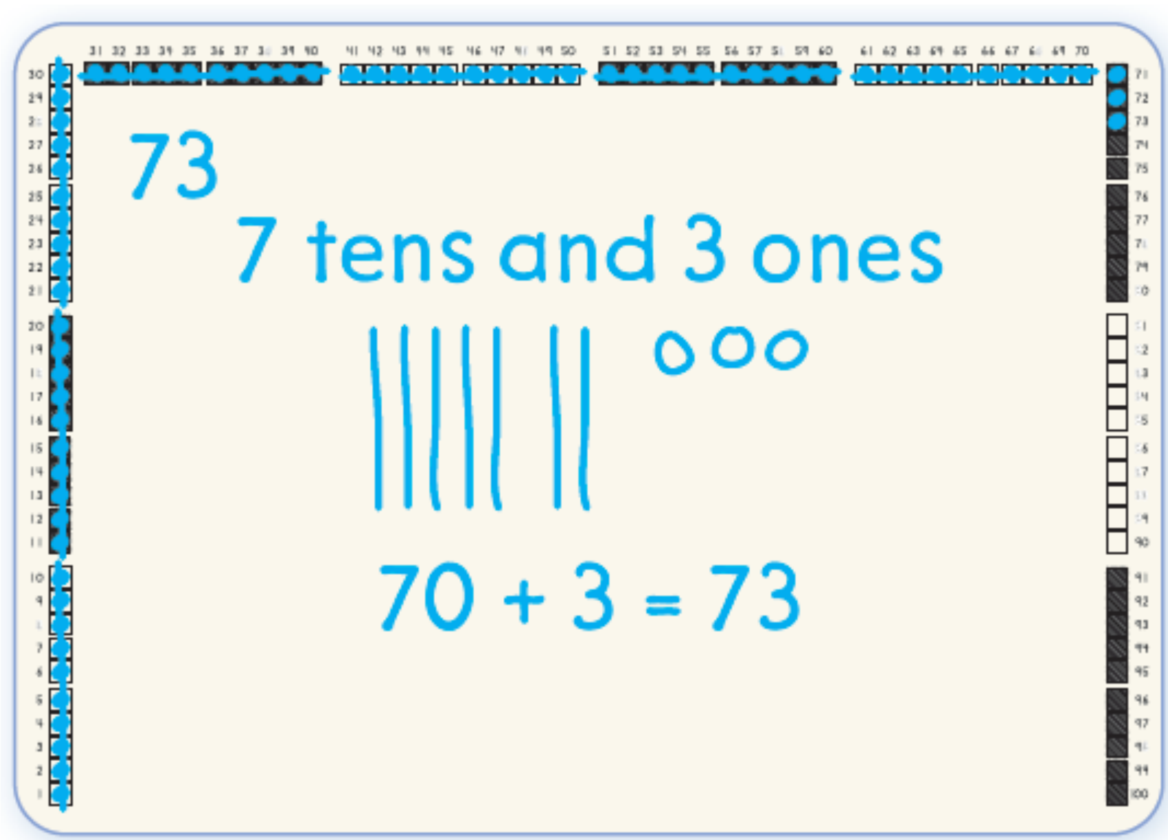
Grade 1 children moved on to making drawings that showed the place-value meanings of 2-digit numbers. The development of the different conceptions of groupings that make the tens-groups from 10 to 100 is shown at the top of Figure 4. The top left ten ones (as a vertical 5 above 5) used for teen numbers in kindergarten can be used initially in grade 1, but soon these ten circles or dots can have a stick drawn through them to make one ten (one group of ten). Then just a stick can be drawn and called a *ten-stick* or a *quick-ten*. The ones on such a ten-stick are known to be there instead of needing to be drawn; children visualize and discuss them, and they can always draw them if needed. Initially children can find the value of such a drawing as shown on the upper right of Figure 4 using a Sequence-Count-by-Tens-and-then-by-Ones conception if they count the tens using decade words such as *ten, twenty, thirty, forty, fifty*, *sixty, seventy,* and then count on the nine ones to get *seventy nine*. Children can also use a Tens-and-Ones conception if they count the tens: *one, two, three, four five, six, seven tens (or one ten, two tens, three tens, …, seven tens)* and then count the nine ones to get *seven tens nine ones*. The five-groups used for the tens and the ones in kindergarten and also as shown here for grade 1 make it possible with experience to look at a drawing and see the values quickly: *5 tens and 2 tens make 7 tens* and *5 ones and 4 ones make 9 ones.* Use of 5-groups also helped the drawings to be more accurate and less messy. Children made drawings of tens and ones using ten-sticks and ones to deepen their place-value conceptions and compare numbers (Common Core State Standards 1.NBT.2 and 3. They related drawings to Secret Code Cards to relate all three multiunit conceptions: Tens-Total-and-Ones, Sequence-Count-by-Tens-and-then-by-Ones, and Tens-and-Ones conceptions. Near the end of the year children used math drawings to support written methods of adding 2-digit numbers with regrouping, as discussed and shown above in the results section.

Grade 1 also used the visual support shown in Figure 5 (see next page). This Number Path was printed on a large (17” by 12”) dry-erase board. It had the numerals 1 to 100 printed alongside squares that were arranged in groups of ten, each composed of two visible groups of 5 squares. Children could draw a ten-stick through each group of ten squares to show ten-sticks. They counted these ten-sticks by tens to develop their Sequence-Count-by-Tens-and-then-by-Ones conception. They could also count the ten-sticks for the Tens-and-Ones conception. As shown in Figure 5, they could also write any quantity drawn on the Number Path as tens and ones using numerals and words and by using ten-sticks and ones circles. Below such words and quantities they could write a Tens-Total-and-Ones equation such as 70 + 3 = 73 or 73 = 70 + 3. Doing all of this repeatedly for different 2-digit numbers supported children to develop and relate the major multiunit conceptual structures.

Note that the Number Path is not a number line. The Number Path is a count model in which squares (or the numbers beside them) are counted. A number line is a length model in which small vertical line segments appear equally spaced along a longer horizontal line segment and are each labeled with a number in order. National Research Council reports have concluded from research and conceptual analyses that number lines are too difficult for young children and should be used in classrooms only beginning in grade 2 when length measurement is introduced (Cross, Woods, & Schweingruber, 2009; Kilpatrick, Swafford, & Findell, 2001). See National

Council of Teachers of Mathematics (2009, 2010) for a fuller explanation. The CCSS reflects these recommendations by only beginning the use of number lines in grade 2 where length is a focus. Number paths (or number lists) are appropriate visual models for children in kindergarten and grade 1. These are the kinds of models often appearing in games for young children, for example in Ramani and Siegler (2008). Numbers may be written on or beside any shape in number paths, as is common in math games for younger children.

Figure 5 Relating conceptions with the Number Path



We and the teachers in the year-long teaching experiments came to refer to the incorrect Concatenated Single Digit (CSD) conception as the Constantly Seductive Digits conception because students would begin to use correct conceptions but fall back to the CSD conception because the digits with their single-digit meanings were so salient. The Secret Code Cards were powerful in overcoming the CSD misconception because we would hear children talking about the “zero hiding under the ones number.” The visual place-value meaning of an English sequence word as 40 in 48 and the spoken English words gradually helped children to overcome the visual 4 and 8 in the written 48 as only single digits.

Discussion and Conclusions

We presented here a simplified framework of the place-value conceptual structures identified in past research. These are

- the incorrect Concatenated Single Digits conception in which 16 means *one six;*
- the Unitary conception in which 16 means the counting word *sixteen*;
- the Tens-Total-and-Ones conception in which 16 means *ten and six (*16 = 10 + 6) and 79 means *seventy and nine* (79 = 70 + 9);
- the Sequence-Count-by-Tens-and-Then-by-Ones conception in which 79 can be made by counting by English tens and and then by ones words: *ten, twenty, thirty, forty, fifty, sixty, seventy, seventy-one, seventy-two, seventy-three, …, seventy-nine;*
- the Tens-and-Ones conception in which 79 means *seven tens nine ones*.

There is a lot of specific detailed place-value knowledge for English-speaking children to learn and relate. Our experience with the learning paths and supports described here is that learning these complex interrelated concepts was a result of intensive repeated experiencing over months. Most children did learn these place-value meanings better than children in the U.S. usually do. The time and effort for this success should not be underestimated. It seems unlikely that a few targeted lessons on place-value standards will be sufficient for the high level of performance shown here.

Although the regular tens and ones structure of East Asian number words does help children learn and use the most advanced Tens-and-Ones conception, East Asian children still do use the three earlier conceptions. In the Miura et al. (1988) study, 59% of the Korean kindergarten children and 6 to 18% of the Chinese, Japanese, and Korean first graders made a unitary collection for 2-digit numbers, using their Unitary conception even though they counted those ones units using tens and ones words. Ho and Fuson (1998) found that only 39% of their Chinese kindergarteners used the Tens-Total-and-Ones conception, i.e., could say immediately that 10 + 6 was 16 even though in Chinese this sounded like *ten and six is ten six*. Here *ten six* is a counting word and the cardinal meaning of the *ten* and the *six* need to be used and connected to the counting meaning *ten six* to be an Understander of Embedded-10 Cardinality. We saw this conceptual connection being made by one of our Spanish-speaking students in the Kamii task. Spanish teen words become regular tens and ones words from 16 to 19; 16 is said as *diez y seis (ten and six)*, though 16 is spelled as *dieciseis.* The student pointed to 10 chips to show the meaning of the 1 on the 16 on the card, saying in a tone of wonderment, "*Dieciseis es diez y sies*!!!" [*Tenandsix* (a counting word coming after 15 *quince*) is *ten and six* (cardinal words meaning numbers of things)!!!]. Our *Math Expressions* English-speaking children did use tens and ones words as well as English words in many activities, so these might have helped them to focus on the groups of ten hidden in the English counting words.

The progression of learning supports outlined in this paper are not the only sequence of activities that could result in the high level of learning of place-value concepts reported here. But given the importance of effectively implementing the Common Core State Standards in schools around the U.S., and especially for children from backgrounds of poverty, learning paths with these or equivalent supports should be emphasized. The Common Core State Standards have at least temporarily freed the U.S. from the past endless discussion in each of the 50 states of what to teach at a given grade level. Now we can focus on ways in which to support all children to learn grade-level content. The learning supports described here could also be analyzed to see what would be useful in other countries in which children speak languages with irregular place-value words, which is most of the Western Hemisphere, Australia and New Zealand, and much of Europe. East Asian children may not need as long or as complex a learning sequence as do children speaking a European language with irregular place-value words. But parts of the progression of learning supports might be considered, especially the use of the New Groups Below method, as suggested by Fuson and Beckmann (2012/13) and Fuson and Li (2014).

The superiority of the *Math Expressions* first graders to the Chandler and Kamii (2009) children in grades 1, 2, and 3 in giving change for a dime for a 6ç purchase emphasizes the importance of the opportunity to learn math concepts. The first graders in our study had opportunities to learn ten as decomposed into addends and a dime as meaning ten pennies. It is time to stop doing studies whose goal is to show that U.S. children or children from poverty do not understand math concepts especially when their learning opportunities are not described in those studies. Now is the time for a substantive national and international discussion of instructional approaches that support the learning of concepts for all children. This paper is intended to contribute to such discourse about visual supports and teaching-learning practices that can support success for all students including those speaking English and other languages with irregularities in the place-value words.

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