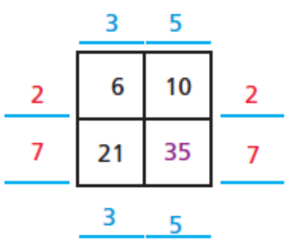
Relating Decimals, Fractions, and Ratios for Deeper Understanding: How Are They Alike and How Are They Different?

Factor Puzzle



Class Multiplication Table

•	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

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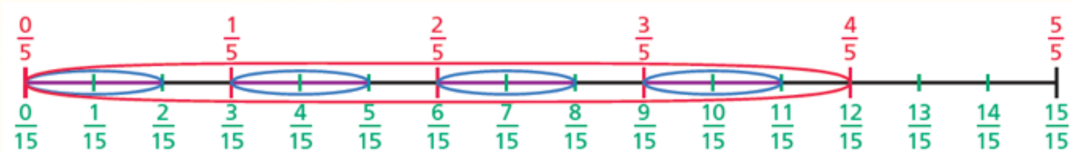

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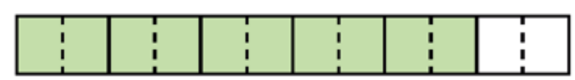
Please see my website karenfusonmath.com for the 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, and presentations including this one.

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$$



$$\frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{25}{30} = \frac{30}{36} = \frac{35}{42} = \frac{40}{48} = \frac{45}{56}$$

$\cdot 2$ $\cdot 3$ $\cdot 4$ $\cdot 5$ $\cdot 6$ $\cdot 7$ $\cdot 8$ $\cdot 9$

Notations

$\frac{2}{5}$ $\qquad 2 \div 5 \qquad 5\overline{)2}$

Notations

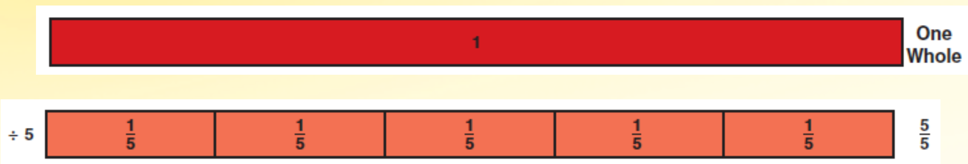
$\frac{2}{5}$ $\qquad 2 \div 5 \qquad 5\overline{)2}$

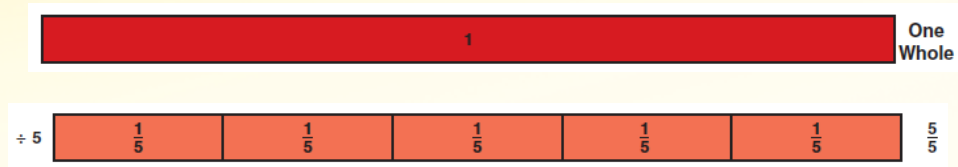
2/5

2 : 5 $\qquad$ 2 : 5 :: 4 : 10

$$2 \div 5$$

$$1 \quad + \quad 1$$





$$\frac{1}{5} \quad + \quad \frac{1}{5}$$

$$\frac{2}{5}$$

$\frac{2}{5}$ **Two fifths means two divided by five** $2 \div 5$

and also <u>the result of</u> two divided by five, the fraction $\frac{2}{5}$

A common error for fraction pictures:

See this as what?

A common error for fraction pictures:

not seeing the fraction <u>embedded in</u> the whole.

See 2 and 3 so say two thirds.

$$\frac{2}{3}$$

How to overcome this error:

Initially make fractions in two steps,

Make all of the unit fractions to show the denominator.

Then shade or encircle the numerator parts.

$$\frac{5}{5}$$

Then $\frac{2}{5}$

Take two of the five fifths.

We used to start fractions by letting the one whole
be a set of things, a number.
1/5 of ten balloons is two balloons.

This was difficult for students unless the unit fraction
divided evenly the number in the set and was not easy even then.

But 1/5 of a set of things is a fraction times a whole number:
1/5 x 10.
Wait and do such multiplications in general in Grade 5.

What is tricky about decimal notation?

Place Value

× 10 (larger) ÷ 10 (smaller)

Thousands	Hundreds	Tens	ONES	Tenths	Hundredths	Thousandths
1,000.	100.	10.	1.	0.1	0.01	0.001

2,000	300	60	1	0 .6	0 .0 3	0.002
2,	3	6	1	.6	3	2

Where is the symmetry for tens and tenths, hundreds and hundredths?

What is tricky about decimal words?

Place Value

× 10 (larger) ÷ 10 (smaller)

Thousands	Hundreds	Tens	ONES	Tenths	Hundredths	Thousandths
1,000.	100.	10.	1.	0.1	0.01	0.001

2,000	300	60	1	0.6	0.03	0.002
2,	3	6	1	.6	3	2

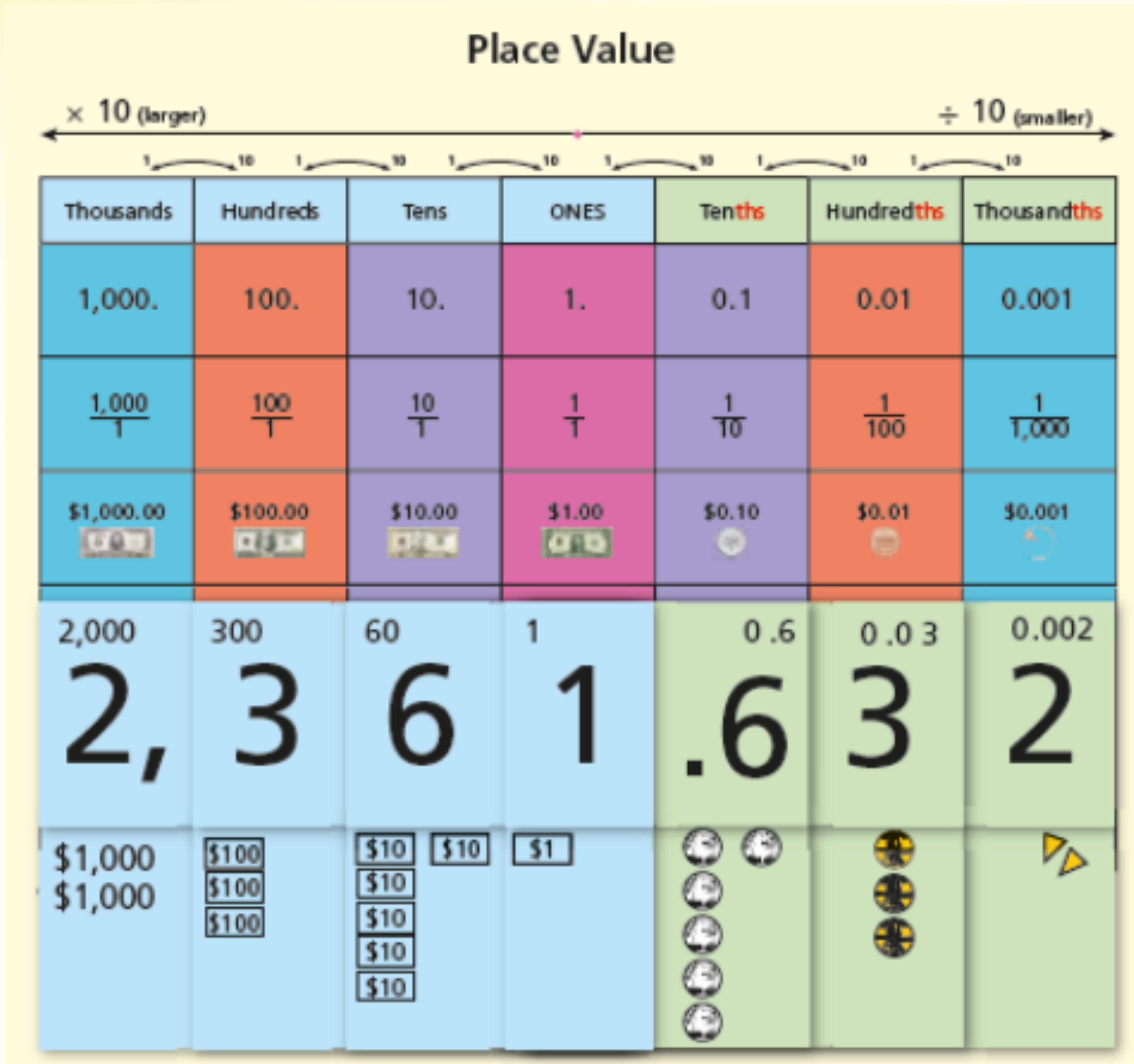
2 thousand 3 hundred 6 ty one and 6 tenths 3 hundredths 2 thousandths

6 hundred 3ty 2 thousandths

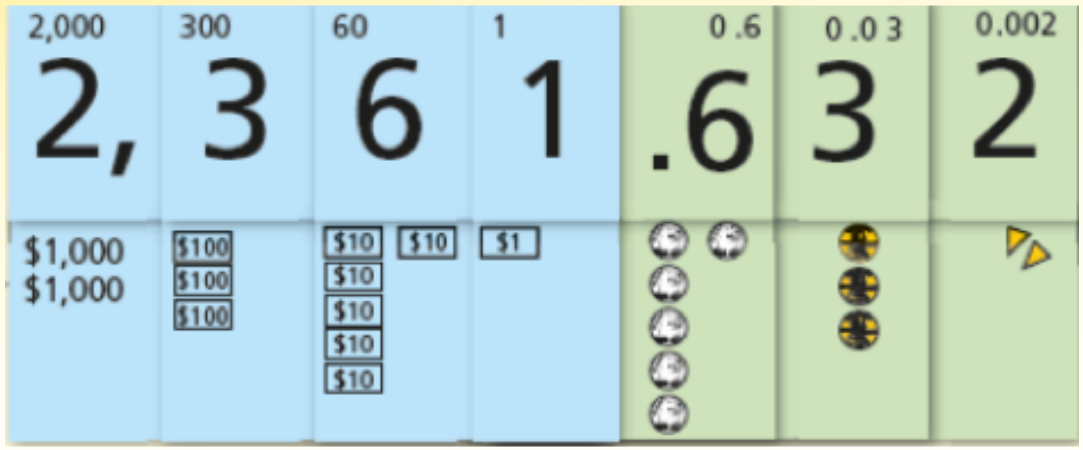
$$\frac{632}{1000}$$

How can we develop meaning for decimal fractions?

How can we help students see the important aspects of place-value notation?



Meanings for each place value part, each place relative to one.



$$2{,}361.632 = 2{,}000 + \underline{300} + \underline{60} + \underline{1} + \underline{0.6} + \underline{0.03} + 0.002$$

$$\begin{array}{ccccc}
 & 0 & .6 & 3 & 2 \\
 & & +\ \frac{6}{10} & +\ \frac{3}{100} & +\ \frac{2}{1{,}000} \\
 & & +\ \frac{600}{1{,}000} & +\ \frac{30}{1{,}000} & +\ \frac{2}{1{,}000} \\
 & 0 & .6 & 0 & 0 \\
+ & 0 & .0 & 3 & 0 \\
+ & 0 & .0 & 0 & 2 \\
\hline
 & 0 & .6 & 3 & 2
\end{array}$$

Why can it be helpful to write 0 before a decimal fraction?

0.6 0.03 0.002

$\frac{6}{10}$ $\frac{3}{100}$ $\frac{2}{1000}$

What is difficult about > and < for fractions?

$$\frac{1}{5} \qquad \frac{1}{10}$$

How can we help students overcome the typical error?

What is difficult about > and < for fractions?

$$\frac{1}{5} \qquad \frac{1}{10}$$

You only see the <u>number</u> of unit fractions and <u>not the size</u> of the unit fractions.

How can we help students overcome the typical error?

NOT $\frac{1}{5} < \frac{1}{10}$

because 5 is less than 10.

See and say the relationship:

If the number for the unit fraction is bigger,

the size of the unit fraction is smaller.

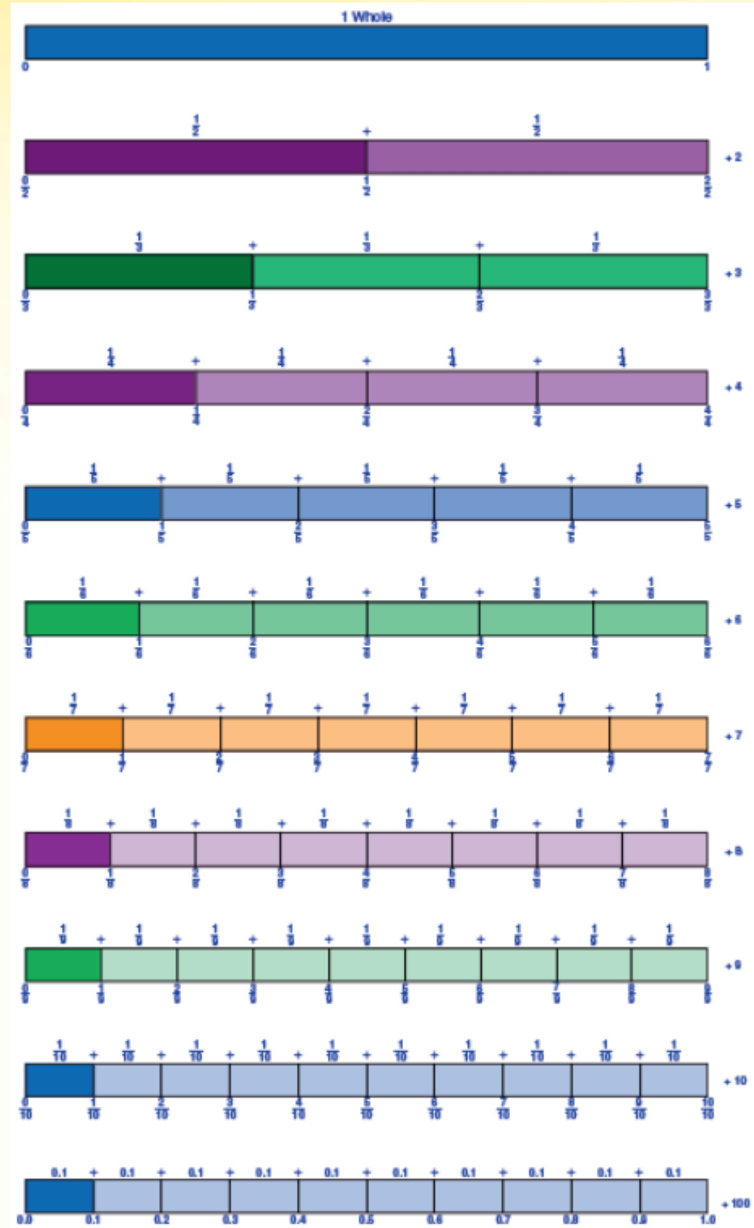
Fold fraction stips to see the size of each unit fraction getting smaller as there as more of them. Use two different lengths for the whole to see that the unit fraction depends on the whole.

Practice with specific examples for unit fractions and for n/a versus n/b.

$$\frac{2}{5} \qquad \frac{2}{3}$$

Two of five equal parts are smaller than two of three equal parts. Two fifths is less that two thirds.

2 fifths compared to 2 thirds

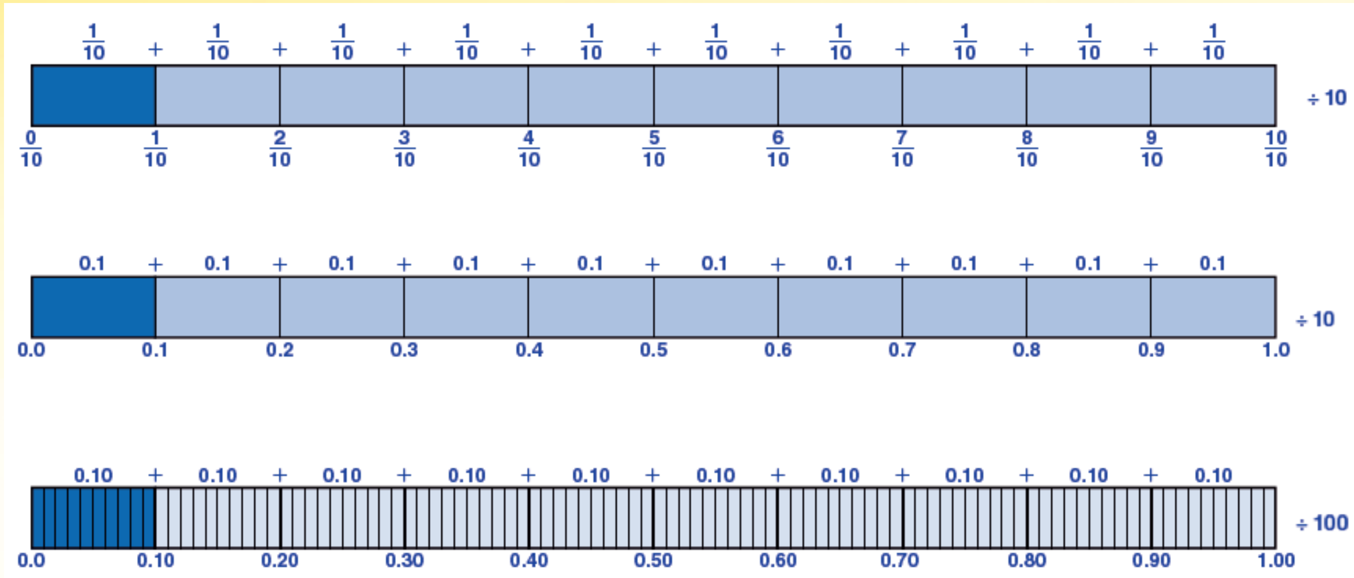


What is difficult about > and < for decimals?

0.4 0.25

How can we help students overcome the typical error?

Understand decimal equivalence:



> or <

0.4   0.25

NOT

0.4 < 0.25

because

4 < 25.

Make the decimals have the same places: 0.40   0.25

Align the decimal places: 0.4

(Know which place is larger.)   0.25

Use fraction equivalents: 40/100   25/100

What is difficult about + and – for like fractions?

$$\frac{3}{7} + \frac{2}{7}$$

How can we help students overcome the typical error?

NOT $\frac{3}{7} + \frac{2}{7} = \frac{5}{14}$

Write fractions as sums of unit fractions to see that the denominator stays the same.

Do this with drawings and with symbols.

9 $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \times \frac{1}{4}$

$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

10 $\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 3 \times \frac{1}{8}$

$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

11 $\frac{5}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 5 \times \frac{1}{5}$

$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

12 $\frac{2}{12} = \frac{1}{12} + \frac{1}{12} = 2 \times \frac{1}{12}$

$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

13 $\frac{4}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = 4 \times \frac{1}{7}$

$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

14 $\frac{7}{9} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 7 \times \frac{1}{9}$

$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Write fractions as sums of unit fractions to see that the denominator stays the same.

Do this with symbols only.

Write each fraction as a sum of unit fractions.

1. $\frac{2}{4} = \frac{1}{4} + \frac{1}{4}$
2. $\frac{5}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
3. $\frac{2}{6} = \frac{1}{6} + \frac{1}{6}$
4. $\frac{7}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
5. $\frac{4}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$
6. $\frac{6}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$
7. $\frac{8}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$
8. $\frac{4}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

Name the fraction for each sum of unit fractions.

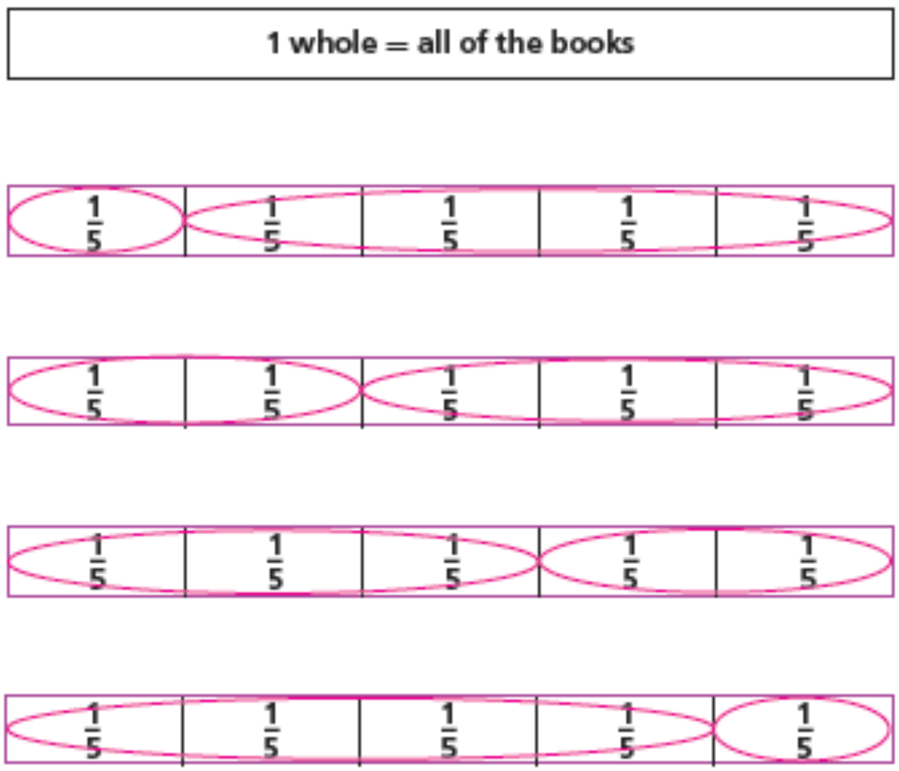
9. $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$
10. $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$
11. $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8}$
12. $\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{7}{12}$
13. $\frac{1}{12} + \frac{1}{12} = \frac{2}{12}$
14. $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$
15. $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$
16. $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{6}{8}$

Write partners of n/n to see the denominator stay the same.

Fifths that Add to One

Every afternoon, student volunteers help the school librarian put returned books back on the shelves. The librarian puts the books in equal piles on a cart.

One day, Jean and Maria found 5 equal piles on the return cart. They knew there were different ways they could share the job of reshelving the books. They drew fraction bars to help them find all the possibilities.

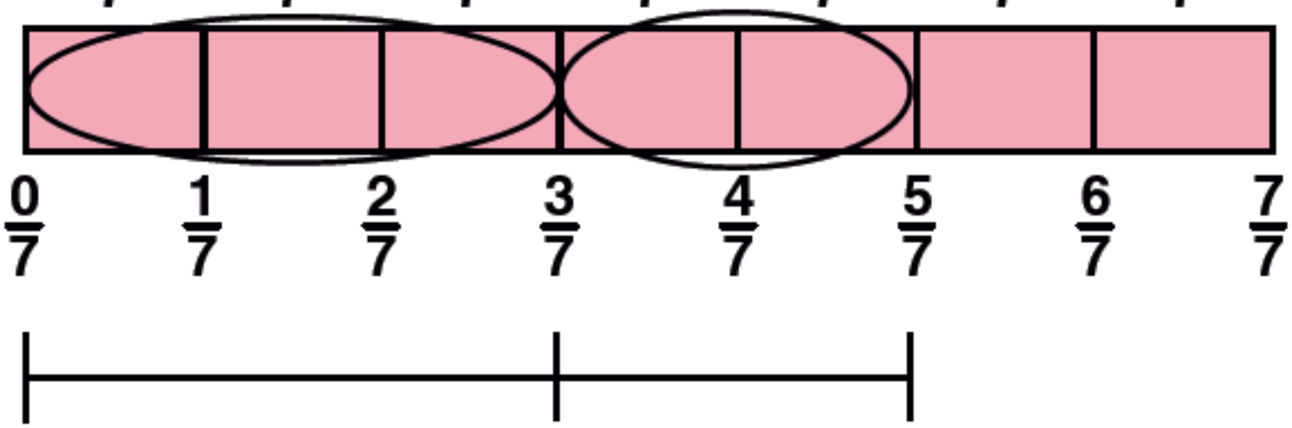


1 whole		Jean's share		Maria's share
$\frac{5}{5}$	=	$\frac{1}{5}$	+	$\frac{4}{5}$
$\frac{5}{5}$	=	$\frac{2}{5}$	+	$\frac{3}{5}$
$\frac{5}{5}$	=	$\frac{3}{5}$	+	$\frac{2}{5}$
$\frac{5}{5}$	=	$\frac{4}{5}$	+	$\frac{1}{5}$

See unit fractions being added in symbols and in math drawings of bars and of number lines.

$$\frac{3}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$$

$$\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{7}{7}$$



$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

5 of the $\frac{1}{7}$

$5 \bullet \frac{1}{7}$

What is difficult about + and – for unlike fractions?

$\frac{1}{5}$ $\frac{1}{10}$

How can we help students overcome the typical error?

NOT $\frac{1}{5} + \frac{1}{10} = \frac{2}{15}$

because 2/15 is tiny and less than 1/5.

As with all previous adding and subtracting,
you have to add or subtract like units.

24 + 3 is 27 because you have to add 4 ones and 3 ones.

Align the place values to see correct adding.

$$\begin{array}{r} 24 \\ +\ \underline{\ \ 3} \end{array}$$

You do not align on the left:

$$\begin{array}{l} 24 \\ +\ \underline{3\ \ } \end{array}$$

2 feet and 3 inches you have to change feet to inches to add them:

24 inches + 3 inches = 27 inches.

4 cm + 3 mm you have to change cm to mm:

4 cm = 40 mm, so 40 mm + 3 mm = 43 mm.

Divide all of the unit fractions to make more but smaller unit fractions.

The fraction stays the same size overall.

The fraction looks bigger because there are more parts and you do not see that these parts are smaller unless you make a drawing.

You can also make equivalent fractions by grouping the unit fractions and dividing symbolically.

Equivalent Fractions

Equivalent fractions are made by:

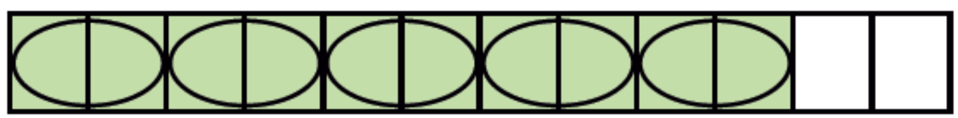
a. more but smaller parts

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$$

$$\frac{5}{6} = \overset{\cdot 2}{\underset{\cdot 2}{\frac{10}{12}}} = \overset{\cdot 3}{\underset{\cdot 3}{\frac{15}{18}}} = \overset{\cdot 4}{\underset{\cdot 4}{\frac{20}{24}}} = \overset{\cdot 5}{\underset{\cdot 5}{\frac{25}{30}}} = \overset{\cdot 6}{\underset{\cdot 6}{\frac{30}{36}}} = \overset{\cdot 7}{\underset{\cdot 7}{\frac{35}{42}}} = \overset{\cdot 8}{\underset{\cdot 8}{\frac{40}{48}}} = \overset{\cdot 9}{\underset{\cdot 9}{\frac{45}{56}}}$$

b. fewer but larger parts

$$\frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$



See many equivalent fractions as two rows from the multiplication table.

There are three cases for finding a common denominator.

4, 8   4, 3   4, 6   5, 10   5, 7   8, 12   6, 8   6, 12   6, 7

List each pair of denominators beside the case that best describes it. Think about the strategy you would use to find a common denominator for the pair.

Case A One denominator is a factor of the other.	Case B The denominators have no common factors other than 1.	Case C The denominators have a common factor that is not one of the numbers and is not 1.
4, 8; 5, 10; 6, 12	4, 3; 5, 7; 6, 7	4, 6; 8, 12; 6, 8

You can always use Case B and multiply each fraction by the n/n of the other fraction's denominator:

Change fourths and thirds to twelfths: $\frac{1 \times 3}{4 \times 3} = \frac{3}{12}$ $\frac{1 \times 4}{3 \times 4} = \frac{4}{12}$

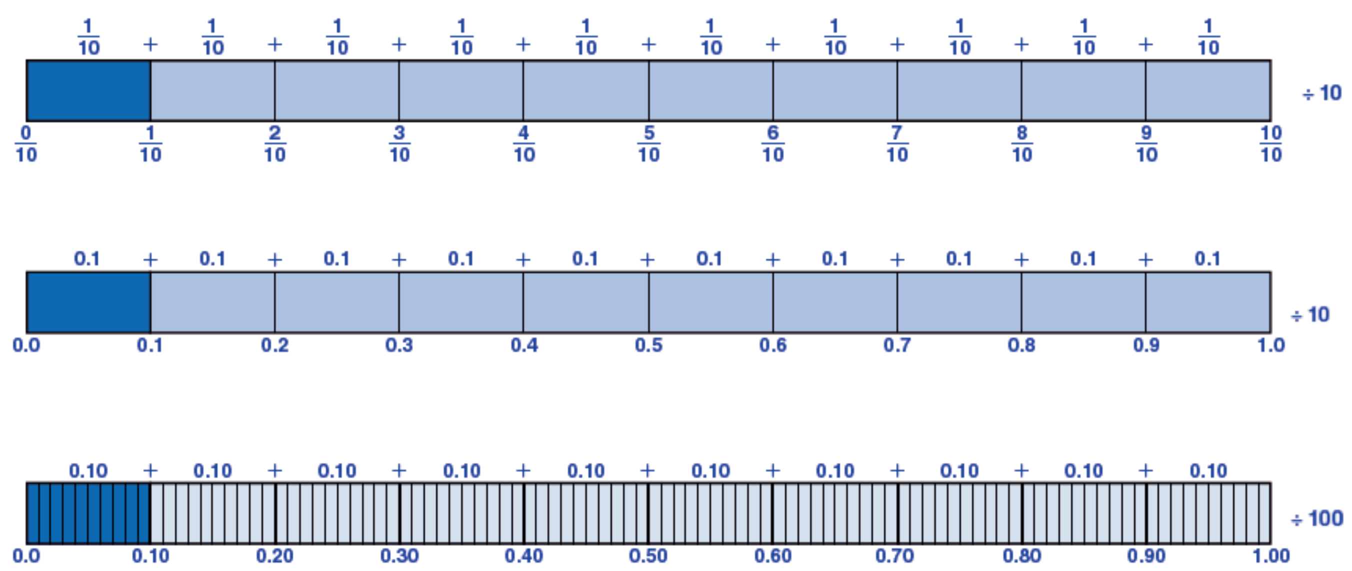
So $\frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

What is difficult about + and – for decimals?

0.4 + 0.25

How can we help students overcome the typical error?

Understand decimal equivalence:



+ or –

0.25
+ 0.4

DO NOT align on the right as for whole numbers:

25
+ 4

Make the decimals have the same places: 0.40 0.25

Align the decimal places with the same value: 0.4
+ 0.25

Use fraction equivalents: 40/100 25/100

What is difficult about + and – for mixed numbers?

1 3/5 + 2 4/5

How can we help students overcome the typical error?

NOT

$1\ 3/5 + 2\ 4/5 = 37/\ 10$

Students may need to write mixed numbers as

$1 + 3/5$ and $2 + 4/5$ to understand what they are.

$1 + 3/5$ and $2 + 4/5 = 3 + 7/5$

which is an acceptable answer and shows the pattern of adding mixed numbers.

But if such answers need to be simplified,

the grouping depends on the unit fraction, here fifths:

$$3 + 7/5 = 3 + 5/5 + 2/5 = 3 + 1 + 2/5 = 4\ 2/5$$

Decimal mixed numbers such as 1.6 + 2.8 just use

regular base-ten grouping:

$$\begin{array}{r} 1.6 \\ +\ \underline{2.8} \\ 1.4 \\ \underline{3.0} \\ 4.4 \end{array}$$

Why does multiplying by a fraction less than one result in a product less than the number being multiplied?

Multiplying by a whole number results in a larger product.

A. $\frac{1}{d} \cdot w = \frac{1}{d} \cdot \frac{w}{1} = \frac{1 \cdot w}{d \cdot 1} = \frac{w}{d}$

$\frac{1}{3} \cdot 4 = \frac{1}{3} \cdot \frac{4}{1} = \frac{1 \cdot 4}{3 \cdot 1} = \frac{4}{3}$

$\frac{1}{3}$ of 4 rectangles is $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$.

B. $\frac{n}{d} \cdot w = \frac{n}{d} \cdot \frac{w}{1} = \frac{n \cdot w}{d \cdot 1} = \frac{n \cdot w}{d}$

$\frac{2}{3} \cdot 4 = \frac{2}{3} \cdot \frac{4}{1} = \frac{2 \cdot 4}{3 \cdot 1} = \frac{8}{3}$

$\frac{2}{3}$ of 4 rectangles is $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$.

C. $w \cdot \frac{n}{d} = \frac{w}{1} \cdot \frac{n}{d} = \frac{w \cdot n}{1 \cdot d} = \frac{w \cdot n}{d}$

$4 \cdot \frac{2}{3} = \frac{4}{1} \cdot \frac{2}{3} = \frac{4 \cdot 2}{1 \cdot 3} = \frac{8}{3}$

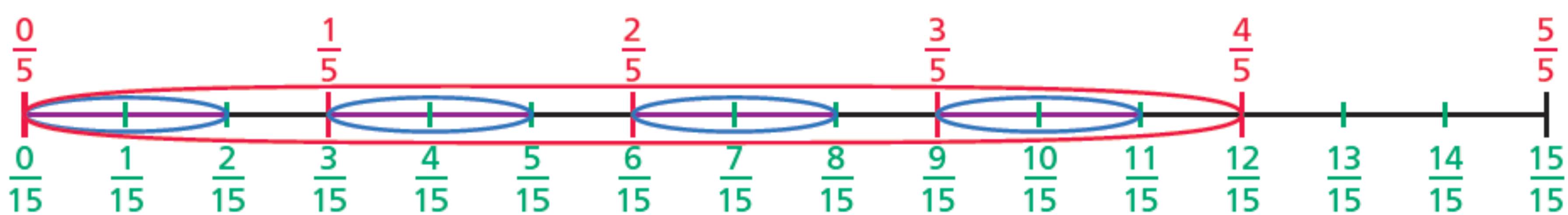
4 groups of $\frac{2}{3}$ is $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$.

Multiplying by a fraction less than one is taking a part of that number.

This is also true when a fraction is being multiplied by a fraction less than one.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \qquad \frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$$

Ginny's Model "I use a number line. First I model $\frac{4}{5}$ (red loop), and then I separate each fifth into 3 equal parts, or thirds (green tick marks). This makes 15 equal parts altogether. Then I take $\frac{2}{3}$ of each fifth (blue loops), which is $\frac{8}{15}$."

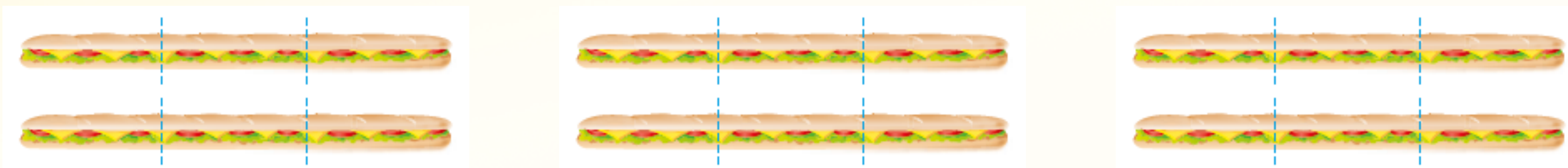


$6 \div 3$ How many threes in 6? Two $6 \div 3 = 2$

The answer (the factor 2) is less than the dividend (product) 6.

$6 \div 1/3$ How many one-thirds are in 6?

3 in each whole, so $3 \times 6 = 18$. $6 \div 1/3 = 18$



The answer (the factor 18) is greater than the dividend (product) 6 because you are finding how many small parts there are in the 6. $6 \div 1/3 = 18$

Dividing by a unit fraction is

multiplying by the denominator of that unit fraction.

Dividing by any fraction less than 1 gives an answer (the factor 3/4) that is greater than the dividend (the product) 6/20 because you are finding how many small parts there are in that dividend.

6/20 ÷ 2/5 = 3/4 3/4 > 6/20

$$\frac{2}{5} \cdot \frac{\boxed{3}}{\boxed{4}} = \frac{6}{20} \qquad \frac{6}{20} \div \frac{2}{5} = \frac{3}{4}$$

Notice that for some problems you can divide by dividing numerators and dividing denominators.
This is because you multiply by multiplying numerators and denominators.

Why can we not always divide fractions by dividing numerators and dividing denominators?

Because the product might have been simplified.

Here we can divide numerators and denominators:

$$\frac{6}{20} \div \frac{2}{5} = \frac{3}{4}$$

But if the numerator or denominator is simplified as below, we can no longer divide numerators and denominators.

$$\frac{6}{20} = \frac{6 \div 2}{20 \div 2} = \frac{3}{10}$$

$$\frac{3}{10} \div \frac{2}{5} = \frac{\boxed{}}{\boxed{}}$$

$\frac{2}{3} \div \frac{5}{7} = ?$

So a general method to divide fractions in which you cannot divide numerators and denominators is to unsimplify the product so that you can divide.

Decide which n/n you use to unsimplify: You want to divide the product numerator by 5 and the product denominator by 7, so use 5/5 and 7/7.

Step 1 $\frac{2}{3} \div \frac{5}{7} = \Big(\overbrace{\underbrace{\frac{2}{3} \cdot \frac{5}{5} \cdot \frac{7}{7}}_{\frac{2}{3}\text{ unsimplified}}}^{\frac{2}{3} \cdot 1 \cdot 1}\Big) \div \frac{5}{7}$

Divide the numerator by 5 and the denominator by 7.

Step 2 $= \frac{2 \cdot \overbrace{5 \cdot 7}^{5 \div 5 = 1}}{3 \cdot 5 \cdot \underbrace{7}_{7 \div 7 = 1}} \div \frac{5}{7}$

This leaves 7 on the top and 5 on the bottom.

Step 3 $= \frac{2 \cdot 7}{3 \cdot 5}$

Step 4 $= \frac{2}{3} \cdot \frac{7}{5}$

So we see a general pattern: Multiply by the reciprocal of the dividing factor.

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5}$$

A shortcut mnemonic is: Flip the factor.

Why does multiplying by a decimal less than one result in a product less than the number being multiplied?

Multiplying by a whole number results in a larger product.

Multiplying by a fraction less than one results in a product less than the number being multiplied.

A decimal less than one can be written as a fraction, so multiplying by a decimal less than one results in a product less than the number being multiplied.

Let's see this using decimal notation:

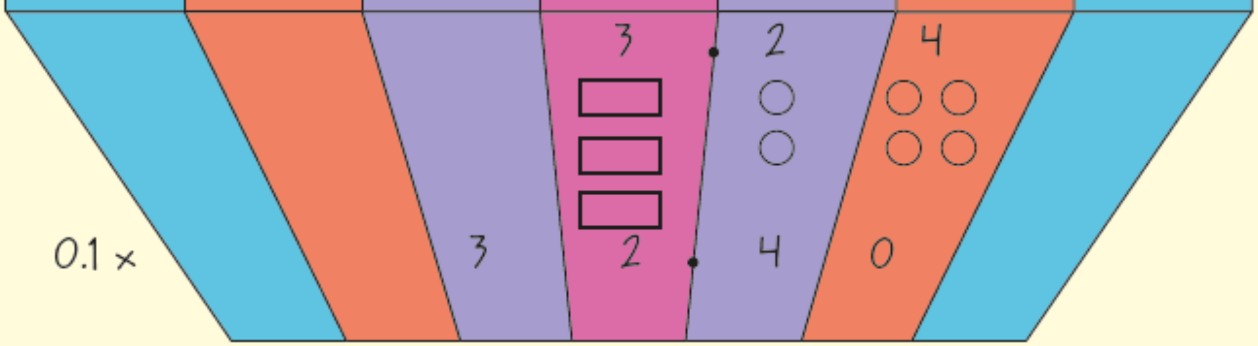
For x 0.1 take 1 of ten parts of every place (think about money values).

The whole number shifts one place to the right (gets smaller).

Place Value

× 10 (larger) ← → ÷ 10 (smaller)

Thousands	Hundreds	Tens	ONES	Tenths	Hundredths	Thousandths
1,000.	100.	10.	1.	0.1	0.01	0.001
$\frac{1{,}000}{1}$	$\frac{100}{1}$	$\frac{10}{1}$	$\frac{1}{1}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
\$1,000.00	\$100.00	\$10.00	\$1.00	\$0.10	\$0.01	\$0.001



Let's see this using decimal notation:

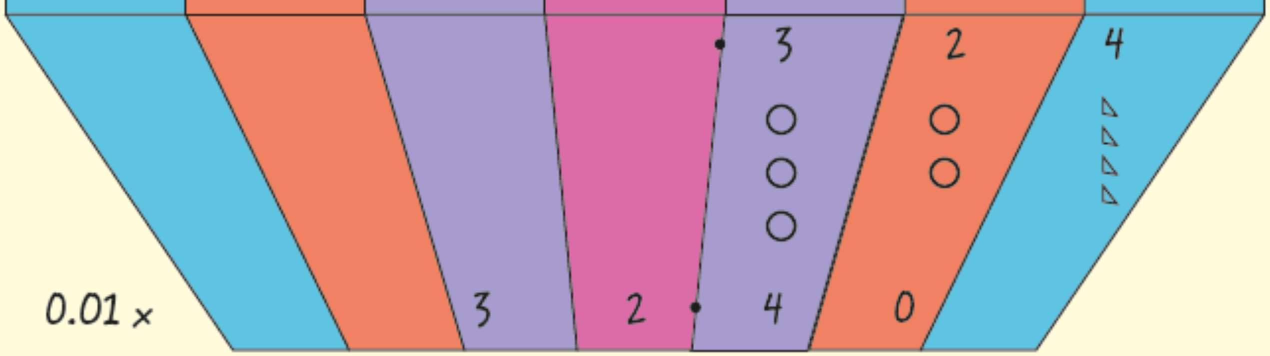
For x 0.01 take 1 of 100 parts of every place (think about money values).

The whole number shifts two places to the right (gets smaller).

Place Value

× 10 (larger) ÷ 10 (smaller)

Thousands	Hundreds	Tens	ONES	Tenths	Hundredths	Thousandths
1,000.	100.	10.	1.	0.1	0.01	0.001
$\frac{1,000}{1}$	$\frac{100}{1}$	$\frac{10}{1}$	$\frac{1}{1}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$
$1,000.00	$100.00	$10.00	$1.00	$0.10	$0.01	$0.001



These shift rules can explain the rule about finding the number of decimal places in a product of two decimals:

Count all of the decimal places and make that many places in your answer.

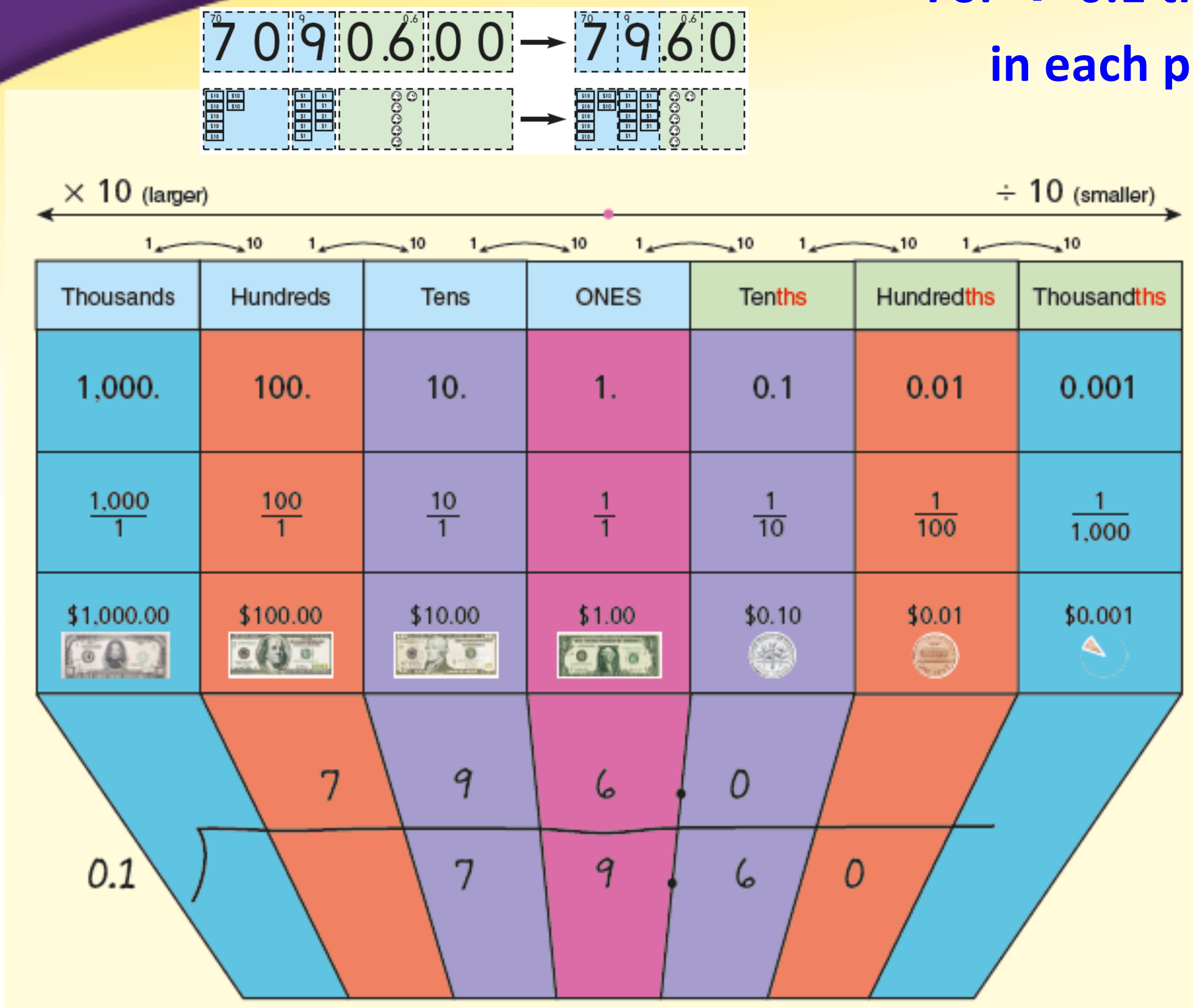
Your original number shifts one or two or three places to the right, so it has its decimal places plus the new decimal places from the shift.

$$0.01 \times 32.4 = 0.324$$

You can also see this by using fractions.

$$\frac{1}{100} \times \frac{32.4}{10} = \frac{324}{1{,}000}$$

For ÷ 0.1 think how many one-tenths in each place (think money values).



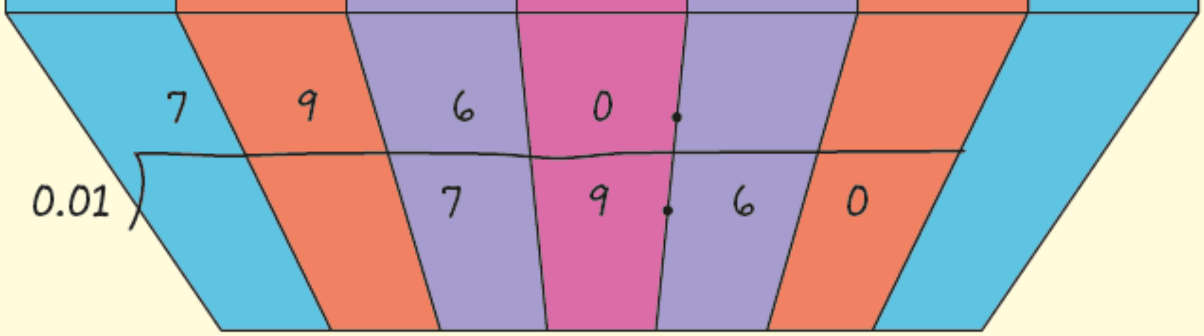
The whole number shifts one place to the left (gets larger): 79.6 becomes 796.

For ÷ 0.01 think how many one-hundredths in each place (think money values).

Place Value

× 10 (larger) ÷ 10 (smaller)

Thousands	Hundreds	Tens	ONES	Tenths	Hundredths	Thousandths
1,000.	100.	10.	1.	0.1	0.01	0.001
$\frac{1,000}{1}$	$\frac{100}{1}$	$\frac{10}{1}$	$\frac{1}{1}$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$
$1,000.00	$100.00	$10.00	$1.00	$0.10	$0.01	$0.001



The whole number shifts two places to the left (gets larger): 79.60 becomes 7960.

Why does the caret method work?

You are multiplying the numerator and denominator of a fraction made by the division problem by 1 = n/n of a form to change the denominator to 1.

a. $0.4\overline{)2.7}$ b. $72.5\overline{)0.39}$ c. $2.7\overline{)9.0}$ d. $6.3\overline{)52.0}$

1 Dividing by 0.1 is the same as multiplying by 10. Discuss the reason for each numbered step.

$$\frac{79.6}{0.1} \underset{①}{=} \frac{79.6}{0.1} \cdot 1 \underset{②}{=} \frac{79.6}{0.1} \cdot \frac{10}{10} \underset{③}{=} \frac{79.6 \cdot 10}{0.1 \cdot 10} \underset{④}{=} \frac{796}{1} \underset{⑤}{=} 796$$

2 Multiplying a problem in long-division format by $1 = \frac{10}{10}$ gives an equivalent problem with a whole number divisor.

$0.1\overline{)79.6} \longrightarrow 0.1\overline{)79.6}$ OR $0.1\overline{)79.6}$

1. ① Multiplying by 1 does not change the value; ② $\frac{10}{10} = 1$; use $\frac{10}{10}$ to change the denominator to a whole number. ③ Multiply fractions by multiplying the numerators and multiplying the denominators; ④ Simplify the numerator and denominator; ⑤ Any number divided by 1 is that number.

To divide by hundredths, multiply both numbers by 100/100

to change the divisor to a whole number.

a. $0.04\overline{)2.70}$ b. $7.25\overline{)0.39}$ c. $0.27\overline{)9.00}$ d. $0.63\overline{)52.00}$

What makes Parts a, c, and d of Exercise 15 tricky?

You have to put one or two zeros on the end of the number to show two more decimal places.

Dividing by 0.01 is the same as multiplying by 100. Discuss the reason for each numbered step.

$$\frac{79.6}{0.01} \underset{1}{=} \frac{79.6}{0.01} \cdot 1 \underset{2}{=} \frac{79.6}{0.01} \cdot \frac{100}{100} \underset{3}{=} \frac{79.6 \cdot 100}{0.01 \cdot 100} \underset{4}{=} \frac{7{,}960}{1} \underset{5}{=} 7{,}960$$

Multiplying a problem in long-division format by $1 = \frac{100}{100}$ gives an equivalent problem with a whole number divisor.

$0.01\overline{)79.6} \longrightarrow 0.01\overline{)79.60}$ OR $0.01\overline{)79.60}$

How are fractions and ratios different?

How are they alike?

What are the notation issues for fractions and ratios?

There are no agreed upon definitions for rate, ratio, or proportion.

They are a muddle and different notations are used for them.

The very common student error for proportions is to use addition instead of multiplication to find unknown values.

Many approaches to ratio begin by using fraction notation for ratios.

This is disastrous because it makes it very difficult for students to understand how ratios differ from fractions.

It is much better to start with different notations for ratios and proportions and then later to discuss why some people use fraction notation for proportions and how that is ok.

The approach outlined here is consistent with the CCSSM. Students achieve at a high rate of accuracy and understanding with proportions in Grade 6. Because it uses the multiplication table, it also helps less-advanced students build more advanced multiplicative reasoning and increase fluency with single-digit multiplication.

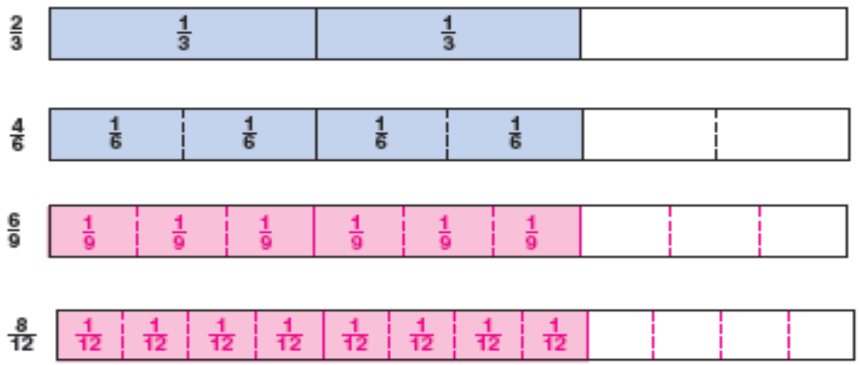
This approach was developed in collaboration with Dor Abrahamson, now a tenured professor at the University of California, Berkeley. Papers about this approach are on my website karenfusonmath.com

There also is a two part Teaching Progression on Ratio and Proportion on the website that describes in detail the approach I am briefly outlining here.

Dr. Sybilla Beckmann, a mathematician at the University of Georgia, has developed a similar approach, and we have reported this together at an earlier NCTM meeting (PPT on my website).

Equivalent Fractions and Equivalent Ratios

4 Show how the pattern of equivalent fractions continues.



$\frac{2 \cdot 2}{2 \cdot 3} = \frac{4}{6}$

b. $\frac{3 \cdot 2}{3 \cdot 3} = \frac{6}{9}$

d. $\frac{4 \cdot 2}{4 \cdot 3} = \frac{8}{12}$

5 Show how the pattern of equivalent ratios continues.

Cups of Juice

Raspberry	2	4	6	8
Blueberry	3	6	9	12

2 cups of raspberry:3 cups of blueberry

4 cups of raspberry:6 cups of blueberry

a. 6 cups of raspberry: 9 cups of blueberry

b. 8 cups of raspberry: 12 cups of blueberry

6 Draw to show the ratio pattern.

7 Discuss how equivalent fractions and equivalent ratios are alike and different.

What do you "see" in the fraction notation and in the strip drawings?

What do you "see" in the ratio notation and in the drawings of objects for the table entries?

Students find patterns in the Multiplication Table.

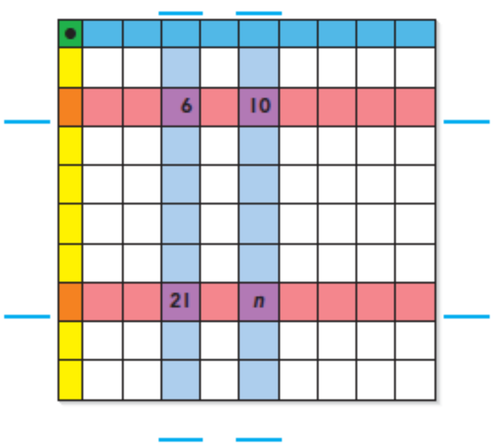
Be sure that they see columns and rows of the table.

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

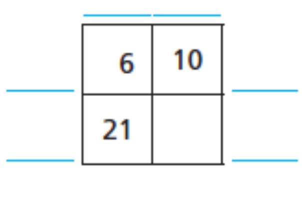
●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

A Factor Puzzle is made from two intersecting rows and columns of the Multiplication Table. Students (and teachers) love solving Factor Puzzles. This gives them experience in proportional thinking, helps with fluency, and is a good way for all students to describe their thinking because there are multiple correct orders of steps in solving a Factor Puzzle.

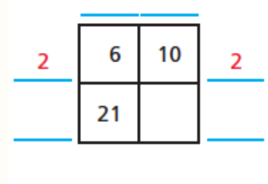
Table 3



Factor Puzzle



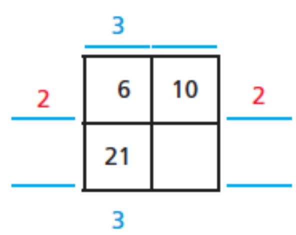
Factor Puzzle



Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Factor Puzzle



Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Find the row and column with only one known value, then you know the unknown value in the Factor Puzzle.

Factor Puzzle

	3	5	
2	6	10	2
	21		
	3	5	

Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Factor Puzzle

	3	5	
2	6	10	2
7	21		7
	3	5	

Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Factor Puzzle

	3	5	
2	6	10	2
7	21	35	7
	3	5	

Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Discuss rate tables from situations, see how the rows build up by adding the same number, the rate.

This **rate table** shows Noreen's savings.

9 Fill in the rest of the table to show how much money Noreen saved each day and how much her total was each day.

Days	Dollars	
1	3	+ 3
2	6	+ 3
3	9	+ 3
4	12	+ 3
5	15	+ 3
6	18	+ 3
7	21	+ 3
8	24	

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Then see how the rate table is two columns from the Multiplication Table and how rows are made by multiplying the unit rate by the quantity in the first column.

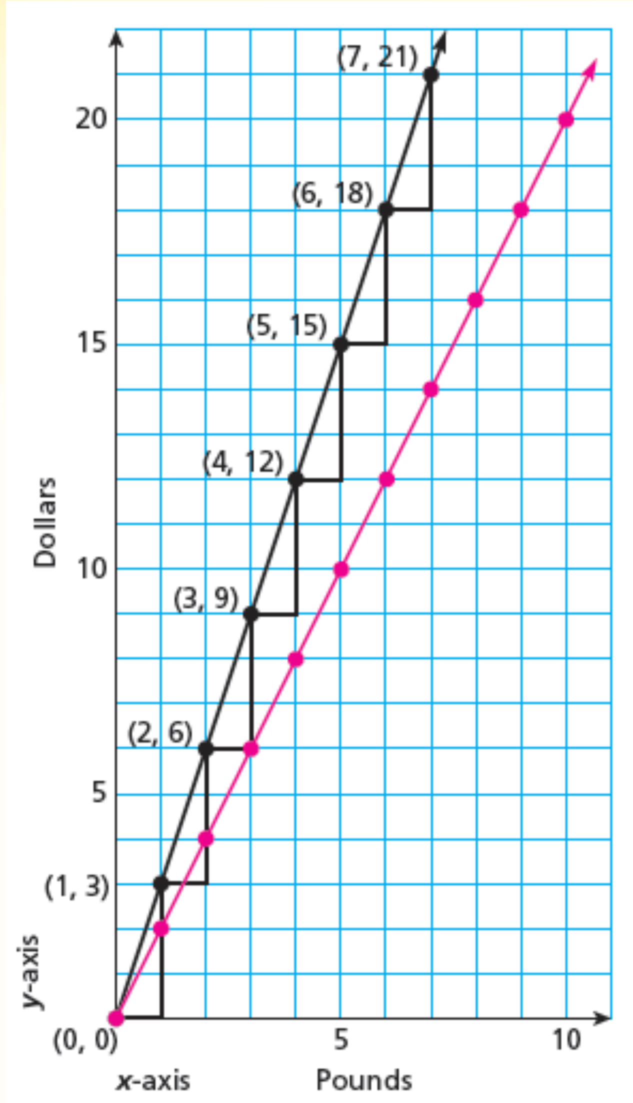
Discuss lots of other rate tables and situations.

See how the first column in a rate table is the ones column and the rate column is made by multiplying by the unit rate.

Number of Pounds	Cost in Dollars
1	3
2	6
3	9
4	12
5	15
6	18
7	21

Number of Pounds	• Unit Rate	=	Cost in Dollars
p	• r	=	C
1	• 3	=	3
2	• 3	=	$\boxed{6}$
3	• 3	=	$\boxed{9}$
$\boxed{4}$	• 3	=	12
5	• 3	=	$\boxed{15}$
6	• 3	=	$\boxed{18}$
$\boxed{7}$	• 3	=	21

Number of Pounds	Cost in Dollars
1 • 3	3
2 • 3	6
3 • 3	9
4 • 3	12
5 • 3	15
6 • 3	18
7 • 3	21



Ratio tables as linked rate tables. They are linked by their common situational ones column.

Noreen saves $3 a day and Tim saves $5 a day. They start saving on the same day. The linked rate table and the ratio table show Noreen's and Tim's savings.

Linked Rate Table

Days	Noreen 3	Tim 5
1	3	5
2	6	10
3	9	15
4	12	20
5	15	25
6	18	30
7	21	35
8	24	40

Ratio Table

	Noreen 3	Tim 5	
	3	5	
+ 3	6	10	+ 5
+ 3	9	15	+ 5
+ 3	12	20	+ 5
+ 3	15	25	+ 5
+ 3	18	30	+ 5
+ 3	21	35	+ 5
+ 3	24	40	+ 5

Each row in the Ratio Table is a multiple of the unit rates shown in the first row, the row with the linked situational ones number.

Proportion problem:

Grandma made applesauce using the same number of bags of red apples and bags of yellow apples. Her red apples cost $6 and her yellow apples cost $14. I used her recipe but made more applesauce. I paid $35 for my yellow apples. How much did my red apples cost?

Two equivalent ratios make a proportion.

All rows in a ratio table are equivalent, so any two rows make a proportion.

Any two rows have the same red apple to yellow apple taste.

Ratio Table

	Bags	R : Y
		3 : 7
	1	3 : 7
• 2	2	6 : 14
	3	9 : 21
	4	12 : 28
• 5	5	15 : 35
	6	18 : 42
	7	21 : 49
	8	24 : 56
	9	27 : 63

Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Factor Puzzle

	Red	Yellow	
	3	7	
2	6	14	2
5	$\boxed{15}$	35	5
	3	7	

Later students work with unit rates that are fractions.

They discuss how fractions and ratios are different.

They observe again that the Multiplication Table rows and columns are symmetric so that the Factor Puzzle relationships will stay the same if rows are exchanged for columns.

So we could write Ratio Tables and proportions from them in vertical form using fraction notation and write Ratio Tables horizontally.

Cups of Trail Mix

	Walnuts	Raisins	
	3	5	
÷ 5	$\frac{3}{5}$	1	÷ 5
	6	10	
	21	35	

÷ 5

Walnuts	3	$\frac{3}{5}$	6	21
Raisins	5	1	10	35

÷ 5

But they continue to emphasize that as ratios increase, they are describing more things, not more parts as with fractions.

Cups of Trail Mix

Walnuts	Raisins
3	5
$\frac{3}{5}$	1
6	10
21	35

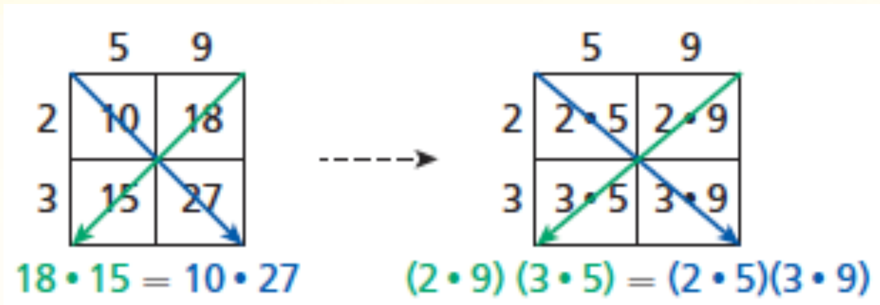
Walnuts	3	$\frac{3}{5}$	6	21
Raisins	5	1	10	35

Fantastically Purple Water

The sixth graders made a small glass of Fantastically Purple water with 2 drops of red and 3 drops of blue food coloring.

Red	2	10	18
Blue	3	15	27

The structure of a Factor Puzzle explains why cross-multiplication works: The opposite corner cells have the same four factors.



Share your questions and answers and things you do to reduce errors and to explain concepts.

Email these to me at

karenfuson@mac.com

I will collect responses, answer questions, and email the resulting Word file to all responders.

I will do this each week until we exhaust new entries or ourselves.

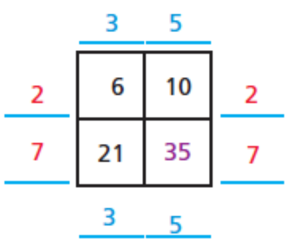
Please put NCSM as the subject for emails you send me.

Please also look at the fraction, decimal, and ratio and proportion Teaching Progressions on my website for more information.

karenfusonmath.com

Relating Decimals, Fractions, and Ratios for Deeper Understanding: How Are They Alike and How Are They Different?

Factor Puzzle



Class Multiplication Table

●	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

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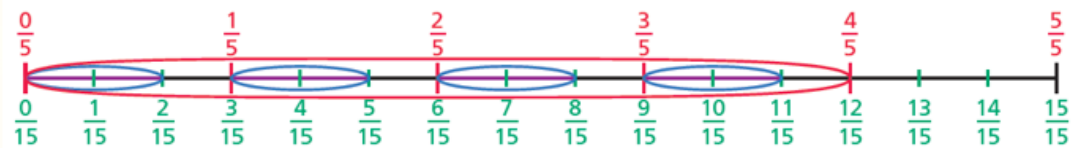

Paper presented at the Annual Conference of the National Council of Supervisors of Mathematics, 2020, Chicago, IL

Please see my website karenfusonmath.com for the 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, and presentations including this one.