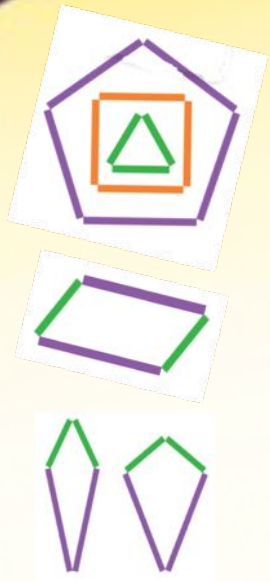
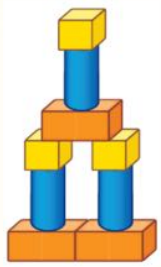
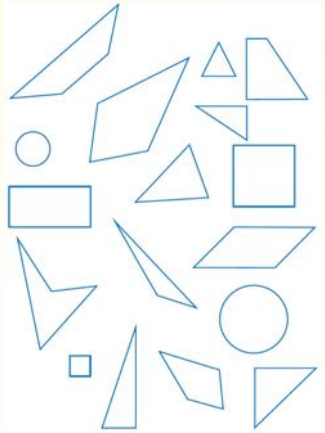
Relating Number and Geometry in the Early Grades



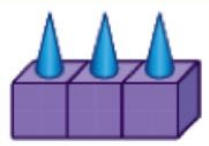
Professor Emerita Karen C. Fuson

Northwestern University





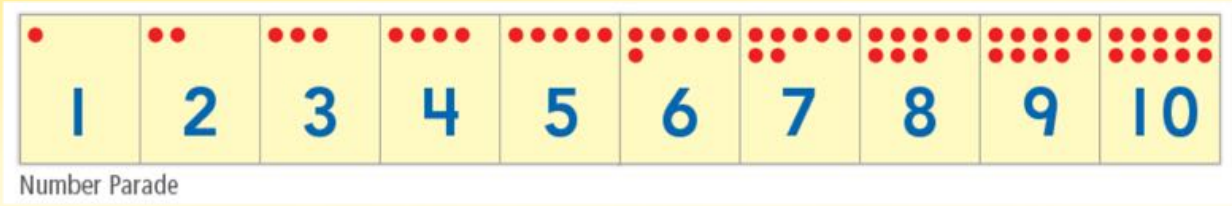
Paper presented at the Annual Conference of the National Council of Teachers of Mathematics, 2019, San Diego, CA

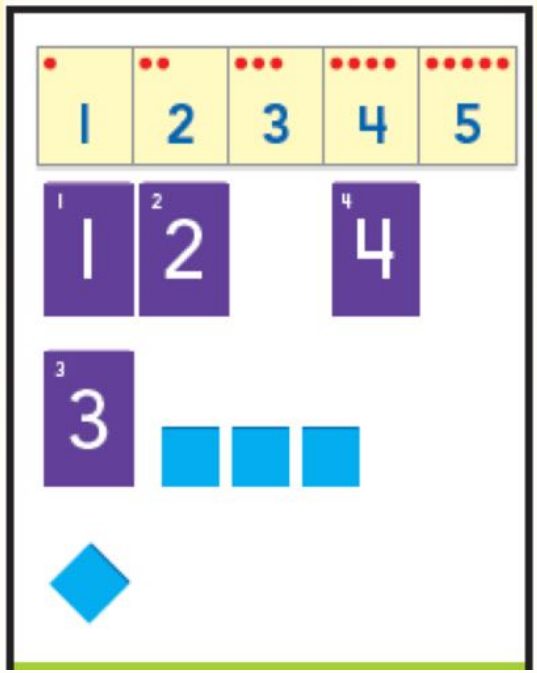


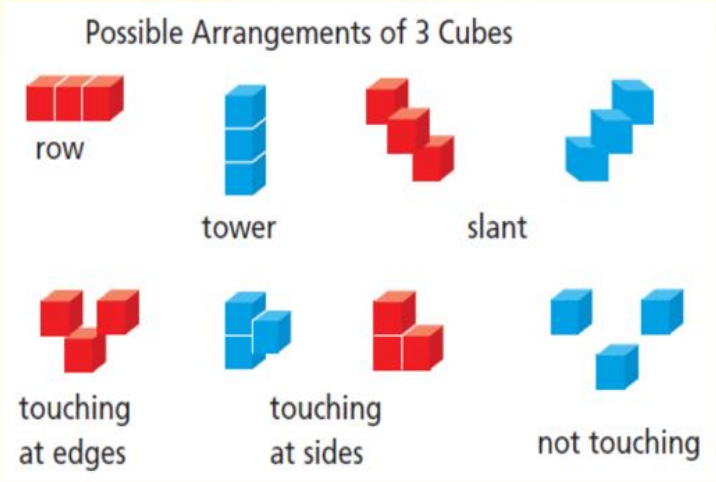
Please see my website karenfusonmath.com for

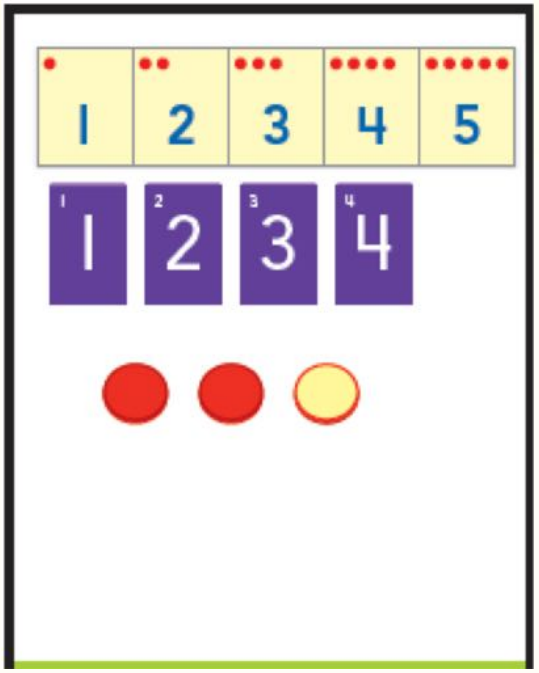
- **the 18 hours of audio-visual Teaching Progressions for all CCSS domains I have made, and**
- **for my papers, classroom videos, and presentations including this one.**

Perceptual and Conceptual Subitizing









Foundational and Achievable Goals for PK Programs

The National Research Council Report

***Mathematics learning in early childhood: Paths toward excellence and equity* (National Research Council, 2009)**

identified two major foci for PK foundational and achievable goals:

the Number, Relations, and Operations Core

the Geometry, Spatial Thinking, and Measurement Core.

The NRC recommendations were a major basis for the CCSS-M standards in K, 1, and 2. I focus here on PK, and we will briefly discuss how K, 1, 2 and later grades build on these PK goals.

I discuss research-based learning activities that allow PK children to move easily into K and be prepared to master K math goals.

Foundational and Achievable Goals for PK Programs

The National Research Council Report

***Mathematics learning in early childhood: Paths toward excellence and equity* (National Research Council, 2009)**

Major NRC recommendations are summarized and exemplified in *Focus in Prekindergarten (2010)*, published by the National Council of Teachers of Mathematics and the National Association for the Education of Young Children.

Classroom Structures Important for PK

Important classroom structures

Teaching/learning Sessions with Math Talk

The NRC Report said that these were crucial.

In this talk, I am emphasizing a learning path for these sessions to build deep connected knowledge; not little units that do not build and practice.

Brief Daily Routines that build the order of counting words and their relationship to written numerals and to cardinal quantities (Number Parades to 10, to 20, to 100)

The calendar obfuscates math structures (it is base 7 not base 10).

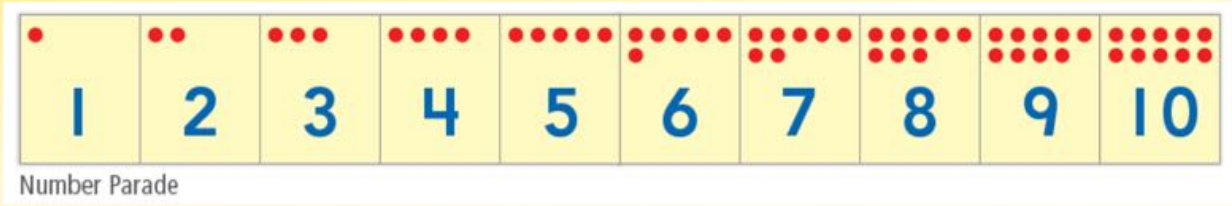
Math Centers for exploration and practice (support Math Talk also)

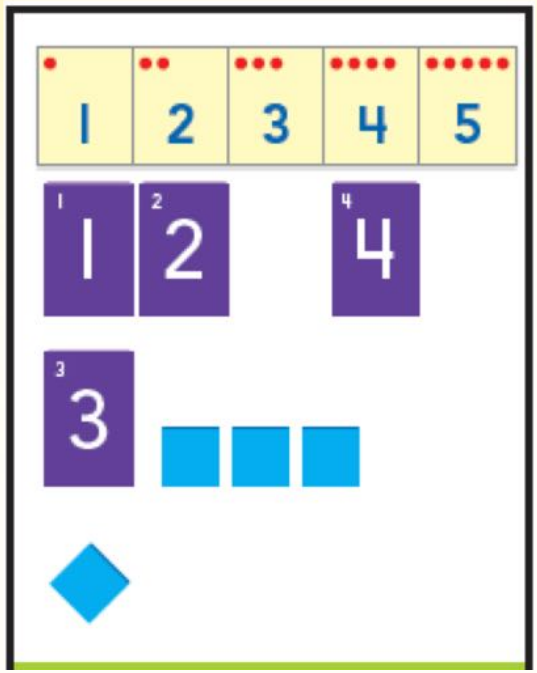
Math Throughout the Day

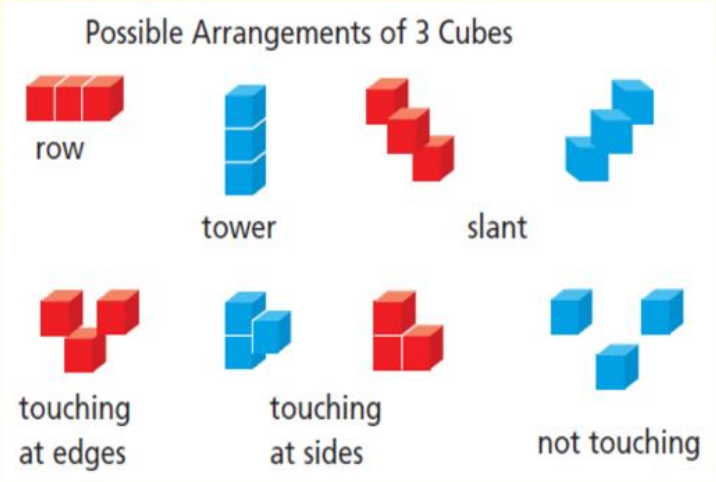
Tutorial Time

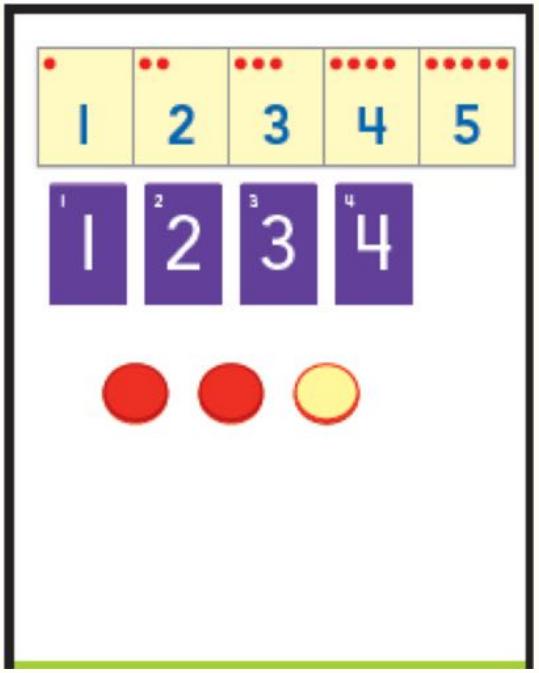
Home Connections

Perceptual and Conceptual Subitizing







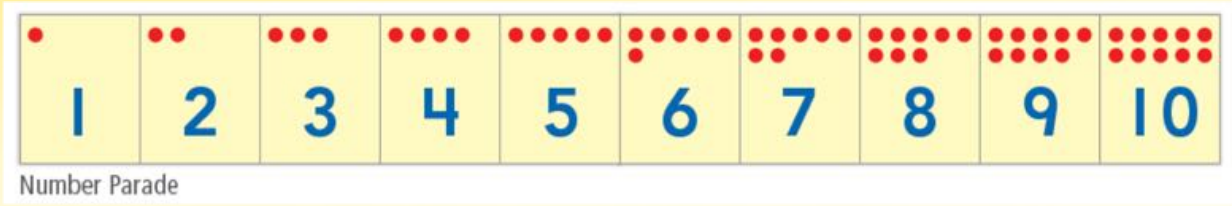


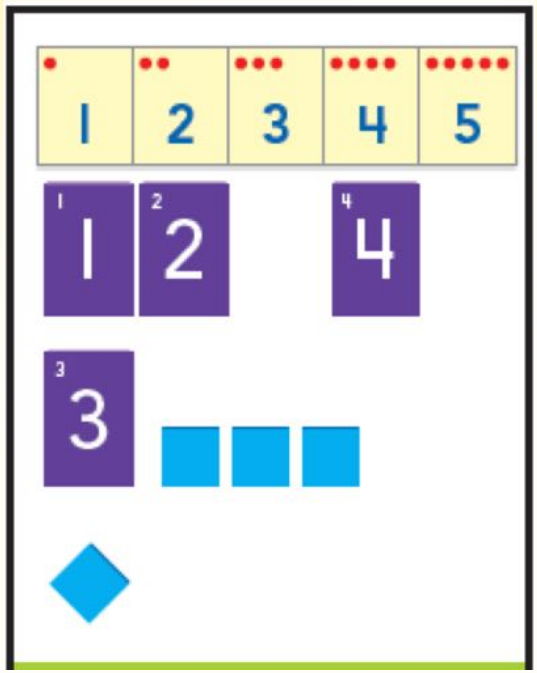
Equity and access benefit from multi-level supported activities

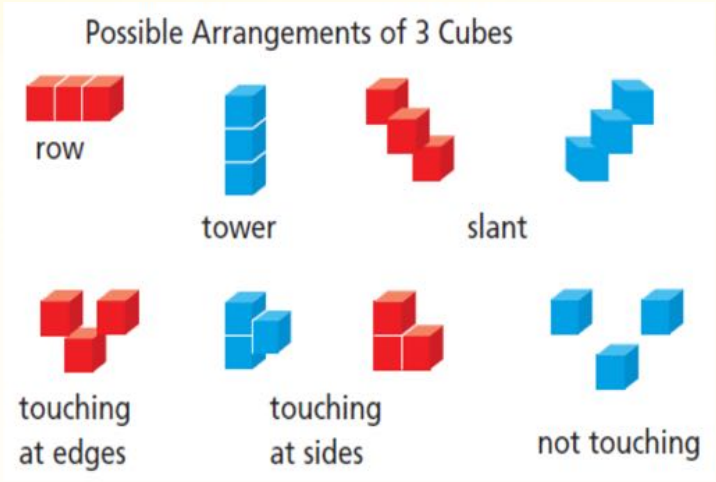
Equity and access benefit from multi-level activities that

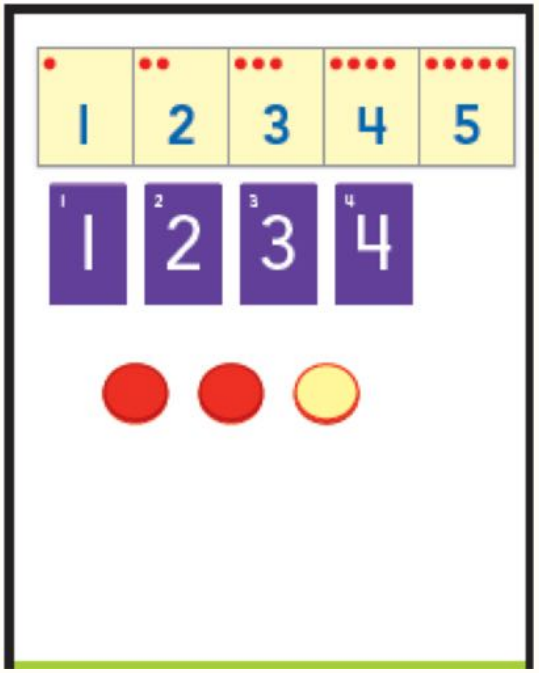
- **support children to relate visual, verbal, and body actions**
- **enable children to enter with more or less knowledge**
- **elicit children's thinking in actions and words in a nurturing Math Talk Community**
- **support children to help each other within and across pairs**
- **repeat the activities (repetitive experiencing) with small steps forward to help everyone learn**
- **and use multiple levels of related knowledge so that children more-advanced in knowledge also find them engaging.**

Perceptual and Conceptual Subitizing

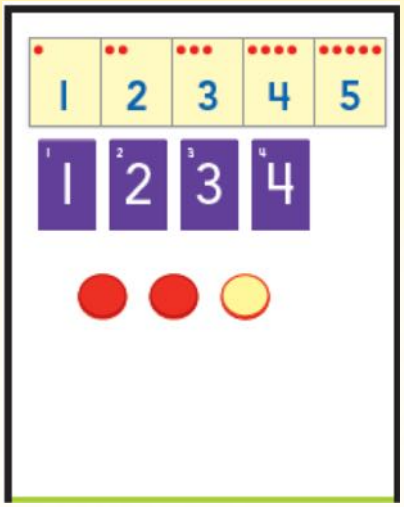


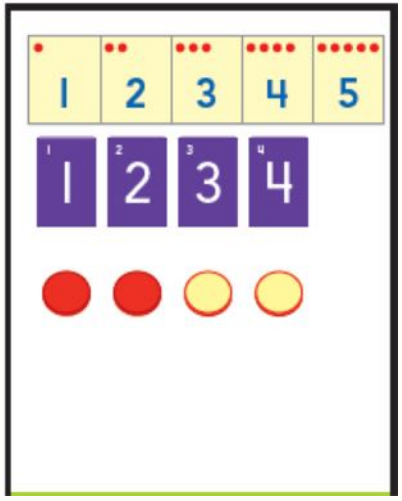


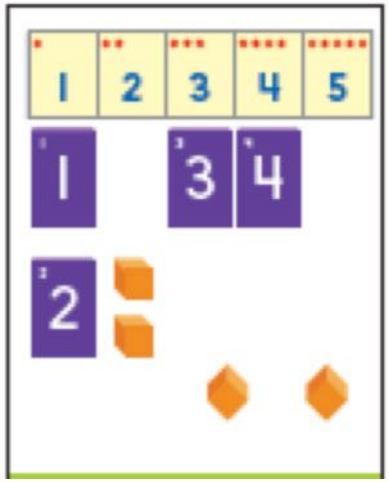


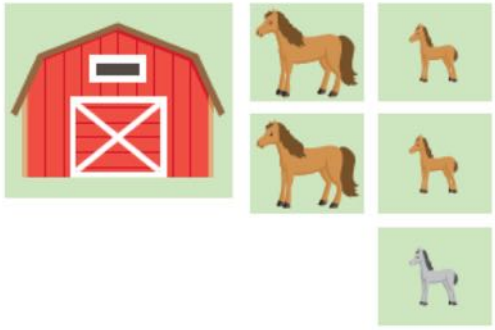


Partners Hiding Inside a Number









Explaining your thinking in a Math Talk Community

A Nurturing Math Talk Community

MathTalk in Action

Lin, tell us about your partners of 4.

Lin: I have 3 red circles and 1 yellow circle to make 4. Erica has the same colors, but hers are in different places.

Yes. Raise your hand if you show partners 3 red and 1 yellow.

Pedro: I have partners 3 and 1, but I have 3 yellow and 1 red.

Who has different partners?

Kiran: I have 2 and 2.

Walter: I do too!

Cathy: Look! I had 3 yellow circles and 1 red circle like Pedro, but I can turn over 1 yellow circle, and now I have 2 red and 2 yellow like Kiran and Walter.

So, do we all have 4 circles altogether? Let's count to be sure. Ready? 1, 2, 3, 4 circles. We all have 4 circles. We have different partners that make 4, but everyone has 4 in all.

MathTalk in Action

Describe how your circles are arranged alike or differently.

Eli: Mine are in a row.

Brianna: My circles are in a tower.

Julie: My 2 red circles are here and my 2 yellow circles are here.

Yes, your 2 red circles are on your left, and your 2 yellow circles are on your right.

Dylan: My 2 red circles are above, and my 2 yellow circles are below.

Odette: My circles go red, yellow, red, yellow, but I still have 2 red and 2 yellow.

Nick: My circles go red, red, red, yellow. I have 3 red circles and then 1 yellow circle.

Mario: Mine show partners of 3 and 1, but my 1 yellow circle is above the 3 red circles.

Gemma: Mine show partners of 3 and 1, but my 1 yellow circle is first.

Fran: I made a tower with my 3 red circles. My yellow circle is on the side.

So, our circles are arranged differently, but we all have 4!

Children detect errors Puzzled Penguin makes and correct the errors

Puzzled Penguin makes counting errors

one two

(point point)

one two

(point point point)

one two three four

(point point point point)

Children can identify errors even when they cannot do it correctly.

Children learn helping and caring behavior.

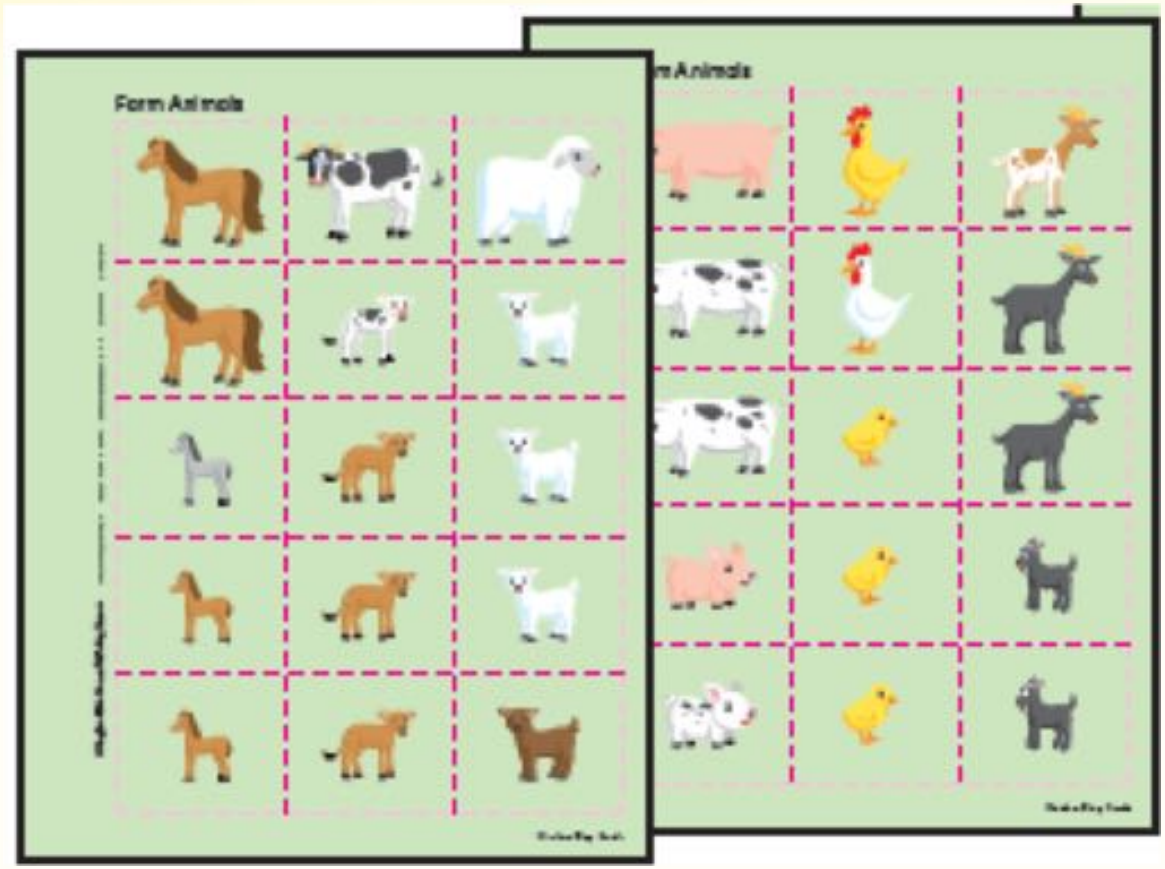
Partners: Seeing, describing, separating to hide some in the barn

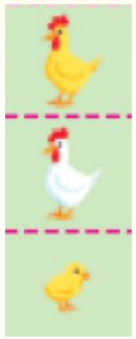
Work with one kind of animal at a time.
Start with any 4 and later move to partners of 5.
Player 1 hides some in (under) the barn.

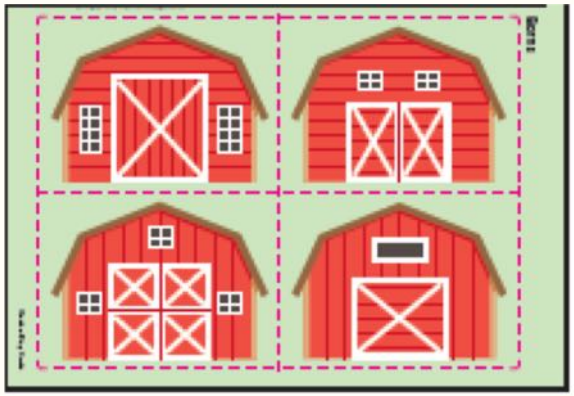
Player 2 looks under the barn and says how many and what kind. Move to:

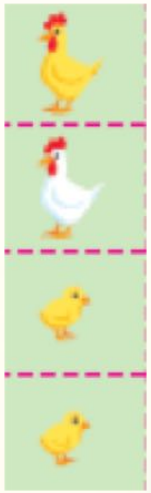
Predicting how many in the barn before looking (use the animals outside the barn)

Tell partners inside and outside the barn (smaller numbers)





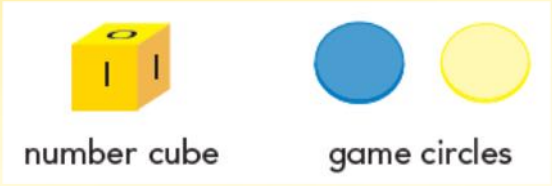




Number Path Games extending Robert Siegler's game

Discuss the pets. Discuss 0. Choose a pet and a Number Path.

Stack 4 squares on the pet area they choose.



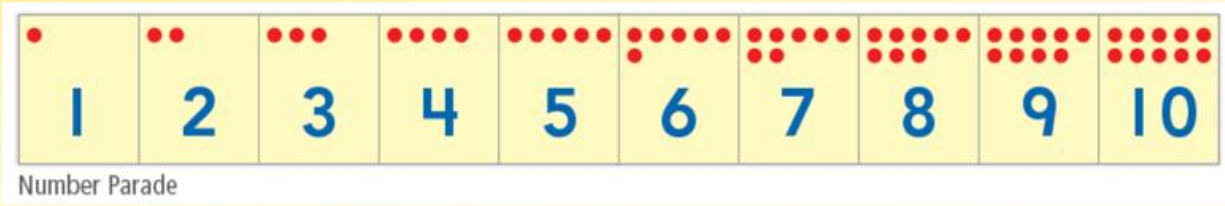
see-through

Play: Predict where the game circle will go if a 1 is rolled.
Roll and say the number on the top of the cube.
Move clear token 1 or 0 spaces on the Number Path.

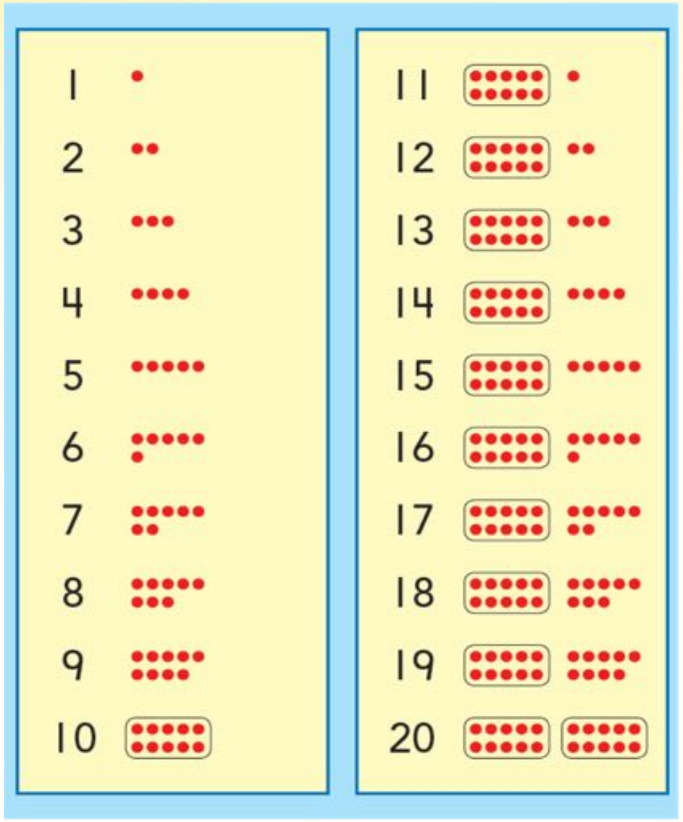
Say the number of that square.
Summarize the move: I was on 2.
I rolled 1 and moved from 2 to 3.
I get 1 new pet: Takes a square from the stack of squares on their pet space and puts it above 3 on his/her Number Path.
Now I have 3 bunnies.

Coordinating counting numbers and cardinal numbers
Focusing on the next number to build those links: 1 2 and 2 3 and 3 4

Daily Routines: Student Leader Points to Each Word as Children Count and Show Cardinalities



Children show with fingers



Part A: 1 to 10 using 5-groups

Part B: 1 to 20 using 10-groups for teen numbers

Part C: 1 to 100 using 10-groups

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

11 is ten and one
12 is ten and two,
etc.

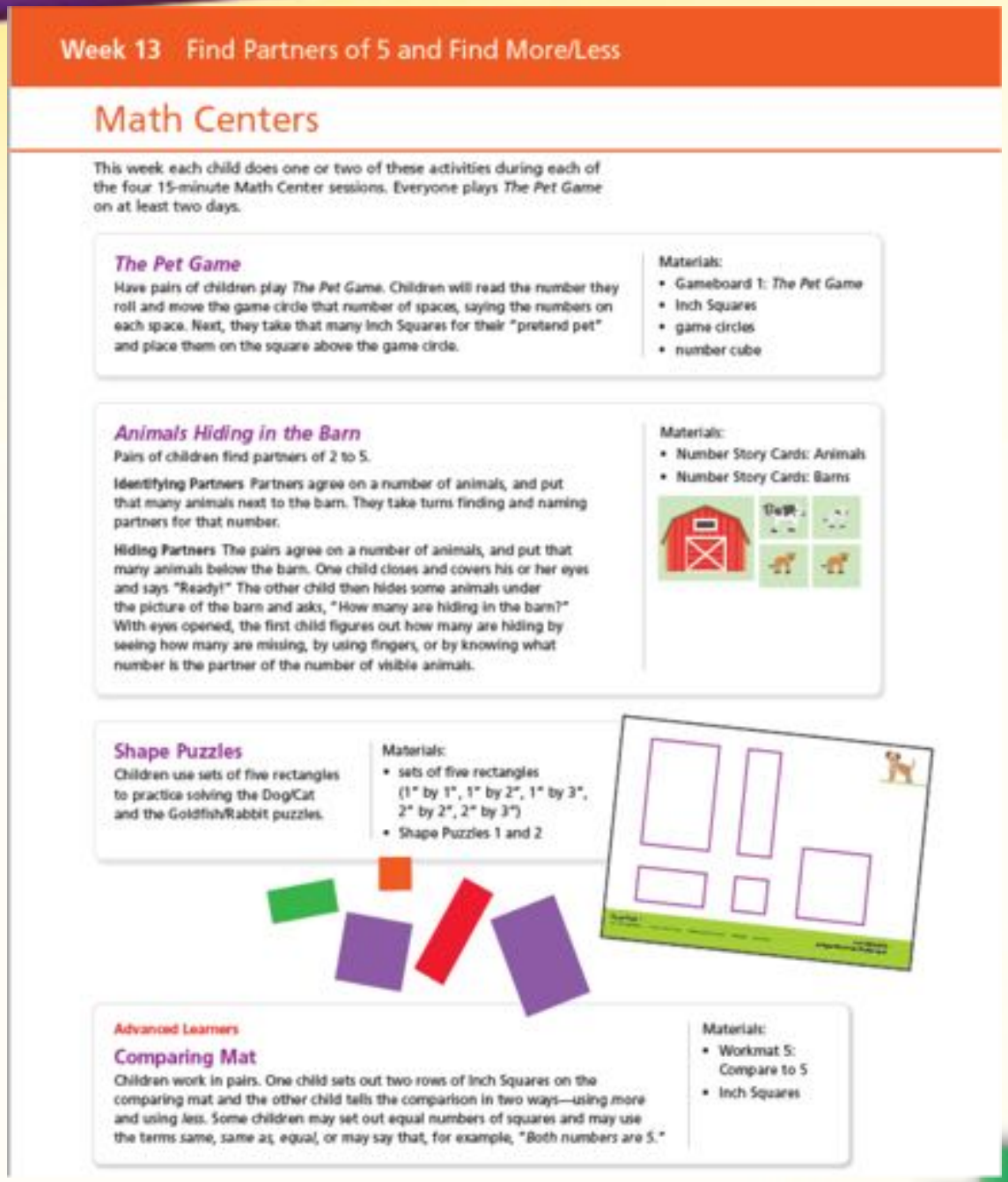
Math Centers for Exploration and Practice

Math Centers change weekly.
Math Centers

allow children to explore materials before they are used in the Learning Sessions

support children to practice activities from the Learning Sessions: children can help each other and even "be the teacher"

Children can use support for Math Talk during the centers.



Week 13 Find Partners of 5 and Find More/Less

Math Centers

This week each child does one or two of these activities during each of the four 15-minute Math Center sessions. Everyone plays *The Pet Game* on at least two days.

The Pet Game

Have pairs of children play *The Pet Game*. Children will read the number they roll and move the game circle that number of spaces, saying the numbers on each space. Next, they take that many Inch Squares for their "pretend pet" and place them on the square above the game circle.

Materials:
- Gameboard 1: *The Pet Game*
- Inch Squares
- game circles
- number cube

Animals Hiding in the Barn

Pairs of children find partners of 2 to 5.

Identifying Partners Partners agree on a number of animals, and put that many animals next to the barn. They take turns finding and naming partners for that number.

Hiding Partners The pairs agree on a number of animals, and put that many animals below the barn. One child closes and covers his or her eyes and says "Ready!" The other child then hides some animals under the picture of the barn and asks, "How many are hiding in the barn?" With eyes opened, the first child figures out how many are hiding by seeing how many are missing, by using fingers, or by knowing what number is the partner of the number of visible animals.

Materials:
- Number Story Cards: Animals
- Number Story Cards: Barns

Shape Puzzles

Children use sets of five rectangles to practice solving the Dog/Cat and the Goldfish/Rabbit puzzles.

Materials:
- sets of five rectangles (1" by 1", 1" by 2", 1" by 3", 2" by 2", 2" by 3")
- Shape Puzzles 1 and 2

Advanced Learners

Comparing Mat

Children work in pairs. One child sets out two rows of Inch Squares on the comparing mat and the other child tells the comparison in two ways—using *more* and using *less*. Some children may set out equal numbers of squares and may use the terms *same, same as, equal*, or may say that, for example, "Both numbers are 5."

Materials:
- Workmat 5: Compare to 5
- Inch Squares

Math Throughout the Day

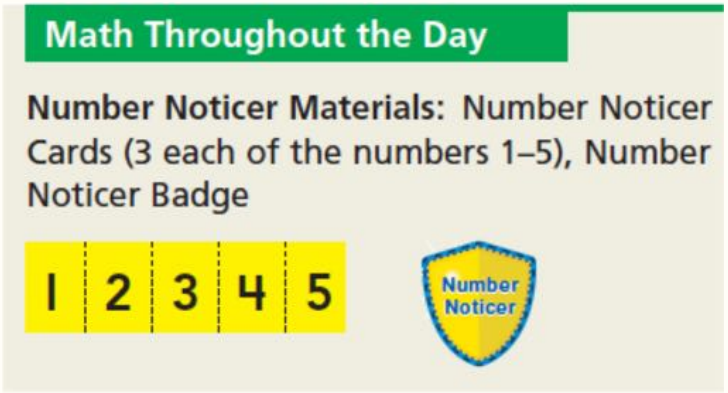
Math Throughout the Day occurs whenever adults or children notice or use math structures in their environment.

This mathematizing is important for bringing a range of meanings to the math structures children are learning and using.

Children who subitize, have counting skills, and notice numerosity do better in school.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Children's preschool subitizing, spontaneous focusing on numerosity and counting skills as predictors of mathematical performance 6–7 years later at school. Mathematical Thinking and Learning, 17, 155–177.

Number Noticer Materials



Children can also choose a Snack Number of the Day and point out and make groups with that number using snack items.

Geometry Part A: 3-Sided and 4-Sided Shapes (6 weeks)

Rectangles and Squares

Compose Rectangles

Triangles

Compose Rectangles with Right Triangles

Recognize and Describe Triangles and Rectangles

Recognize, Describe, and Match Shapes

Students need considerable experience with square units and the related rectangles, right triangles, and isosceles triangles made from these units. These are units and visual subunits for MD Geometric Measurement. The NRC Report recommended more time with such shapes. I developed such a special 2D shape set.

Six Coordinated Colors to Make Many Patterns and Generalize Across Color

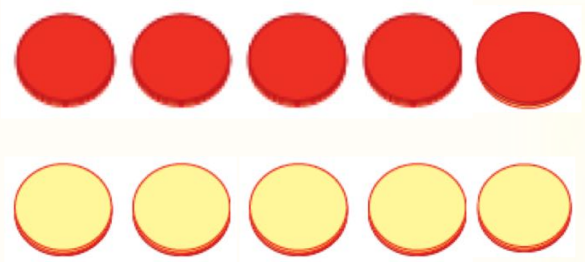
Children have square inch squares and square inch cubes in six colors.

They also have matching right triangles 1 inch on a side.

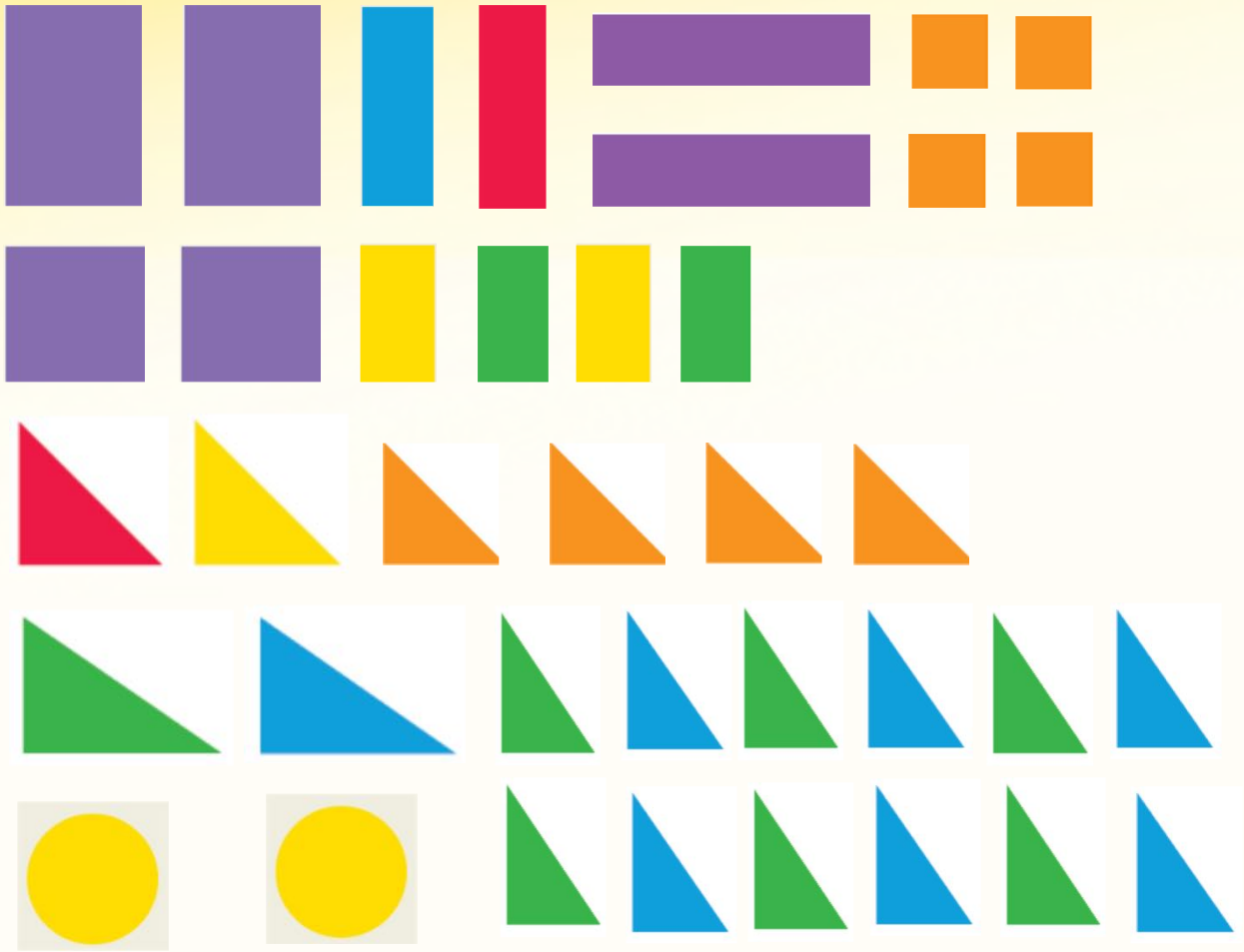
These shapes are used in number and in geometry activities.

These are manipulite and thick and easy to work with. There are enough for Math Centers also.

Children also work with two-sided red and yellow circles for number activities.



Math Expressions 2D Shape Set



Rectangles were selected to compose to make each other in multiple ways and to make many other shapes.

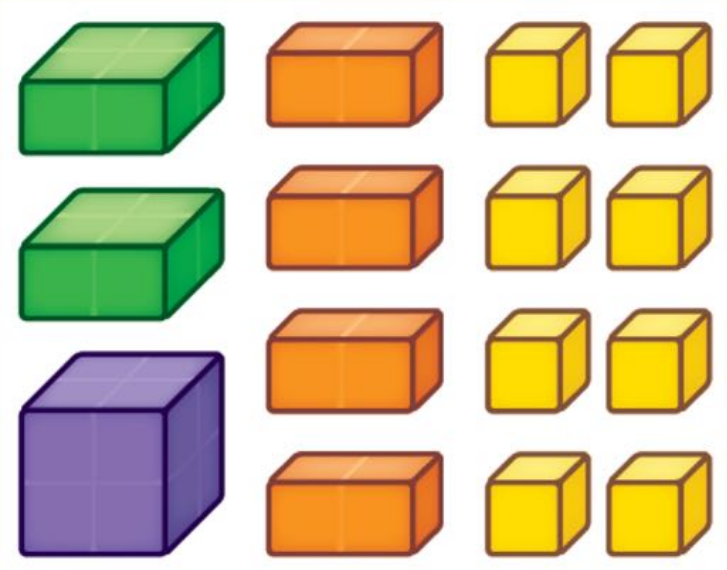
Right triangles compose to make rectangles and other shapes. They also work with the rectangles and with the 1”x1” right triangles of 6 colors.

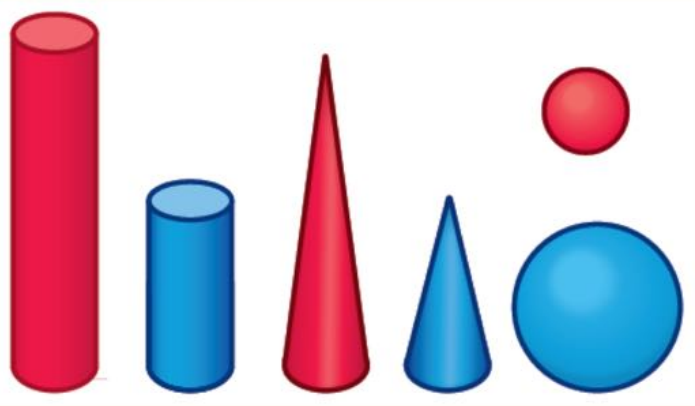
Math Expressions 3D Shape Set

Cubes, cylinders, cones, and spheres

Small yellow cubes make the orange, green, and purple shapes.

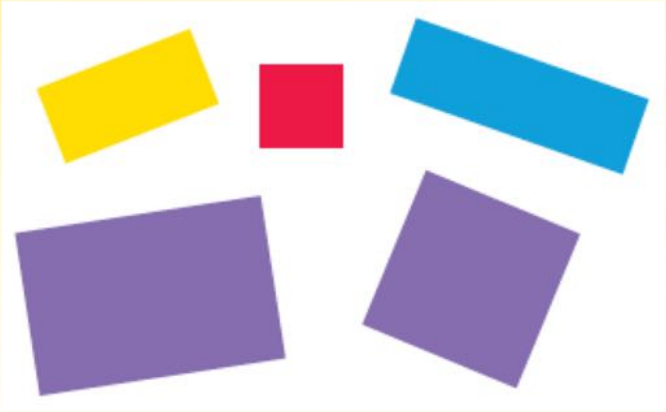
Orange and green build to make the big purple cube.





Rectangles and Squares

Manipulite shapes



Children discuss attributes of rectangles.

They count 4 sides. They count 4 corners.

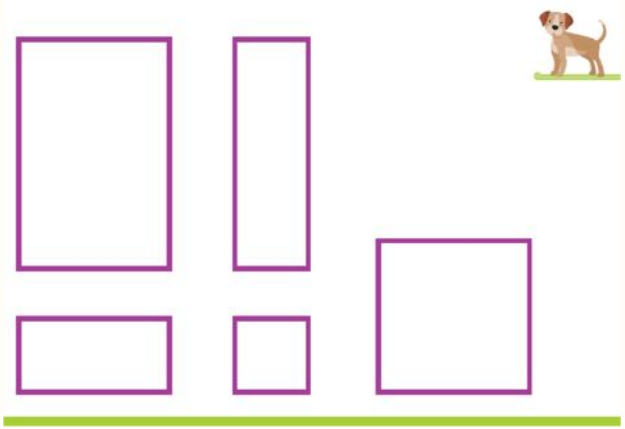
They discuss and model square corners by bending their arm up to make a square corner.

They open and close the arm to make angles that are not a square corner.

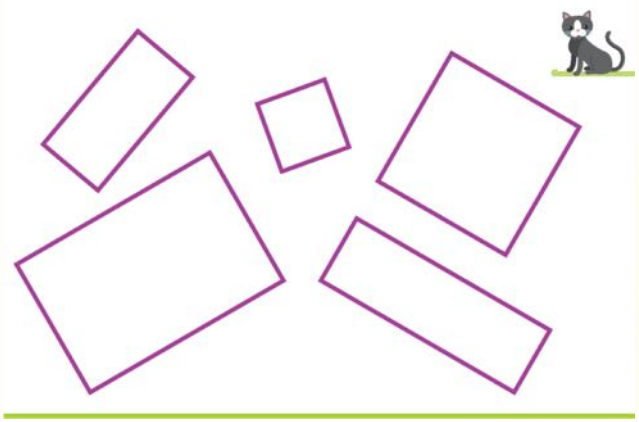
They point to square corners on a blank sheet of paper.

They match rectangles to puzzle shapes.

They explain what they are doing to the animal on the puzzle.



Dog, this is a purple rectangle. It has 4 sides and 4 square corners.



Squares are Special Rectangles

Children discuss attributes of the two squares.

They discuss why squares are special kinds of rectangles.

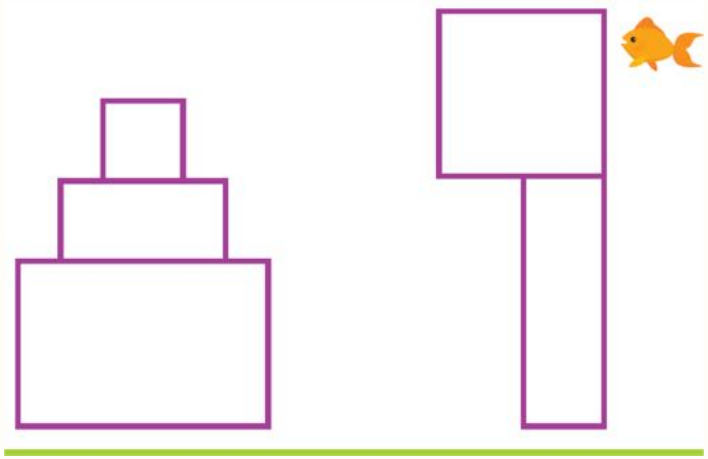
They count 4 sides and 4 angles, so it must be a rectangle.

But the squares are special because

all 4 sides are the same length.

They can use a second red square and move it around to see that all 4 sides are the same length.

They continue to match shapes to puzzles and to tell the goldfish or rabbit what they are doing. They can turn the puzzle pages around to see the shapes in different orientations.



Goldfish: This is a big square but it is also a rectangle because it has 4 sides and 4 square corners.

Squares and Rectangles That Are Not Squares

Children look at rectangles that are not squares and discuss why they are not squares.

This blue rectangle has a shorter side and a longer side.

There are not 4 equal sides, so it is not a square.

The red rectangle does not have all 4 sides the same length.

It has 4 sides and 4 square corners, so it is a rectangle.

Rectangles have square corners, but that does not make them be a square.

Puzzled Penguin says that this is a square, so it is not a rectangle.

This has 4 sides and 4 square corners, so it is a rectangle.

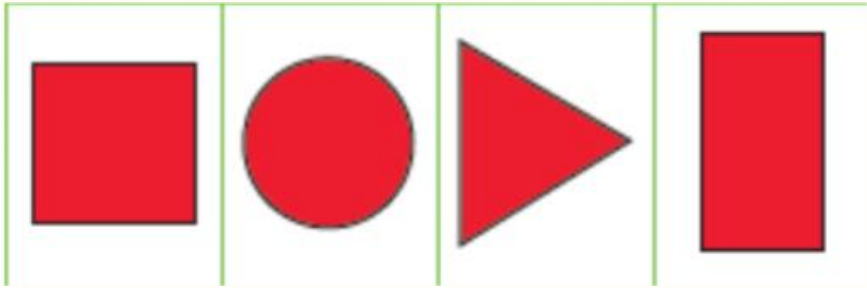
A square is a special kind of rectangle.

It is a square but it is also a rectangle.

It is actually a square and a rectangle.

K: A Common Error: Seeing Very Limited Examples of a Shape

In some programs children used to see only
what were sometimes called "the four basic shapes":
a square, a circle, an equilateral triangle, and a rectangle.

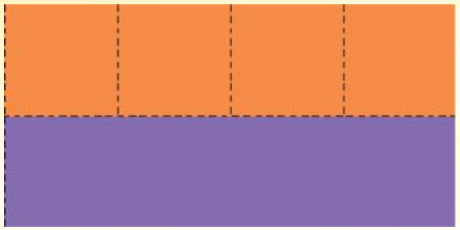


Seeing and discussing only limited shapes led many children
and even some teachers to think that
a square is not a rectangle.
But a square is a special rectangle with all sides of equal length.

Compose Rectangles to Make Rectangles

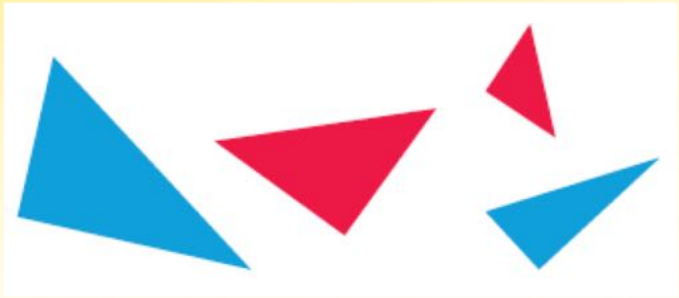
Composing allows children to see many different rectangles.

We show shapes beside each other, but children can put shapes on top.



Children identify rectangles and squares in the classroom and think of examples outside the classroom. This is done with all shapes.

Discussing Triangles



Children discuss what they notice about triangles.

They look pointy.

They have straight sides.

They have corners.

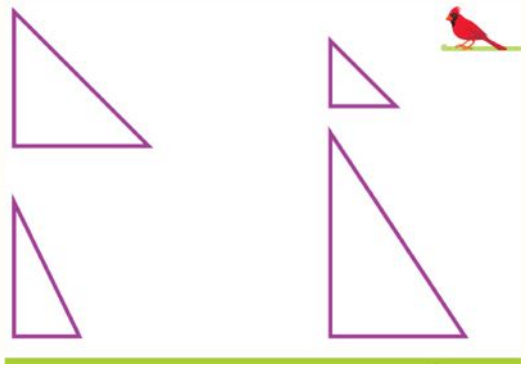
They have 3 sides.

They have a square corner.

They are not rectangles.

They are not squares because they do not have 4 sides or all sides the same length.

A triangle has 3 sides and 3 corners.

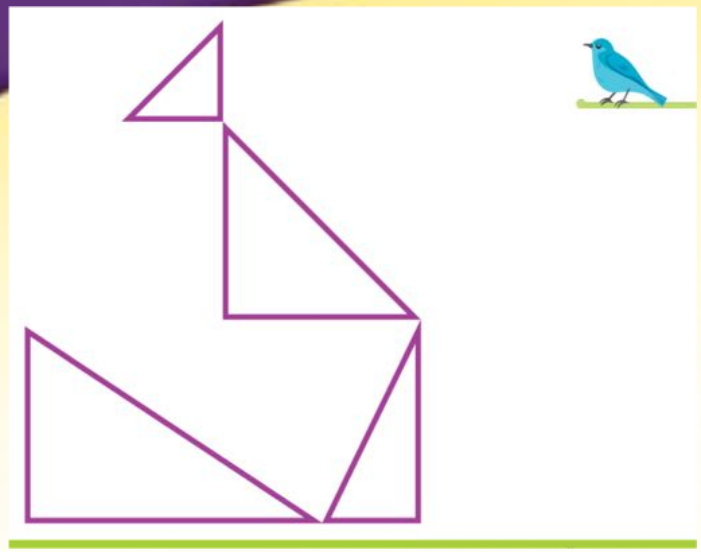


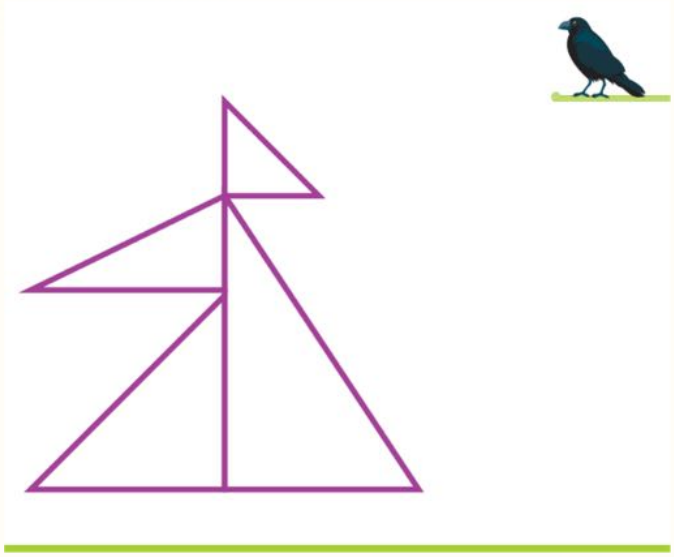
These triangles have one square corner.

They are called right triangles.

Red bird: I am putting the biggest triangle here. It is a right triangle. See its square corner.

Solve and Discuss Right Triangle Puzzles





For triangles, children may need to flip a shape over to fit on the puzzle.

They can turn triangles and turn the puzzles to see the shapes in different orientations.

MathTalk in Action

Who would like to tell us what you told Crow about one of the shapes on the puzzle? Rosa, please start.

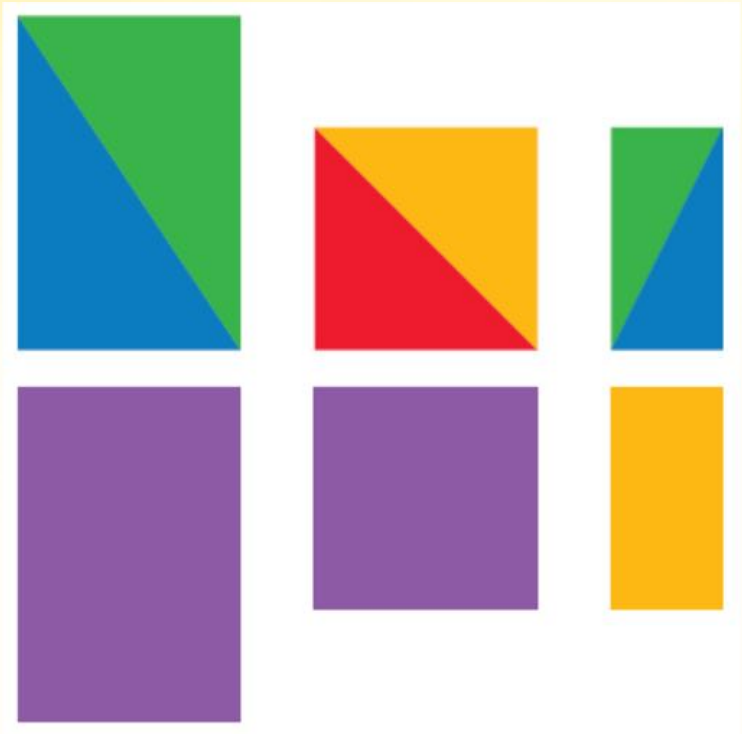
Rosa: I said, "Crow, we are making a picture this time. I am putting this red triangle at the bottom. It has two sides that are the same and a long side."

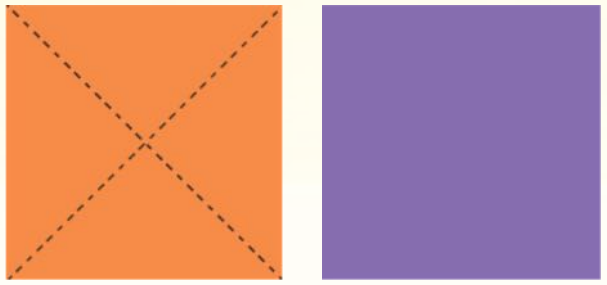
Jack: I told Crow that I thought the picture was going to be a boat. I took the small red triangle and put it at the top. It looks like a flag.

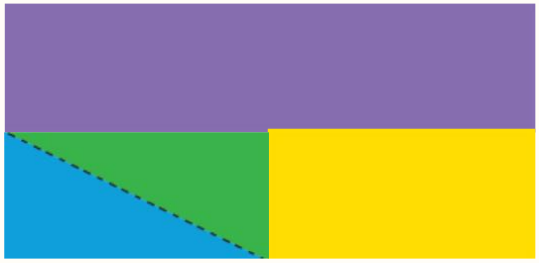
Manuel: I picked the big blue triangle because I like blue. Then I said, "Crow, I have to turn and flip the big blue triangle to make it match the puzzle."

Compose Right Triangles to Make Rectangles

Composing is like subitizing shapes hiding inside other shapes.



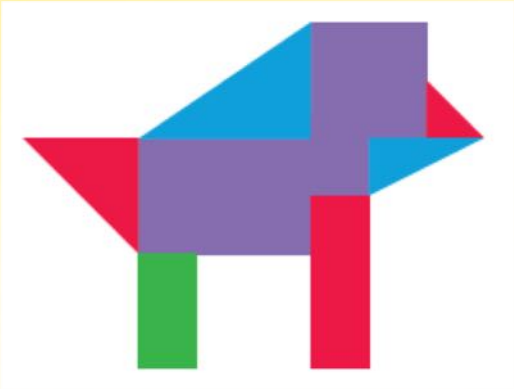




Composing right angled shapes prepares for many future math topics.

We show shapes beside each other, but children can put shapes on top.

Math Centers Provide Lots of Explorations with Shapes

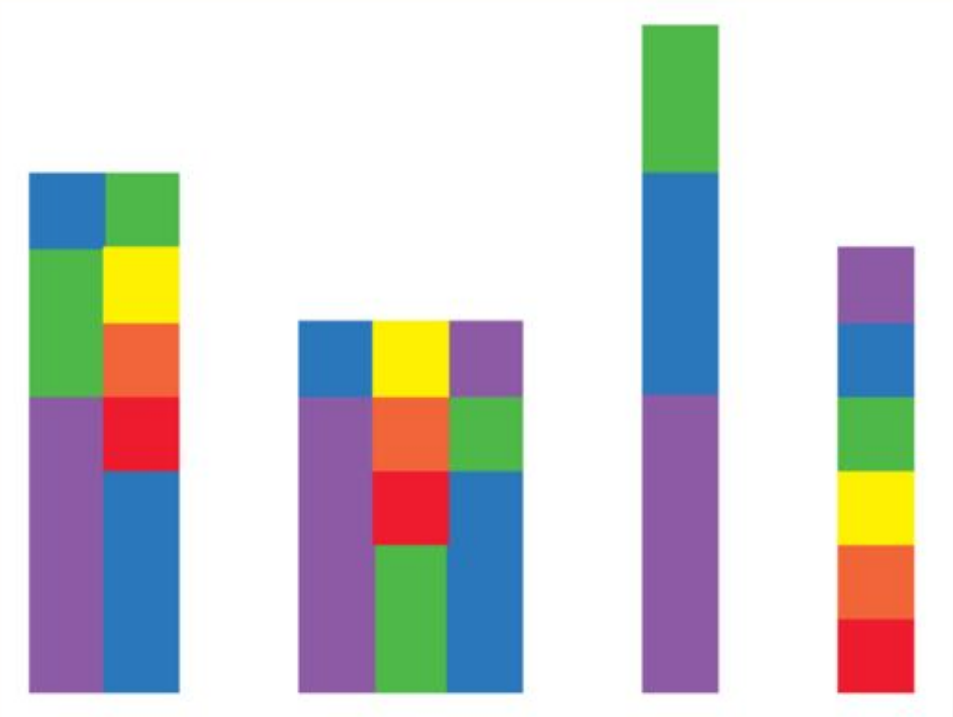


I made a long rectangle.

Look at my tall rectangles.

This is my pet dog.

And this is my submarine.



Compose 2 Identical Right Triangles

Children work together to discuss how they can put 2 identical right triangles together.

They describe what shapes they made.

Look, we made two different triangles. One is tall and one is short.

They look like tents.

This has 4 sides and 2 right angles. It looks like a kite.

And another tilty shape with 4 sides.

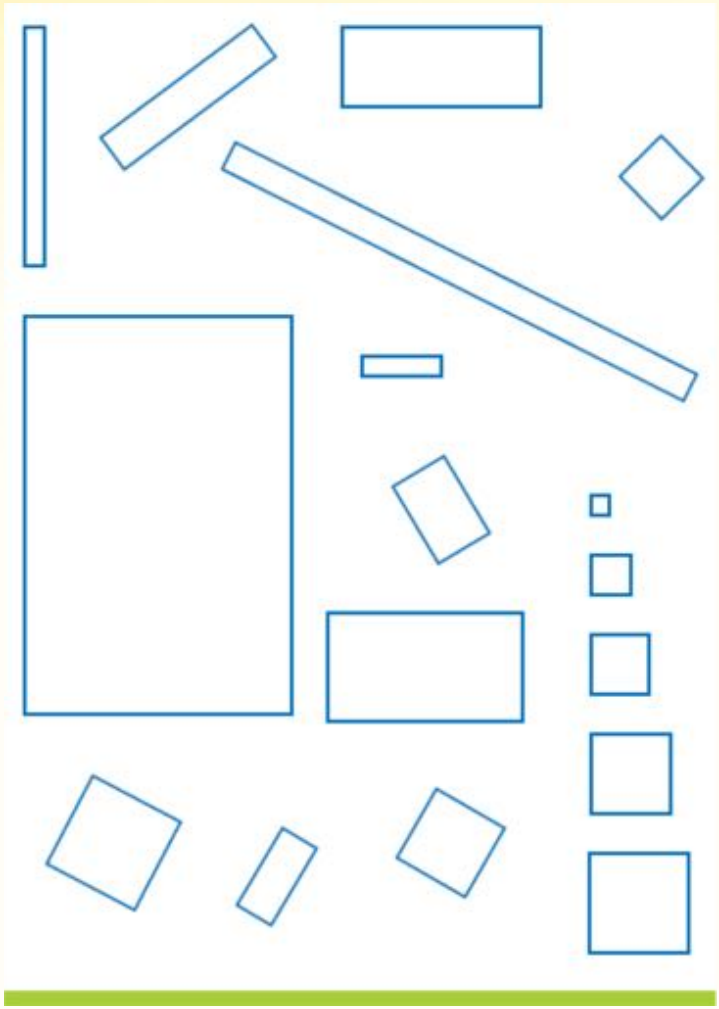
And we have a kind of tilty rectangle. It has 4 sides but it does not have any right angles. It maybe is going to fall over.

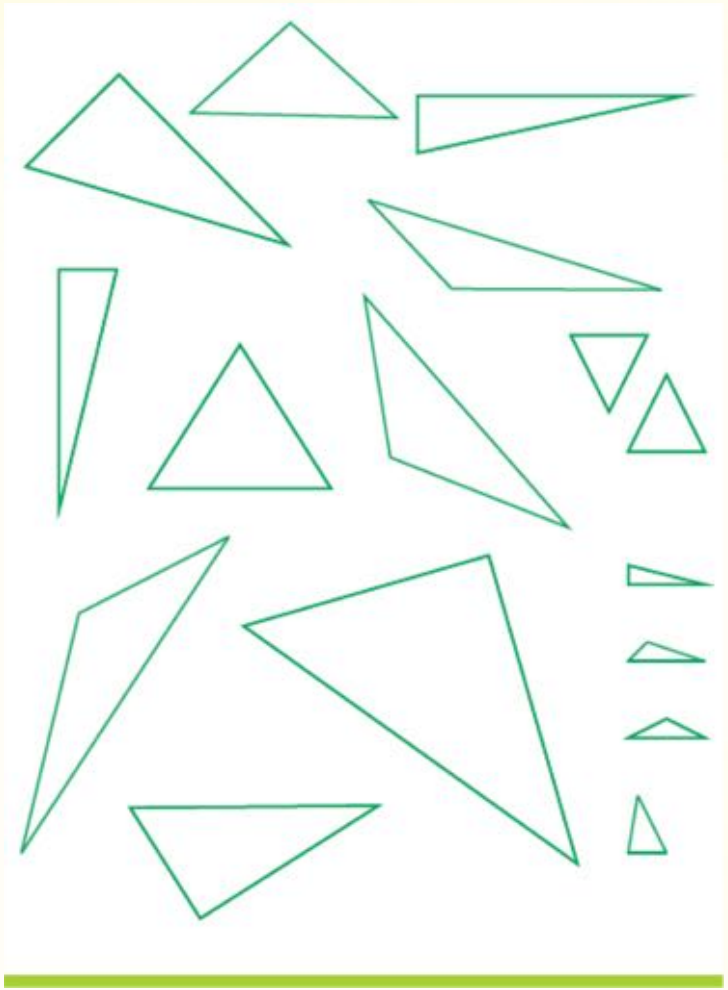
We made a rectangle. It has 4 sides and 4 right angles.

Generalize Rectangles and Triangles

Children discuss the different examples on Discussion Cards.

They relate these shapes to examples in the real world.

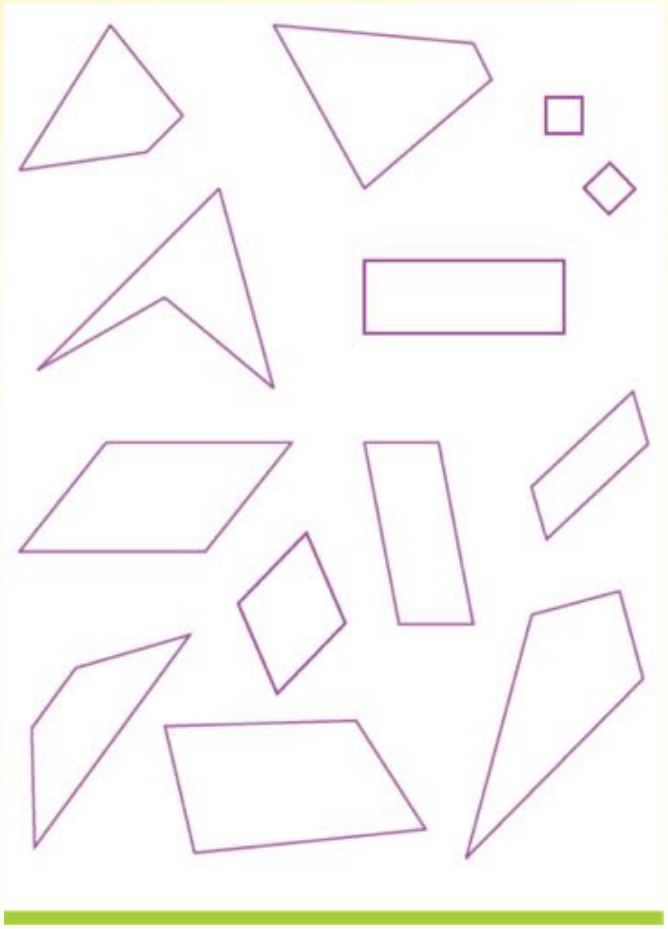


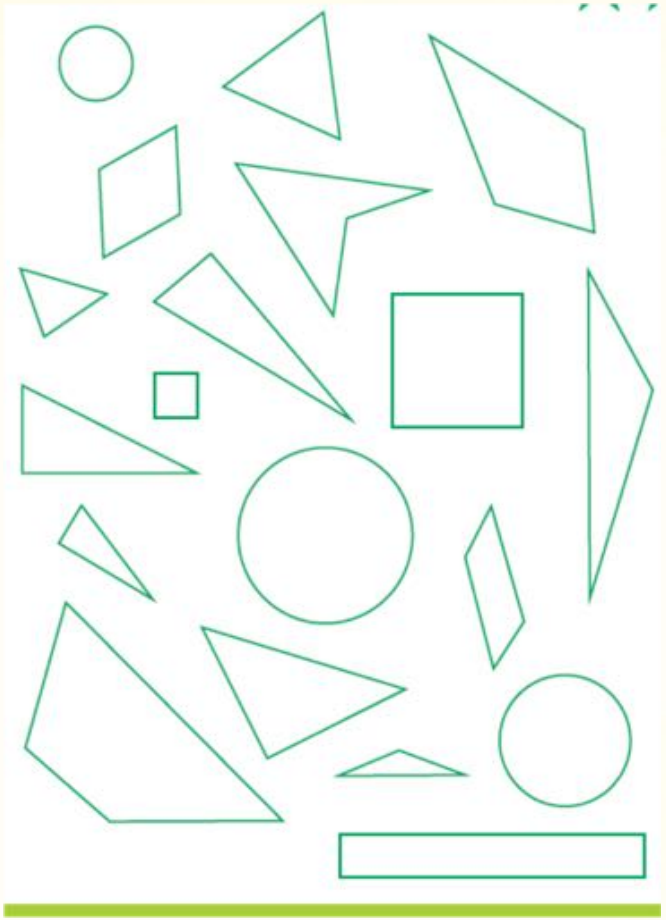


Generalize 4-Sided Shapes and Identify 3-Sided and 4-Sided Shapes

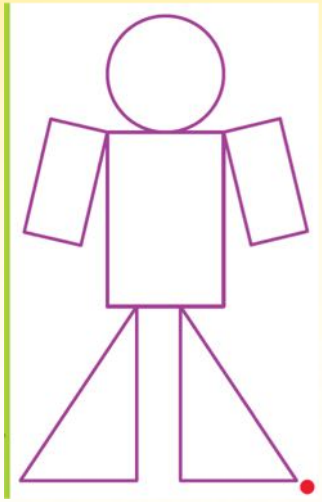
Children discuss the different examples on these Discussion Cards.

They relate these shapes to examples in the real world.

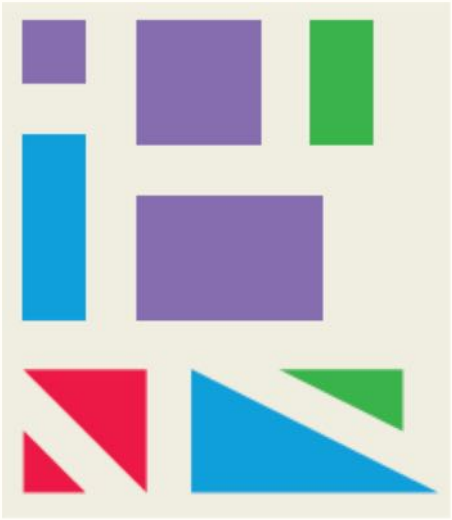


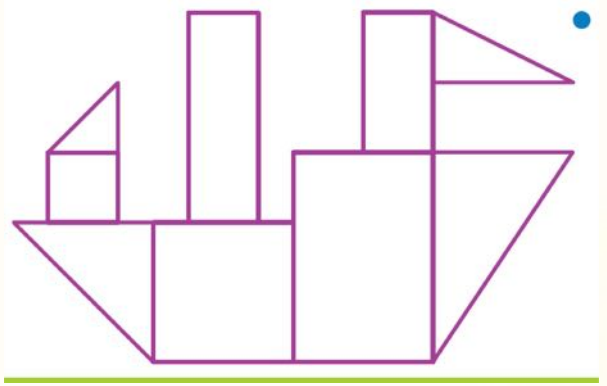


Match Shapes to Picture Puzzles

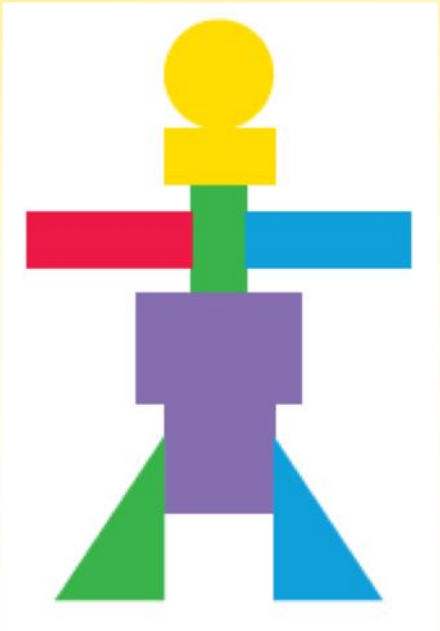


Children match shapes from their shape set to make puzzle designs.



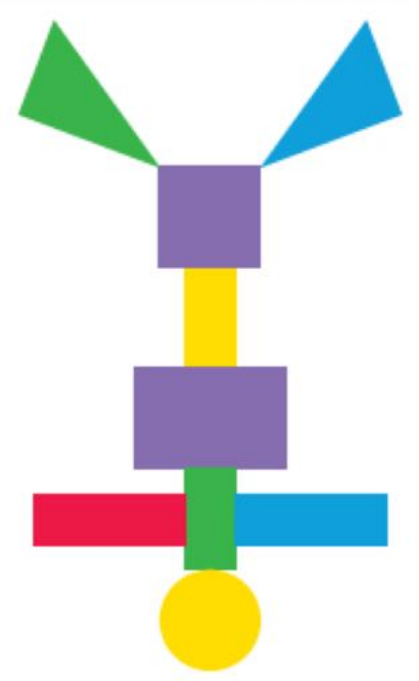


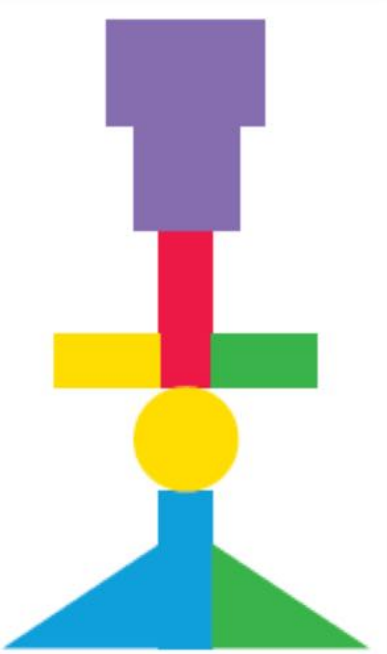
Math Centers: Children Build and Discuss Their Own Designs



Look at our robots!!!

We'll tell you what they do.

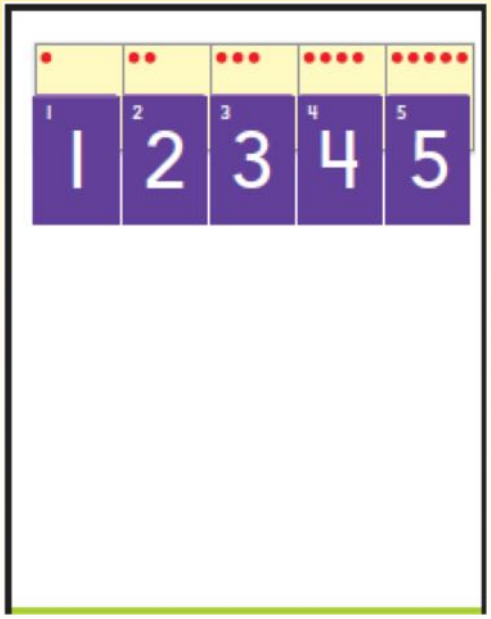


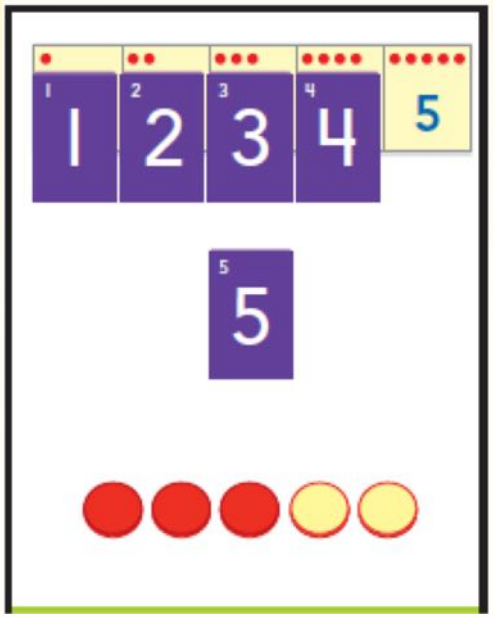


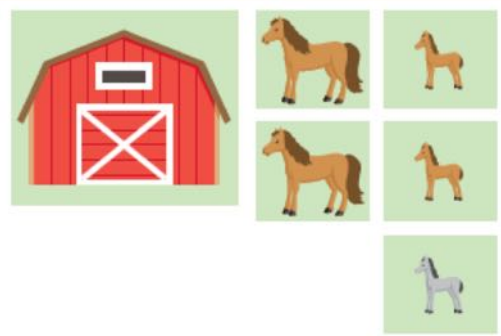
Part B: Partners of 5, Match, Compare, Add Numbers Through 5, Make Repeated Patterns

Partners of 5 use all earlier number knowledge.

Partners are now related to adding and the + sign.

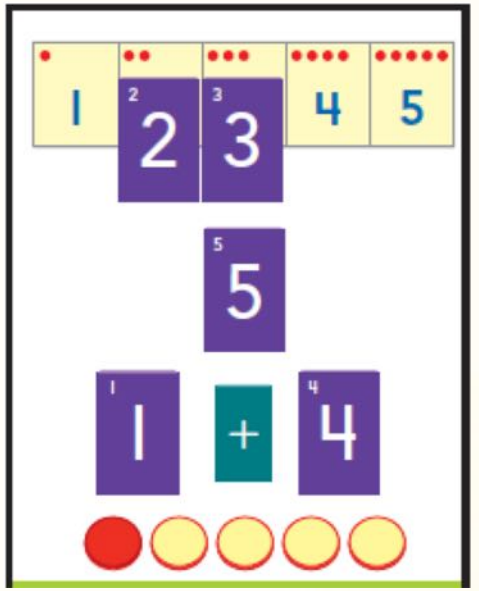






5		
1	4	[Ramona] made 1 and then 4.
4	1	[Cory] made 4 and then 1.
3	2	[Chen] made 3 and then 2.
2	3	[Jim] made 2 and then 3.

5		
1	+	4
4	+	1
3	+	2
2	+	3



1 and 4 make 5.

Partners of 5 Use Knowledge of 1, 2, 3, 4

Relating partners and adding words.

MathTalk in Action

Okay, who can tell me what partners you have found in your pictures?

Cho: On the dish with carrot sticks, I see 4 on this plate and 1 on this plate. 5 has partners 4 and 1.

Maya: And on that plate with 4 carrot sticks, there are 2 long sticks and 2 short sticks. So, 4 has partners 2 and 2.

Josh: Look, the apple pieces show that 5 has partners 3 and 2. 3 and 2 are on different plates.

Nora: And on this dish with apples, I can see that 2 has partners 1 and 1. 1 red and 1 green make 2.

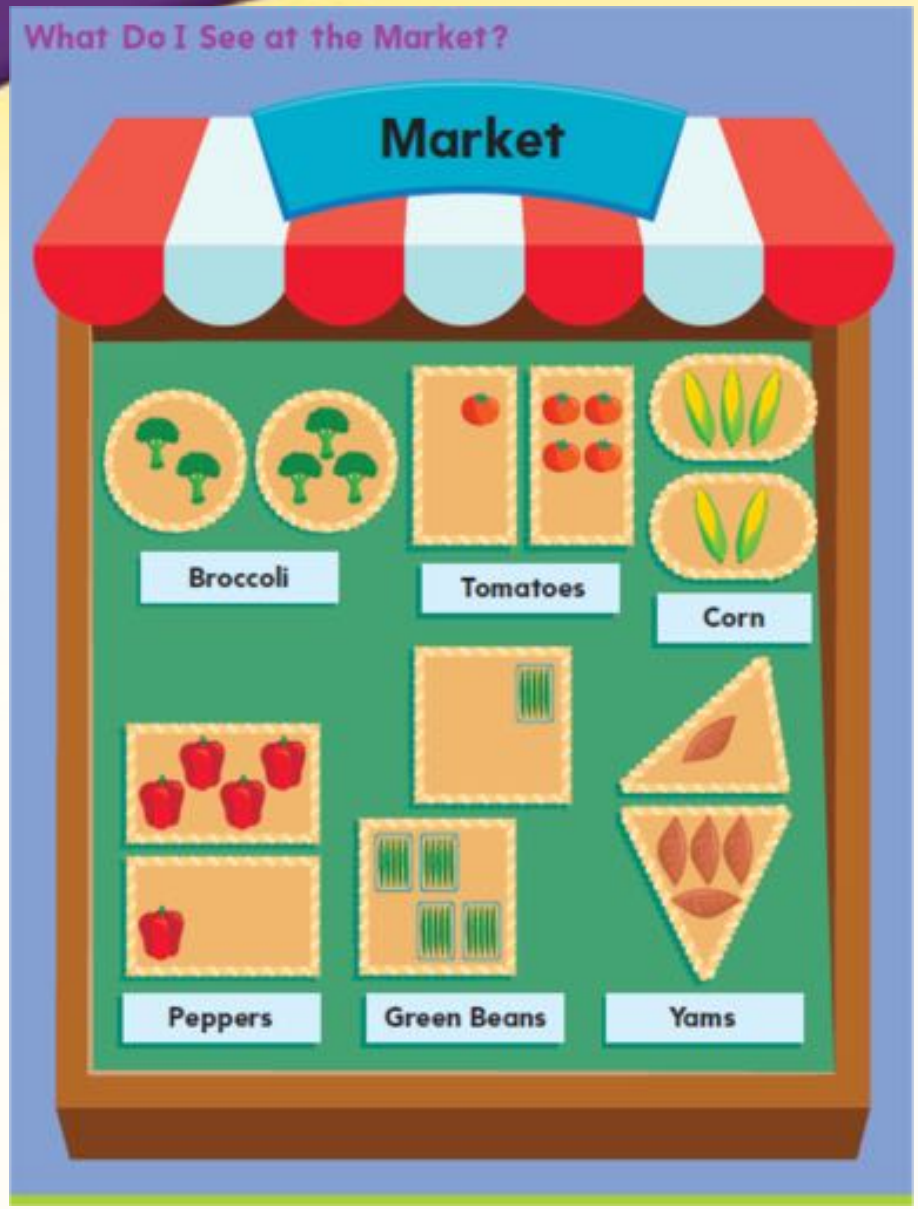
Now say the partners with adding words.

4 and 1 make 5.

4 plus 1 makes 5.

4 carrot pieces and 1 carrot piece make 5 pieces.

Telling Adding Stories Is Crucial for Children of Poverty



Discuss being at the market.

Tell adding stories about the pictures.

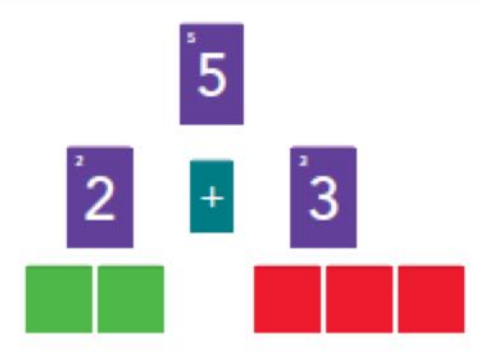
I see 4 yams and 1 yam. That makes 5 yams in all.

At the market there were 2 broccolis and 3 broccolis. How many broccolis?

Anne's story (no apples are present):

I buy 2 green apples and then I buy 3 red apples. How many apples do I buy in all?

Let's pretend that our squares are apples.



Simple Counters Versus Complex Real-World Things

Simple shapes help children subitize perceptually and conceptually.

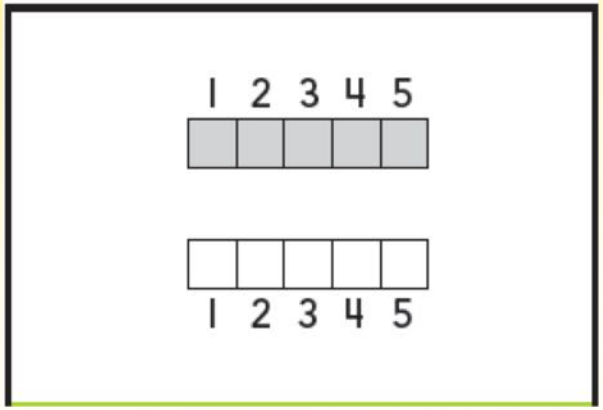
Simple shapes help children form mental images that are prototypes that provide meanings for number relations and operations

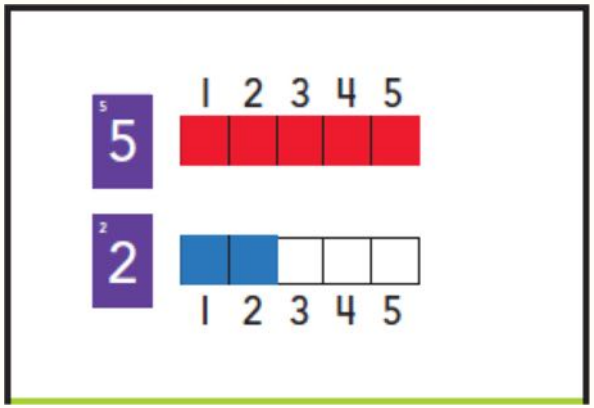
and that can be used in problem solving.

Working with circles, squares, cubes, and right-angle triangles can help children form general-enough initial concepts and support geometry learning.

Children need to see and discuss a range of objects to generalize number meanings, but complex objects are not needed for problem solving and may interfere with visualizing the situation or numbers.

Children Compare Using Compare Mats





Which is more, 5 or 2?

Which is less, 5 or 2?

5 is more than 2.

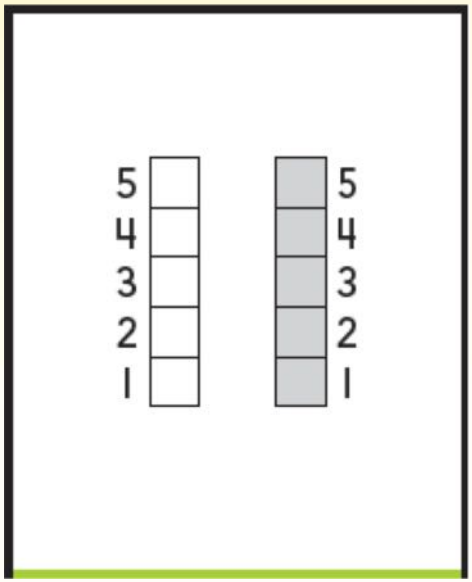
2 is less than 5.

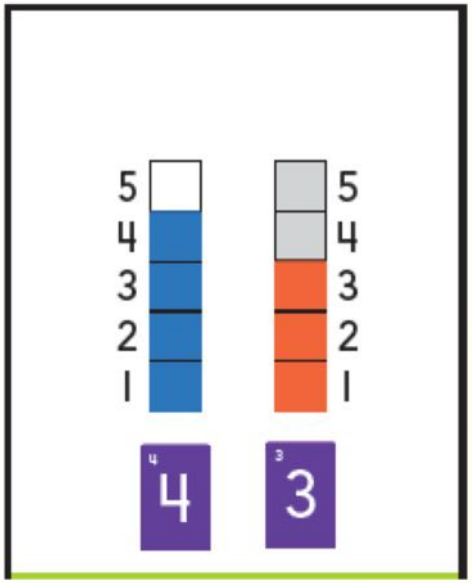
The row with 5 is longer than the row with 2.

The row with 2 is shorter than the row with 5.

The tower with 4 is taller than the tower with 3.

The tower with 3 is shorter than the tower with 4.





Generalizing Repeating Patterns

Rittle-Johnson, B., Fyfem, E. R., Loehr, A. M., & Miller, M. R. (2015). Beyond numeracy in preschool: Adding patterns to the equation. *Early Childhood Research Quarterly,* 31, 101–112

These authors identified four levels of working with repeating patterns and emphasized the importance of the last two levels: duplicate, extend, abstract, and explicitly recognize the smallest repeating unit of a pattern.

The *Math Expressions* repeating pattern work emphasizes these last crucial levels using color and language. Children also count the rows made for the patterns to gain practice with counting between 5 and 10 objects.

Generalize repeating patterns and identify the smallest repeating unit

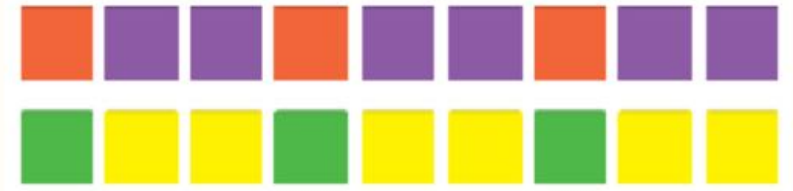
Say patterns with colors and with first/second and with hand motions.

Red blue red blue red blue

First color, second color, first color, second color, …

Red blue is the smallest repeating unit.

Red blue blue red blue blue red blue blue



Counting rows within 10

Children count their row and say how many there are.
They take turns hiding their eyes while their partner takes 1 or 2 from their row.
They count again and say how many.

Number Path Games

Ramani, G. B., & Siegler, R. S. (2008). Promoting broad and stable improvements in low-income children's numerical knowledge through playing number board games. *Child Development,* 79, 375-394.

Number board games increased children's performance on several number tasks. Children used a spinner with a 1 and a 2 on it and moved a token along a row of squares with the numbers from 1 to 10 in the squares.

The key step in his board game is that the children said the numbers on the squares on which they moved instead of counting 1 or 1, 2 as they moved forward on the row of squares.

So if they started on 3, they would say 4, 5 or 4, 5, 6.

Note: These are number paths and NOT number lines.

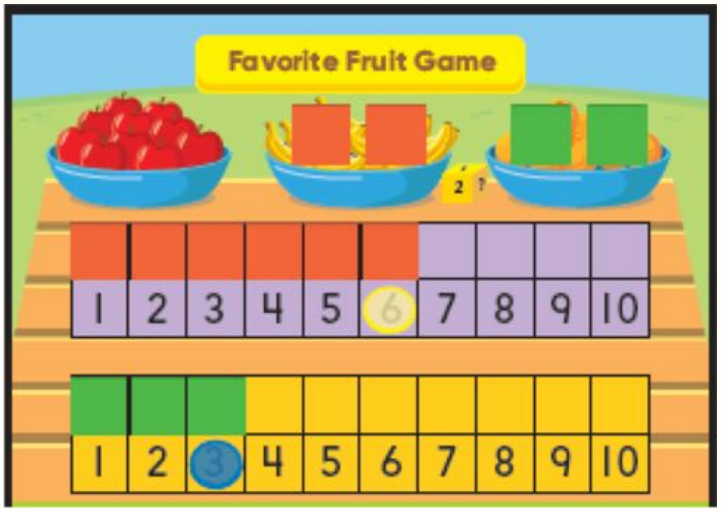
Math Expressions Number Path Games

Features I added to the Number Path Games

A row of blank squares is above the number path; children place squares on this row to show the cardinality of the number their token is on.

The tokens are transparent so that children can see all of the numbers.

Children make stacks of square inches so that they can see how the stack decreases as their row increases. These use 5-groups and 10-groups for the number paths to 10 and to 20.

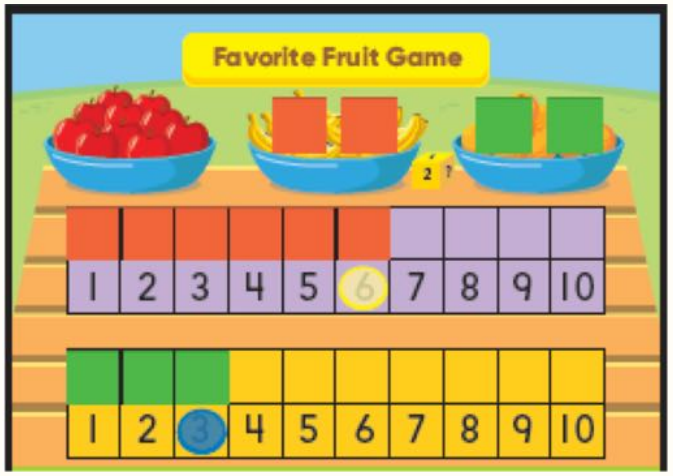


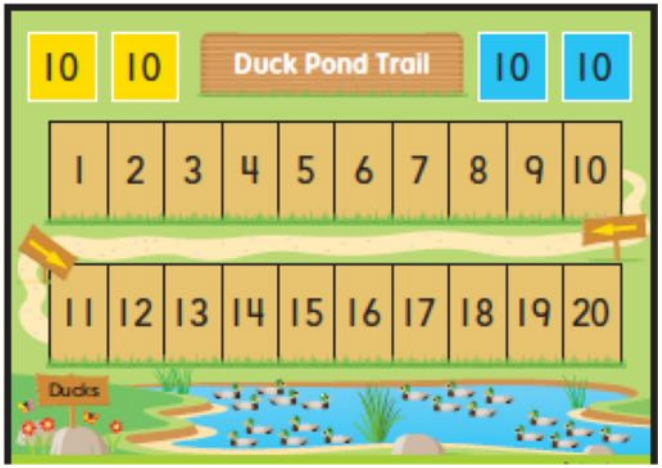
Cube has three 2s, two 1s, and one ? for which children choose 1 or 2.

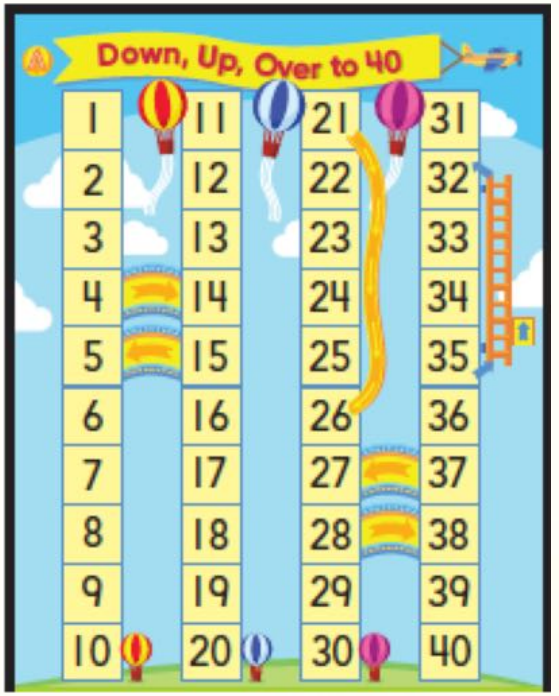
Children predict where they go if they roll a 1 or a 2 and then summarize after the move to focus on consecutive numbers. They say the counting on as adding: 6 and 2 more make 8. After each move, they compare the numbers on which the tokens are and say the comparisons both ways: 6 is more than 3 and 3 is less than 6 (build cardinal and counting meanings).

Four Different Number Path Games to 4, 10, 20, and 40

Children make stacks of square inches so that they can see how the stack decreases as their row increases. These use 5-groups and 10-groups for the number paths to 10 and to 20. Children pretend that these squares are pets, fruits, or duck food they are collecting.







Area Patterns, 3-Dimensional Shapes, and Shapes with Right Triangles (5 weeks)

Rows and Patterns

Make Buildings with Cubes

Compose Shapes with Square Corners

Compose with Right Triangles and Rectangles

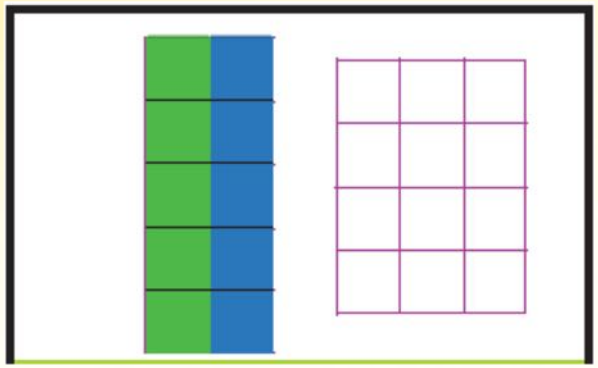
Cubes, Cones, Cylinders, and Spheres

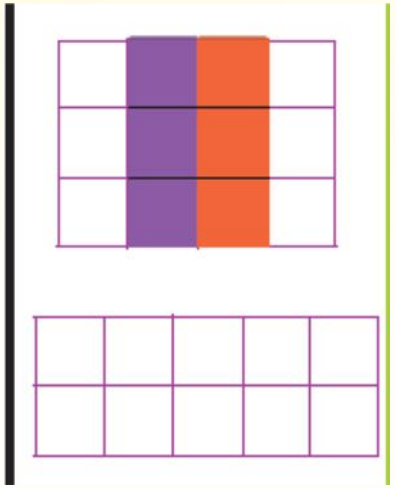
Students continue to experience square units and the related rectangles, right triangles, and isosceles triangles made from these units. Building and seeing area grid patterns with square units is an important step.

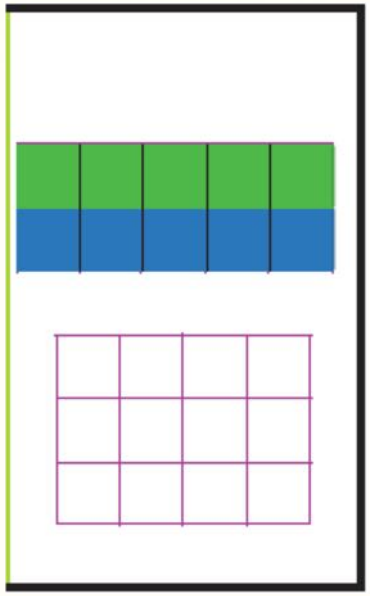
Rows and Patterns: Experiencing Area as Filling with Squares

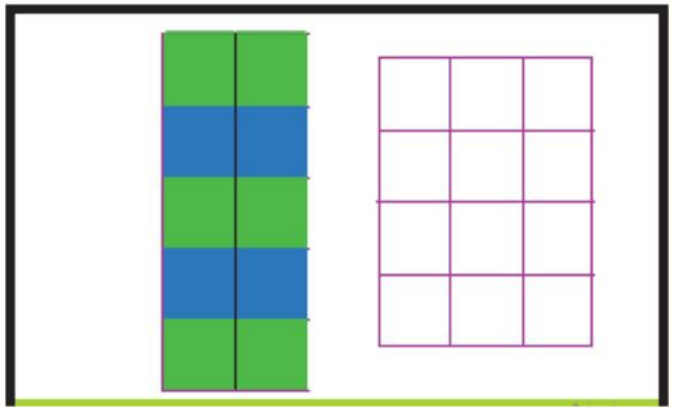
Children make and analyze patterns on square grids.

This helps them build visual images of square grids.

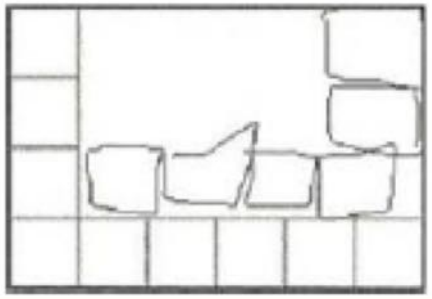


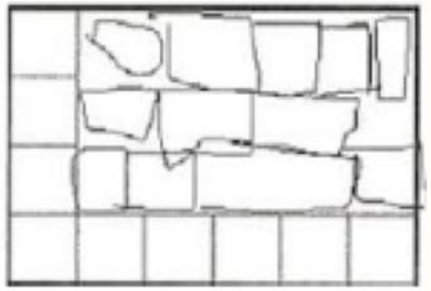


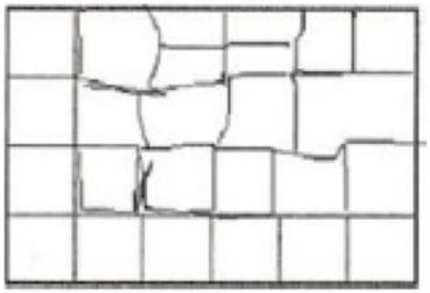




Math Centers: Make Patterns on Square Grid Cards







Clements, D. H., Sarama, J.. Van Dine, D. W., Barrett, J. E., Cullen, C. J., Hudyma, A., Dolgin, R., Cullen, A., Eames, C. L. (2018). Evaluation of three interventions teaching area measurement as spatial structuring to young children. *Journal of Mathematical Behavior, 50*, 23-41. https://doi.org/10.1016/j.jmathb.2017.12.004

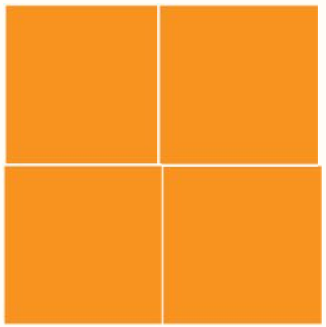
Elementary school children have difficulties visualizing and drawing an area grid. Drawing around squares or drawing rows or columns helps.

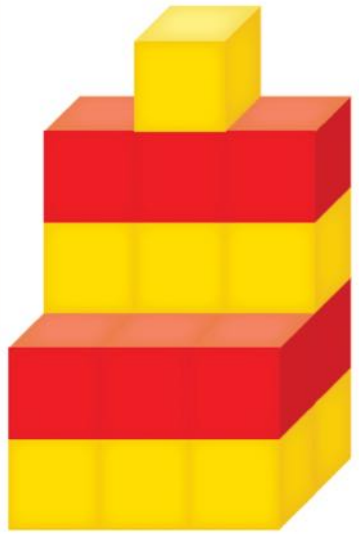
Working with grids earlier can build visualizations.

Make Buildings with Cubes: Experience Volume as Layers

Children build on rectangles making floors and apartments.

They focus on how many in a layer.



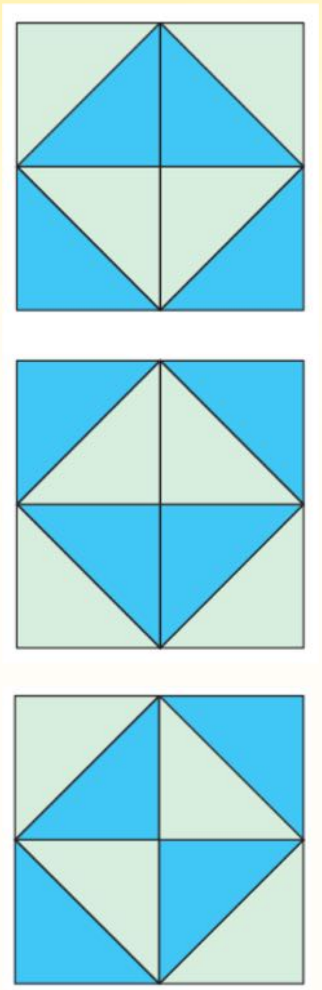


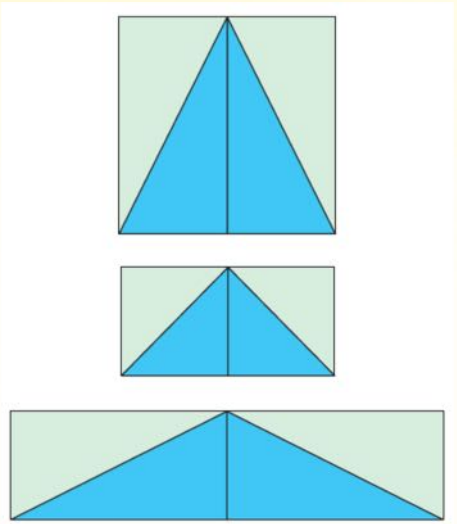
Compose Shapes with Square Corners

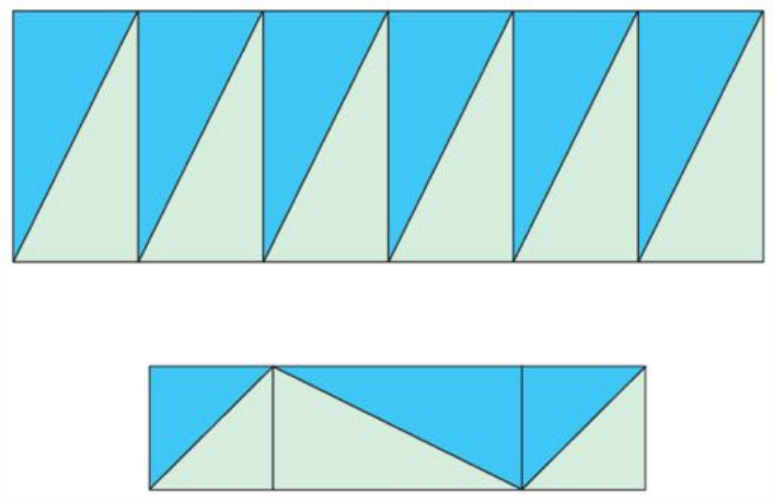
Children use the right triangle shapes to match puzzles.

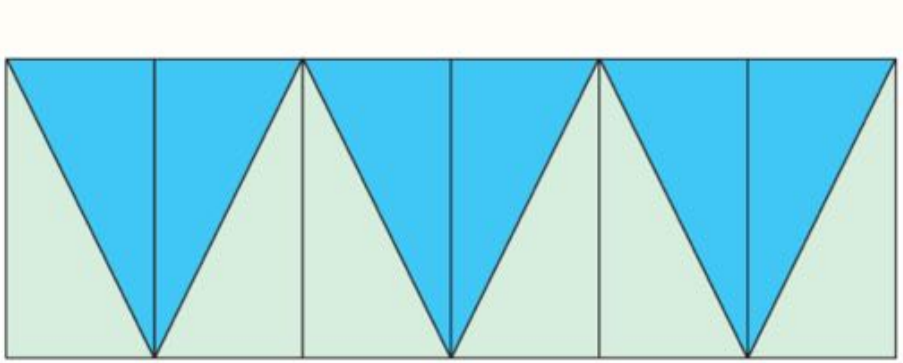
They discuss what they see.

They make their own designs.

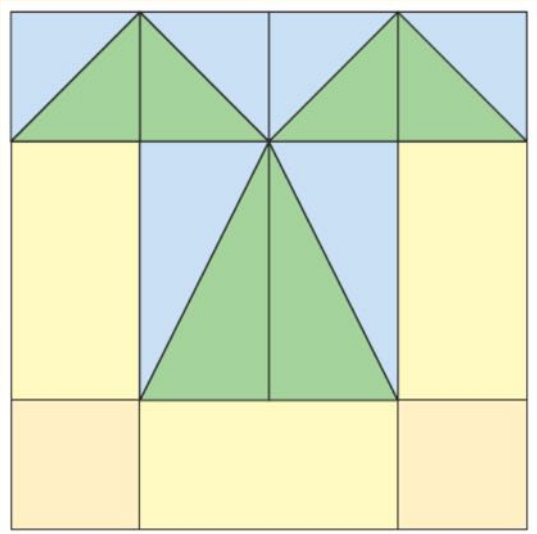




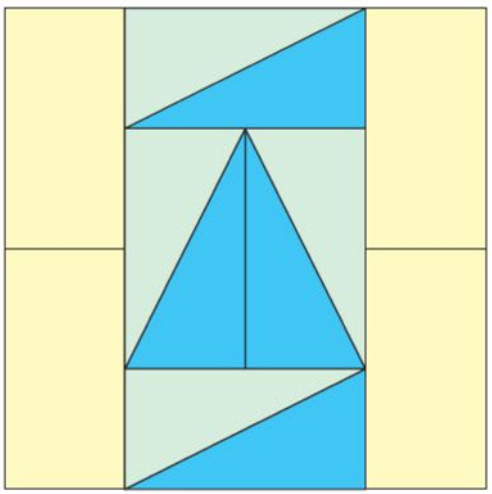




Compose Rectangles and Triangles

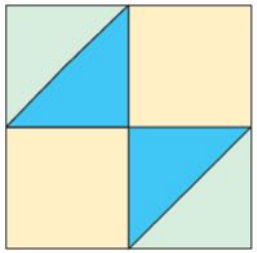


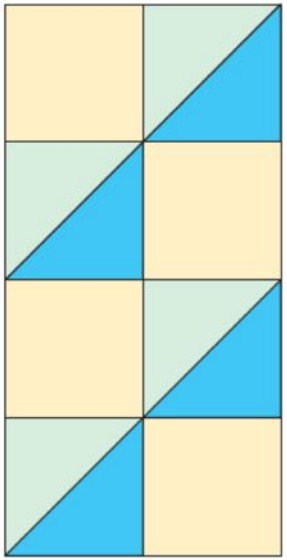
Children put shapes on puzzles and discuss what they see and how they fit the shape on the puzzle.

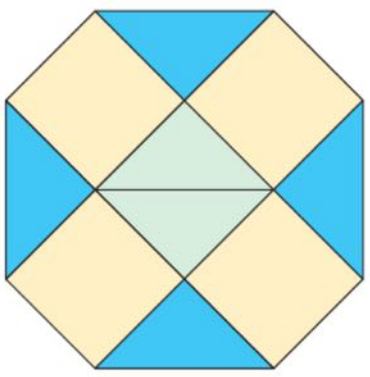


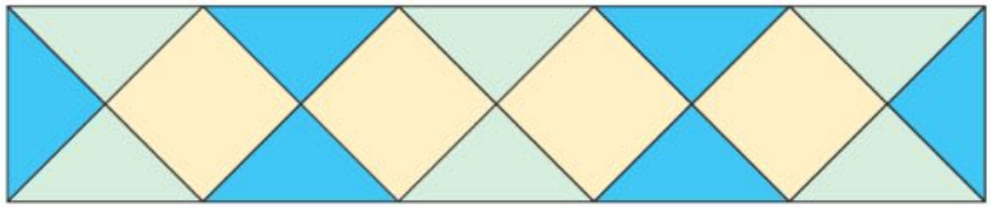
They make their own designs.

They copy designs their classmates make.



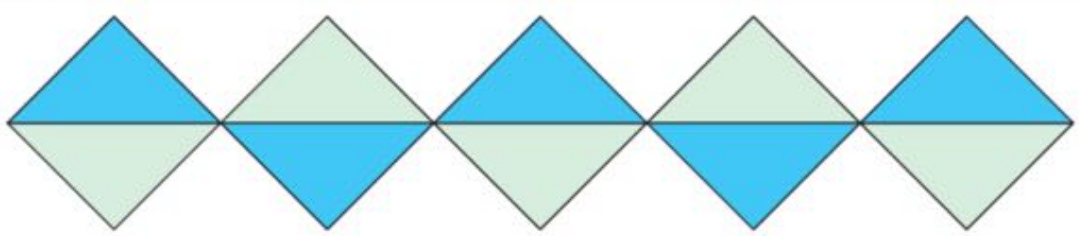


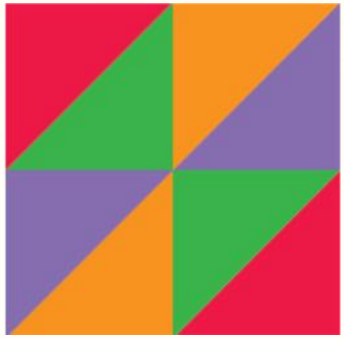


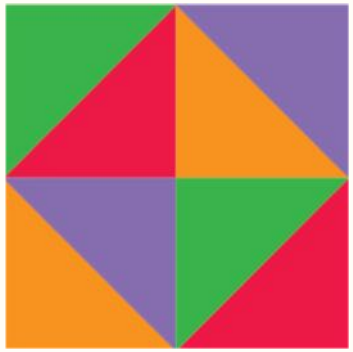


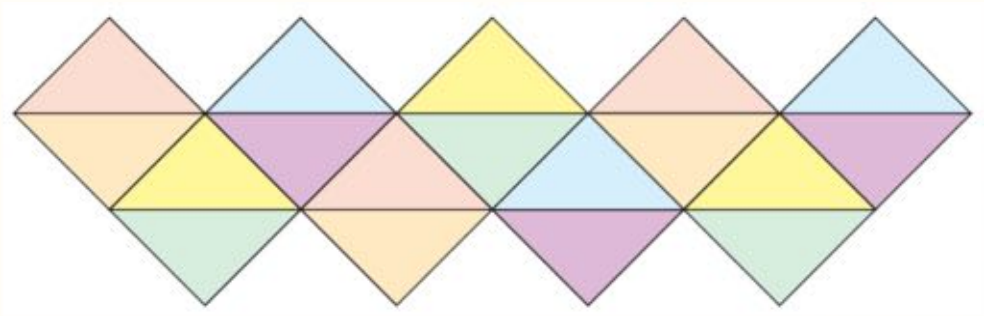
Make Designs with Small Right Triangles of 6 Colors

Children use the small right triangles of 6 colors on puzzles, and they make their own designs.





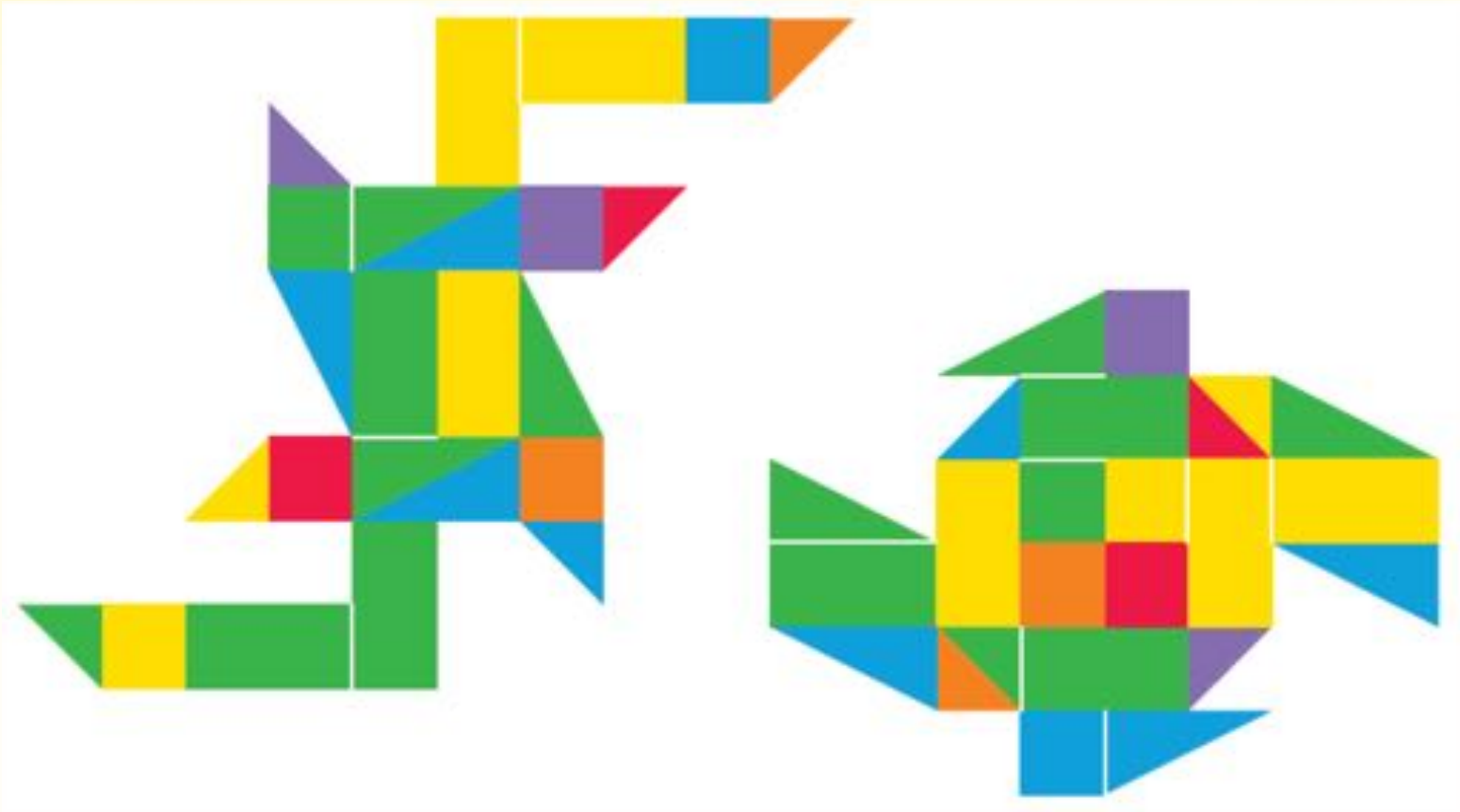




A Math Center Design Challenge: Use Six of Each Shape

Children can choose which shapes they use.

They can also use 3 or 4 or 5 of each shape.

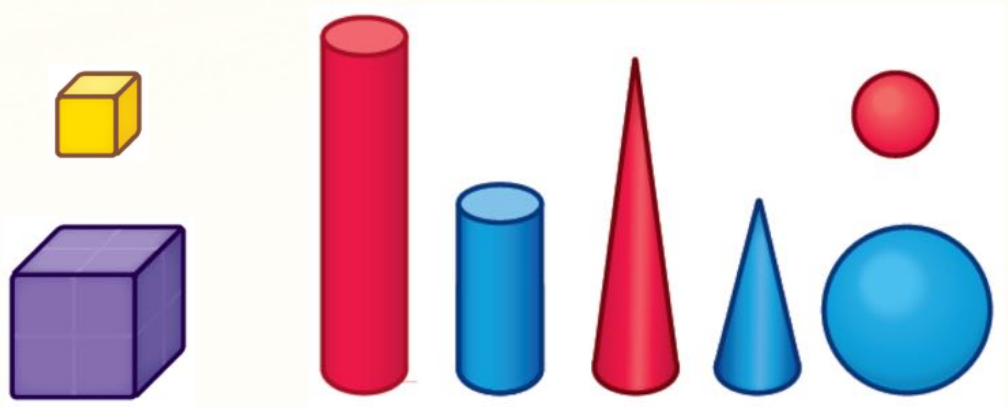


Cubes, Cylinders, Cones, and Spheres

Children discuss these 3-dimensional shapes

They find these shapes in the classroom or think of things that are these shapes.

They build with these and the 1” cubes of 6 colors.

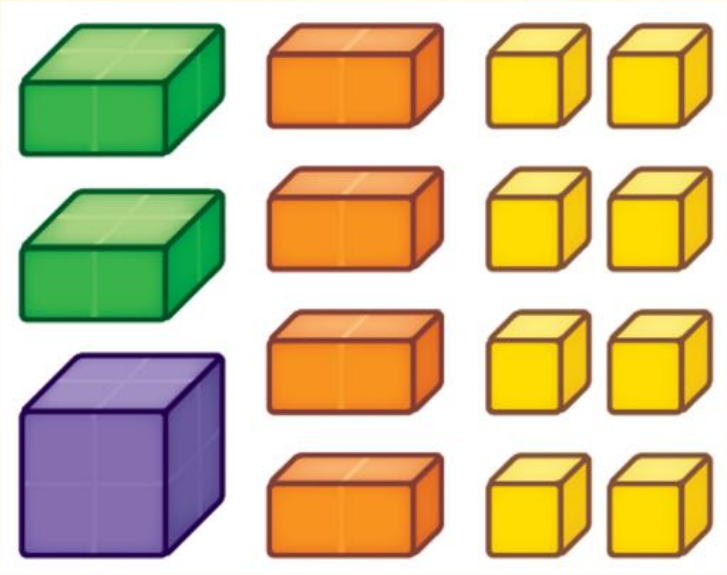


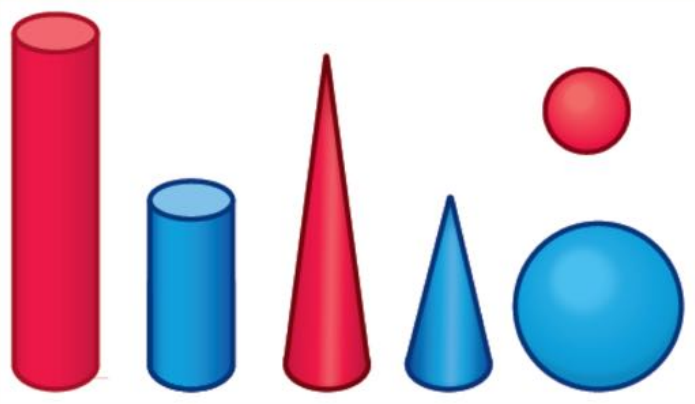
Building with Cubes to Make 3-D Shapes

Small yellow cubes make the orange, green, and purple shapes.

Orange and green build to make the big purple cube.

Children build with all 3-D shapes and discuss what is made.



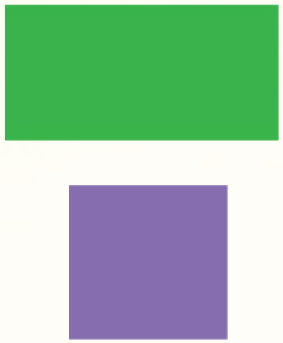


Children Learn and Use Relative Position Words With Shapes

above and below

The green rectangle is above the purple square.

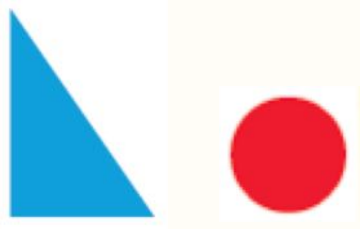
The purple square is below the green rectangle.



in front of and in back of

The purple rectangle is in front of the yellow rectangle.

The yellow rectangle is in back of the purple rectangle.



beside and next to

The triangle is next to the circle.

The circle is beside the triangle.

Number Part C: Add, Subtract, Compare, and Make Repeating Patterns

Children add and subtract to 5 with objects, number tiles, + and = tiles, and fingers.

They tell and represent adding and subtracting stories.

They extend comparing to 10.

They continue making repeating patterns and count objects in rows to 10, in circles, and counting only a particular color in a row.

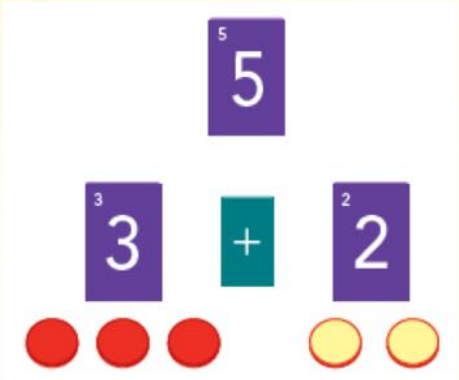
Number path games go to 20 and 40.

Relate Adding to Subtracting and to Partners

Show this adding story with your circles, numbers, and plus tile.

There are 3 orange butterflies. There are 2 yellow butterflies.

How many butterflies in all?



Now we are going to do subtracting stories.

Subtracting is about taking apart a number to make 2 groups.

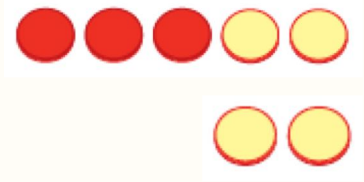
There are 5 butterflies here in the garden. 3 are orange.

How many are yellow? 2.

Subtracting is also about taking away one group.

There are 5 butterflies.

3 fly away. How many are left?



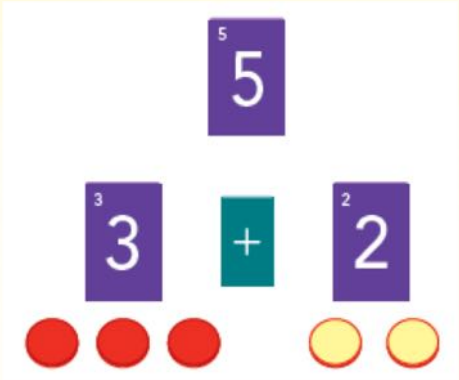
We need to see the partners in subtraction.

Relate Adding to Subtracting and to Partners

Show this adding story with your circles, numbers, and plus tile.

There are 3 orange butterflies. There are 2 yellow butterflies.

How many butterflies in all?



Now we are going to do subtracting stories.

Subtracting is about taking apart a number to make 2 groups.

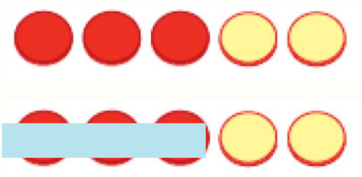
We see the partners here.

Subtracting is also about taking away one group.

There are 5 butterflies.

3 fly away. How many are left?

We need to see the partners when taking away. Use a take-away strip.



Solving More Subtracting Stories

Here is a subtracting story.

I have 5 carrot sticks. I eat 4 of them. How many are left?

Who can tell a different subtracting story?

MathTalk in Action

Who can tell us about subtracting?

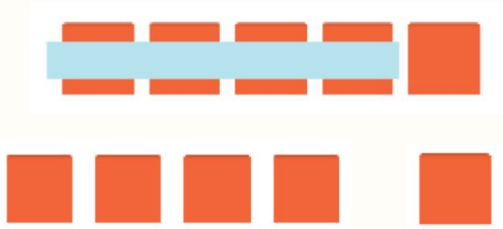
Simone: We can take away one group and leave the other group.

Harry: We can show taking away with a Take-Away Strip.

Elena: We can take apart a number.

Li: We can see subtracting on a picture or show it with our circles.

Yes, those are all ways to tell about subtracting. And today, we are going to learn more ways to talk about subtracting.



Use a Take-Away Strip in Pictures and with Shapes so Both Partners Remain at the End

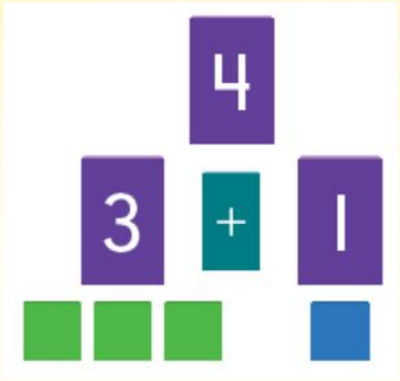
Show this story with your numbers and squares.

At the farm three sheep are brown. 1 sheep is black.

How many sheep in all?

Say the adding in words. 3 plus 1 is 4.

Show the story with your fingers.



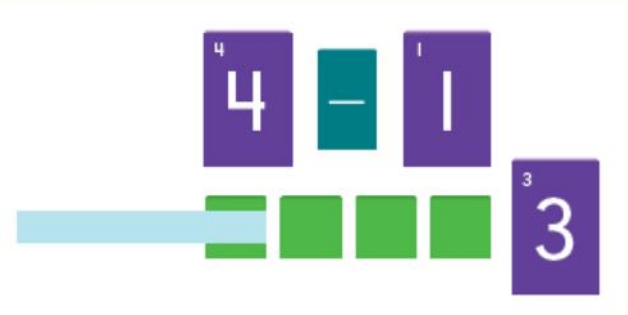
Now show this subtracting story.

Four sheep are at the farm. One is black. The rest are brown. How many are brown?

Say the subtracting in words; show with fingers.

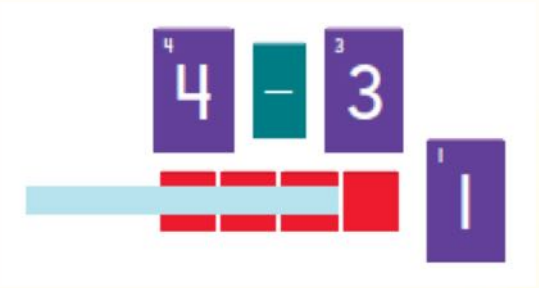
4 take away 1 is 3. 4 minus 1 is 3.

4 breaks apart to make 1 and 3.



Now we'll subtract the other partner.

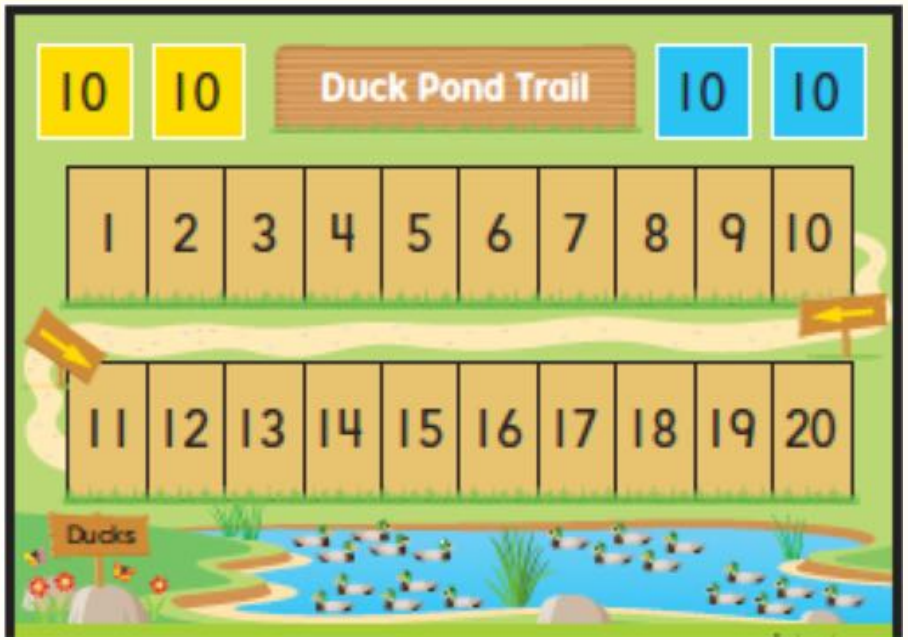
Four sheep are at the farm. Three are brown. The rest are black. How many are black?

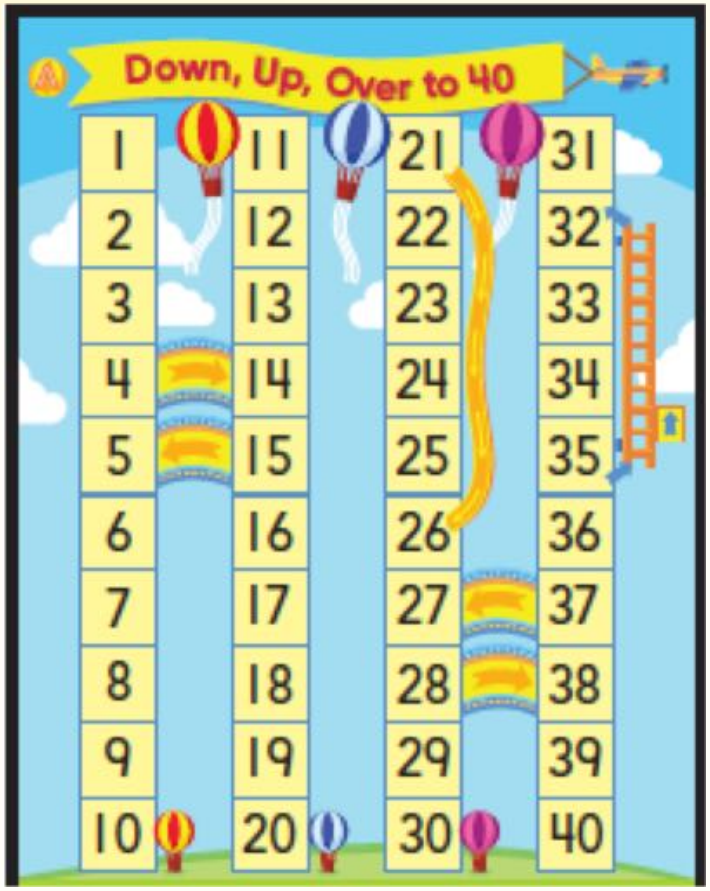


4 makes 3 and 1. 4 take away 3 is 1. 4 minus 3 is 1.

Children Play Number Path Games to 20 and to 40

Children build knowledge of teen numbers with the Duck Pond Trail Game.
They stack two groups of 10 squares on the yellow or blue squares.
They see the teen numbers as 10 and some more.





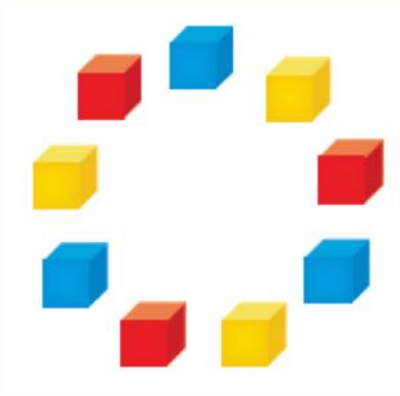
Children extend number path knowledge to 40.

Use Cubes to Make Repeating Patterns, Count One Color, and Count Cubes in a Circle

Children say their and other repeating patterns in color words and with first, second, third to generalize.

Children count the cubes in the row and say how many.

Then they count only one color of cubes and say how many.

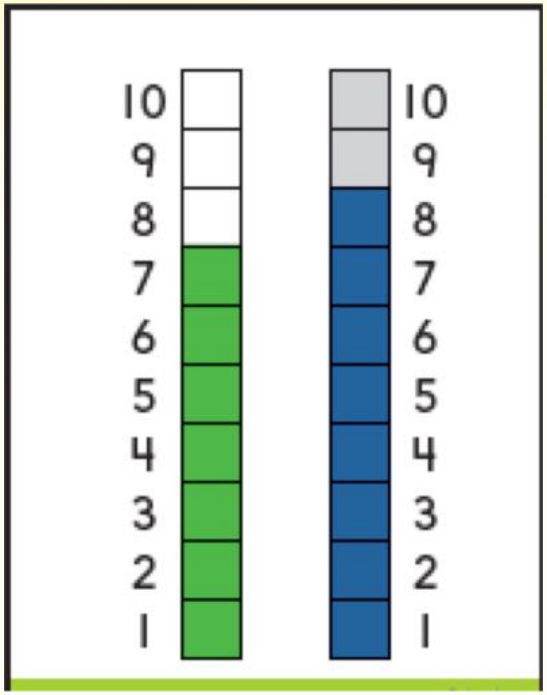


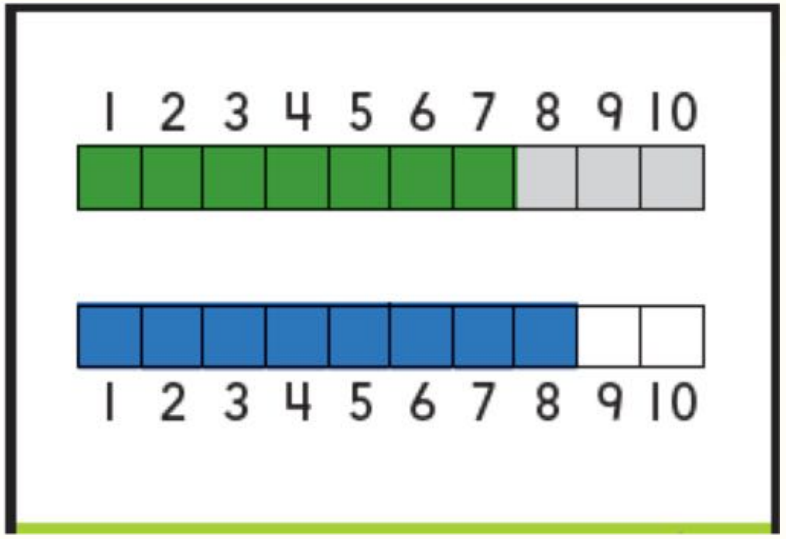
Children count the cubes in the circle and say how many.

Puzzled Penguin keeps counting around and around the circle. Children help.

Extend Comparing to 10

Children continue to use more/less, taller/shorter, and longer/shorter comparing language.





Geometry Part C: Measure, Compose Shapes, and Look Back

Measure, Compose Shapes, and Look Back (4 weeks)

Measure Lengths and Make Shapes

Make Shapes and Discuss Attributes

Solve Picture Puzzles

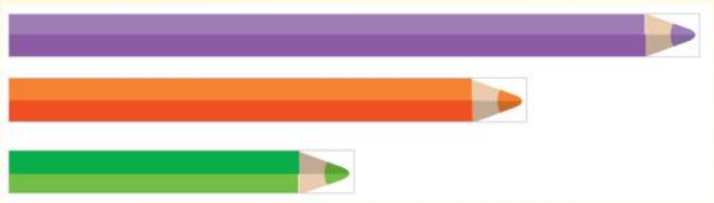
Look Back at What Was Learned

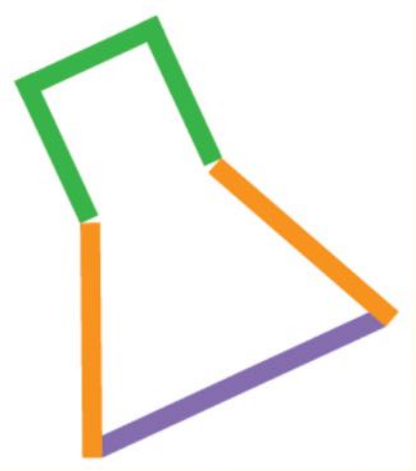
Children work with lengths to measure and make shapes.
They use counting and subitizing knowledge they have been building all year.

Measure Lengths and Make Shapes

Children use punch out strips of 2”, 3”, and 4” to measure and make shapes.

Pencils are on one side and colored strips on the other side.





Make Shapes and Discuss Attributes

The length strips allow children to see "empty" shapes and focus on the sides and their relationships.

MathTalk in Action

Now tell us about the 4-sided shapes you made.

Ann: I made a square with all orange sides.

Pedro: I made a rectangle with green and purple sides.

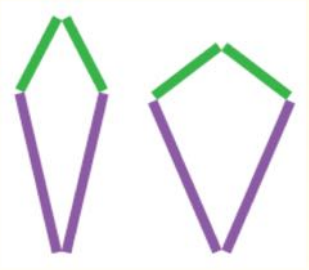
Swati: We used 2 purple down here and 2 little green up here at the top. It looks like a kite.

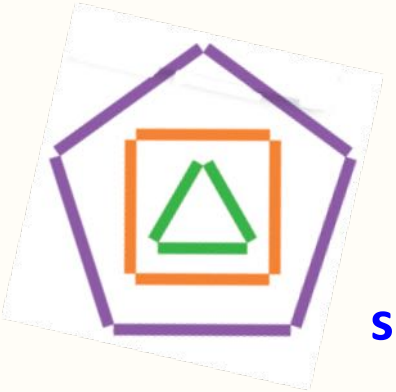
Yoshi: Ours has all orange sides. It looks like a diamond.

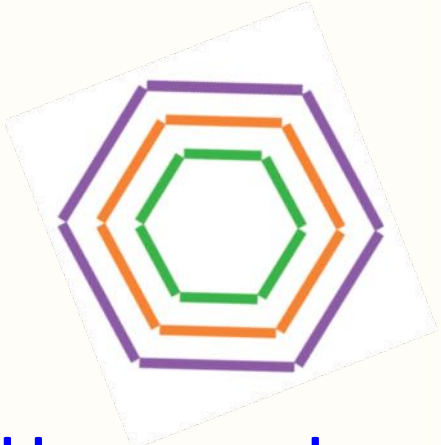
Jane: Look, I made a tilty rectangle with purple and green; it does not have square corners.

Mike: Then it is not a rectangle, because rectangles have square corners. So, it is just a 4-sided shape.

Swati: Mike's right.





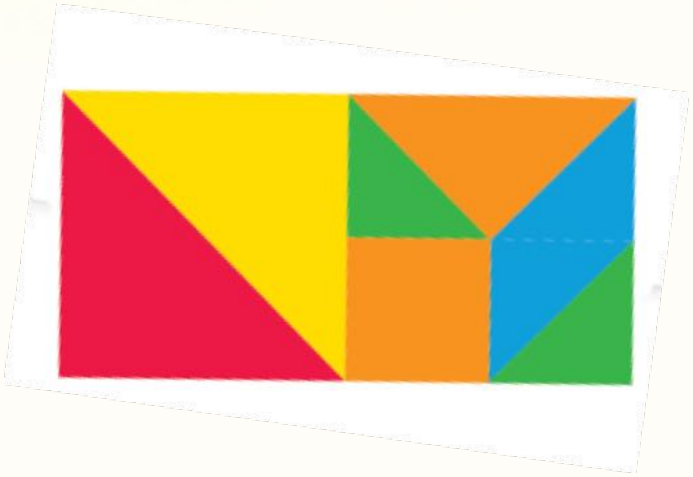


Children can make shapes with 3, 4, 5, 6, 7, 8, 9, 10 sides.

These can have equal or unequal sides.

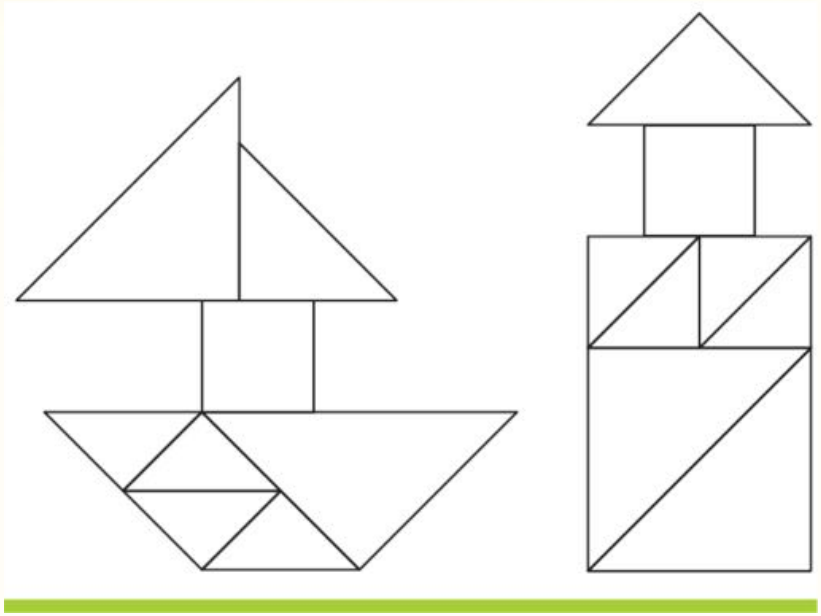
Tangram Pieces Make a Rectangle and a Square

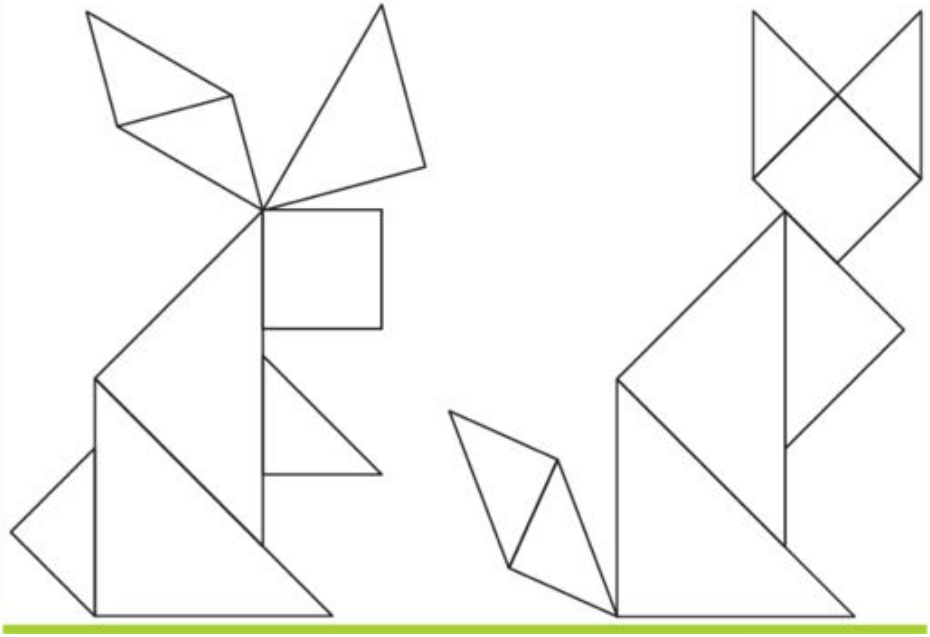
The 2-D shape set has tangram pieces if 2 small right triangles are put together to make a parallelogram (here we show this in blue).



Solve Picture Puzzles

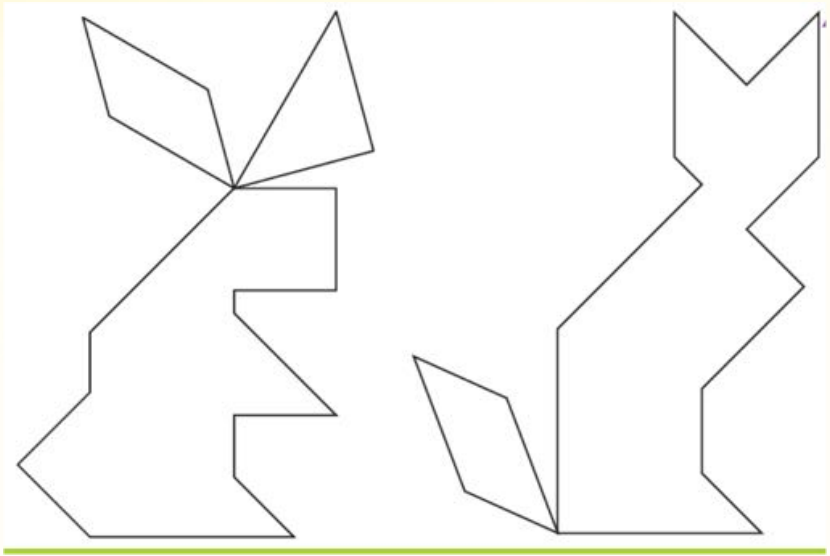
Children use the tangram shapes on picture puzzles that show the shapes.





The Same Picture Puzzles Also Have No Lines In Them

Puzzles without internal lines are more difficult to solve, but children can use the puzzles with internal shapes if they need help.



Children can also cover the puzzle with the small right triangles to see how the shapes are made by right triangles.

MathTalk in Action

Fumi and Tom are discussing how they solved the second puzzle on Picture Puzzle Card 3.

Fumi: I knew right away that the second puzzle was a cat. But I had to look at it carefully to see where to put my first puzzle piece.

Tom: You put the big right triangle at the bottom. That was a good move. I could tell that the ears were the small triangles so I put one on one ear for my turn.

Fumi: And I used another one for the other ear.

Tom: The two ears made a square corner, so I put the square there. It made the cat's head.

Fumi: Then, the orange triangle had to go under the head.

Tom: There was only one place left to put the other big triangle.

Fumi: So, then one part of the tail was a small triangle.

Tom: And the other part of the tail was the other small triangle. And we finished it!

Children can have many productive times as they solve and discuss these puzzles.

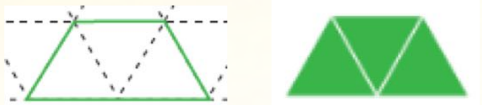
G1: Children Compose Small Equilateral Triangles

Pattern blocks make these restricted shapes.

Triangles with all 3 sides of equal lengths are called equilateral triangles.

Equilateral triangles compose to make certain shapes.

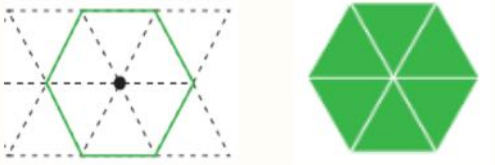
2 triangles make a special parallelogram called a rhombus because all 4 sides have equal lengths.



3 triangles make a trapezoid. A trapezoid has 1 pair of parallel sides. Parallel sides go on and on and never meet; they stay the same length apart.

4 triangles make a bigger equilateral triangle.

4 triangles also make a parallelogram. A parallelogram has 2 pairs of parallel sides; opposites sides are parallel.



6 triangles make a hexagon (kindergarteners saw these).

Which 2-D Shapes Are in Which Grade Standards?

2-D shapes specifically named in grade-level standards:

K: squares, circles, triangles, rectangles, hexagons (why hexagons now?)

1: rectangles, squares, trapezoids, triangles; half-circles, quarter-circles
(why trapezoids and not parallelograms?)

2: triangles, quadrilaterals, pentagons, hexagons (number of sides/angles)

3: rhombuses, rectangles, squares, quadrilaterals
(why rhombuses and not parallelograms?)

These shapes are pattern block shapes, but the examples are restricted.
If a shape is introduced, it should be treated generally.
Parallelograms should be introduced with trapezoids to see relationships.

4: Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

5: Classify two-dimensional figures in a hierarchy based on properties.

A learning path approach to 2D shapes:

Emphasize right-angled shapes because they are crucial in much of geometry.

Provide shapes from which children can compose related shapes and reflect on the relationships.

Help children focus on equal lengths in shapes because lengths are easy to see.

If a shape is introduced, treat it generally by seeing a range of examples.

Parallelograms can be introduced with trapezoids in Grade 1 (or even earlier) to see relationships and compose from rectangles and right triangles.

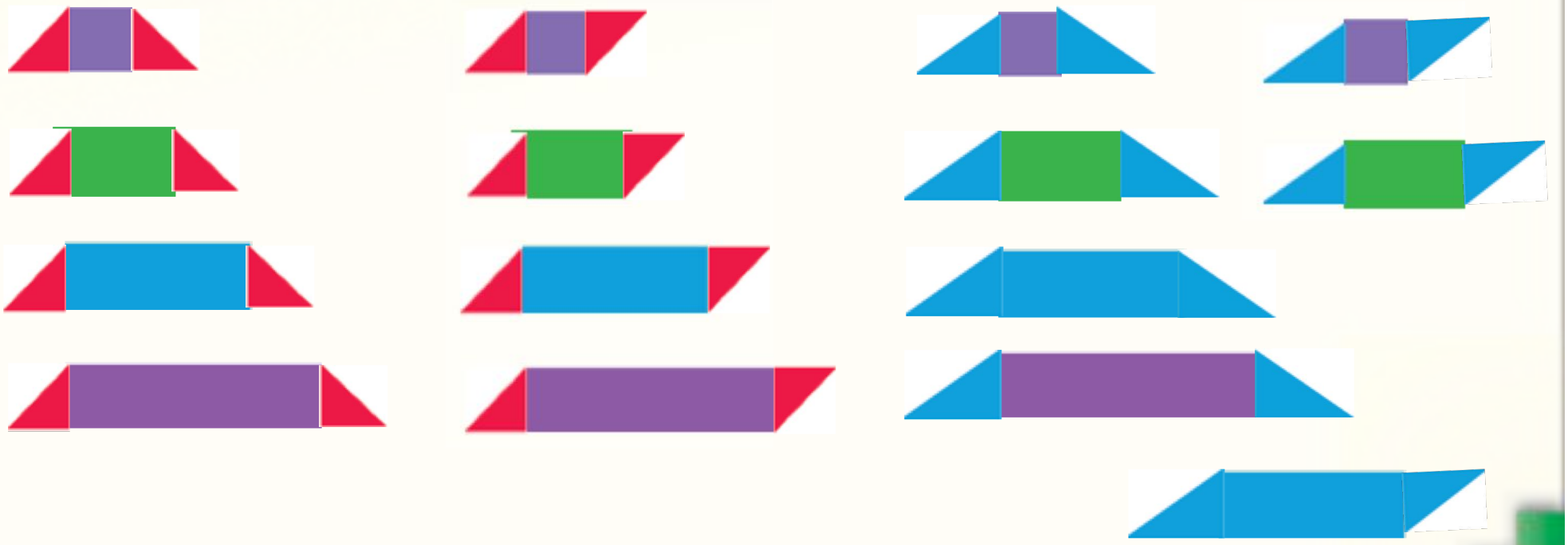
The terms for trapezoid and parallelogram do not have to be learned, but many young children enjoy big words.

G1: Relating Trapezoids and Parallelograms by Flipping One Right Triangle

Parallelograms are more important shapes than trapezoids, so it is not clear why parallelograms are not listed with trapezoids in Grade 1. These two shapes have a lovely relationship that children can discover by making these shapes and also discussing related drawings of these shapes.

Children can see and discuss trapezoids and parallelograms that have the same rectangles in the middles and have one triangle flipped to extend the top of the rectangle to the right.

trapezoids parallelograms trapezoids parallelograms

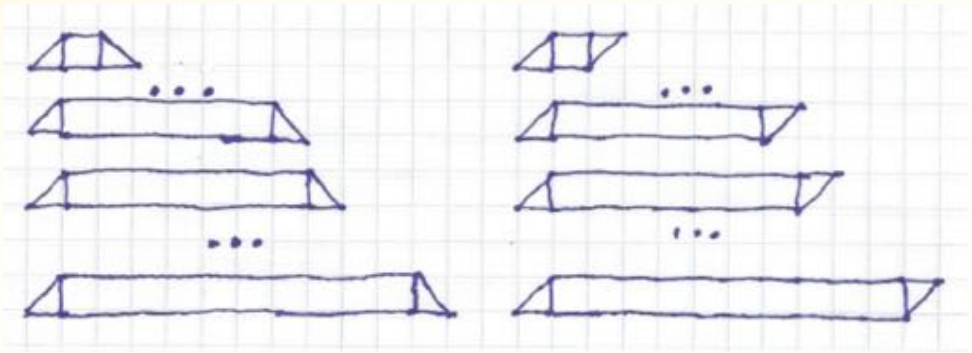


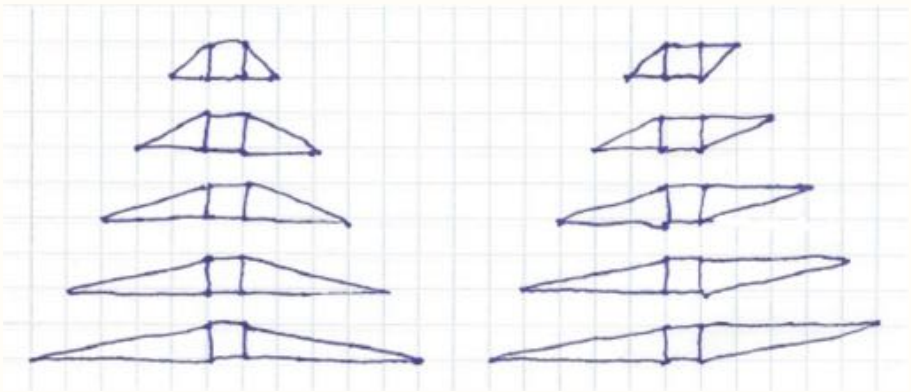
G1: Generalize by drawing on grid paper

Children can generalize and, by drawing on grid paper, see even more trapezoids and parallelograms made by flipping the right triangle.

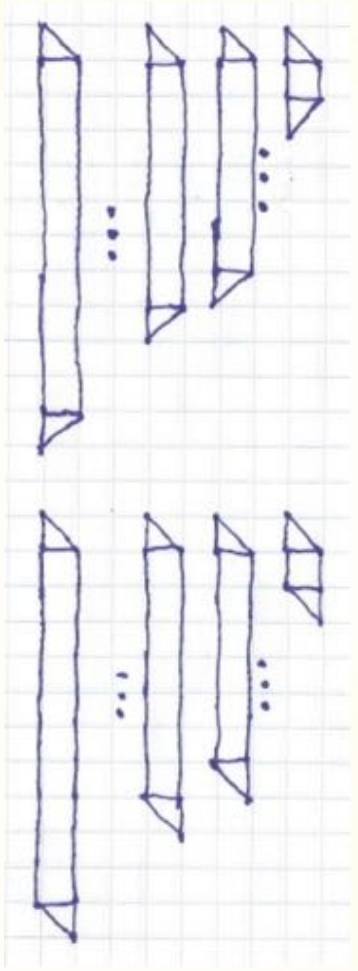
They can draw extreme cases with long rectangles or long sides on the right triangles to see many different cases of these shape.

They can turn around the grid paper to see the shapes in different orientations.

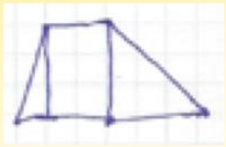




Children are counting length units as they draw.

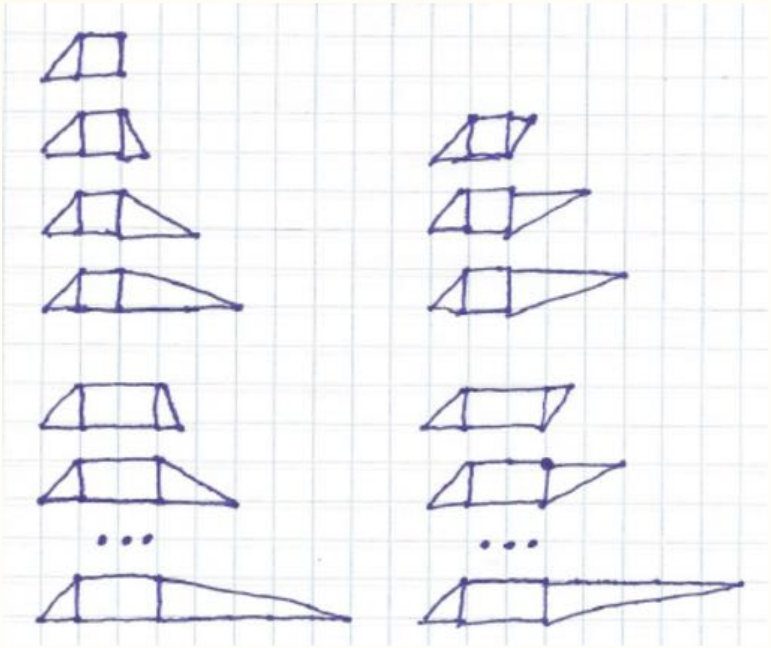


G1: Trapezoids but not parallelograms can have 2 different right triangles.



Trapezoids can have slanting sides of different lengths. But if you flip one unequal triangle, you do not get a parallelogram.
This is an early experience of slope: over n and up m. You count these lengths as you draw.
The slants are different so those sides are not parallel.

trapezoids **not parallelograms**



So a trapezoid is composed from a rectangle and one or two right triangles. Seeing the right triangles gives early experiences of the height of a trapezoid. The compositions help to prepare for all of the fun ways to show the area of a trapezoid that depend on various decompositions or compositions.

Geometry Content in MD Measurement/Data

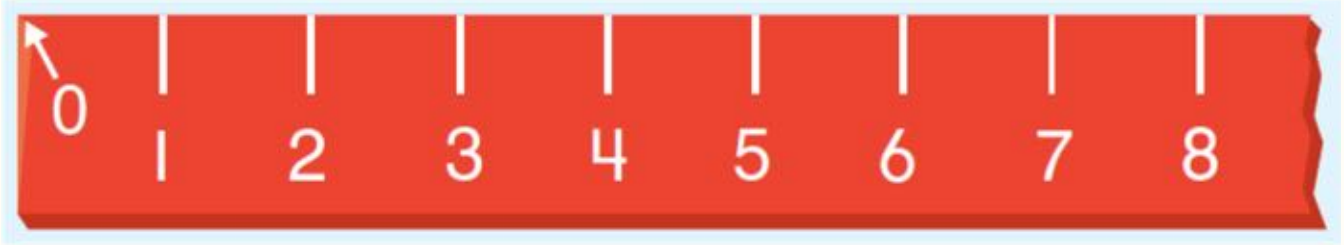
	K	1	2	3	4	5
	MD Measurement and Data: K to 5					
Length	**Geometric Measurement: K to 6** Use length to make area and volume units					
	(Describe attributes)	(Length)	Length	Area	**Angles**	Volume [G6 geometry: surface area and area of triangles, special quadrilaterals, and polygons
	Other Measures: K to 5					
Various	(Describe attributes)	(Time)	Time Money	Time Liq volume Mass	Larger to smaller units x	Convert units both ways x ÷
Base ten				Metric liq vol, mass are multiples of ten		
	Represent and interpret data: K to 5					
			Line plots	½ ¼	½ ¼ 1/8	Use fraction operations
Things	Classify into categories, count	Up to 3 categories compare	Picture & bar graphs all problems	Picture & bar graphs scale multiple unit 1- and 2-step compare		

2.MD.1: Measure Lengths in Standard Units Using Tools.

Length tools are visually difficult.

Children are wired to see things, so they see the marks on rulers.

Numbers by the marks draw the eye even more to marks.



All length tools share this problem. It is a HUGE PROBLEM and takes a lot of teaching to overcome the problem.

Make Shapes and Discuss Attributes

The length strips allow children to see "empty" shapes and focus on the sides and their relationships.

MathTalk in Action

Now tell us about the 4-sided shapes you made.

Ann: I made a square with all orange sides.

Pedro: I made a rectangle with green and purple sides.

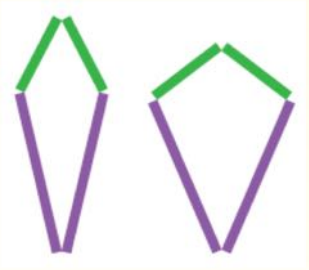
Swati: We used 2 purple down here and 2 little green up here at the top. It looks like a kite.

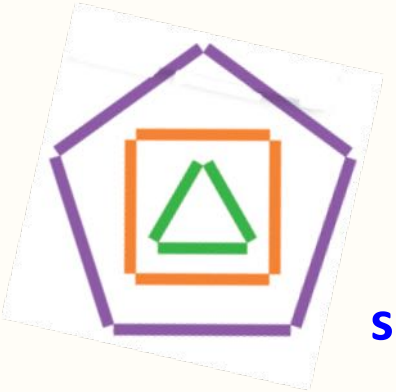
Yoshi: Ours has all orange sides. It looks like a diamond.

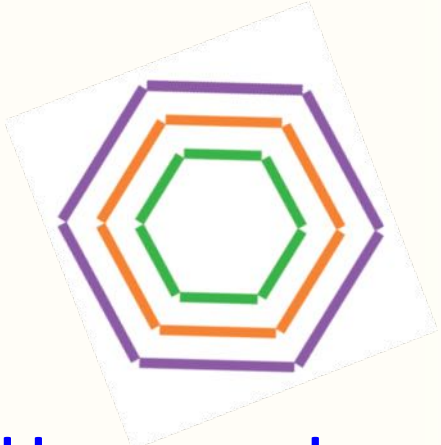
Jane: Look, I made a tilty rectangle with purple and green; it does not have square corners.

Mike: Then it is not a rectangle, because rectangles have square corners. So, it is just a 4-sided shape.

Swati: Mike's right.







Children can make shapes with 3, 4, 5, 6, 7, 8, 9, 10 sides.

These can have equal or unequal sides.

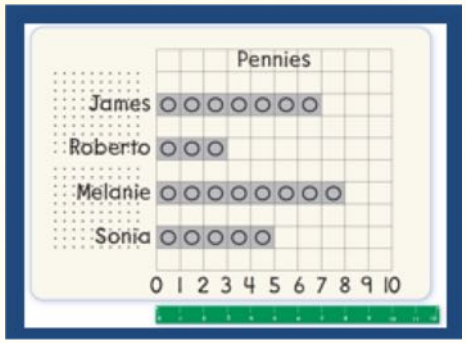
Conceptual Tools in Measurement, Data, and Fractions

Length Tools

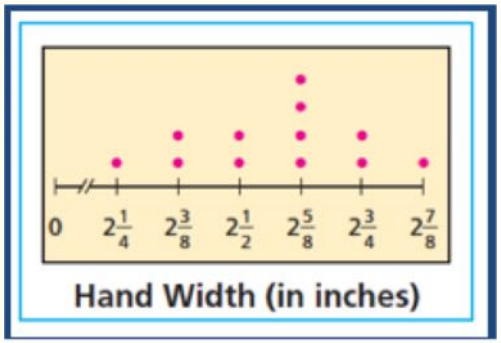
Rulers

Measurement scales for liquid volume and mass

Bar graph scales

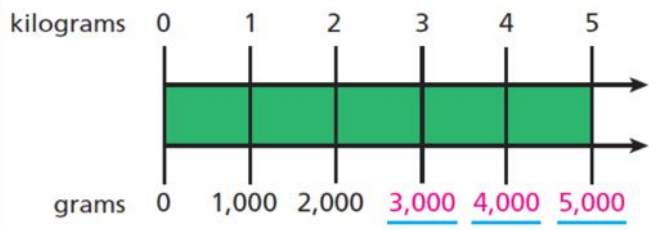


Line plots



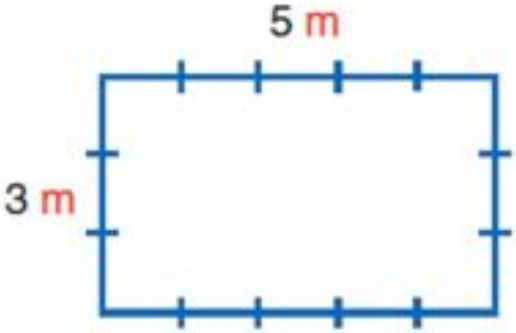
Number-line diagrams

Double number-line diagrams



Emphasize the length units for perimeter and and the square units for area

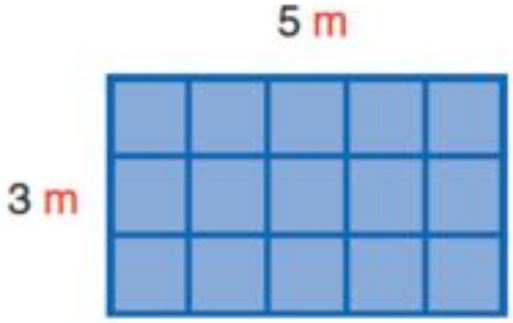
Perimeter and Area



$P = l + w + l + w$ or $2l + 2w$

$P = 5\text{ m} + 3\text{ m} + 5\text{ m} + 3\text{ m} = 16\text{ m}$

Perimeter is the distance around a figure. You add the side lengths to find the total distance.



$A = l \cdot w$

$A = 3\text{ m} \cdot 5\text{ m} = 15$ square meters

Area is the number of square units that cover a figure. You multiply the length and the width to find the total number of square units.

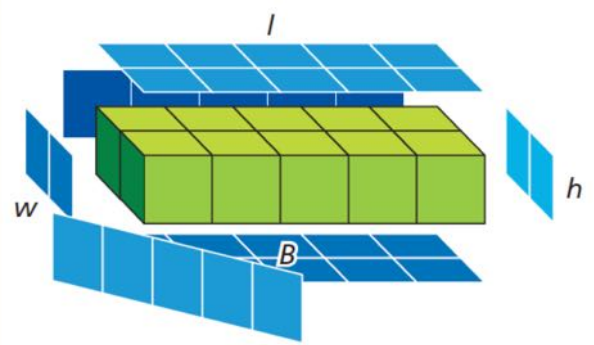
Differentiate the length units in perimeter and the square units in area.

For area, check that the side lengths have the same units so you can make the square units.

G6: Students Differentiate Surface Area and Volume of Prisms

Students see and identify the kinds of units used to measure surface area and volume.

- They see the **square units** that make the surface area and review that they write the answer as unit^2.
- They see the **cubic units** that make the volume and review that they write the answer as unit^3.

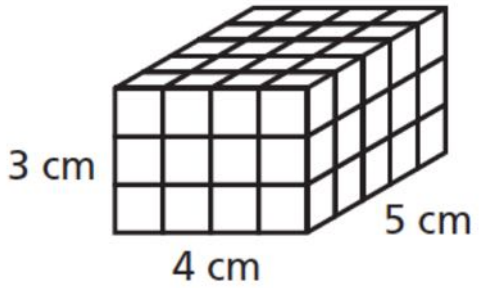


11. What is the surface area and volume of the prism you made?

$SA =$ 34 cm^2 2x5 + 2x2 + 2x2x5 cm^2

$V =$ 10 cm^3 2x5x1 cm^3

16.

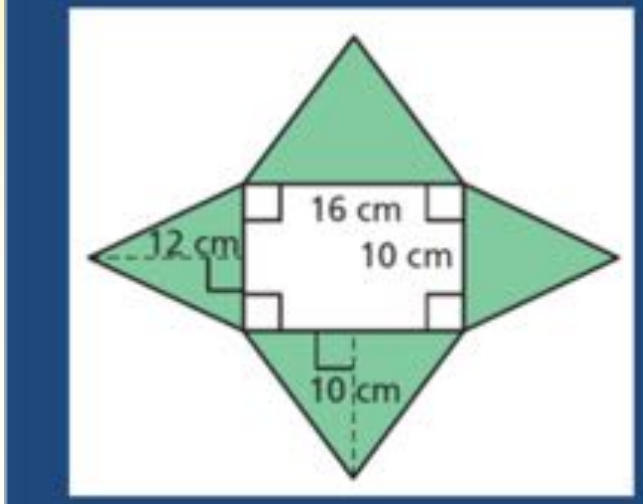


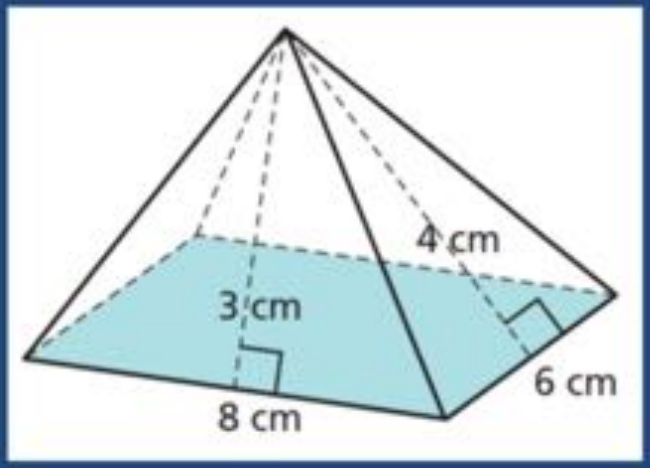
$SA =$ 94 cm^2 2(3x4 + 3x5 + 4x5) cm^2

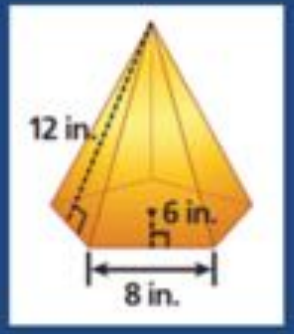
$V =$ 60 cm^3 3x4x5 cm^3

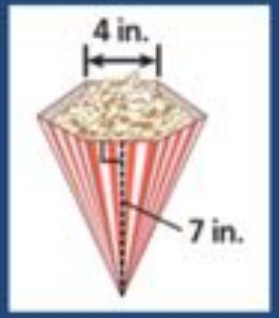
Seeing square units on different shapes and composing shapes

Surface Area Grade 6







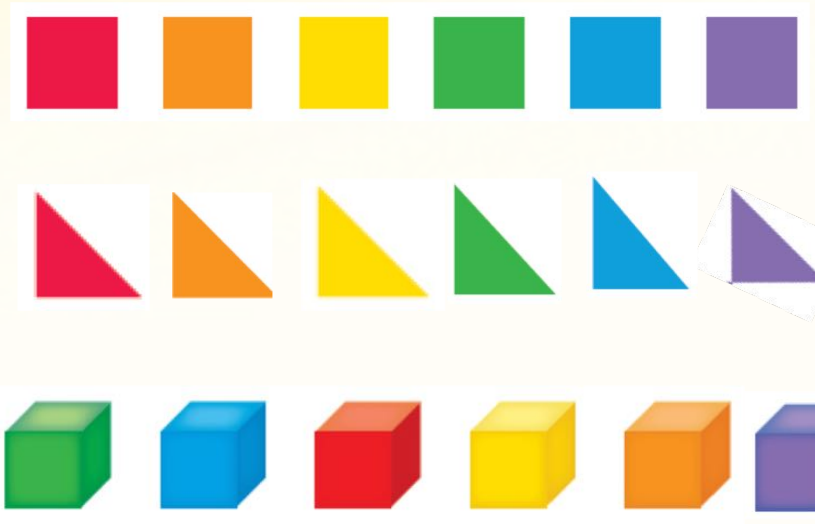


Six Coordinated Colors to Make Many Patterns and Generalize Across Color

Children have square inch squares and square inch cubes in six colors.

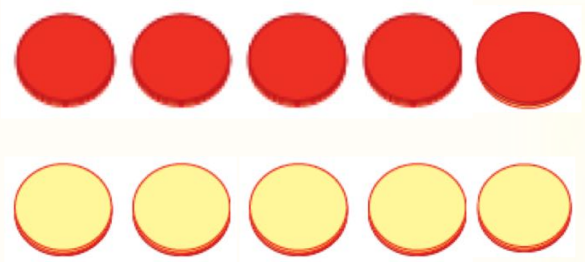
They also have matching right triangles 1 inch on a side.

These shapes are used in number and in geometry activities.



These are manipulate and thick and easy to work with. There are enough for Math Centers also.

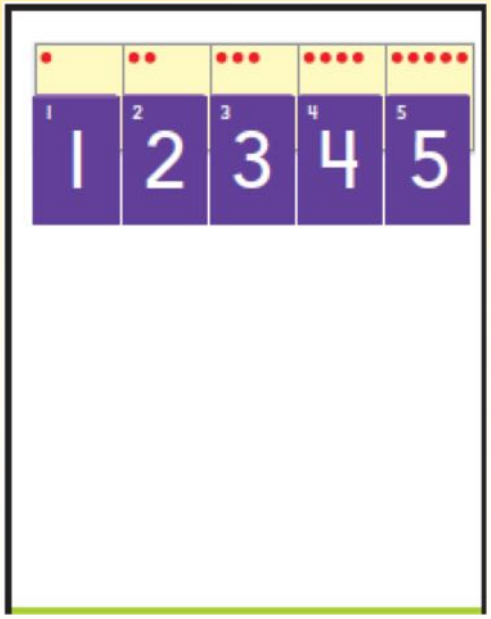
Children also work with red and yellow circles for number activities.

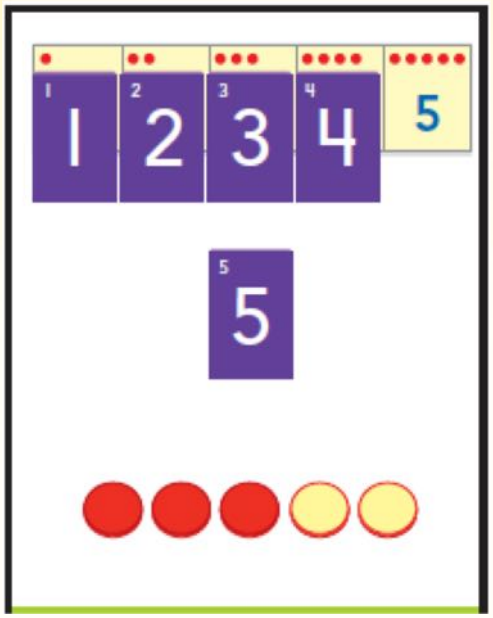


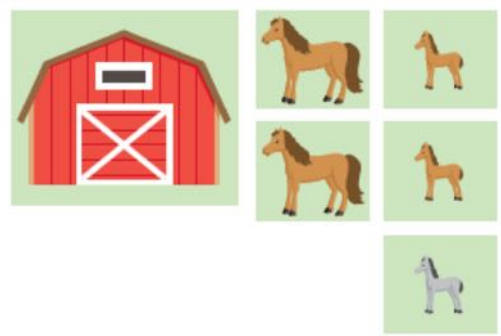
Part B: Partners of 5, Match, Compare, Add Numbers Through 5, Make Repeated Patterns

Partners of 5 use all earlier number knowledge.

Partners are now related to adding and the + sign.

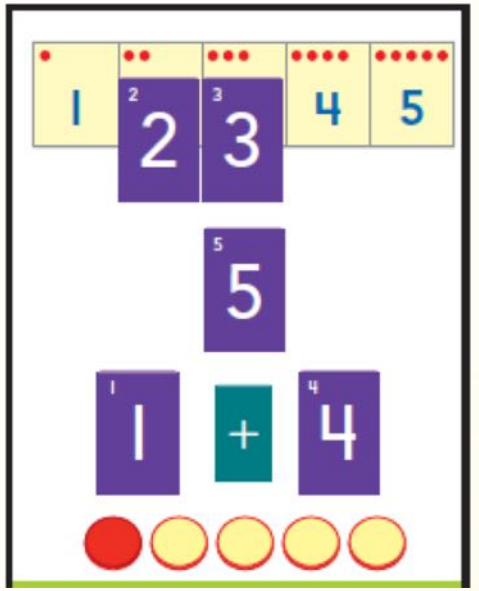






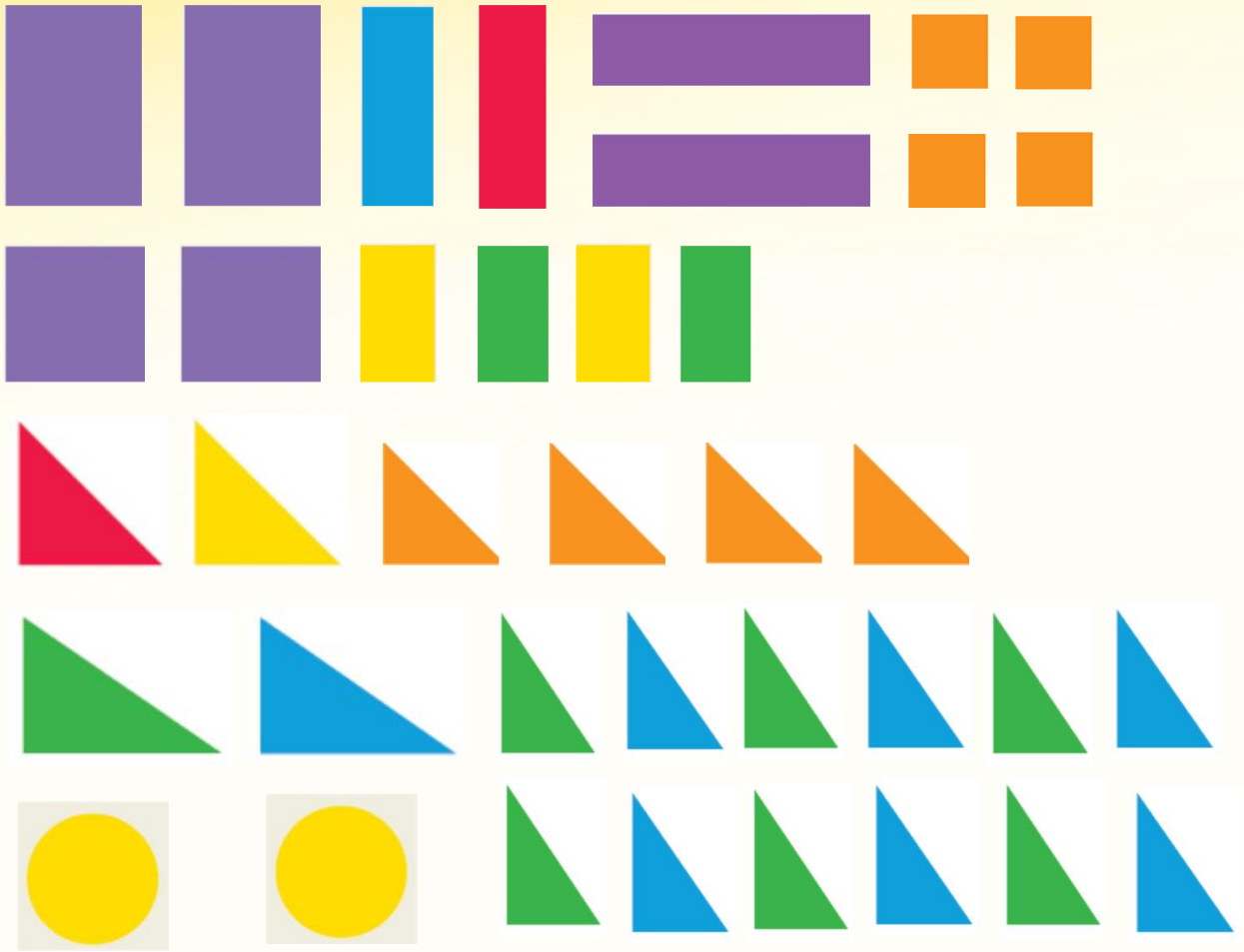
5		
1	4	[Ramona] made 1 and then 4.
4	1	[Cory] made 4 and then 1.
3	2	[Chen] made 3 and then 2.
2	3	[Jim] made 2 and then 3.

5		
1	+	4
4	+	1
3	+	2
2	+	3



1 and 4 make 5.

Math Expressions 2D Shape Set



Rectangles were selected to compose to make each other in multiple ways and to make many other shapes.

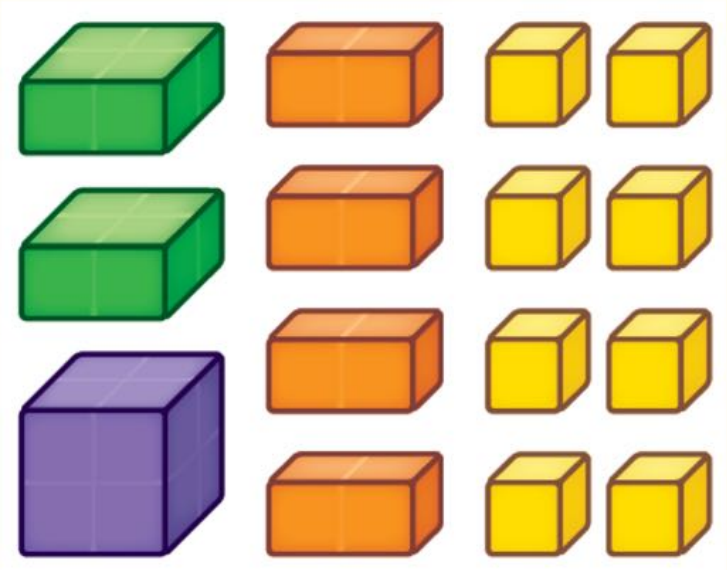
Right triangles compose to make rectangles and other shapes. They also work with the rectangles and with the 1”x1” right triangles of 6 colors.

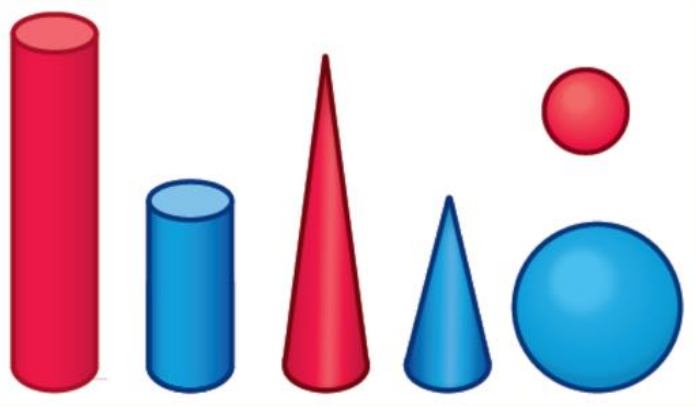
Math Expressions 3D Shape Set

Cubes, cylinders, cones, and spheres

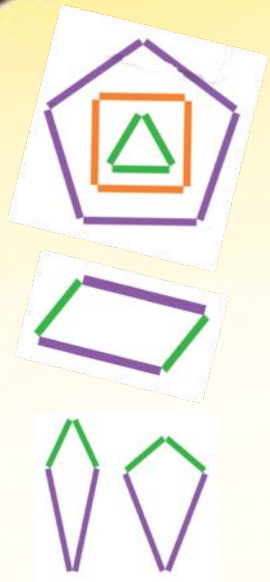
Small yellow cubes make the orange, green, and purple shapes.

Orange and green build to make the big purple cube.



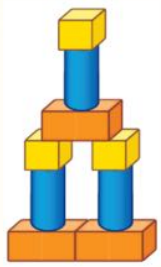


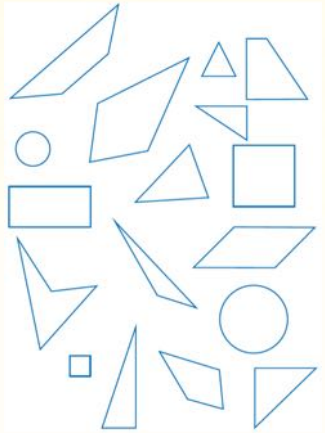
Relating Number and Geometry in the Early Grades



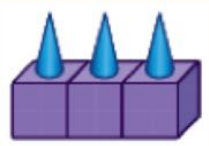
Professor Emerita Karen C. Fuson

Northwestern University





Paper presented at the Annual Conference of the National Council of Teachers of Mathematics, 2019, San Diego, CA



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