Overview of Teaching in *Math Expressions*

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Introduction to Learning Path Teaching in a Math Talk Community

Math Expressions uses **a research-based approach to teaching** supported by recommendations of the National Council of Teachers of Mathematics, the National Council of Supervisors, and multiple reports by the National Research Council. This approach **can result in high levels of mathematics learning by students and teachers from all backgrounds**. I am overviewing this approach here to help educators reflect on and improve their understanding of teaching in *Math Expressions*. Summarizing aspects of this teaching in tables and figures allows the reader to understand and relate aspects of this teaching. This paper is long because I wanted to bring together the various aspects of teaching I have discussed over the years. It can be helpful to read and discuss parts of the paper in Professional Learning Communities or with a colleague. The sections of this overview are listed below so that you can get a sense of the parts. The later parts do depend a bit on the earlier parts but you can skip to something in which you are particularly interested. Some of these tables and figures can be read again and reflected upon and discussed as the year goes along.

It can be helpful to look at videos on my website **karenfusonmath.com** to see classrooms in action. Also on my website are overviews I developed of the Common Core State Standards in each math domain. **Two of these overview the Math Talk Community**. And many of my papers about teaching specific topics and about teaching in general are available for downloading on my website under Publications (click on a paper and then click on the right column to download that paper).

What is a Nurturing Math Talk Community?

Table 1 below describes three crucial teaching tasks for building a Nurturing Math Talk Community. These tasks give a picture of what is happening in the classroom and what a teacher needs to do to create such a classroom. Later sections will give more detailed information about how to implement these teaching tasks and build a classroom community. Building a community is important so that all students feel accepted and that they have a voice. Belonging to a community can reduce anxiety, free the mind to think more deeply and independently, and build interdependence and empathy as students learn to help each other understand.

Table 1 *Use the Continuing Teaching Tasks 1, 2, 3 to Create the Year-Long Nurturing Meaning-Making Math Talk Community to Relate Students' Informal Knowing to Formal Mathematical Knowing*

Formal mathematical vocabulary, ideas, and methods: Bring students up to the higher mathematics in meaningful ways and by small supported coherent steps

Via a Nurturing Meaning-Making Math Talk Community

Teaching Task 1: Teacher builds the nurturing teaching-learning community: Co-creates an inclusive and participatory classroom culture in which the class co-constructs emerging related understandings for all by providing multiple levels of access (everyone can participate) through mathematizing (seeing the math in children's worlds); making math drawings; using rich language by validating all children's language and experiences while connecting them to standard language and symbols; and facilitating listening, speaking, writing, and helping competencies to make problems accessible to all

Teaching Task 2: Teacher creates a cognitively supportive meaning-making classroom by using coherent visual, sensory-motor, linguistic, and situation learning supports along with math modeling to create interest and accessibility of ideas: Mathematical words and symbols are linked to coherent meaningful referents by mathematizing known contexts or by providing new experiences to be mathematized; rich language use by all (see Teaching Task 1); everyone makes Math Drawings or uses other visual or sensory-motor supports to facilitate reflection, discussion, analysis, and understanding of everyone's thinking

Teaching Task 3: Teacher develops a collaborative Math Talk (instructional conversation) culture of understanding, explaining, questioning, justifying, and helping that elicits, values, and discusses student ideas and methods while relating visual quantities to steps in each method and discussing mathematical attributes of methods; talkers and listeners can understand each other because Math Talk connects to referents (see Teaching Task 2); all teachers are learners and all learners (students) are teachers of themselves and of others (peer helping); all participants help to develop coherent networks of knowledge by relating ideas and experiences within instructional conversations (Math Talk)

Informal preexisting vocabulary, ideas, and methods: Start where students are and keep learning meaningful

Note. The vertical arrow indicates that the formal and informal vocabulary, ideas, and methods continually relate to each other via the Teaching Tasks 1, 2, 3. Mathematizing means focusing on the mathematical aspects of a situation.

This table is slightly modified from Table 3 in Fuson, K. C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a Class Learning Path Model that individualizes within whole-class activities. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 10* (1), 72–91.

An effective Math Talk Community **uses mathematical practices** as summarized in the sentence:

Teachers continually assist students to do **math sense-making** about **math structure** using **math drawings** to support **math explaining**.

The bold terms come from pairing mathematical practices as shown in the table below. I developed the terms for these pairs and the sentence summarizing their use on the classroom. These practices come from the Common Core State Standards, but all high-quality state standards have similar standards for mathematical practices in the classroom. Keeping this sentence in mind while one is preparing for teaching and while actually teaching can be very helpful.

Mathematical Practices			
Math Sense-Making	**Math Structure**	**Math Drawings**	**Math Explaining**
Make sense and use of appropriate precision.	**See structure and generalize.**	**Model and use tools.**	**Reason, explain, and question.**
MP1 Make sense of problems and persevere in solving them. **MP6** Attend to precision.	**MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	**MP4** Model with mathematics. **MP5** Use appropriate tools strategically.	**MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others.
Teachers continually assist students to do math sense-making about math structure using math drawings to support math explaining.			

Figure 1. How eight mathematical practice coalesce into 4 pairs and one sentence to guide teaching. From the Math *Expressions* Teacher Editions p. I 6.

My classroom research found that two kinds of Solve, Explain, Question, and Justify classroom activity structures are effective in engaging all students in math-talk (see Table 2). In both structures, all students solve the same problem simultaneously. In the first structure, as many students as possible go to the board to solve the same problem while the rest of the students solve that same problem at their seats using Math Boards or paper. Then the teacher selects two or three students to explain their solution on the board and answer questions from classmates. The explainers can be those who have interesting solutions or need the chance to explain their work. Only two or three students need to explain their work because students usually cannot maintain concentration for more than two or three discussions of the same problem. Everyone solves the same problem so that they will have thought about it before the explanation by a student. Students often solve in different ways, and this stretches everyone's thinking.

Next, a different group of students goes to the board to solve the next problem while the rest of the class also solves the problem at their seats. This process is very motivating to students. Most students enjoy solving problems at the board even if they do not get the chance to explain their work. Two or three students from this second group explain, so the class loops through Steps 2, 3, and 4 in Table 2. The discussion can now also contrast and compare the first and second solutions as well as others in the past. While the students are working at the board, the teacher has a chance to see details of how they are writing their explanations, and for example, what gets written first and what gets erased and modified. The teacher also gets a good sense for how individual students are doing. In one class period, many or even all of the students can get a turn at the board.

The Solve, Explain, Question, and Justify activity structure can also be used in small groups so that more students can practice explaining. Any student at any time can ask for help from anyone including help explaining. Small group solving and explaining can sometimes be followed by whole-class sharing and discussion, but such discussion of methods should be limited in number so that students do not have to sit through "too many" explanations. I suggest two or three explanations, but sometimes more can be shared. The emphasis should be on the more-advanced methods that are in the lessons so that students can move forward to these. The methods shown in lessons have positive and important mathematical features (they are mathematically desirable), and my years of classroom research found that they are accessible to students, teachers, and parents.

Table 2 *The Solve, Explain, Question, and Justify Classroom Structure*

Step 1 Solve	All students solve the same problem so they can have thought about it before another student explains their thinking.
Step 2 Explain	One student explains their thinking while pointing to parts of the drawing and/or problem. The explaining student ends by asking, "Questions?" or "What do notice or wonder about?"
Step 3 Questions	Other students ask questions to clarify or extend or they add things they noticed or wondered about.
Step 4 Justify	The original explainer responds to classmate's thinking.

This table is adapted from Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2015). Describing levels and components of a Math-Talk Learning Community. In E. A. Silver & P. A. Kenney (Eds.), *More lessons learned from research: Volume 1: Useful and usable research related to core mathematical practices* (pp. 125-134). Reston, VA: NCTM.

The Importance of Math Drawings and Other Visual Images for Conceptual Understanding and Explaining

How can students who do not know concepts come to understand them and even develop solutions before they are explicitly taught any solutions? The key to this question is that **students make math drawings** to show the quantities in the problem situation and/or the problem structure (how different quantities relate to each other) and/or the problem solution. With such visual supports for thinking and explaining, students can access new concepts and work with the ideas involved. Kinds of drawings can be developed by students or shared by the teacher if they are research-based drawings that are powerful for students. The math drawings introduced in *Math Expressions* are based on ten years of classroom research that elicited math drawings from students and worked to find the most powerful drawings. Faster and better ways to introduce and support these drawings were also developed and are presented in the lessons.

The research-based most powerful drawings for real-world situations I found in my research are shown in Figure 2. There are three kinds of addition/subtraction and three kinds of multiplication/division real-world situations. I pooled array and area situations because they are so similar and only vary in whether they use count or measure units. The CCSS takes this same approach. The grade level for the initial introduction of each kind of situation is in a box at the top right. All of the situations relate three quantities. Any of the three quantities can be the unknown quantity. The *Math Expressions* Teacher Editions give examples of these problem types for all three unknowns and describe the problem solving process on Introduction pages I13 to I15.

For the **Add To/Take From** (also often called Change) situations, I tried various drawings and nothing worked as well as equations. Students already used equations before we did anything with them because their first meaning for the = is a change meaning. This meaning is shown below the drawings by an arrow. In the situation the two quantities on the left disappear and become (are changed into) the third quantity. The middle column shows how the total is **taken apart** to make the two addends (T = A + A) or how the addends are **put together** to make the total (A + A = T). For the middle column the quantities on the left **are identical to** the quantities on the right. For T = A + A, one can focus at a given moment on the total or the addends, and one can shift one's focus back and forth. For F x F = P, one can focus on the two Factors or the Product, and for P ÷ F = F, one can focus on the division Product ÷ Factor or the other Factor. The right column shows additive or multiplicative **comparison situations**. For additive comparison situations (top right), the only quantities in the situation are the Big quantity and the Small quantity. The Difference quantity is created by the question that concerns the difference between the quantities. Likewise, for multiplicative comparison situations (bottom right), the only quantities in the situation are the Big and Small quantities. The multiplication or division operations are created by the comparing sentence. For both kinds of comparisons, the **number** on the **left** of the equation **is the same number as** on the **right**. But the quantities are not the same quantities in the situation.

The equations in Table 2 are situation equations that show the relationship of the quantities in the situation. Students may think about their situation equation and write a solution equation to help them solve the problem. So for example the **situation equation** ☐ – 27 = 48 shows a Take From situation in which the Start number is unknown. Using understandings of relationships between the addends and the total, the situation equation might be rewritten as a **solution equation** 48 + 27 = ☐.

When students are first learning about the situation drawings, it is important that **they show the situation** with their drawing or equation. They can then reflect on that drawing or equation and decide how to solve it. For some simple problems, situation equations and solution equations are the same. For example, Add To Total Unknown would have both equations as A + A = ☐. So students need to see some problems in which the situation equation and solution equation is not the same. For example, Add To Addend Unknown has a situation equation A + ☐ = T but the solution equation is T – A = ☐. This is true for large numbers but for small numbers, students sometimes solve unknown addend or subtraction situations by counting on to find the unknown addend.

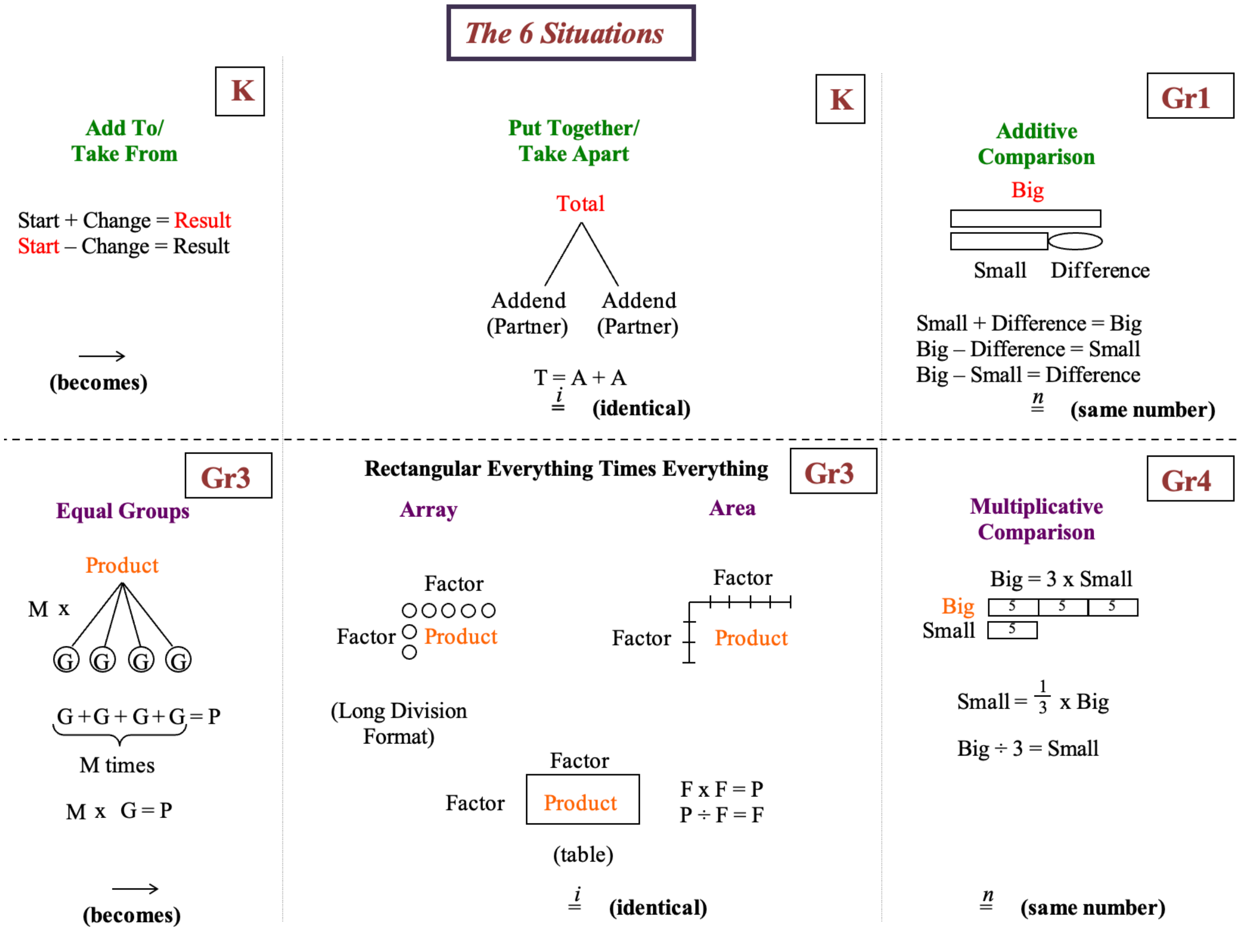


Figure 2 Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each

This figure appeared in Fuson, K. C., Murata, A., Abrahamson, D. (2015). Using learning path research to balance mathematics education: Teaching/learning for understanding and fluency. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford Handbook of Numerical Cognition* (pp. 1020-1038). Oxford, England: Oxford University Press. [Also appears online in *Oxford Handbooks Online, July, 2014.* DOI: 10.1093/oxfordhb/9780199642342.013.003]

Differences in how students represent a word problem situation are shown in Figure 3. Students can focus on the quantities in the situation or how the quantities relate numerically or what needs to be done with the quantities to find an answer. Some students may make more than one representation as they think through, represent, and solve the problem. The bottom right shows a drawing in which the student drew the 5 golf balls Yolanda had left and then the golf balls that Eddie took. This student also wrote a **situation equation** ☐ – 7 = 5 that began with the unknown start number from which the other two quantities were made. The bottom left shows a drawing showing the **relationships among the quantities**: The unknown start is separated into its two quantities shown in a Math Mountain drawing. The label shows that the student understands that these two addends can be put back together to make the beginning quantity. The top problem is a **computation solution equation** that finds the number in all. These solutions move from being embedded in the situation to operating on the quantities more abstractly.

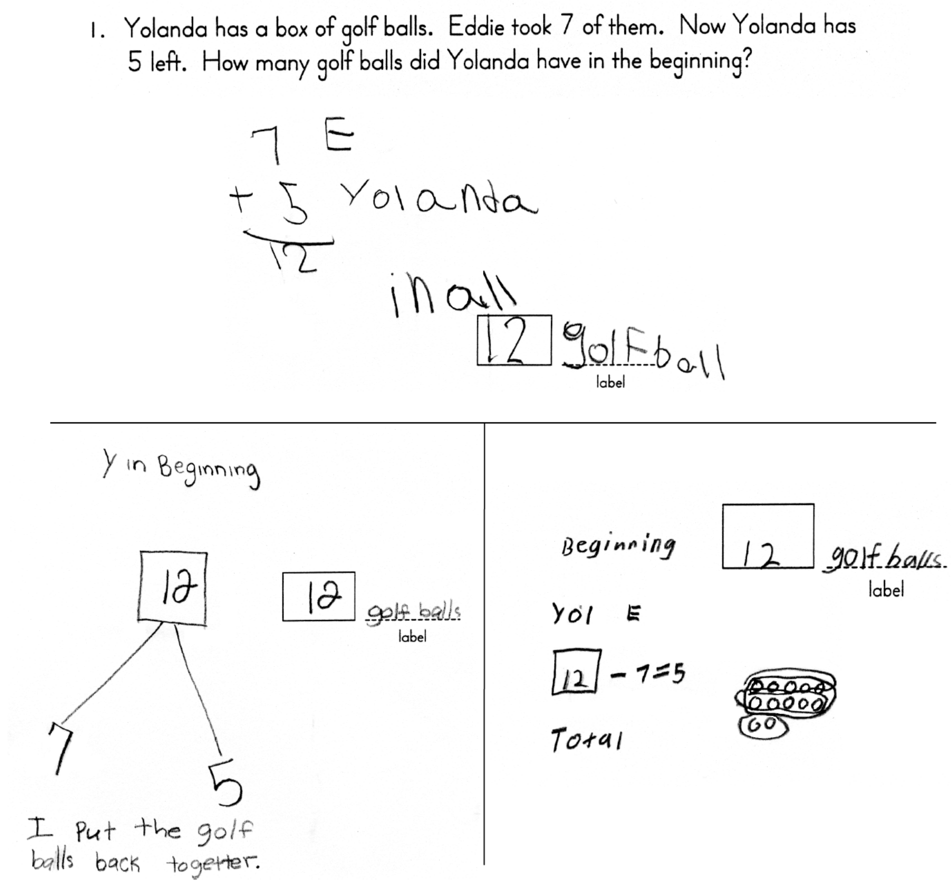


Figure 3 Varying student representations of an unknown start problems showing the importance of labeling the parts of math drawings

This figure was used in the presentation Fuson, K. C. (2006, April). *An algebraic approach to word problem solving for all learners using ambitious problems, drawings, and Math Talk.* Paper presented at the annual meeting of the National Council of Supervisors of Mathematics, St. Louis, MO.

In Figure 4 problems with the same unknown start structure are solved for 3-digit numbers and for fractions. Students can use the same approaches as for the single-digit numbers: a computation solution equation, a Math Mountain showing the relationships among the quantities, and a situation equation from which the student can write the needed solution computation. The **difference between a situation equation and a solution equation** is especially important to understand because too many teachers can think a situation equation is incorrect. But it is for some students a necessary step in representing and solving a problem. Figures 3 and 4 show how the same situational math drawings can be used for different quantities so that **problem solving**

can be related across all grades. For example, math drawings and written methods are shown for fractions in Figure 3. Situational math drawings can be used for different kinds of quantities.

The key to solving word problems is **understanding the situation**, which students can represent in different ways. The research-based situational math drawings in Figure 2 focus on the situation and help students represent it in various ways.

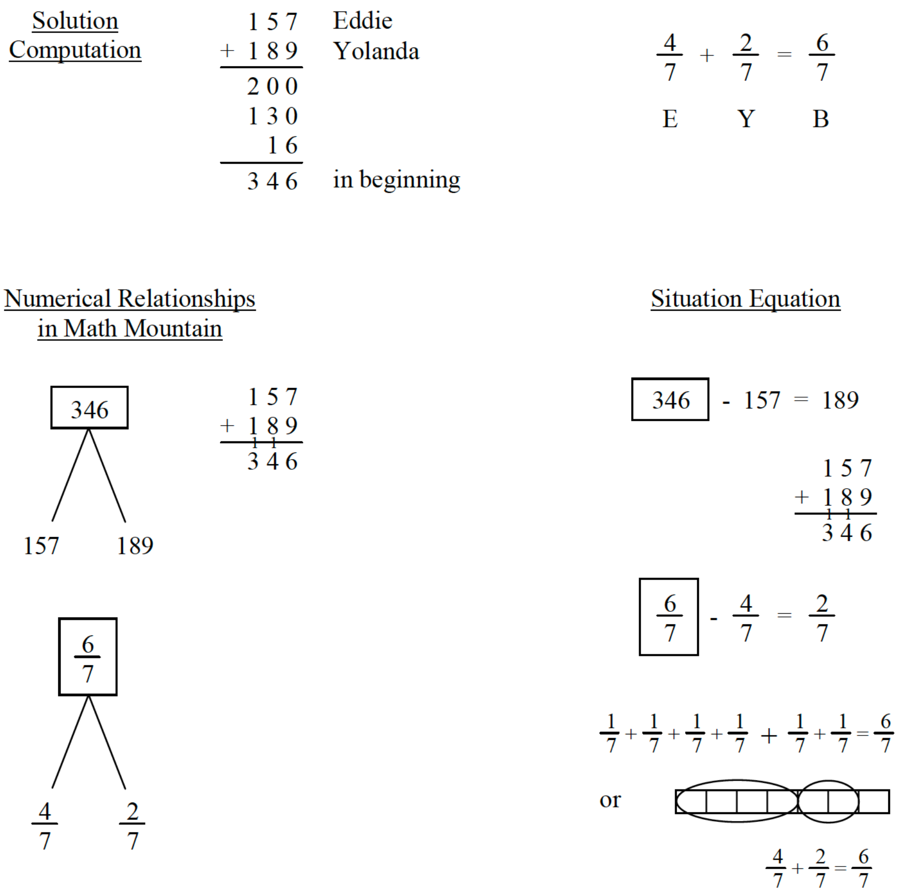


Figure 4 Unknown start solution approaches for multidigit numbers and fractions

This figure was used in the presentation Fuson, K. C. (2006, April). *An algebraic approach to word problem solving for all learners using ambitious problems, drawings, and Math Talk.* Paper presented at the annual meeting of the National Council of Supervisors of Mathematics, St. Louis, MO.

Using math drawings. When explaining a solution method students need to stand at the side of their math drawing and point to parts of the drawing as they explain. Steps in the math drawing need to be related to steps in the written method. The goal is for students to do written methods without drawings but with understanding of each step in the written method. Relating written method steps to the math drawing steps helps them do this. In Figure 5 we see a step-by-step math drawing and student explanation with questions from the class for the mathematically-desirable and accessible new-groups-below method. This method is better than the Current Common method because you can see the teen number you are writing (see the 16 and 14 below), you can write the teen number in the usual way writing the tens number first, and when adding you add the two larger numbers first and then just increase that total by one. In the Current Common method you add the new one on the top to the top number and add that number you do not see to the number below ignoring the top number that you still do see.

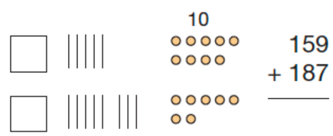
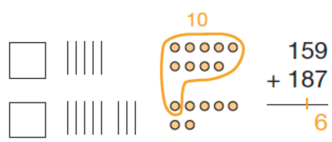
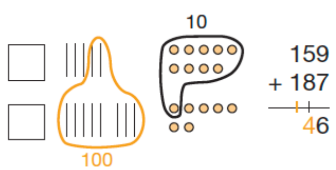
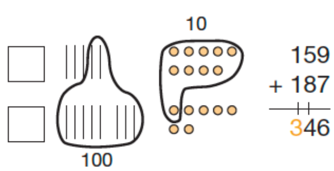
Math Drawing and Problem	**Explanation Using Place-Value Language About Hundreds, Tens, and Ones**
a. 	I drew one hundred, five tens, and nine ones to show one hundred fifty nine, and here below it I drew one hundred, eight tens, and seven ones for one hundred eighty seven. I put the ones below the ones, the tens below the tens, and the hundreds below the hundreds so I could add them easily.
b. 	See here in my drawing, nine ones need one more one from the seven to make ten ones that I circled here, and I wrote 10. That leaves six ones here. With the numbers the seven gives one to the nine to make ten that I write over here in the tens column, see one ten. And I write six ones here in the ones column.
c. 	With the tens, I start with eight because it is more than five so it is easier. I get two tens from five tens to make ten tens, see here, and I write one hundred here to remind me that the ten tens make one hundred. There are three tens left in the five tens and I have one more ten from my ones (see here in my drawing and the one ten at the bottom of the tens column). That makes four tens and the one hundred. So in my problem I write the one hundred below in the hundreds column and the four tens in the tens column.
d. 	There are three hundreds, two in the original numbers I'm adding and one new hundred from the ten tens. I write three hundreds here in the hundreds column. Are there any questions? Yes, Stephanie.
Student Question	**Explainer Answer**
Stephanie: For the tens, you never said fourteen tens as the total of the tens. Why not?	Because when I'm making ten tens, I just can write that one hundred over here with the hundreds and just think about how many tens I need to write. But I can think eight tens and five tens is thirteen tens and one more ten is fourteen tens, so that is one hundred and four tens. You can do it either way. (Aki)
Aki: Do you still need to make the drawings or did you just make them so you could explain better?	I don't have to make the drawings, but I can explain better with a drawing because you can see the hundreds, tens, and ones so well. (Jorge)
Jorge: Do you do make-a-ten in your head or just know those answers?	I just know all of the nine totals because of the pattern: the ones number in the teen number is one less that the number added to nine because it has to give one to nine to make ten. So nine plus seven is sixteen. I just know that pattern super fast. For eight plus five, I do make-a-ten fast, sort of just thinking five minus two is three, so thirteen. (Sam)
Sam: I know five and eight is thirteen, so why did you write a four in the tens column, Karen?	Because I had one more ten from the ones. See here in the drawing: nine ones and one one from the seven ones make ten ones. I wrote 10 here to remind me, and here in the problem I wrote the new one ten below where I can add it in after I find thirteen. You have to write your new one ten big enough to be sure you see it.
Sam: Oh yes, I see it now. I can see the new one ten when I write it, but I couldn't see yours.	OK, thanks. I'll write it bigger next time so everyone can see it.

Figure 5 Step-by-step math drawing and student explanation with questions from the class for the mathematically-desirable and accessible new-groups-below method
This figure is used with permission from National Council of Teachers of Mathematics (NCTM) (2011). Focus in Grade 2: Teaching with Curriculum Focal Points. Reston, VA: NCTM.

Multidigit computation. In the left column of Figure 6 are math drawings that show the quantities for multidigit computation. The middle column shows the one or two mathematically-desirable and accessible methods that are in lessons to be introduced to students. These mathematically-desirable and accessible methods are the methods that my years of classroom research found to be the best mathematically and most accessible to students. These methods were all invented by students in Math Talk classrooms. They are introduced after the initial sharing of student thinking about a topic.

The last column shows the Current Common method in the United States. This is the method that many people think has to be taught even though each of these Current Common methods has disadvantages. For example, for multiplication when you multiply 60 x 3 you get 180. The 1 hundred is written above the problem in <u>the tens column</u> instead of in the hundreds column, and you add that <u>1 ten</u> to the <u>4 hundreds</u> in 2400 (coming from 60 x 40) and get <u>5 hundreds</u> in 2500, which makes no sense. **There is no one "standard algorithm" that must be taught.** Over the years there have been many methods taught in the United States, and many methods are taught around the world. The standard algorithm is **a collection of different ways of writing a procedure that uses single-digit operations and concepts of place value**. The research-based mathematically-desirable and accessible methods are such procedures but every step makes sense and can be related to a drawing of multi-digit quantities (see Fuson and Beckman, 2012/2013, for more discussion of these points).

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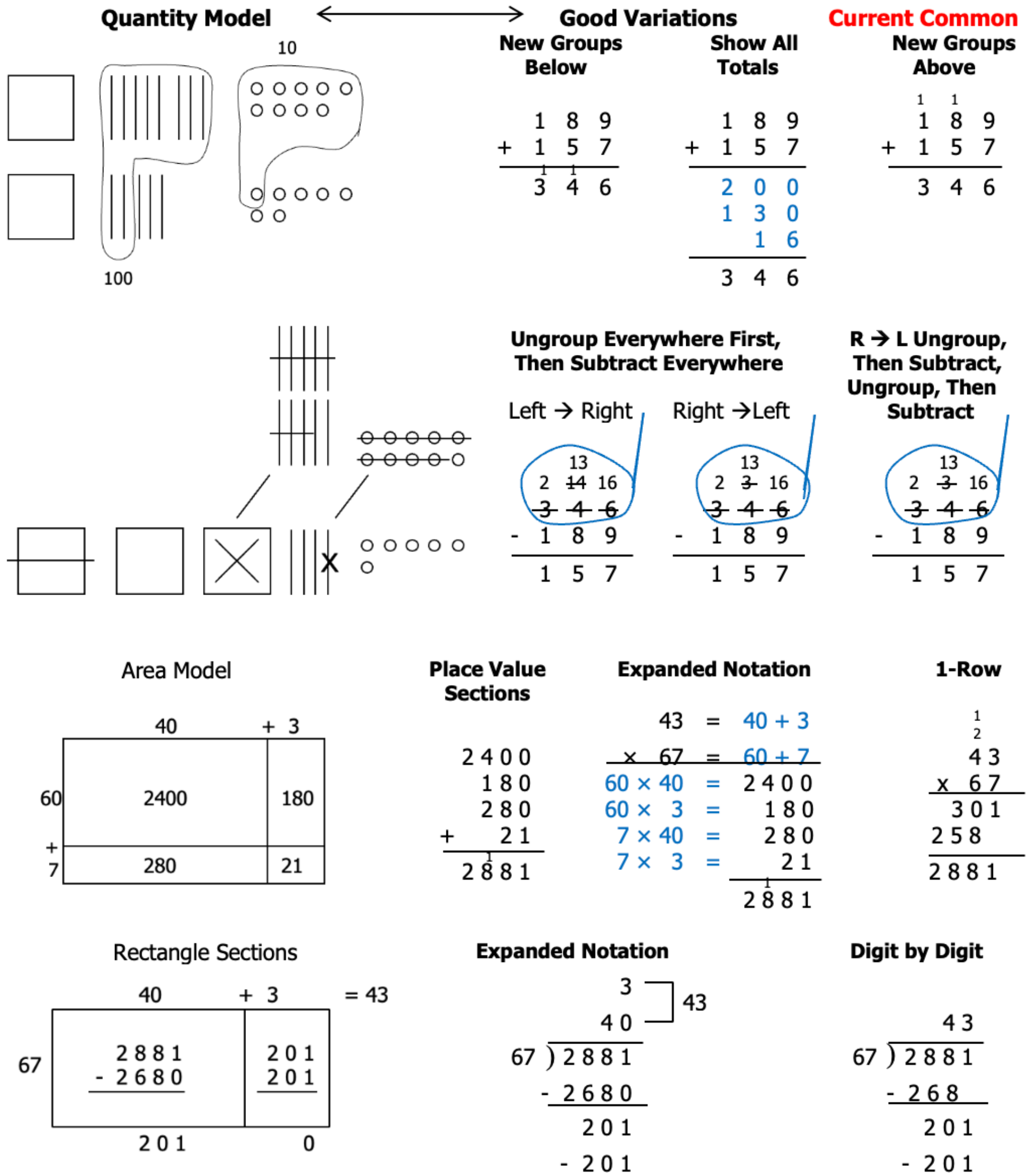

Figure 6 Math drawings, good variations of "the standard algorithm" that are mathematically-desirable and accessible, and Current Common methods that are neither

This figure is adapted from Fuson, K. C., Murata, A., Abrahamson, D. (2015). Using learning path research to balance mathematics education: Teaching/learning for understanding and fluency. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford Handbook of Numerical Cognition* (pp. 1020-1038). Oxford, England: Oxford University Press. [Also appears online in *Oxford Handbooks Online, July, 2014.* DOI: 10.1093/oxfordhb/9780199642342.013.003]

The advantages of the New Groups Below method on the top left were discussed above. The Show All Totals method shows the expanded form for each total, which is meaningful and needed for some students. In the subtraction methods a big circle with a stick is drawn around the top number and the ungrouping. This is to inhibit students from making the common "top from bottom" error, for example, subtracting 6 from 9 to get 3 in the ones column even though the 6 is part of the top number from which 9 must be taken. Even when students can subtract with ungrouping correctly, they sometimes look at the single digits in a column and just write the difference because they know those answers so well. This is the reason that the Current Common method seduces so many students to make errors. In this method students must alternate two kinds of steps: ungrouping and subtracting. When they get to a next left column they have just been subtracting, so they are more likely to subtract again instead of ungrouping. The mathematically-desirable and accessible methods both ungroup everywhere that needs ungrouping and then subtracts everything. Both the ungrouping and the subtracting can be done in either direction. To inhibit the "top from bottom" errors, we have students first draw that big circle with a stick around the top number, saying that it is a magnifying glass that lets them check each column to see if the top number is bigger than the bottom number. They then do any necessary ungrouping without subtracting yet.

The current common method of multidigit multiplication alternates multiplying and adding, which is more difficult than doing all of the multiplying and then all of the adding. Furthermore, it records a new group in the wrong place. In Table m when you are multiplying 60 x 3, you get 180, but the 1 hundred is written above the problem in <u>the tens column</u> instead of in the hundreds column, and you add that <u>1 ten</u> to the <u>4 hundreds</u> in 2400 (coming from 60 x 40) and get <u>5 hundreds</u> in 2500, which makes no sense. The two methods in Table m do all multiplying first and then add the partial products. The method in the middle was developed in a fourth-grade classroom in a high poverty school. Student said that they were misaligning the partial products if they started from the right, but if they started from the left and wrote the biggest product first, it would be easier to align under that number. And they wanted to write the tens and ones in the factors so they could see what they were multiplying and write the factors they were multiplying on the left to make sure they had all of the products. Seeing the written tens made it easier to multiply correctly. With experience most students dropped the extra steps and just wrote the partial products. The method on the left was carried out by students who had difficulty with all of the multiplications in the middle method. They could see what to multiply in the area model, so they wrote each product in its section and then added up those products out at the side.

The important aspect of the division methods is that they relate division to multiplication by using the same area drawing. The two rows are collapsed into one to show the divisor, and the task is to find the unknown factor that is the length of the rectangle. Partial products are written inside each rectangle section and subtracted from the amount left at that time. The other written methods use the common long division format in which the rectangle sections continue below each other but thereby lose the sense of division as the area of a rectangle. The common digit-by-digit method is not as bad as the other three common methods, but it is so easy to write the places within the long division as in the expanded notation method that it seems better for students to do that, at least initially.

Advantages of student math drawings are summarized in Table 3. Math drawings are keys to success in a *Math Expressions* classroom. These tables and the Figures 3, 4, 5, 6 above show math drawings and how they can help teachers do the fundamental goal in teaching:

Teachers continually assist students to do **math sense-making** about **math structure** using **math drawings** to support **math explaining**.

Table 3 *Advantages of Student Math Drawings of Problem Situations, Problem Quantities, and/or Solution Methods*

Young children aged 2 through kindergarten benefit from **using physical objects to show mathematical ideas.** Manipulatives also are important for introducing some math ideas for older students (e.g., making fraction strips by folding unit fractions). But many ideas beginning in grade 1 benefit from having students make math drawings and relate them to the formal math symbols. **Math drawings are simplified drawings** that show **quantities** (e.g., 238 or 6 dogs) in simple ways (e.g., drawing 2 hundreds-squares, 3 tens-sticks, 8 circles or drawing 6 circles for the dogs). Or they might show the math structure of a situation as in figure N used to show the relationships among the quantities in word problems.

Student math drawings of problem situations and/or solution methods enable students to **explain their thinking more clearly and explicitly** by pointing to parts of their drawing as they explain and enable listeners to understand because of the relating of language and visuals. Student math drawings can be especially helpful to students learning English.

Students can relate parts of the drawing to the problem situation by **labeling those parts** with a word or letter.

Students can **relate in the drawing** (e.g., by using an arrow) a step with quantity drawings (e.g., making 1 new ten from ten ones) to that same step in the numerical method (e.g., writing the new 1 ten in the tens column).

Math drawings are **windows into the minds of students** that allow teachers to understand student approaches and errors on homework and classwork. They enable teachers to do **continual assessment for instruction**. Teachers can always follow up on a math drawing by asking a student to explain it, but this frequently is not even necessary to understand student thinking.

Math drawings are **easier to manage than are manipulatives**. They are not dropped on the floor, thrown at other students, lost, mixed up, taken from the school by last year's teacher, or lost during summer school.

Math drawings are **cheaper than are manipulatives**. Drawings can be made on classroom boards, individual dry erase boards, sheet protectors, or on recycled paper from businesses.

Math drawings **remain after the problem is solved**. They can show the whole action, whereas actions with manipulatives may be over by the time the teacher gets to a particular student or group. Their continuing presence **enhances explanation and reflection**.

The teacher **can collect all math drawings** made on paper after class to overview and reflect on student methods shown that day. This cannot happen with manipulatives.

Many **students take pride in their math drawing creations** and use care in making and in editing their drawings. They have a product at the end of their solution. Math drawings can be collected in student notebooks and used later to review a topic.

Low literacy students need **experience with 2-D representations on paper** (like math drawings) to help understanding pictures and drawings in books.

This table is slightly modified from Fuson, K. C., Atler, T., Roedel, S., & Zaccariello, J. (2009). Building a nurturing, visual, math-talk teaching-learning community to support learning by English language learners and students from backgrounds of poverty. *New England Mathematics Journal, XLI*, 6–16.

Although student math drawings are crucial, there are also other kinds of visual supports that are helpful in classrooms: manipulatives, charts, visuals on Student Activity Book pages, and posters. Benefits of using all kinds of visual supports are outlined in Table 4.

Table 4 *Benefits of a Visual Math-Talk Community*

Math becomes a time for **community building** through Math Talk as students help each other solve problems in partners, in small groups, and the whole class.

Speaking helps the **speaker to clarify his/her own thoughts.**

Listeners understand more by hearing another idea or an explanation; an explanation by a peer may be more adapted to their thinking than a teacher explanation.

Because Math Talk is communication to build understanding together, students are **more involved and engaged** in the class.

Students learn to **speak to a large group**: They learn to speak loudly, clearly, and articulately and stand beside their work and point to parts of it with a pointer as they explain.

Listeners learn to **listen** attentively and thoughtfully, **prepare to assist** when needed, and **prepare to ask questions** (Good Thinker Questions) that support everyone's learning. Helping or asking good questions extends the thinking of the listeners as they adapt their thinking to that of a peer.

The Visual Math-Talk Community expects and enables students to become better **problem solvers and explainers** because peers and the teacher model this.

Students are **empowered** as they learn from each other, ask each other questions to clarify meaning, and have their own thinking valued in the community.

The classroom teacher orchestrates discussions but also directs attention, introduces new vocabulary and notation, models, clarifies and restates, probes and questions, extends, summarizes, and sets classroom expectations. These all **facilitate deeper and more relational learning** by the students (and by the teacher)

Assessment for instruction is on-going as the teacher hears and sees student thinking expressed. This enables teachers and students to provide learning assistance as a student needs it.

Teachers deepen their understanding of math and of various aspects of student thinking about math as they listen, see, understand, and assist problem solving and explaining by the range of students in their class.

The Visual Math-Talk Community provides **differentiated instruction in the whole-class setting** because the methods explained by students range from less to more-advanced. Peers and the teacher help students move to more-advanced methods.

All of these features **support equity**—high expectations and strong support within an accepting and nurturing teaching/learning community where everyone's thinking is valued.

This table is slightly modified from Fuson, K. C., Atler, T., Roedel, S., & Zaccariello, J. (2009). Building a nurturing, visual, math-talk teaching-learning community to support learning by English language learners and students from backgrounds of poverty. *New England Mathematics Journal, XLI*, 6–16.

How Does a Nurturing Math Talk Community Develop Over Time From a Traditional Teacher-Focused Classroom?

We have identified five crucial components in a classroom that work together to describe the math talk community a given teacher builds: teacher role, questioning, explaining mathematical thinking, mathematical representations, and building student responsibility within the community. In this section the focus is on Phase 1 Guided Introducing and Phase 2 Learning Unfolding that use a lot of math talk (see the next section for discussion of these phases). If a particular lesson is Phase 3 building fluency, there will be less math talk.

It takes time and experience to build and maintain a nurturing math talk community. Many teachers begin their learning path by starting in a traditional teacher-directed classroom with only brief answer responses from students. Table 5 shows how such a Level 0 traditional classroom appears with respect to the five components. From a traditional classroom there is a shift in which the classroom community grows to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking. Teachers gradually begin to elicit and support more student thinking and expand student responsibility for the conceptual and communication aspects of the classroom. Levels 1, 2, and 3 are outlined below showing the steps forward for each of the five components. Many teachers have found the descriptions of Level 2 and Level 3 to be helpful in lifting their classrooms to more fully discuss mathematical thinking.

Continue on the next page.

Table 5 *Levels of Math-Talk Learning Community Components*

Levels of Math Talk Learning Community: Teacher and Student Action Trajectories				
Components of the Math Talk Learning Community				
Teacher Role	Questioning	Explaining mathematical thinking	Mathematical representations	Building student responsibility within the community
Overview of shift among Levels 0-3: The classroom community grows to support students' acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking				
Shift from teacher as leader of conversation to students/teacher as co-leaders.	Shift from teacher as questioner to students and teacher as questioners.	Students increasingly explain and articulate their math ideas.	Students increasingly explain their math thinking relying, as needed, on math drawings/ representations.	Students increasingly take responsibility for learning and evaluation of others and self. Math sense becomes the criterion for evaluation.
Level 0: Traditional teacher directed classroom with brief answer responses from students				
Teacher is at the front of the room and dominates conversation.	Teacher is only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher questions focus on correctness. Students provide short answer-focused responses. Teacher may tell answers.	Representations are missing or teacher shows them to students.	Students believe they need to keep ideas to themselves or just provide answers when asked.
Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the Math Talk Community				
Teacher encourages sharing of math ideas and directs speaker to talk to the class, not to the teacher only.	Teacher questions begin to focus on student thinking and less on answers. Only teacher asks questions.	Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in an explanation. Students provide brief descriptions of their thinking in response to teacher probing.	Students learn to create math drawings to depict mathematical thinking.	Students believe their ideas are accepted by the classroom community. They begin to listen to each other supportively and to restate in their own words what another student said.
Level 2: Teacher modeling and helping students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases. Teacher physically begins to move to the side or back of room away from being the sole focus of students.				
Teacher facilitates conversation between students, and encourages students to ask question of one another.	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to teacher probing and	Students label their math drawings so others are able to follow their mathematical thinking.	Students believe they are math learners and that their ideas and the ideas of classmates are important. They listen actively so that they can

	prompting from teacher.	volunteer their thinking. Students begin to defend their answers.		contribute significantly.
Level 3: Teacher as co-teacher and co-learner. Teacher monitors all that occurs, still fully engaged. Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).				
Students carry conversation themselves. Teacher only guides from the periphery of the conversation. Teacher waits for students to clarify thinking of others.	Student-to-student talk is student initiated. Students ask questions and listen to responses. Many questions ask 'why' and call for justification. Teacher questions may still guide discourse.	Teacher follows student explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from the teacher.	Students follow and help shape descriptions of others' math thinking through math drawings and may suggest edits in others' math drawings.	Students believe they are math leaders and can help shape the thinking of others. They help shape others' math thinking in supportive, collegial ways and accept the same.

This table was adapted by Hufferd-Ackles, Fuson, and Sherin (2016) from two versions of related tables in Hufferd-Ackles, Fuson, and Sherin (2004, 2015). S. Friel gave helpful feedback on the adapted table in January 2016.

For Each New Math Topic, Learning Path Teaching in a Math Talk Community Goes Through Four Phases That Move Students From 1) Incomplete Knowing to 2) Knowing to 3) Fluency to 4) Remembering Over Time

There are four phases in teaching a new math topic. For some simpler topics these phases might all be done over one or two days. For more complex topics like multidigit computation, these four phases might require two weeks or more and then months for some students to reach fluency. These four phases are outlined in Table 6 along with the means of responsive assistance that teachers can use in each phase. These Learning Path Topic Phases begin with introductory **Phase 1 Guided Introducing** of a topic where visual models including math drawings are introduced or developed by students and the teacher elicits knowledge that students already have about the topic. In the **Phase 2 Learning Unfolding** phase students explain, discuss, and compare methods with drawings so that the mathematical aspects become explicit. This Phase 2 is the most crucial and often the longest. The Solve and Explain classroom structure (see Table 2) is used frequently in this phase to support student methods, reflection, and discussing commonalities and differences between methods. In **Phase 3 Kneading Knowledge** students develop fluency, usually solving problems without using a visual math drawing. In **Phase 4 students maintain fluency and relate to later topics**. The parts of the *Math Expressions* program used in each phase are listed in Table 6. The Teacher Edition summarizes errors and helpful visual models for discussions.

Continue on the next page.

Table 6 *Learning path phases and the means of responsive assistance primarily used in each phase*

1a. Learning Path Topic Phases

Phases for a Math Topic

1. Guided introducing

Introduce topic, visual models, very short phase

Students share methods.

Teacher elicits solution methods and addresses common errors when necessary.

2. Learning unfolding

Student Activity pages, Solve and Explain classroom structure

Students explain methods with drawings to stimulate correct relating of concepts and symbols.

Model/Show & Instruct/Explain

Students discuss and compare methods so the math aspects become explicit.

Focus: Clarify, Question

Extend: Question, Give Feedback

Teacher models and explains only when necessary.

3. Kneading knowledge

Student Activity Pages, Homework, Quick Practice

Students gain fluency by practicing woth some reflection and some explaining as needed

4. Maintaining fluency & relating to later topics

Remembering pages are cumulative review.

Students review during the rest of the year and occasionally discuss and relate old problems. To new probleems.

Means of Responsive Assistance*

Engage and Involve

All students participate in developing understanding.

Coach

Guide student learning with more-explicit supports.

Model/Show, Instruct/Explain

Clarify, Question, Give Feedback

Formative assessment question helps you decide how to respond – more coaching or move to Phase 3 and Managing

Manage and Coach

Help students monitor, be responsible for, and take ownership of their own learning.

Manage

**Teacher initially models responsive assistance, then EVERYONE builds the classroom community in these ways. Any means can be used at any time.*

Table 6 is adapted from Fuson and Decker (2017) which related the learning path topic phases to the means of assistance. These Four Topic Phases and the Means of Responsive Assistance were introduced and discussed in Murata, A., & Fuson, K. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using ten. *Journal for Research in Mathematics Education. 37* (5), 421-456. Modifications as described above were in Fuson, K. C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a Class Learning Path Model that individualizes within whole-class activities. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 10* (1), 72–91.

The four phases and the means of responsive assistance were introduced by Murata and Fuson (2006) in an analysis of Japanese first graders learning the make-a-ten method, e.g., 8 + 6 can be thought of as 8 + 2 + 4 which is 10 + 4 which is 14. This method is valued in Japan, and considerable time is spent on moving through the four phases in teaching as all children learn this method. The *means of assistance* came from a model of teaching literacy by Tharp and Gallimore (1988, p. 41, 44-70). We found that their definition of teaching fit what I had been doing in the Children's Math Worlds project and what Aki Murata was seeing in Japanese classrooms. We inserted the italicized words to make this definition be: "**Teaching can be said to occur when *responsive* assistance is offered *by more capable others* at points in the Zone of Proximal Development (ZPD) at which performance *and understanding* require**

assistance". The ZPD was introduced by Vygotsky and can be defined as "the distance between the child's actual developmental level and his or her potential development under the guidance of or in collaboration with a more-experienced partner." *Responsive assistance* means that assistance is decreased over time as students need it less, and also that assistance varies across students because some students need more assistance than others.

Two aspects of our definition of teaching above are important to explain. We added the words *by more capable others* to make clear that **a lot of teaching can come from fellow students.** It is the responsibility of the teacher to create a classroom community in which it is clear that students are to help each other understand. The teacher may also need to help students learn to help each other. We have found that in all classrooms some students know how to help sensitively and all others can improve in their understanding of what helping is and is not. For example, some students need to learn that just doing a solution for another student or just telling the answer is not helping. Helping means to support the student to do it alone and understand why. Asking questions is a crucial part of helping. These questions can be as part of the Solve, Explain, Question, Justify activity structure or part of helping a classmate individually. In both of these settings what we call "teacher questions" are important. These are leading questions that help the listener(s) focus on important aspects of the problem or the solution. They are questions that might be asked by a teacher. On my website there is a lovely segment where a student asks a question to focus the explainer on an error in the explanation. When the explainer corrects the error and said what he forgot when he was solving initially, the helping student looked at the teacher triumphantly as if to say, "We are teachers together and I helped him to understand that point."

A second important aspect of the concept of the Zone of Proximal Development is that the **existence in the classroom of students at different places in their understanding contributes to the learning of everyone.** This is pictured in Figure 7 where the class ZPD can be seen as composed of the ZPDs of all of the students. Although each student does have an individual ZPD, many of these are so close together than they can be considered the same in what the students need to learn to advance. We see that students can learn more when different levels are learning together in the same classroom. Less-advanced students can hear and see more-advanced students show and explain methods that are mathematically desirable. Sometimes the explanation of a peer is more understandable by other students than is that of the teacher because some students have just created/learned a given method. Everyone also can learn that their classmates may use different methods and explain with different words. This can deepen the understanding of more-advanced students as they consider how to help classmates understand some method or conceptual issue and as they compare that new method or words to their own usual method or words.

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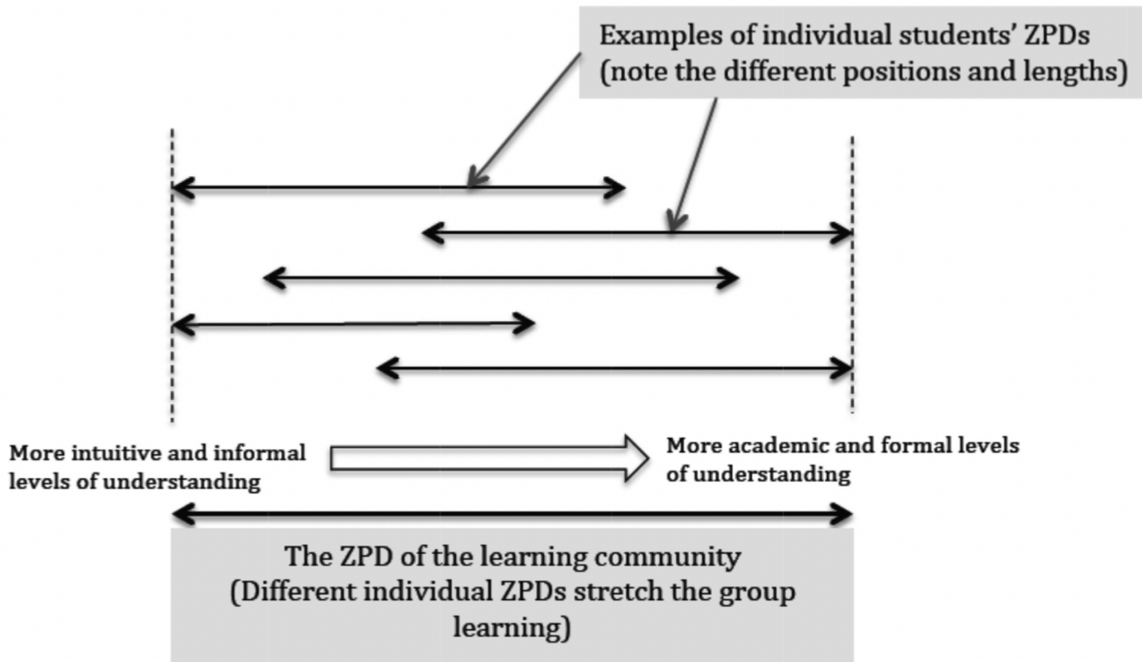

Figure 7 Individual and classrooms Zones of Proximal Development (ZPDs)
This figure is used with permission from Murata, A. (2013). Diversity and high academic expectations without tracking: Inclusively responsive instruction. *The Journal of Learning Sciences, 21* (2). 312-335.

In their literacy work Tharp and Gallimore identified six means of assistance: *managing, modeling, giving feedback, instructing, questioning, and cognitive structuring*. We added a seventh means of assistance for the first Guided Introducing phase: *engage and involve* (Murata and Fuson, 2006). *Engage and involve* includes inviting all students to share ideas and questions, promoting analysis and discussion, and expecting that all students participate in developing understanding together in the community. *Managing* includes helping students monitor, be responsible for, and take ownership of their own learning. We made several other minor modifications to these categories, resulting in the terms shown in Table 5. The definition of *teaching as responsive assisting* and the range of the means of responsive assisting identified in Table 6 emphasizes how much beyond traditional teaching our Math Talk classrooms are. The teacher does a lot more than show and tell.

Table 6 above shows how each of the first three phases has a major means of assistance: *engage and involve* in Phase 1 Guided Introducing, *coaching* in Phase 2 Learning Unfolding, and *managing* in Phase 3 Kneading Knowledge and in Phase 4 Maintaining Fluency and Relating to Later Topics. However any means of assistance can be used at any time. The teacher initially models these means of assistance, and then everyone in the classroom builds the classroom using these. Some students enter the classroom ready to do coaching, but all students can learn these five aspects with support in the Nurturing Math Talk Community.

How Does Learning Path Teaching in a Math Talk Community Differentiate Within Whole-Class Lessons?

To many people differentiating means that groups of students do different things, often doing different sets of learning experiences responding to their perceived levels of understanding. This requires additional time for teachers in planning and in instructing the different groups in the classroom. And this necessarily means that the teacher can only help a few students at a time while other students get no teacher time. Also in many programs one method of solving a problem is taught to students. This taught method may not be the best method for some students, who may do better with a different method.

A different approach to both of these issues is created by the Nurturing Math Talk Community in which the whole class works together much of the time solving and explaining one problem after another. The teacher elicits methods from students, and these methods are discussed and compared. **So students experience a range of different methods from incorrect to slow to mathematically-desirable methods.** Table 7 shows how this range of methods arises within different phases and how the incorrect and slow methods drop out and the mathematically-desirable methods predominate.

A major result of my years of classroom research was identifying **methods that are mathematically desirable but also accessible to students and teachers**. All of the mathematically-desirable and accessible computation methods used in *Math Expressions* were initially developed by students and then tried in a range of classrooms to see how accessible they were. In all cases they are better than the Current Common method, often called the traditional method or the Standard Algorithm. Many people think that the Current Common method is the only correct or acceptable method, but it is not. As discussed above with respect to Figure 6, **there is no one "standard algorithm" that must be taught**. The methods commonly taught have varied over time and places in this country and also worldwide. The standard algorithm is **a collection of different ways of writing a procedure that uses single-digit operations and concepts of place value**. The mathematically-desirable and accessible methods introduced in *Math Expressions* lessons are standard algorithms but without the confusions or difficulties in the Current Common method. The Current Common methods are introduced in *Math Expressions* because many families may know them and show them to their children. Various methods may enter the classroom from families, and these methods all need to be related to drawings so that they are understood and distinguished from other methods. Otherwise students may use an incorrect blend of a method from home and a method from the classroom. Students are welcome to use a method from home as long as they can relate it to a drawing and explain it.

In Learning Path teaching, everyone advances within their own learning path at their own rate. But the methods students develop are limited so they can all be discussed over phases. The major means of assistance are shown on the left with each phase, but refer to Table 6 for more details about the teaching means of assistance.

Continue on the next page.

Table 7 *Differentiated Learning Within the Class Learning Zone Phases: Student Methods for Each Phase and Shifts Across Phases*

Student Methods Used in Learning Path Topic Phases

Topic Phases with Responsive Assistance	Student Methods Used
Phase 1. Guided introducing **with Engage and Involve** *Introduce topic, visual models, very short phase* **Students** share methods. *Teacher elicits solution methods and addresses common errors when necessary.*	Phase 1 methods Methods-with-Errors Concrete & Slow methods Phase 2 methods are possible
Phase 2. Learning unfolding **with Coaching** *Student Activity pages, Solve and Explain classroom structure* **2a, Students** explain methods with drawings to stimulate correct relating of concepts and symbols. **Coach: Model (show) & Instruct/Explain** **2b. Students** discuss and compare methods so the math aspects become explicit. **Coach to focus: Clarify, Question** **Coach to extend: Question, Give Feedback** *Teacher models and explains only when necessary.*	Phase 1 methods begin to disappear Phase 2a Mathematically-Desirable and Accessible (MD&A) methods are the focus Phase 2b Mathematically-Desirable and Accessible methods predominate and become more fluent 2b Current Common ("the standard algorithm") methods discussed
Phase 3. Kneading knowledge **with Manage and Coach** *Student Activity Pages, Homework, Quick Practice* Students gain fluency with reflection and some explaining.	Phase 3 and Phase 4 Fluency with one 2a good MD&A method or one 2b less-good CC method Uses method without a visual model, and some students are fluent with more than one method.
Phase 4. Maintain fluency & relate to later topics **Manage** *Remembering pages are cumulative review* Occasionally discuss and relate old and new methods/concepts.	Methods in Phases 1, 2a, 2b are initially linked to a visual model/math drawing to support understanding.

Students advance along learning paths across lessons, but **they also may advance within a lesson**. We call this the Solve and Discuss Escalator that advances students within a lesson:

Students solve and discuss problem 1.

Then they solve and discuss problem 2 where some students may try a more-advanced method.

Then they solve and discuss problem 3 by which time more students may be ready to try a more-advanced method or may be faster or more secure with the method they chose at first.

Teachers help these advances by making sure that **the mathematically-desirable and accessible methods are explained, discussed, and clarified each time**. As different student methods are shared, discussed, and corrected, students advance in their thinking and also may advance in the method they use. Students follow individual learning paths, but these are connected by the class instructional conversations and methods. Everyone advances in smaller or larger steps. Because there are only a few problem solutions to any given problem, the complexity is limited and is usually not overwhelming to students or the teacher. Because good Math Talk takes time, it is strongly preferred to do fewer problems rather than many except at the end stage of learning a topic, where more problems may be needed to build fluency. During all of the Math Talk, the teacher builds, leads, and focuses the instructional conversation using the Mathematical Practices: Teachers continually assist students to do **math sense-making** about **math structure** using **math drawings** to support **math explaining**.

The Class Learning Path Model is **effective with students from all backgrounds**. But it especially simplifies the teacher's complex tasks in teaching students who must learn English as they are learning math. Learning for everyone initially has context-embedded problems and concrete solution strategies. Students higher in math knowledge will introduce **more-advanced methods** into the classroom discourse, and students higher in English will provide **more-advanced explanations** (that still may need to be extended by the teacher for full explanations). Thus, knowledge of English is modeled by classmates, and all students then need opportunities in class to produce the relevant English words either in oral drills or other vocabulary-focused whole-class activities or in explanations of a solution method. As students gain experience in the topic, the problems become context-reduced and less concrete so as to generalize the math topic concepts. Everyone moves toward being able to solve these context-reduced problems with less concrete solutions and becomes better able to explain their thinking.

The **effectiveness of the Class Learning Path Model** in increasing English performance about math topics was exemplified when students in a school with many students identified as needing bilingual support were interviewed using the state interview of English speaking in academic areas and in everyday language. The outside interviewers were struck by the high levels of students who could explain 2-digit ungrouping in English when they did not even know English words for parts of the body and other everyday English language. The teachers explained that in their Math Talk classrooms **all students were expected to be able to learn to explain their thinking in English**, and that with considerable modeling and help **and belief** that they could, they learned to do so.

The Learning Path Unit Cycle: The Mastery Learning Loop (MLL) to Target Periodic Intervention

It can be rare for all students in a class to have the same level of understanding of a topic at the end of a given lesson. I knew this as I was writing and so I grouped lessons into Big Ideas (lesson chunks) so that later lessons can deepen and extend understanding of earlier lessons. If teachers stop to bring everyone to the same level of mastery after each lesson, the on-level students will be deprived of important grade-level concepts because there will not be time to get to them. So it is very important that **teachers read 5 lessons ahead** to help see how the lessons build on each other and reduce the need for mastery of each lesson before going on. Some concepts and methods take multiple days for most students to understand, so you <u>do not need to stop and review</u>. **JUST KEEP GOING** using Math Talk in the lessons until you reach the Quick Quiz for a Big Idea.

Then give the **Big Idea Quick Quiz** at the end of the last lesson for that Big Idea.

- Analyze the quiz results to plan differentiation and intervention.
- Flexibly group to **differentiate instruction for one day** during which:
  - above-level and on-level students work in self-directing groups on various tasks;

  - o the teacher works with strugglers in a small group using the Quick Quiz results and formative assessment observations from teaching that Big Idea;
  - o students who need more help also attend teacher-led Response-to-Intervention (RtI) meetings 3 times a week.

Differentiate for one day after each Big Idea lesson chunk.

Repeat for each Big Idea in the unit.

Then **give the Unit Test** that is in the Student Activity Book without any review.

- Based on what students missed on the test, **differentiate as above for 1 or 2 days**. Do any more needed remediating during the teacher-led Response-to-Intervention sessions.
- If needed for grading, give Form A of the test in the Teacher Assessment Guide. Use the results on the first Unit Test for those students who did well on that test.
- Students who need help beyond 1 or 2 days attend teacher-led Response-to-Intervention meetings 3 times a week (see below).

Repeat the above for each new unit.

The detailed description of the Mastery Learning Loop has many suggestions for activities for students on level and above to do in their self-directing groups. See the end of this overview of teaching for this detailed description of the Mastery Learning Loop. The Mastery Learning Loop was developed in collaboration with Pam Richards and Robyn Decker.

When students demonstrate a need for **sustained and in-depth Tier 2 or Tier 3 intervention**, then schools need to provide for additional time outside of class for students to receive support and more time with the *Math Expressions* lessons and concepts with which those students are experiencing difficulty. This will usually be most effective when done by the regular classroom teacher, who knows the individual students and their difficulties because of the on-going observation of student work on student activity book pages, homework, and formative assessment questions. Such Tier 2 and Tier 3 out-of-class intervention can be a combination of post-teaching concepts to meet the needs remaining after the Mastery Loop intervention, pre-teaching class lessons to reduce the need for the Mastery Loop intervention, and teaching and practicing concepts from earlier grade levels to support on-level learning. Such interventions usually require at least three 20-minute periods with 3 to 6 students at a time. Tier 3 students may require even more time. After 3 units it is also a good idea to revisit any questions students missed on the unit test even after reteaching and also revisit some early Quick Practices. The latter are likely to be easier than they were originally, giving a good sense of accomplishment. The former usually can be corrected fairly quickly after students have had just a bit of time on them.

Quick Practice and Student Leaders

Figure 8 shows five core structures identified as crucial within *Math Expressions*. Of these, Math Talk, Building Concepts, and Helping Community have been discussed in many parts of this overview especially as they function in the first two phases. Quick Practice and Student Leaders are a bit different. As I was writing, there were key small components of a topic which all students need to be fluent. Some of these needed to be built before they were required in a lesson such as understanding math drawings for hundreds, ten, and ones need to be fluent before using these in computations. Others need to be brought to fluency after initial teaching, and require time for that, such as telling time to five minutes. At all grade levels I identified these special components and designed short 5-minute class activities that practiced these activities. These activities **begin a class and are called Quick Practice**. These involve a crucial role for students called **a Student Leader.** Quick Practices are simple activities that can be led by students even by those in kindergarten. Some more complex Quick Practices use several student leaders doing different steps. Standing up in front of the class and leading it by directing attention and eliciting responses can build confidence in students. We have found that even the shyest students will eventually be able to be good student leaders. It is crucial to build fluency in these chosen components, so **it is important not to skip them**. The activities change every few days because there are many concepts to learn in a given year, and most of them can become fluent enough in a few days.

Daily Routines are used in Kindergarten, Grade 1, and Grade 2. They focus primarily on place value concepts, symbols, and words. They extend over months with gradual extensions because there is so much to learn about place value. Just learning the English number words takes time. Learning what number words mean and how they relate to the written numerals is complex because there are so many of each and because their mathematical structures sometimes differ. We say *fourteen* but write the *four* second as 14. We do not say *four tens* but say *forty,* where the meaning of the *ty* is not clear. We name the place values in words, but just write numerals using relative positions to name their values: *three hundred eighty two* is written 382. Comparing numerals is complex. Why is 82 more than 28 and 346 is more than 289? The Daily Routines bring all students to fluency with these crucial concepts. In Grades 1 and 2 after place value to 1000 is fluent enough, some other concepts are the focus of Daily Routines. Student Leaders play important roles leading these activities. The Daily Routines provide a familiar community activity in which choral responding is often used.

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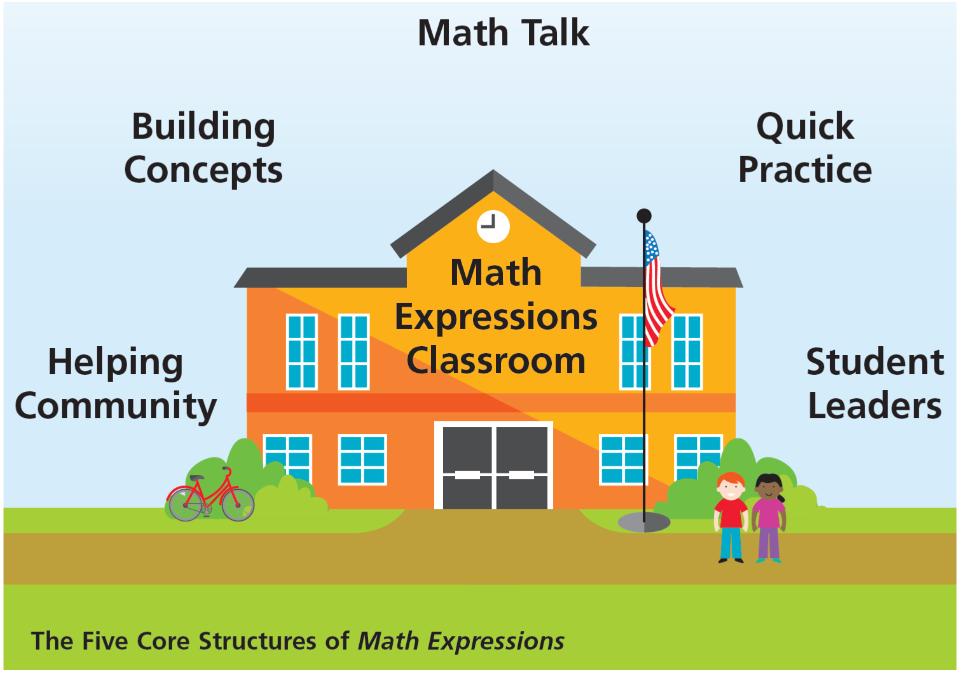

Figure 8 *Five Core Structures of Math Expressions*

Student Leaders can also help other students during classwork. But leading an activity is a special growth experience for students. In one inner-city school a new teacher was given all of the disruptive students in that grade level. Losing the privilege of being a Student Leader was the only thing she found that could enable some of her students to control their behavior. In another class a student who was very shy and had stuttering problems finally volunteered to be a Student Leader. He did a good job, and the class spontaneously applauded when he finished.

Advice From Teachers About Starting, Developing, and Extending a Nurturing, Visual, Math-Talk Community

I worked with experienced *Math Expressions* teachers Todd Atler, Sherri Roedel, and Janet Zaccariello to describe aspects of a nurturing math-talk community in the paper

> Fuson, K. C., Atler, T., Roedel, S., & Zaccariello, J. (2009). Building a nurturing, visual, math-talk teaching-learning community to support learning by English language learners and students from backgrounds of poverty. *New England Mathematics Journal, XLI*, 6–16.

Tables 8 and 9 below contain many of their ideas. This paper is available on my website **karenfusonmath.com**. Tables and figures here from more recent papers of mine show tables that have been extended after this early paper.

Table 8 Starting a Visual Math-Talk Community

Make it safe: Emphasize that Math Talk is not a test. It is helping everyone learn more by talking about their thinking. Emphasize: No making fun of anyone ever.

Community assist: The teacher can relieve pressure for an explainer who appears 'stuck' by asking the explainer if s/he would like help from the group or from a specific student. Students can then begin to ask for such assistance.

Explain with a helper: Shy students can be helped by the presence of a friend with them at the board (the friend may or may not help explain).

Handling mistakes: Emphasize that everyone makes mistakes. The important thing is to learn from our mistakes. Point out or describe your own mistakes and how you learn from them. If a student makes a mistake, you can thank them for helping everyone learn more by discussing that mistake.

Make it safe for the teacher: Remember that you as well as students can say, "I need to think more about that. Let's talk about it tomorrow."

Make the math thinking visible: Children must make some kind of math drawing to show their thinking. This supports understanding by the listeners and promotes meaning. This is very important for equity: less-advanced students and English learners are helped by the math drawing linked to the explanation by pointing.

Use a "Solve, Explain, Question, Justify" structure: As many as possible solve at the board; the rest solve at their seats. Only call on two or three to explain because attention spans for a single problem are limited. Students often explain to each other in pairs before the whole-class explanations and discussion.

Emphasize and assist close and supportive listening: Ask children to repeat what someone said either exactly or in their own words. Lots of rephrasing helps build and practice vocabulary and language.

Provide continual teacher assistance: Engage and involve, manage, and coach by using mixtures of modeling, clarifying, explaining, questioning (probing), and giving feedback. The goal is to help students move to being able to assist others by doing similar coaching in pairs, groups, or whole-class discussions.

Start simple and build up from there: Some students may start out with explaining only one part of their thinking using a few words. The teacher or other students can assist by asking about steps before or after that part and by expanding the sentence and then checking with the student to see that the expansion was correct. It is crucial for students to own their own thinking and to be validated for it.

Table 9 is on the next page.

Table 9 Developing and Extending a Visual Math-Talk Community

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A Visual Math-Talk Community is an inquiry-based learning environment whose continual focus is on sense-making by all participants: Students are expected to understand what they are doing, come to be able to explain their thinking, understand the thinking of other students, learn to seek help when they need it, and help others who need it.

Math Talk is an instructional conversation about the math: It is not just taking turns talking. Teachers need to be sure that the mathematics is clear for everyone, and sometimes stimulate higher-level discussion such as discuss advantages and disadvantages of various methods or how methods are alike and different. Students can often help their peers understand better and can learn to do so even more.

"Bite your tongue" to provide enough wait time: Many students need time to think or develop questions. Many will ask a question or add a comment if you wait.

Help students speak to their classmates rather than to you: Maintain eye contact and attention while moving away to encourage louder speaking. Move to the back of the room so that as the explainer looks at you, and s/he will see and start looking at their classmates. Later, remind students to address each other (not the teacher). This can be done with a silent gesture so as not to interrupt.

Questioners may ask genuine questions they have; they may need assistance to make these clear enough. They may ask "Teacher-y Questions" whose response will help the explainer or the listeners (playing teacher). Or they may participate by asking questions they have heard others ask, either generic for any topic or specific for a topic. Create a "How to Ask Good Questions" Poster with the students and add to it as you go. Include generic questions and questions for particular topics.

Practice Math Talk in pairs/small groups before the whole-class explanations. Sometimes talk about what the room should look/sound like when good math talk is happening (it starts with good listening). Create a Math Talk Poster and add to it.

Post vocabulary: Post a list of relevant vocabulary words and core sentences (e.g., I group ten ones to make one new ten.) to prompt and focus students. Practice reading and saying the words/sentences chorally and individually to build fluency.

Expect and assist students to coach themselves and others by using mixtures of modeling, clarifying, explaining, questioning (probing), and giving feedback.

Students must speak and not just listen: Structure opportunities to explain to a partner and repeat what the partner says, if needed. This repetition can be exact or paraphrased. Students eventually find their own words, but may initially need the security of saying an explanation they know is correct.

This paper summarizes various aspects of a nurturing visual math-talk community within the learning path teaching. The phases within the learning path for each big topic area balance conceptual understanding and fluency by focusing on conceptual understanding first and then on fluency. Fluency is also interwoven throughout the year in the Quick Practices for all grade levels, in the Daily Routines for K, 1, 2, and in the Remembering pages for all grades. The nurturing math-talk community supports all students to develop independence and interdependence as all students have a chance to explain their thinking and to help their classmates with their methods. The visual aspects of the math-talk community support everyone to solve and explain at their own level, thus individualizing within whole-class activities where everyone has the support of the teacher. Students learn to take on the perspective of other students and benefit from seeing a range of solution methods. The research-based accessible and mathematically desirable methods that are seen and discussed by all students in Phase 2 of the learning path enable everyone to progress in their learning paths to good methods. Finally, the Mastery Learning Loop describes how interventions at certain times can give the teacher time to give more support to those who need it and the chance for independent and interdependent work by those who do not need such extra support. My responses to frequent teacher questions at specific grade levels are in the paper Grade Level Teaching Comments on this website in the section *Math Expressions Users*.

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The *Math Expressions* Mastery Learning Loop: Keeping All Students on the Grade-Level Learning Path by Giving More Time and Support in Periodic In-Class Interventions and Out-of-Class Tier 2 & Tier 3 Follow-Up Interventions

Karen Fuson, Pam Richards, and Robyn Decker

Differentiating During Whole-Class Lessons

Math Expressions provides high quality classroom instruction using **a Four-Phase Learning Path Teaching-Learning model implementing a Math Talk Community** in the classroom (see Figure 1 below). This model allows for considerable differentiation within whole-class instruction as students solve and explain a variety of solutions to problems. Teachers in Phase 1 help students learn math drawings for concepts and elicit student methods. In Phase 2 *Math Expressions* provides research-based mathematically desirable and accessible methods that enable less-advanced students to advance rapidly. Math drawings made by students help with meaning-making and explaining. In Phase 3 students practice a general math method without math drawings to obtain fluency. Phase 4 focuses throughout the year on maintaining and integrating fluency as students use the Remembering pages and relate old topics and concepts to new topics/concepts.

Student Methods Used in Learning Path Topic Phases

Phases for a Math Topic

Phase 1. Guided introducing

Introduce topic, very short phase, visual models

Students share methods.

Teacher elicits solution methods and addresses common errors when necessary.

Phase 2. Learning unfolding

Student Activity pages, Solve and Explain classroom structure

2a. Students explain methods with drawings to stimulate correct relating of concepts and symbols.

Model (show) & Instruct/Explain

2b. Students discuss and compare methods so the math aspects become explicit.

Focus: Clarify, Question

Extend: Question, Give Feedback

Teacher models and explains only when necessary.

Phase 3. Kneading knowledge

Student Activity Pages, Homework, Quick Practice

Students gain fluency with reflection and some explaining as needed.

Phase 4. Maintaining fluency & relating to later topics

Remembering pages are cumulative review

Occasionally discuss and relate old and new methods/concepts.

Student Methods Used

Phase 1 methods

Methods-with-Errors

Concrete & Slow methods

Phase 2 methods are possible

Phase 1 methods begin to disappear

Phase 2a good Helping Step methods

and good Compact methods appear

Phase 2b good 2a Helping Step methods

and good Compact methods are still used

2b less-good Compact methods appear

Phase 3 and Phase 4 Fluency with one 2a good Compact method or one 2b less-good Compact method

without a visual model (some students are

fluent with more than one method)

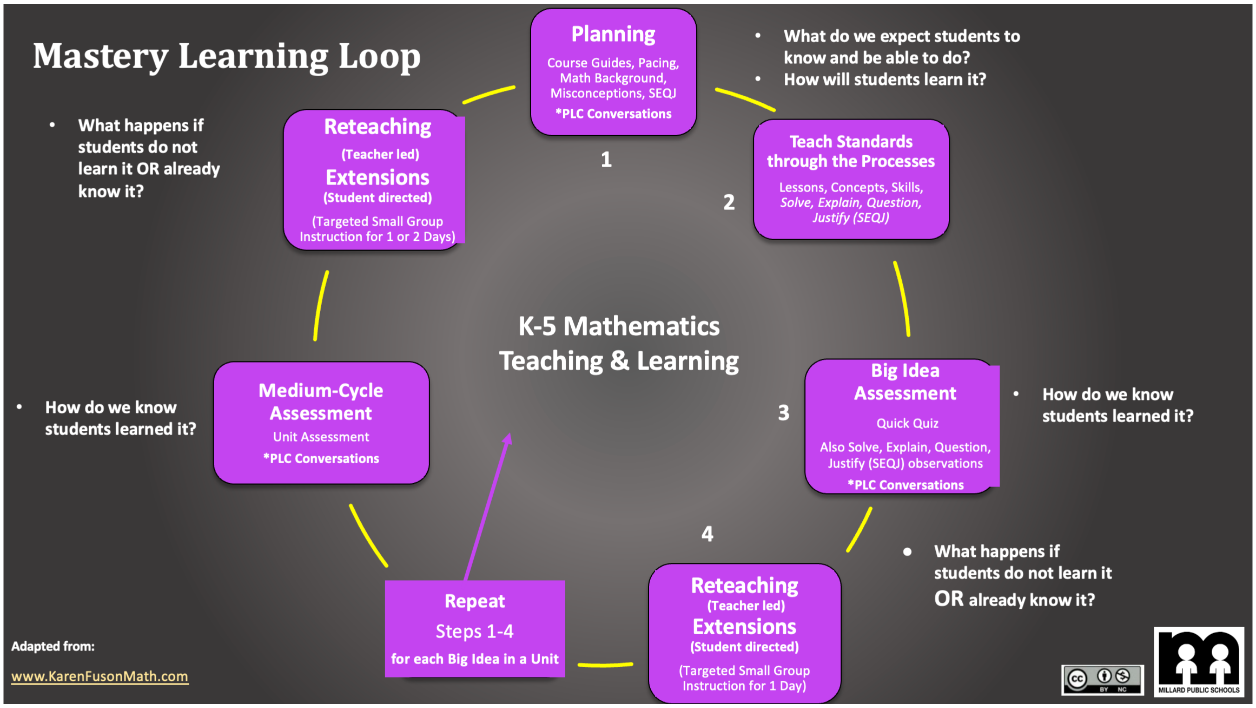
Methods in Phases 1, 2a, 2b are initially linked to a visual model/math drawing to support understanding.

Use the Mastery Learning Loop to Target Intervention Help

The Math Talk Community allows the teacher to do continual formative assessment to modify instruction so as to address errors and extend good mathematical thinking during lessons. But some students need more time and help than others. The *Math Expressions* Mastery Learning Loop was designed to provide **periodic in-class interventions** for students who need this additional support **without slowing down instruction so much** that on-level students cannot learn all of the grade-level content. This approach is usually sufficient to keep most students on grade level throughout a unit.

Math Expressions units are divided into Big Ideas of 3 to 6 lessons. A Quick Quiz is given at the end of each Big Idea. This quiz provides data for the teacher to differentiate the class **in one intervention day at the end of that Big Idea**. The Unit Test in the Student Activity Book given without review provides data for **one or two more intervention days at the end of each unit.**

After each Big Idea Quick Quiz differentiate for one day.
After the unit test differentiate for one or two days.



Teachers need to read 5 lessons ahead to help see how the lessons build on each other and reduce the need for mastery of each lesson before going on. Some concepts and methods take multiple days for most students to understand, so you do not need to stop and review. **JUST KEEP GOING** using Math Talk in the lessons until you reach the Quick Quiz for a Big Idea.

Then give the **Big Idea Quick Quiz** at the end of the last lesson for that Big Idea.
- Analyze the quiz results to plan differentiation and intervention.
- Flexibly group to **differentiate instruction for one day** during which:
  - o above-level and on-level students work in self-directing groups on various tasks (see details below in Differentiation Days and Response to Intervention);
  - o the teacher works with students who need more support in a small group using the Quick Quiz results and formative assessment observations from teaching that Big Idea;
  - o students who need more help also attend teacher-led Response-to-Intervention (RtI) meetings 3 times a week (see below).

Repeat the above steps for each Big Idea in the unit.

Then **give the Unit Test** that is in the Student Activity Book without any review.
- Based on what students missed on the test, **differentiate as above for 1 or 2 days**. Do any more needed remediating during the teacher-led Response-to-Intervention sessions.
- If needed for grading, give Form A of the test in the Teacher Assessment Guide. Use the results on the first Unit Test for those students who did well on that test.
- Students who need help beyond 1 or 2 days attend teacher-led Response-to-Intervention meetings 3 times a week (see below).

Repeat the above for each new unit.

Students who have trouble on the Phase 4 Cumulative Review Remembering pages can attend the Response-to-Intervention teacher-led meetings or get help from peers on those items during the intervention days.

In-Class Intervention Days and Extra Response to Intervention (RtI) Sessions

In the Mastery Learning Loop, an in-class intervention day occurs after each Big Idea and for one or two days after giving the Unit test without review. **On-level and advanced students will work in self-directed activities** alone, in pairs, or small groups on appropriate activities in the learning stations as described below. **The teacher will work with students who need more support** as identified through an item analysis of the quiz and/or by ongoing formative assessment in the classroom. *Math Expressions* has summative unit assessments with multiple forms that identify similar content and strategies. The unit test in the Student Activity Book is given at the end of the unit, and the results—as well as all of the teacher's knowledge about individual students for that unit—are used to determine the intervention levels for another day or two days as needed. Form A of the test in the Teacher Assessment Guide can be used to assess learning after the intervention days. Students who did well on the first unit test do not need to take it again.

When students demonstrate a need for **sustained and in-depth Tier 2 or Tier 3 intervention**, schools need to provide for additional time outside of class for students to receive support and more time with the *Math Expressions* lessons and concepts with which those students are experiencing difficulty. This will usually be most effective **when done by the regular classroom teacher**, who knows the individual students and their difficulties because of the on-going observational of student work on student activity book pages, homework, and formative assessment questions. Such Tier 2 and Tier 3 out-of-class intervention can be a combination of post-teaching concepts to meet the needs remaining after the Mastery Loop intervention, pre-teaching class lessons to reduce the need for the Mastery Loop intervention, and teaching and practicing concepts from earlier grade levels to support on-level learning. Such interventions usually require at least three 20-minute periods with 3 to 6 students at a time. Tier 3 students may require even more time.

The Mastery Learning Loop in-class intervention sessions will contain a mixture of high needs students who often need extra learning time and support and students who missed class or had special difficulties with the given Big Idea concepts on the Quick Quiz. Some students will be able to move fairly quickly during the intervention time to independent practice while others will need more problems and support directed at particular aspects of difficulty (and perhaps later additional out-of-class support). **The teacher plans systematic and explicit instruction focusing on the contextual representations and math talk during the intervention period.** This facilitates diagnosing individual student problems and aspects of needed support. The instruction can use earlier lessons and problems quite closely because many intervention students just need to consider and solve the same or additional problems when help is available.

Small Group and Learning Station Implementation on the Intervention Days

Math Expressions **provides a wide range of resources to support teachers to arrange differentiated activities for on-level and advanced students** in flexible groups using learning stations. During the first Mastery Learning Loop intervention day, the teacher focuses on supporting all students in learning how to engage in purposeful practice and/or enrichment activities. On subsequent intervention days, **on-level and advanced students manage themselves with minimal teacher support** while **the teacher works with struggling students**. Having a whole-class period for the interventions allows students to understand how the learning stations function and experience more than one such station during the class period if they desire. If regular math class is more than 60 minutes, such differentiating can be done more frequently to enable the teacher to work with students who need it between quizzes. This approach should not slow up the rate at which lessons are done by extending lessons over to a second day.

Teachers who have followed the pacing guide for *Math Expressions* and used the Mastery Learning Loop rather than reteaching or differentiating more frequently **have time to finish all needed units before spring testing dates.** Teachers then can use the rest of the year to catch up students who need additional work and to extend and deepen learning for students already on grade level using learning stations.

Math learning stations can occur anywhere in the classroom: small clusters of desks, tables, and even on the floor. Students can work independently, with a partner, or a small group of 3 to 4 collaboratively to use materials that will expand their mathematical thinking. The purpose of each station is to provide activities that reinforce concepts, extend prior instruction, and/or allow students to deepen mathematical understandings.

The number of stations set up varies with class size and student needs. In the first unit there may be two or three stations with the same activity in order to establish routines, management, and teach students how to function independently in work stations. The station time can be shorter, with perhaps one or two rotations for the first few times. With experience, students will become familiar with the station structure, and the amount of independent time and number of rotations can be increased. Students can and should be involved in setting up the stations by finding the resources they need to complete the activities. Additional ideas for using stations or centers in the classroom can be found in the Center Planning Guide in the *Math Expressions* Math Activity Center Tri-fold and in the online version of the Math Activity Center.

On the first day of use of the learning stations**, the teacher introduces the work stations** and can use much of the class period to allow for explanation of the stations, establish the expectations for self-management, and teach students the routine of what to do if they have a question while the teacher is working with other children. This first lesson should be used to:

- model how to use and find materials for using the leveled Activity Cards from the Math Activity Center;
- model how to write a good response to one of the writing prompts found on the back of the leveled Activity Cards from the Math Activity Center;
- discuss where to find materials for completing station work;
- discuss what to do if they finish before it is time to switch to a new station.

After students can function well independently, 10 to 20 minutes of a regular class period can be used with such activities while the teacher catches catch up and helps struggling students. Fewer learning stations might be used in this case. New kinds of work stations used later on in the year may take some extra time to introduce, but the initial expectation is that students will figure out and manage themselves in stations and can propose and design new stations for future days.

Learning Station Ideas

Station activities should allow students to practice problem solving, communicating, and making connections between big mathematical ideas as well as representing mathematics in many ways. Students also need opportunities for choice in how they engage in the mathematics. Many of the resources listed below are included in the *Math Expressions* Math Activity Center Tri-fold and in the online version of the Math Activity Center.

Literature Station

- The Math Readers from the Math Activity Center or the *Math Expressions* Literature Library books can help students broaden their understanding of the mathematical idea or even develop a stronger conceptual understanding of the big idea. Teachers can either use whole group time to read aloud the book (depending on the difficulty of the book) or have the students read the book in the station.
- The Math Readers include questions and activities at the end of each book. The Teacher Guides that come with the Literature Libraries offer suggestions or activities that students can do to engage with the text or content of the book.

Enrichment Station

- Challenge Activity cards and
- Challenge copy masters from the Math Activity Center.

Fluency Station (choose some and vary as needed):

- Strategy Cards
- Check ups
- Quick Practice activities
- Fact fluency triangle cards (Math Mountain cards)
- Board games from the Math Activity Center
- Interactive Games and Fluency Builders from the online Math Activity Center

Vocabulary Station

- Record vocabulary words in a math dictionary, use graphic organizers (for example, the Frayer Model) to define, draw, find similarities and differences, etc.
- Play matching/concentration games with vocabulary and definitions.
- A Vocabulary Game can be found in the Math Activity Center and additional activities are in the back of the Student Activity Book.
- Vocabulary Cards are at the beginning of every unit in the Student Activity Book.
- The free downloadable Study Pop App can also be used to practice vocabulary.
- Use suggestions in the lessons for English Learners.

Writing and Problem Solving Station

- Use the Math Writing Prompts found on the backs of the leveled Activity Cards in the Math Activity Center.
- Create your own word problem and make a drawing and explain the solution on the back.
- Solve or write multi-step problems.

- Anytime problem (Gr. 3-6): Students solve and then write 2 more related Problems of the Day.
- Solve problems written by classmates.
- Use questions from the PARCC, Smarter Balanced, or High Stakes test prep books online (Grades 3 to 6).

Catch up or Helping Station

- Student Activity book page
- Homework or Remembering page (past or present)
- Solve and Explain activities with word problems
- Reteach and Practice copy masters in the Math Activity Center.
- Interactive Response-to-Intervention Tier 1 or 2 in the online Math Activity Center

Place Value Station

- Hundreds board activities- patterns, hundred chart puzzles
- Math Board drawings representing various numbers
- Secret Code Card activities
- Daily Routines for K, 1, 2

The In-Class Interventions

Working with intervention students at the board where each can solve is one good approach because teachers can see student work easily, everyone can group around a given problem as it is explained, and the teacher can offer corrective feedback as students solve the problems. On-level or advanced students can help individuals or ELL students during this activity, differentiating their own thinking as they do the challenging task of helping someone else with their own ways of thinking.

Tier 2 and Tier 3 students may need to solve more problems than the on-level and faster learners, and **they often need help at different individual critical points** where they do not understand some specific concept or idea. Some children just need the extra practice with the teacher available to help (even if the teacher is not needed for most problems), and then the problem solving begins to come together and flow more smoothly and confidently. The intervention can reuse the same problems used in class. This gives confidence to students. Teachers can also make up similar problems or have students begin doing the homework with the teacher available to help. Resources for Tier 2 and 3 intervention can also be found in the online Math Activity Center.