Relating Fractions, Measurement, and Data with Meaningful Number Lines

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Paper presented at the Annual Conference of the National Council of Teachers of Mathematics, 2018, Washington, D.C.

For more details about all CCSS domains, please see the 18 hours of audio-visual Teaching Progressions I have made. You can find links to these and to papers and other presentations and to classroom videos at karenfusonmath.com

This presentation is also posted there.

K	1	2	3	4	5

MD Measurement and Data: K to 5

Geometric Measurement: K to 6

Uses length to make area and volume units

(Describe attributes)	(Length)	Length	Area	Angles	Volume [G6: surface area; and area of triangles, special quadrilaterals, and polygons is in geometry]

Other Measures: K to 5

(Describe attributes)	(Time)	Time Money	Time Liq volume Mass Metric liquid volume, mass are multiples of ten	Larger to smaller units x	Convert units both ways x ÷

Represent and interpret data: K to 5

		Line plots	½ ¼	½ ¼ 1/8	Use fraction operations
Classify into categories, count	Up to 3 categories compare	Picture & bar graphs all problems	Picture & bar graphs scale multiple unit 1- and 2-step compare		

NF Number and Operations–Fractions: 3 to 5

General:	Unit fractions	Compare any frs find eq frs	+- any fr eq frs fr x WN, fr [G6 fr ÷ fr]
Special cases:	Compare like n or d	+- like denom WN x fr	n ÷ d = fr WN÷fr fr÷WN

Length Tools

Geometric measurement: Rulers, tape measures

Other measures: Measurement scales for liquid volume and mass

Data: Bar graph scales and line plots

Fractions: Number-line diagrams

Problem solving and ratio: Double number-line diagrams

Coordinate graph scales

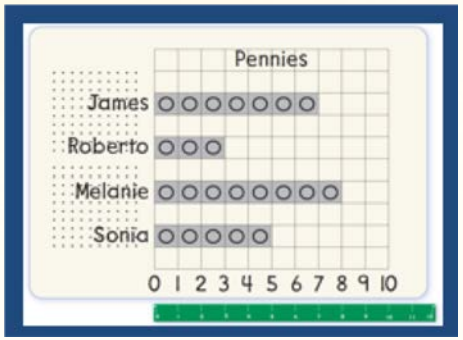
Conceptual Tools in Measurement, Data, and Fractions

Length Tools

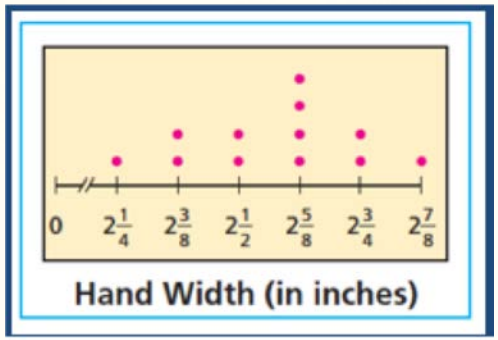
Rulers

Measurement scales for liquid volume and mass

Bar graph scales

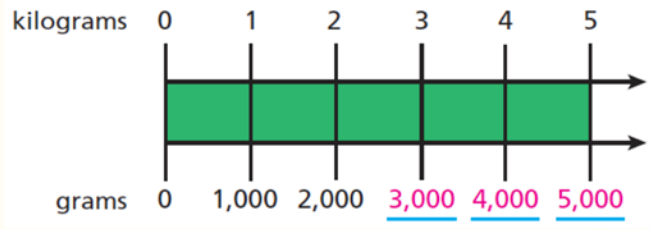


Line plots



Number-line diagrams

Double number-line diagrams



Length Tools in Measurement, Data, and Fractions

How are these tools alike?

How are these tools different?

Conceptual Tools in Measurement, Data, and Fractions

Length tools that are number-line diagrams	Equivalent easier length tools
Rulers, tape measures	
Measurement scales for liquid volume and mass	
Bar graph scales	Picture graph count scale
Line plots	
Number-line diagrams	Number bar/strip models
Double number-line diagrams	Tables
Coordinate graph scales	

G3 NF Standards: 3.NF.1, 2, 3

Grade 3: Develop understanding of fractions as numbers.

1. Understand a unit fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

3. Explain equivalence of fractions in special cases [small numbers], and compare fractions by reasoning about their size [same denominator or same numerator].

G3 Number Line Standard 3.NF.2

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a. Represent a fraction 1/b on a number line diagram by defining the **interval** from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size 1/b and that the **endpoint** of the part based at 0 locates the number 1/b on the number line.

b. Represent a fraction a/b on a number line diagram by marking off a lengths 1/b from 0. Recognize that the resulting interval has size a/b and that its **endpoint** locates the number a/b on the number line.

Length Tools in Measurement, Data, and Fractions

What is difficult about these endpoint-label tools even the ”simplest” such tools?

What errors do students make with these tools?

Why?

How can we reduce these errors?

2.MD.1: Measure Lengths in Standard Units Using Tools.

Length tools are visually difficult.

Children are wired to see things, so they see the marks on rulers.

Numbers by the marks draw the eye even more to marks.

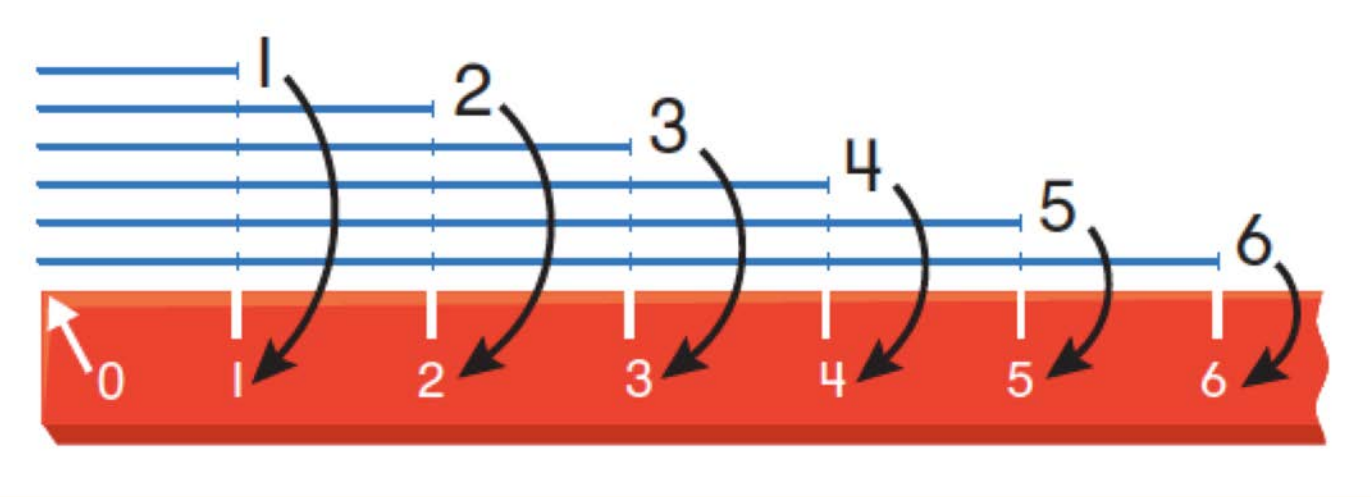
All length tools share this problem. It is a HUGE PROBLEM and takes a lot of teaching to overcome the problem.

G2: The Ruler as a Stack of Length Units

Composing length units to make a ruler.

Draw length units one by one marking the ends.

Move a finger along each unit as count the units.



After drawing all of the smaller length totals close to each other, imagine sliding them all on top of each other. All of the lengths are now embedded within the ruler.

2.MD.1: How to Help Students See and Count the Length Units

See and feel the length units.

- Draw length units one by one marking the ends.
- Move a finger along each unit as count the units.
- Color alternating unit lengths to see the lengths.
- Imagine a unit walker has shoes of the unit size and is walking heel to toe to make those units.
- Show length units by holding fingers apart that much (see the invisible length).
- Work with partner lengths to focus on the units.
- Discuss the pattern of one more mark than lengths.

G2: Work With Partner Lengths to See Length Units

See and feel the length units.

Draw length units one by one marking the ends.

Move a finger along each unit as count the units.

Work with partner lengths.

Find all of the partners for some small numbers so students see the partner lengths, here 2 cm and 4 cm.

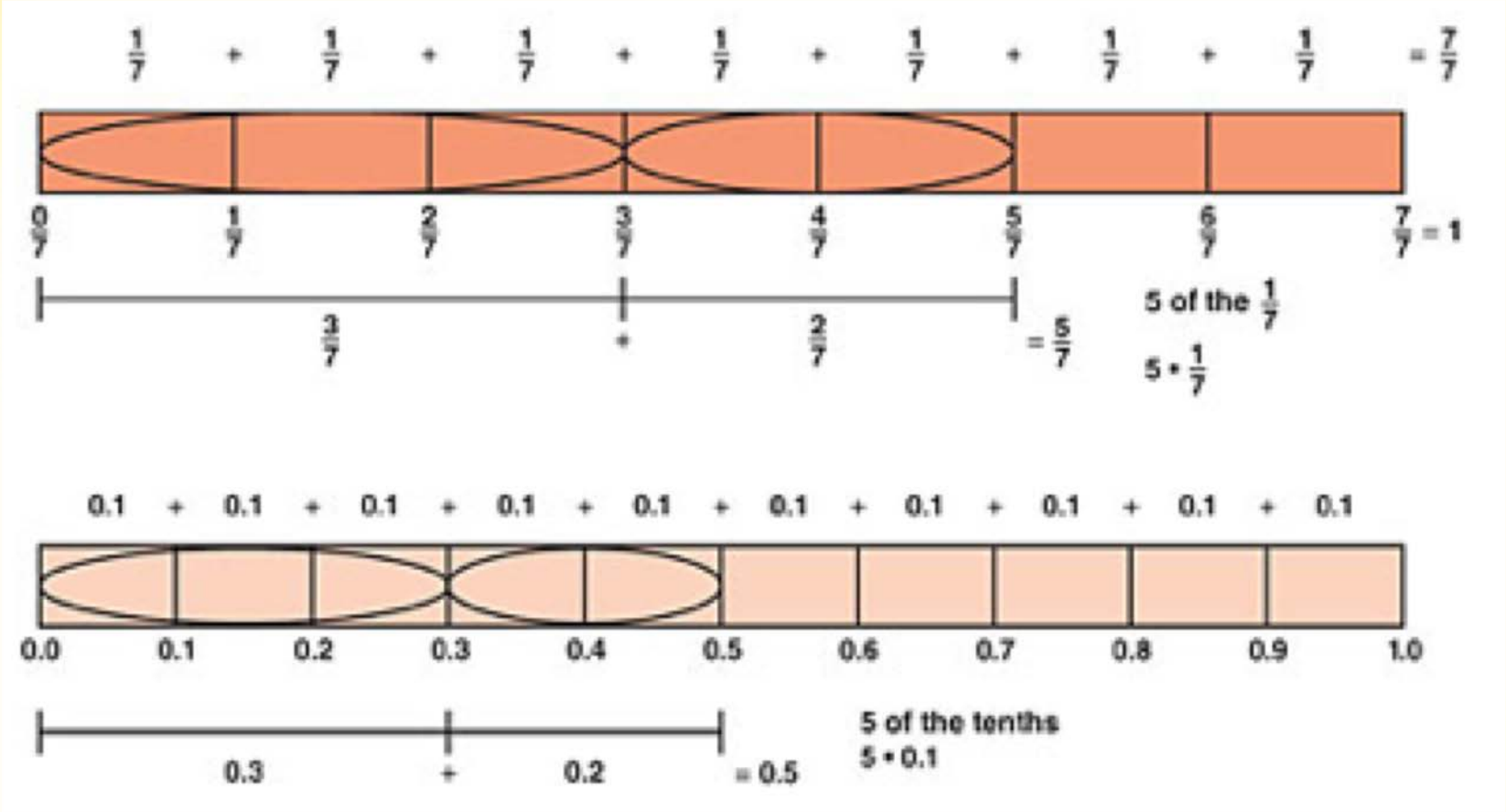
G2: There Is One More End-Mark Than Length Units

Find the pattern of one more mark than unit lengths.

Centimeters	Marks	1-cm Lengths
4	5	4
7	8	7
11	12	11
16	17	16

So counting end-marks instead of unit lengths will give a wrong answer.

See lengths by fraction bar labelling and by encircling



G6 Seeing Division as Finding the Unknown Factor in an Equal Groups Situation

2. The mugs at a restaurant hold $\frac{2}{3}$ cup of hot chocolate. The restaurant has $\frac{8}{15}$ cup hot chocolate left in its pot. How many servings of $\frac{2}{3}$ cup are in the pot?

$\frac{4}{5}$ serving

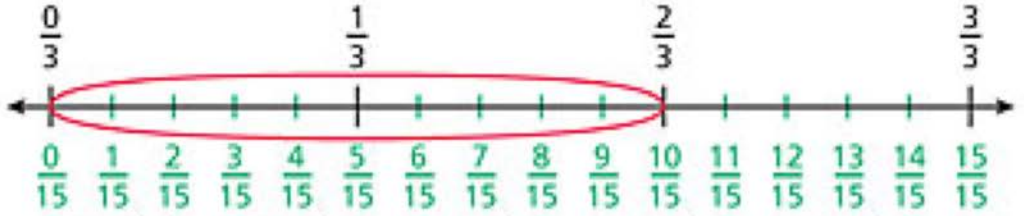
Step 1 Write an equation.

$\frac{?}{?} \cdot \frac{2}{3} = \frac{8}{15}$

Step 2 Look at the denominators.

Divide each $\frac{1}{3}$ into 5 equal parts to make fifteenths.

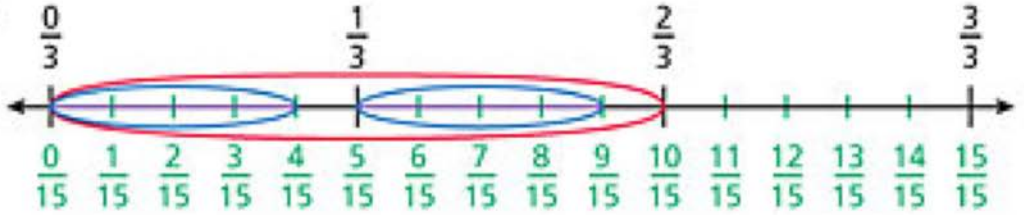
$\frac{?}{5} \cdot \frac{2}{3} = \frac{8}{15}$



Step 3 Look at the numerators.

Take 4 fifteenths from each of the 2 thirds to make $\frac{8}{15}$.

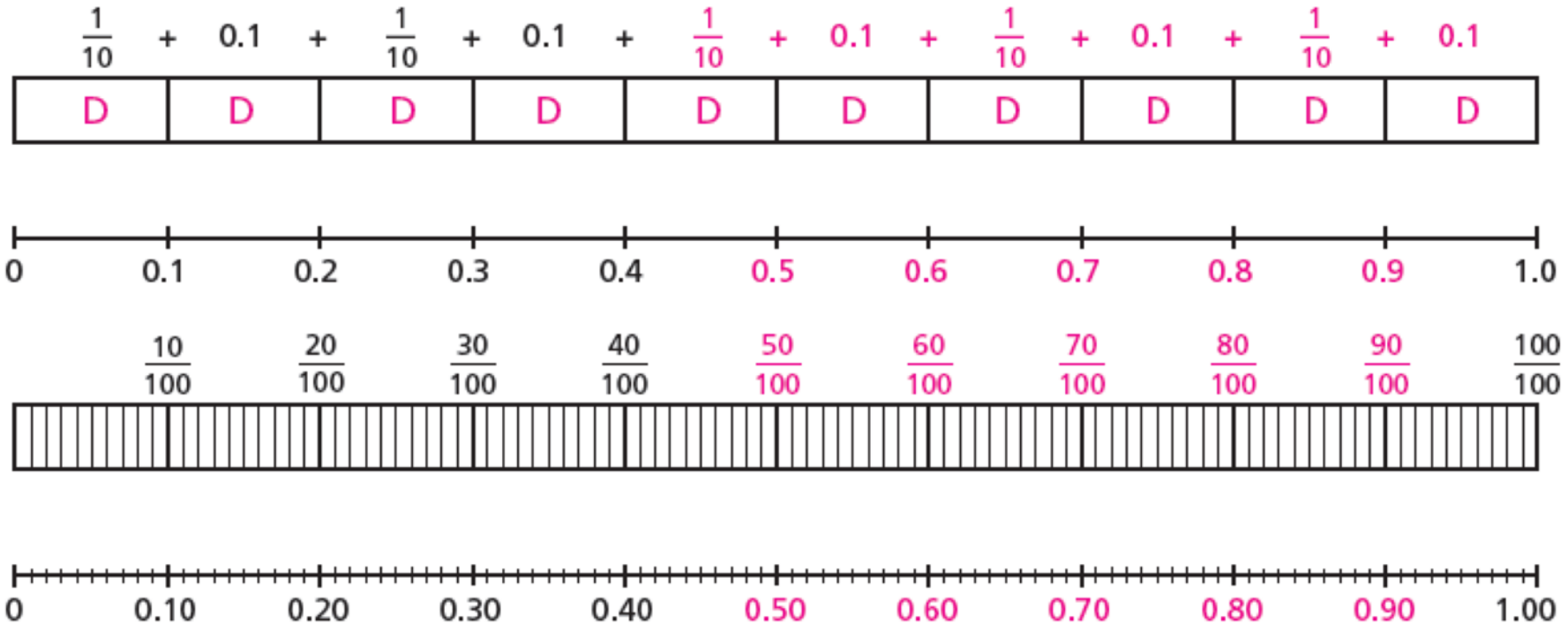
$\frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$



G4 Length Models for Decimals and Fractions

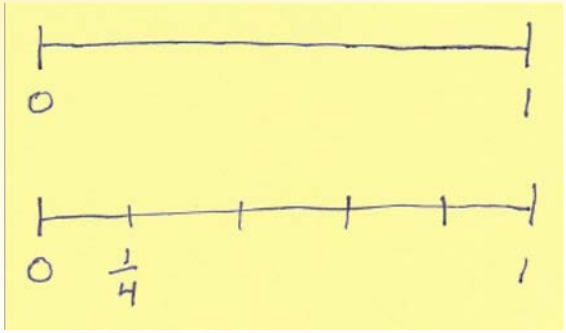
► Understand Tenths and Hundredths

Answer the questions about the bars and number lines below.



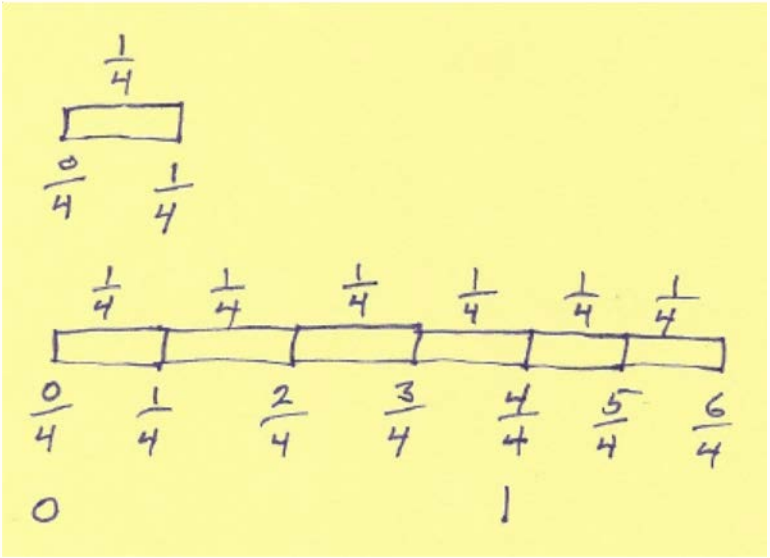
Building Up Fractions Drawing Unit Lengths

Students often make fractions by equally dividing one whole.



They may make 4ths by making 4 marks instead of 4 lengths.

Have students build up fractions by drawing unit fractions.



This also helps them see fractions greater than one as sums of unit fractions.

Length Tools in Measurement, Data, and Fractions

What endpoint-label tools have special difficulties?

What errors do students make with these tools?

Why?

How can we reduce these errors?

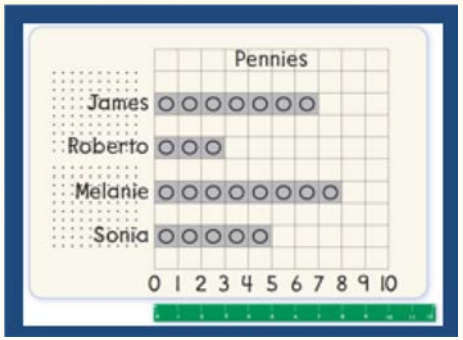
Conceptual Tools in Measurement, Data, and Fractions

Length Tools

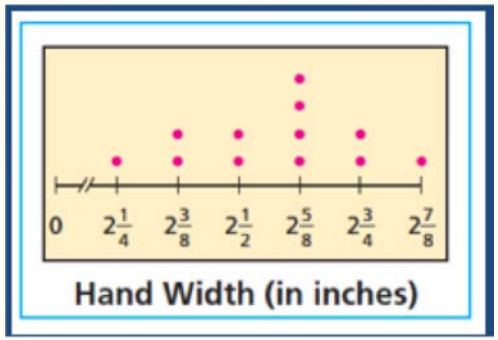
Rulers

Measurement scales for liquid volume and mass

Bar graph scales

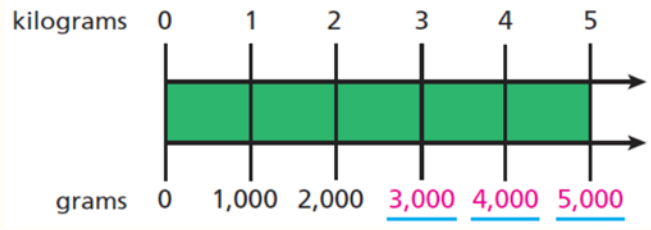


Line plots



Number-line diagrams

Double number-line diagrams

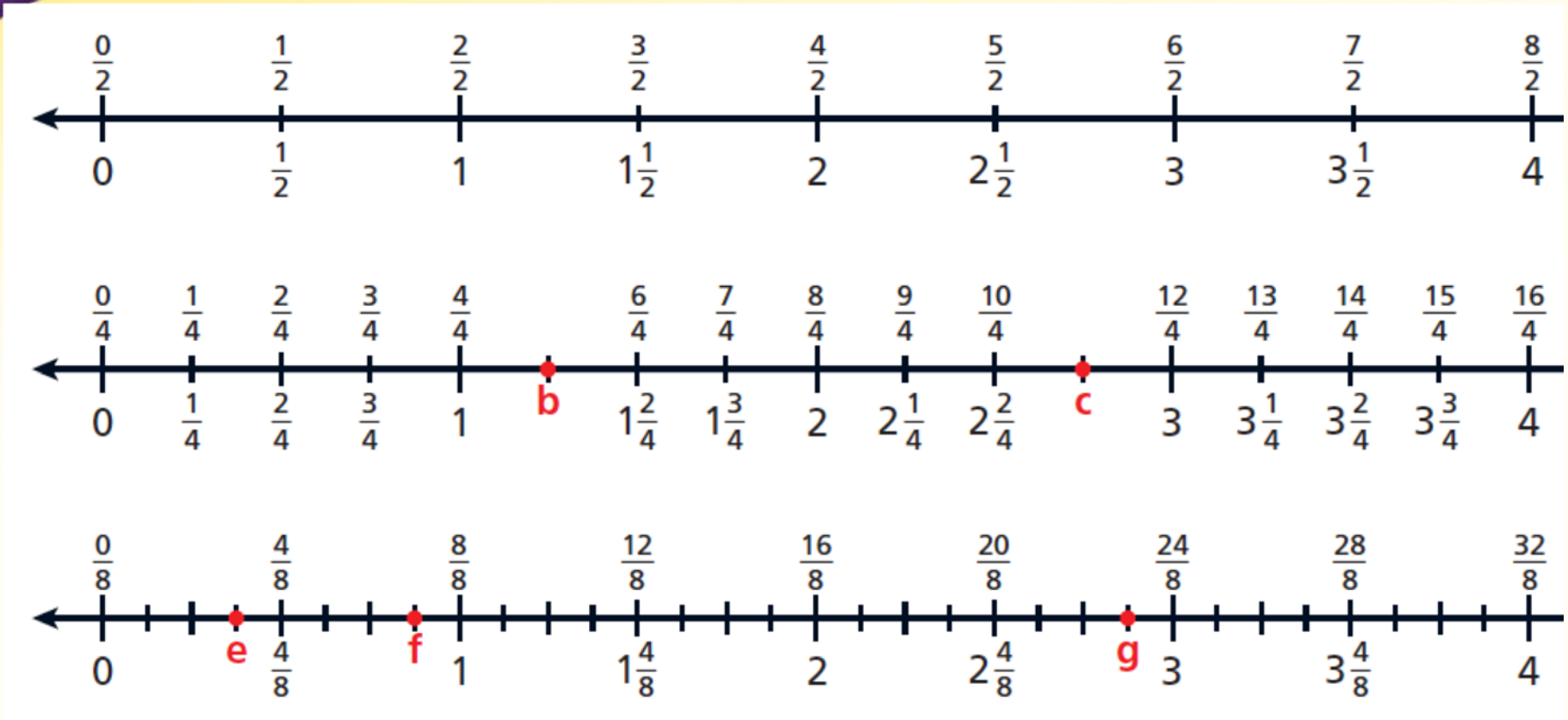


G

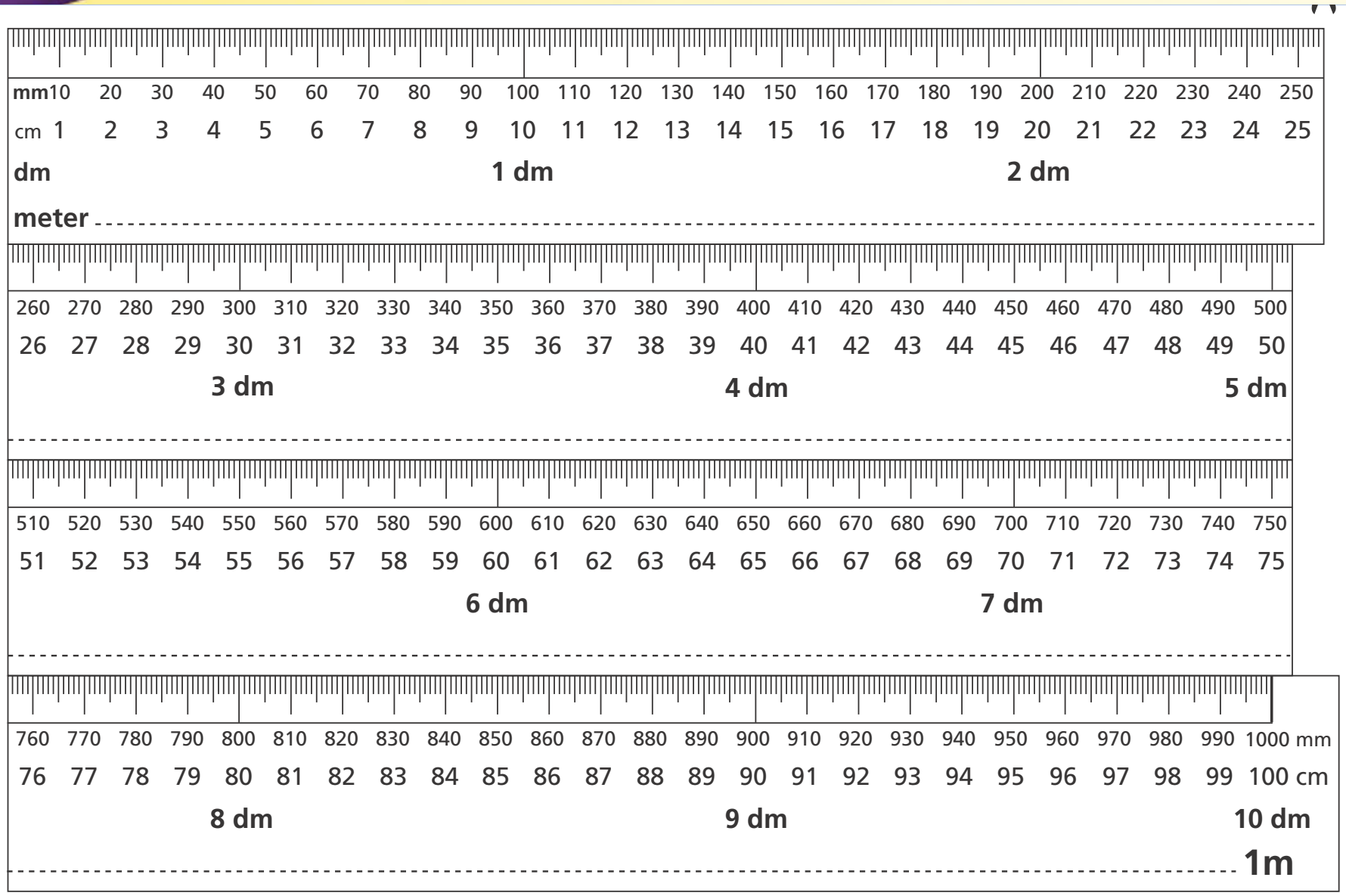
Inch ruler with 16ths

Inch ruler with 8ths

Relate rulers with different fractional units



G5 Using Metric Length to Understand Decimals



G5: Review Metric Units for Liquid Volume and Mass and Relate These to Units of Length

Emphasize the main units for liquid volume and mass.

Size of unit gets larger.

Length

Kilometer (km) 1,000 m	Hectometer (hm) 100 m	Dekameter (dam) 10 m	METER (m) 1 m	Decimeter (dm) 0.1 m or $\frac{1}{10}$ m	Centimeter (cm) 0.01 m or $\frac{1}{100}$ m	Millimeter (mm) 0.001 m or $\frac{1}{1{,}000}$ m
10 × 1 hm	10 × 1 dam	10 × 1 m	10 × 1 dm	10 × 1 cm	10 × 1 mm	
	$\frac{1}{10}$ or 0.1 × 1 km	$\frac{1}{10}$ or 0.1 × 1 hm	$\frac{1}{10}$ or 0.1 × 1 dam	$\frac{1}{10}$ or 0.1 × 1 m	$\frac{1}{10}$ or 0.1 × 1 dm	$\frac{1}{10}$ or 0.1 × 1 cm

Liquid Volume

Kiloliter (kL) 1,000 L			LITER (L) 1 L			Milliliter (mL) 0.001 L or $\frac{1}{1{,}000}$ L

Mass

Kilogram (kg) 1,000 g			GRAM (g) 1 g			Milligram (mg) 0.001 g or $\frac{1}{1{,}000}$ g

Size of unit gets smaller.

G5: Relate Metric Units of Length, Liquid Volume, and Mass

Metric units of length, liquid volume, and mass have two lovely relationships.

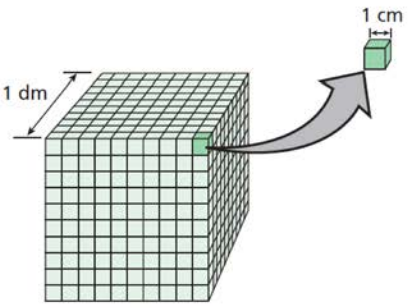
One relationship involves a cube that is 10 cm x 10 cm x 10 cm.
This **cubic decimeter** holds **1 liter** of water with a **mass of 1 kilogram**.

The other relationship involves a tiny cube that is 1 cm x 1 cm x 1 cm.
This **cubic centimeter** holds **1 milliliter** of water with a **mass of 1 gram**.

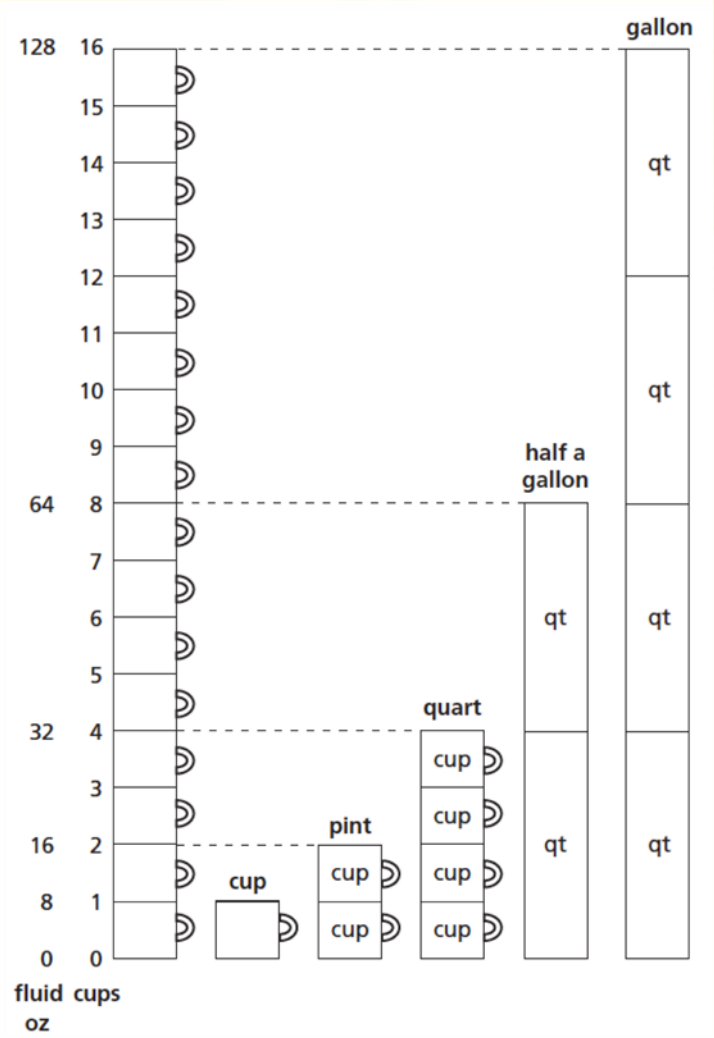
Teachers can make a cubic decimeter by cutting a half-gallon container in half. This is about 10 cm x 10 cm x 10 cm and shows how much a liter is.

Relate Metric Units of Length, Liquid Volume, and Mass

1 cubic decimeter can hold 1 liter.	1 cubic centimeter can hold 1 milliliter.
1 liter of water has a mass of 1 kilogram.	1 milliliter of water has a mass of 1 gram.



G4: Customary Units for Liquid Volume



The cup is the basic unit for liquid volume.

Here is a pictograph showing how many cups are in a pint, quart, half a gallon, and gallon.

The vertical scale on the left shows the number of cups and the number of fluid ounces (oz.) in each of these liquid volume units.

We can also see the relationships among the units, for example,

2 pints = 1 quart 2 quarts = 1 half gallon

2 half gallons = 1 gallon

If we include a half gallon in the sequence of units, we get a nice regular doubling relationship across the units. Things are often sold in a half gallon, so this is a meaningful unit to add to make the relationships coherent.

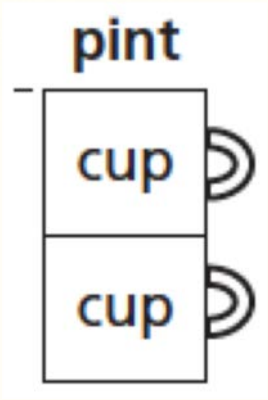
Students can bring in containers to show these units. They can discuss relationships they see and discuss this chart of the liquid volume units.

G5: Relate a Customary Unit of Liquid Volume to a Customary Unit of Weight

There are not nice relationships between the customary systems of measurement as there are for metric systems.

There is one relationship that relates liquid volume and weight in a rhyme:
A pint's a pound the world around.

One pint of liquid (16 fluid ounces) weighs about a pound (16 ounces).

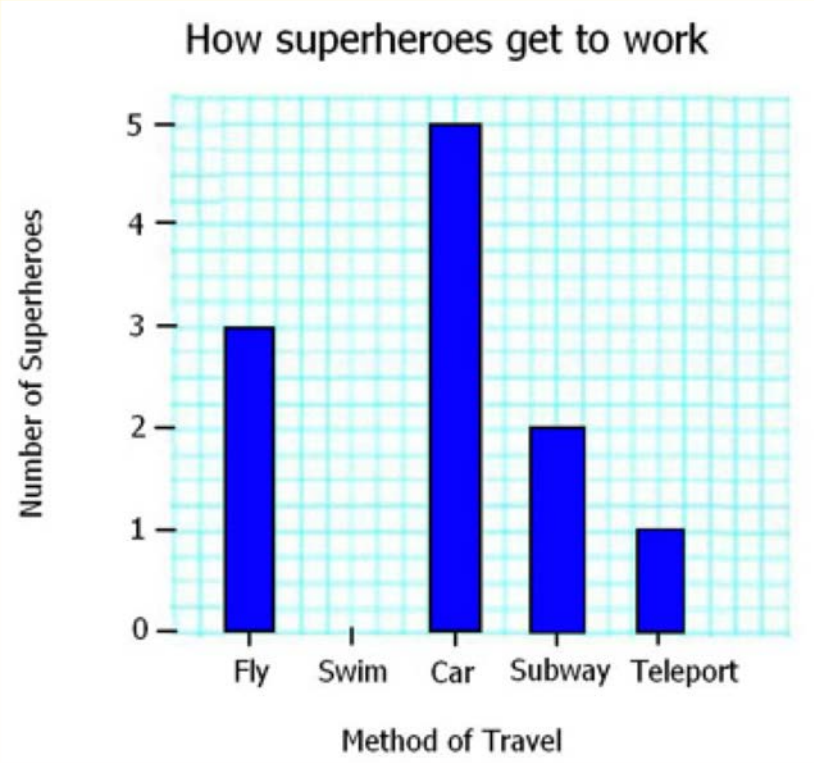


However, this is only true for the 16 fluid ounce pint in the United States.
In Britain a pint is 20 fluid ounces, which weighs 1 1/4 pounds.

So the rhyme needs to say:
A U.S. pint's a pound the world around.

weighs a pound

Graph grids can help read bar graph scales but you have to use lengths not the points



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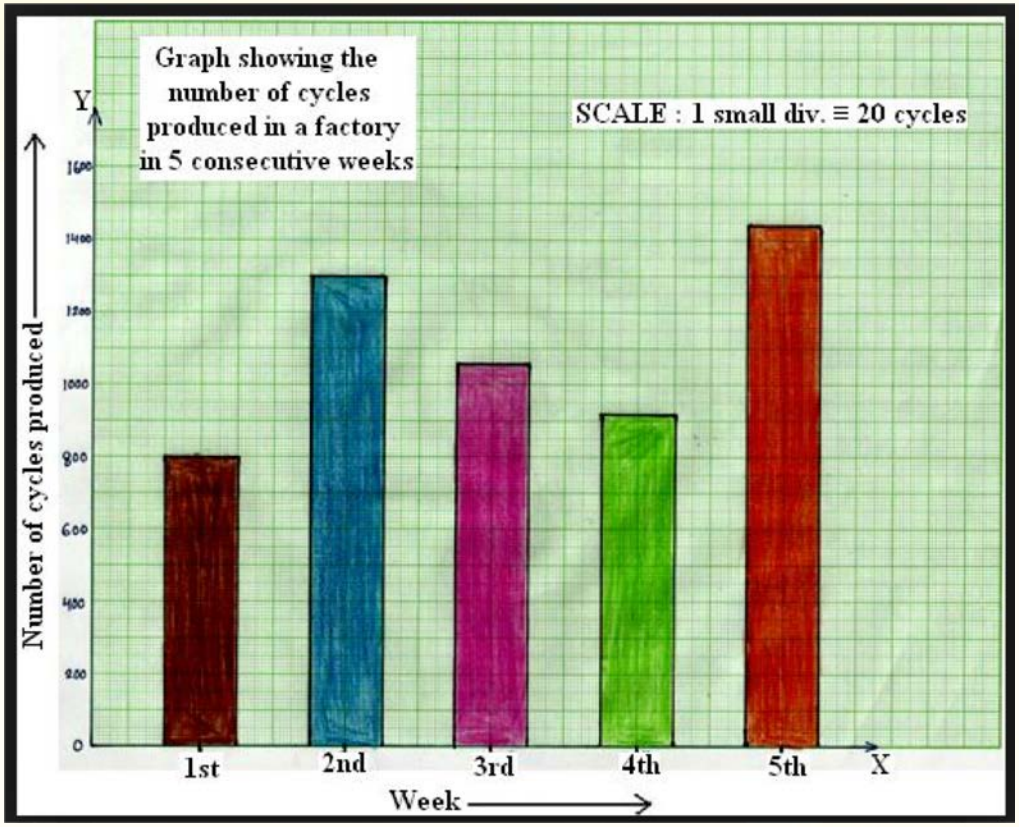
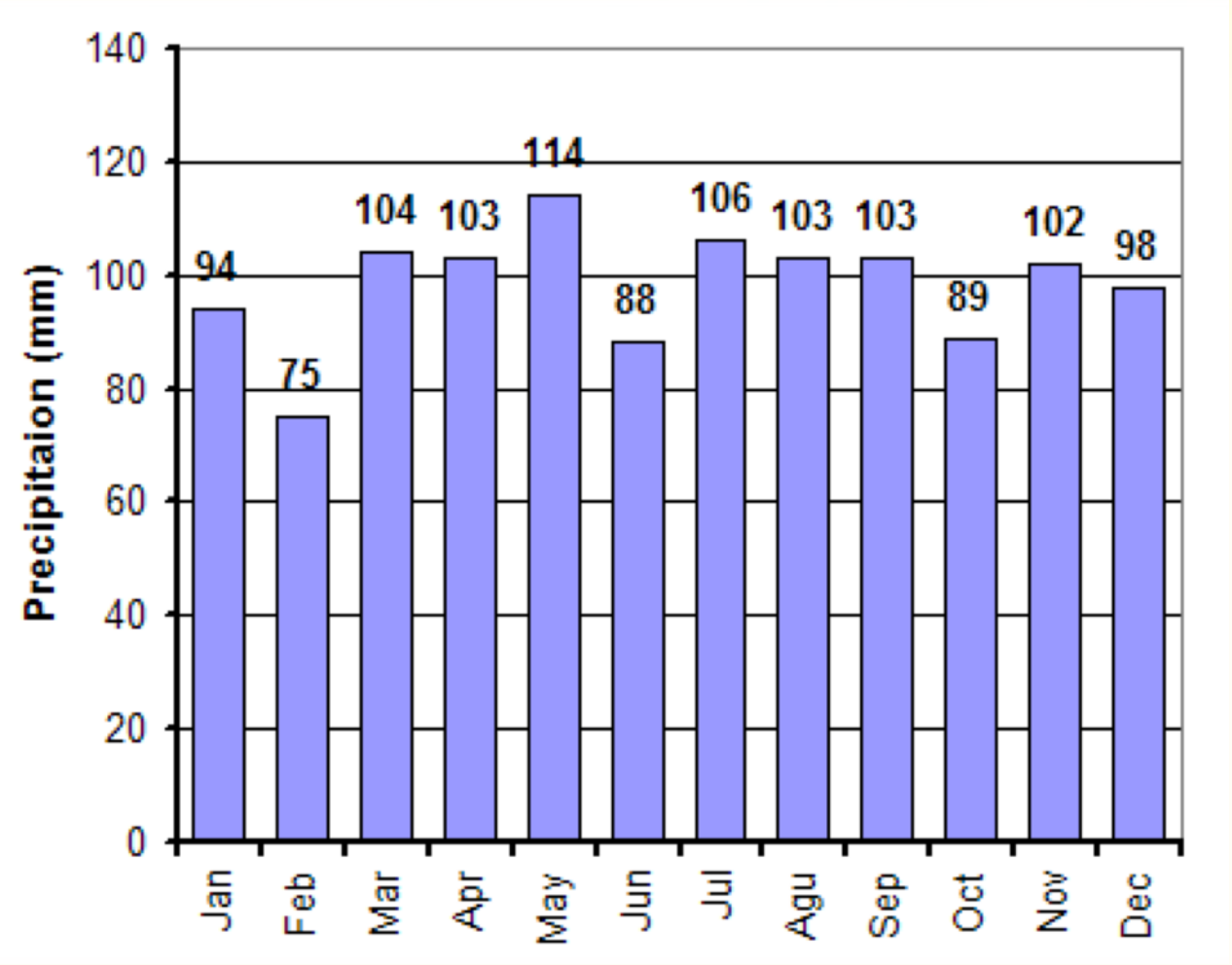


Math-only-math.com

Students can write the totals on a bar; is this better than a table?



http://ciese.org/curriculum/weatherscope/popup/graph1b/

Geometric measurement has different units built up by lengths.

So measuring tools need to be well learned.

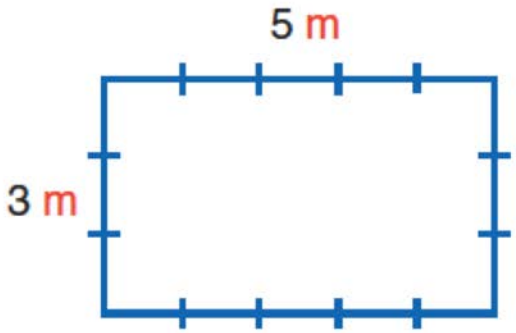
What errors do students make with these units?

Why?

How can we reduce these errors?

Emphasize the length units for perimeter and and the square units for area

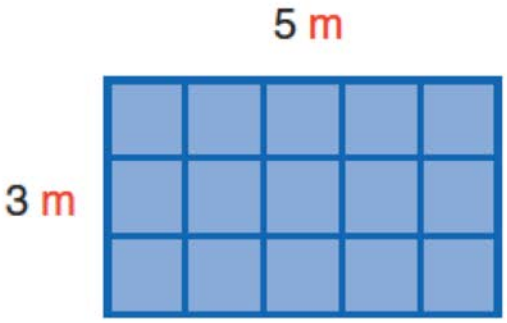
Perimeter and Area



$P = l + w + l + w$ or $2l + 2w$

$P = 5\text{ m} + 3\text{ m} + 5\text{ m} + 3\text{ m} = 16\text{ m}$

Perimeter is the distance around a figure. You add the side lengths to find the total distance.



$A = l \cdot w$

$A = 3\text{ m} \cdot 5\text{ m} = 15$ square meters

Area is the number of square units that cover a figure. You multiply the length and the width to find the total number of square units.

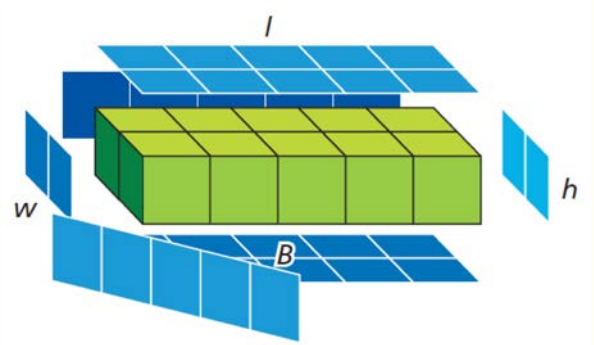
Differentiate the length units in perimeter and the square units in area.

For area, check that the side lengths have the same units so you can make the square units.

G6: Students Differentiate Surface Area and Volume of Prisms

Students see and identify the kinds of units used to measure surface area and volume.

- They see the **square units** that make the surface area and review that they write the answer as unit^2.
- They see the **cubic units** that make the volume and review that they write the answer as unit^3.

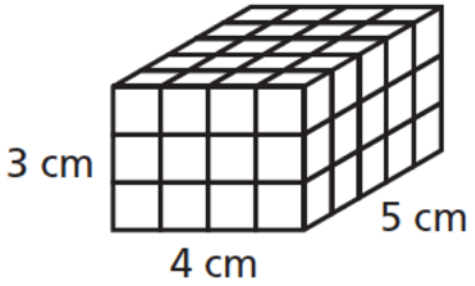


11. What is the surface area and volume of the prism you made?

$SA =$ 34 cm^2 2x5 + 2x2 + 2x2x5 cm^2

$V =$ 10 cm^3 2x5x1 cm^3

16.



$SA =$ 94 cm^2 2(3x4 + 3x5 + 4x5) cm^2

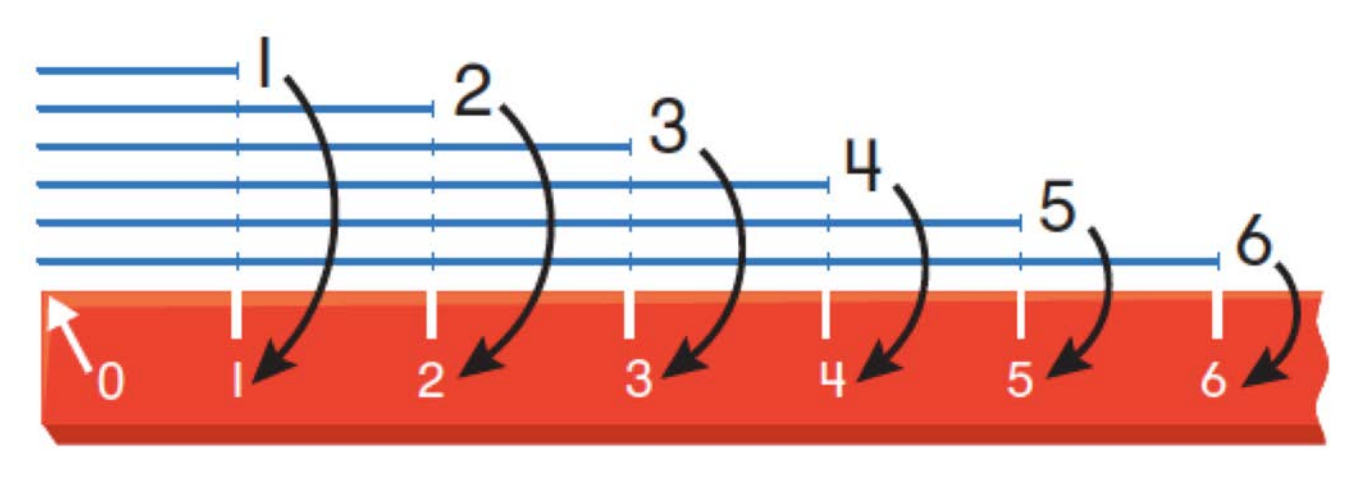
$V =$ 60 cm^3 3x4x5 cm^3

It all starts in Grade 2. Grade 2 teachers need to emphasize the lengths.

Composing length units to make a ruler.

Draw length units one by one marking the ends.

Move a finger along each unit as you count the units.



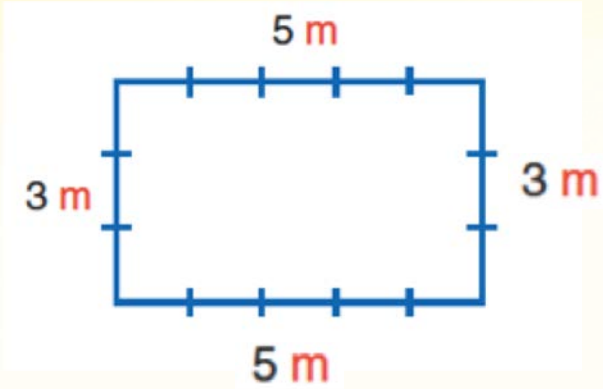
Encircle lengths.

Hold fingers apart to show the lengths.

Say 1 inch, 2 inches, 3 inches, 4 inches as count the lengths.

Grade 3 is crucial. Grade 3 teachers need to emphasize the lengths.

Grade 3 teachers need to emphasize the lengths in perimeter, in bar graph scales, and in rulers, mass, and liquid volume scales.



Perimeter:

Students show the lengths of all 4 sides to visualize perimeter correctly.

Also all work with fraction number-line diagrams needs to emphasize the lengths by relating these to number bar diagrams and in other ways discussed.

In all grades teachers frequently say,
"Show me the lengths."

"Show yourself the length units."

"Are you counting objects/things or length units?"

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