Children Living in Poverty Can Solve CCSS OA Word Problems

Karen C. Fuson and Steven T. Smith

School of Education and Social Policy, Northwestern University, Evanston, IL, USA

karenfusonmath.com

Paper presented at the Annual Conference of the National Council of Teachers of Mathematics, San Francisco, CA, April, 2016.

The PPT used for this presentation is in the section on this website karenfusonmath.com called Visual Presentations. This paper is a written extended version of the data and concepts presented in that presentation. Please also see that PPT and the Teaching Progressions on this website karenfusonmath.com for Operations and Algebraic Thinking, Parts 1 and 2, for more details of the teaching approaches used for this paper. You can also see on karenfusonmath.com classroom videos of some of these learning supports and how they are used in the Classroom Videos (select MENU and Classroom Videos and then C Longer Classroom Teaching Examples Part 3 Grade 2 comparison word problems and D Kindergarten and E Grade 1 single-digit addition and subtraction the parts that say word problems).

Abstract

U.S. children from backgrounds of poverty do more poorly on word problems than do children not from such backgrounds (see research discussed in Cross, Woods, & Schweingruber, 2009). The Common Core State Standards (CCSS) require an ambitious progression of word problem solving in the OA standards. Year-long design experiments found that kindergarten, grade 1, and grade 2 children from backgrounds of poverty could do well on the OA CCSS types of word problems including the new kindergarten decomposing problem K.OA.3 (e.g., $5 = 2 + 3$ and $5 = 4 + 1$). The roles of this problem situation in moving to Level 2 counting on solution methods are discussed, and how grade 1 and grade 2 children used Level 2 decomposing drawings to solve intermediate and difficult word problems is shown in their drawings. These children showed considerable variability in how they used equations and math drawings to represent and solve word problems. This high performance required extended learning path opportunities using and relating different forms of equations, drawings, and decomposing diagrams (math mountains). This learning path is described to permit others to support such learning and to contribute to a dialogue about effective instructional supports for understanding particular math concepts.

Acknowledgements: This research was supported in part by grants from the National Science Foundation REC-9806020 and ESI-9816320. Many thinks to the teachers and children involved in these studies who gave generously of their time and their math thinking and to the hard-working and sensitive research staff involved in this project.

The Common Core State Standards for Operations and Algebraic Thinking (OA) in kindergarten, grade 1, and grade 2 draw heavily on the world-wide research about types of word problem situations that give meanings for addition and subtraction and about the learning progression of solution methods children go through in adding and subtracting single-digit numbers (CCSSO/NGA, 2010). Table 1 displays these OA standards.

Table 1

Kindergarten, Grade 1, and Grade 2 Operations and Algebraic Thinking (OA) Standards

---

K.OA.A Understand addition as putting together and adding to,
and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings[2], sounds (e.g., claps), acting out situations, verbal explanations, *expressions*, or *equations*.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or *equation* (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or *equation*.
5. Fluently add and subtract within 5.

[2]Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

1.OA.A Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and *equations* with a symbol for the unknown number to represent the problem.[2]
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and *equations* with a symbol for the unknown number to represent the problem.

[2]See Glossary, Table 1.

1.OA.B Understand and apply properties of operations and
the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.[3] Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)
4. *Understand subtraction as an unknown-addend problem*. For example, subtract 10 – 8 by finding the number that makes 10 when added to 8. Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).
[3]Students need not use formal terms for these properties.

1.OA.C Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if *equations* involving addition and subtraction are true or false. For example, which of the following *equations* are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.
8. Determine the unknown whole number in an addition or subtraction *equation* relating to three whole numbers. For example, determine the unknown number that makes the *equation* true in each of the *equations* 8 + ? = 11, 5 = □ – 3, 6 + 6 = □.

2.OA.A Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using <u>drawings</u> and *equations* with a symbol for the unknown number to represent the problem.[1] Add and subtract within 20.
2. Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.
[1]See Glossary, Table 1.
[2]See standard 1.OA.6 for a list of mental strategies.

Note. In this table we underlined the word *drawing*, and references to equations are italicized to draw attention to these features in the OA standards. Table 1 referenced above in footnote 2 for 1.OA.A and in footnote 1 for 2.OA.A is shown in Figure 4 of this paper.

Underlinings and italics in Table 1 show three aspects that were not widely present in earlier state standards:

1. Drawings are emphasized throughout as representations of problem situations and of solution methods.
2. Equation forms are varied and are related to situations that give meanings to these varied forms. In particular, the unknown in the usual word problem situations is to be varied across the three situational quantities rather than always having the unknown alone on the right side as the result. There is also a new problem type not emphasized in the earlier U.S. research: Take Apart: Both Addends Unknown (CCSSO/NGA, 2010, Table 1, page 88).
3. The single-digit learning progression of levels of solution methods for adding and subtracting single-digit numbers is included as specific methods (1.OA.6, 2.OA.2), and the three prerequisites for the Level 3 make-a-ten methods are included as specific

kindergarten standards (K.OA and K.OA.4 in Table 1 and K.NBT.1: Compose and decompose numbers from 11 to 19 into ten ones and some further ones).

These aspects of the OA standards raise important interrelated design questions and require modifications of earlier conceptual analyses and proposals for teaching these topics.

This paper focuses on four such questions, each involving conceptual analyses of representations of situations and of solution methods.

- First, what kinds of drawings can be used in the classroom, and how can these conceptual tools be related to various situations and equations?
- Second, how do children use these tools in representing and solving situational and numerical problems and what levels of conceptual relationships among problem situations do these uses reveal?
- Third, given the research about the relatively low levels of performance on word problems by children from backgrounds of poverty (e.g., see the research in Cross, Woods, & Schweingruber, 2009), can children from such backgrounds do well on the OA standards including the new Take Apart: Both Addends Unknown situation if they have a rich and extended opportunity to experience the mathematical content of these standards?
- Fourth, how could the tools and teaching approaches used in this design study be improved and extended to produce higher levels of learning especially for more difficult problems? The conceptual analyses here build from kindergarten to grade 1 to grade 2, and data are presented at each grade level.

Fuson has discussed in various papers the first three questions. She suggested drawings for word problems situations (Fuson, 1988, 1992; Fuson & Willis, 1989); these varied somewhat from paper to paper. She also identified different meanings of the equals sign that varied with the type of problem situation (Fuson, 1988, 1992). She studied and overviewed how to help children move from Level 1 counting all to Level 2 counting on methods to add to find the total and to subtract by finding the unknown total (Fuson, 1986, 1988, 1992; Fuson & Fuson, 1992; Fuson & Secada, 1986; Fuson & Willis, 1988; Secada, Fuson, & Hall, 1983). To update this research, in this paper we summarize more recent approaches and drawings that appear in the math program *Math Expressions* (Fuson, 2006). This program *Math Expressions* was used in year-long design experiments with children from backgrounds of poverty whose results we describe here. We describe aspects of the program relevant to these questions about OA to support understanding of our results. More recent editions of *Math Expressions* have appeared (2009, 2011, 2012, 2013, 2018); these all have used the major aspects of OA approaches discussed in this paper. Certain parts of the approaches were made more extended in these later editions, for example more extensive work on comparison bars was included.

Kindergarten

First we report performance of kindergarten children from backgrounds of poverty on addition and subtraction numeral and word problems including use of drawings and of equations and on the new type of problem Take Apart: Both Addends Unknown (K.OA.3 and 4). Comparison data from Stigler, Lee, and Stephenson (1990) will be given for Japanese, Chinese, and U.S. children from a range of backgrounds for the first tasks. We include the East Asian children because they have on such tasks performed better than U.S. children (Geary, Bow-Thomas, Fan, & Siegler, 1993; Geary, Bow-Thomas, Liu, Siegler, 1996; Stigler, Lee, & Stephenson, 1990), and one goal of the Common Core State Standards is to raise U.S. performance to high international levels. A conceptual analysis of the learning tasks and learning tools for kindergarten follows the presentation of the data results.

Methods

Setting and participants. All *Math Expressions* kindergarten participants came from three schools in a large urban area with high levels of children receiving free lunch, a common criterion for backgrounds of poverty and low socio-economic status (SES) (100%, 96%, and 68% for Schools A, B, and C). All children in School A were native Spanish-speakers; the children participating in the study were from half-day kindergarten bilingual classrooms primarily conducted in English. Children in the full-day classroom in School B were from a range of ethnic and linguistic backgrounds including native English speakers, but they spoke English well enough to be in an English-speaking classroom. Most children in the full-day classrooms in School C were native English-speakers, but some were native speakers of Spanish or other languages. English was the classroom language, but teachers used Spanish occasionally to clarify issues. Schools A and B were from a large urban school district; school C was in a small heterogeneous city bordering on that district. All participants were in classrooms using the research-based program *Math Expressions* (Fuson, 2006) published by Houghton Mifflin.

Tasks, Procedures, and Results

Addition and subtraction word and numeral problems (K.OA.1, 2 and 1.OA.1). The data for addition and subtraction word and numeral and word came from three half-day classrooms in School A, $n = 68$. The Stigler et al. (1990) U.S. grade 1 children were from 20 public and private urban and suburban schools from a range of economic levels in the Chicago area ($n = 240$). The Stigler et al. (1990) Japanese and Chinese (East Asian) grade 1 children were from cities and school districts chosen to be comparable to their U.S. sample ($n = 120$ and 120). So the SES level of all three samples was above that for the *Math Expressions* low SES sample from backgrounds of poverty.

The addition and subtraction word problems and numeral problems used are shown with the results in Table 2. Problems with totals ≤ 10 were given to kindergarten *Math Expressions* children. All items for the *Math Expressions* children were on a whole-class written test given in the spring. The teacher read each item aloud. For the other three samples (Stigler, Lee, & Stevenson, 1990) the two numeral problems were on a fall written test with other items ($n = 750$, 1037, and 976, respectively), so these Grade 1 results are roughly comparable to the spring *Math Expression* kindergarten results. The word problems were given in an individual spring interview and were read to the students, and paper and pencil were available ($n = 120$, 120, and 240, respectively).

Performance is given in Table 2 along with significance levels for the chi-square tests. For the four tasks at the top on problems with totals ≤ 10, chi-square analyses comparing the number of *Math Expressions* kindergarten children answering correctly to the number of the Stigler et al. grade 1 U.S. children answering correctly indicated significantly more correct answerers in the low SES *Math Expressions* sample: 100% vs. 77%, 97% vs. 89%, 81% vs. 52%, and 90% vs. 73%. The performance of the *Math Expressions* kindergarten children was equal to that of the grade 1 East Asian (Japanese and Chinese) children on numeral and word problems, 92% vs. 90% correct averaged over the four tasks.

Table 2

Percentage Correct on Addition and Subtraction Tasks for Grade 1 Stigler et al. Japanese, Chinese, and U.S. Children and Math Expressions Half-Day Kindergarten and Grade 1 Children

Task	Stigler, Lee, Stephenson			*Math Expressions* Half-Day Kinder-garten
	Japanese Grade 1 n = 120	Chinese Grade 1 n = 120	U.S. Grade 1 n = 240	n = 68
Addition 5 + 4	99	96	77***	100
3 + 2 word problem: Joey had 3 marbles and then found 2 more. How many marbles does Joey have now?	98	97	89***	97
Subtraction 9 - 1	80	74	52***	81
6 – 2 word problem: Jan's father gave her 6 cookies. She ate 2 of them. How many did she have left?	93	81	73*	90
				Math Expressions Grade 1 n = 90
Addition 9 + 4 word problem: Some squirrels picked up 9 nuts yesterday and 4 nuts today. How many nuts do they have altogether?	88	76**	64***	90
Subtraction 15 – 9 word problem: There were 15 bunnies. 9 hopped away. How many bunnies were left?	66***	38***	30***	89
Totals ≤10: 5 + 4 and 9 – 1				98

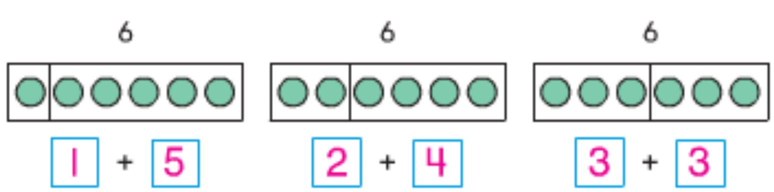
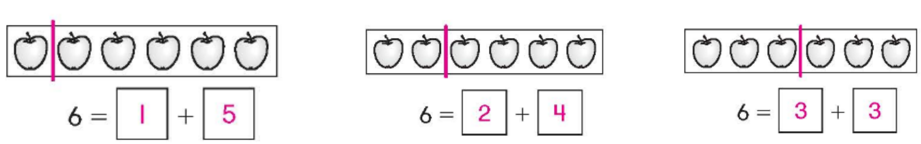
Note. These tasks fall centrally within the following CCSS: K.OA.1, 2 and 1.OA.1. Chi-square analyses on each starred percentage were significant in a 2x2 table of frequencies with the *Math Expressions* frequencies at the 0.001, 0.01, and 0.025 levels for ***, **, and * respectively.

Most (73%) of the *Math Expressions* kindergarten errors were made by children in one of the three classes. The other two kindergarten classes were 99% correct on the addition tasks and 95% correct of the subtraction tasks. In these latter two classes, all of the children made a math drawing for all four problems. In the class making many errors, some children made drawings. Drawings will be discussed in the following section. Of those children giving correct answers on the word problems, 91% of the addition answers and 100% of the subtraction problems had a correct equation written (the directions did not say to write an equation). So kindergarten children can write equations for word problems.

Taking apart a number in more than one way (K.OA.3 and 4). The data for the new type of problem Take Apart: Both Addends Unknown (K.OA.3) were from one kindergarten class in School B and two kindergarten classes in School C. Six items on taking apart a number in more than one way (K.OA.3) were given on end-of-unit tests for Units 3, 4, and 5: $n = 71$ for Units 3 and 4 and $n = 69$ for Unit 5. These tests were given mid-winter, mid-spring, and near the end of the year. The items are shown in Table 3 where the results are given. Items 1 to 5 concern K.OA.3, and item 6 is K.OA.4. Correct performance required that all parts of each multi-part item be correct. Because this task conceptually breaks apart a number into two addends, equations show the total on the left and the addends on the right, as in K.OA.3. Results were high especially considering that all responses for a given item had to be correct.

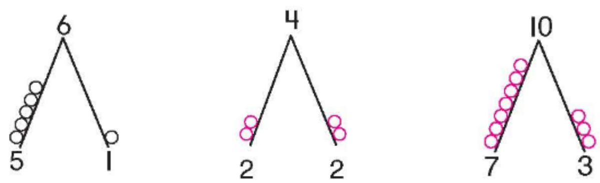
Table 3

Percentage Correct on Partner (Addend) Tasks for Kindergarten Children

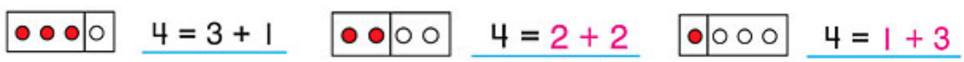
Unit	%	Task
3	90	1. Write the Partners 
4	92	2. Draw a line to show the partners. Write the partners. 

(Table 3 continued)

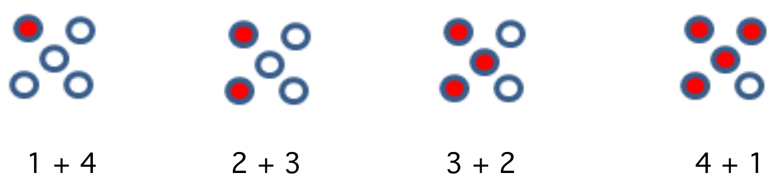
4	92	3. Draw Tiny Tumblers on the Math Mountain



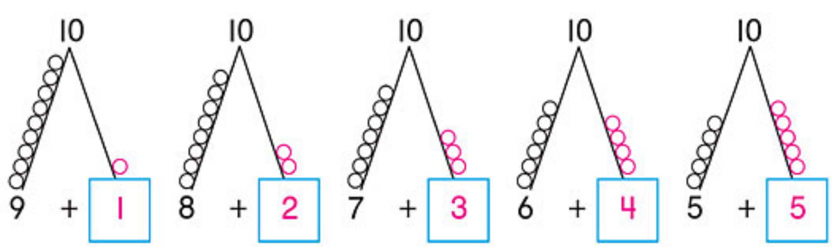
4	85	4. Write the partner equation.



5	88	5. Shade to show all the 5-partners in order. Write the 5-partners.



5	83	6. Draw Tiny Tumblers on the Math Mountain and write the partner.



Note. Items 1 to 5 falls centrally within CCSS K.OA.3 and the sixth item concerns K.OA.4.

Learning Paths and Visual Learning Supports

Teaching-learning practices. The Common Core State Standards for kindergarten and grade 1 drew heavily on the research summarized in the National Research Council's report *Mathematics learning in early childhood: Paths toward excellence and equity* (Cross, Woods, & Schweingruber, 2009). Effective teaching-learning practices from the report are summarized in Table 4 (see next page). Some aspects of this table and of the National Research Council report are discussed in Fuson (2009, 2011). The *Math Expressions* program drew heavily on the research contained in this report, and so used the same teaching-learning practices. This program also used the eight mathematical practices in the Common Core State Standards. These were paired and given names and can be summarized by this sentence: In the classroom focus on *sense-making* about *mathematical structure* using *math drawings* (visual models) to *support math explaining* (MP.1 + 6, MP.7 + 8; MP.4 + 5, MP.2 + 3).

Table 4

Effective Teaching-Learning Practices in Kindergarten and Grade 1

A. The teacher expects and supports *meaning making and mathematizing of* the real world by—

- providing *structured visual settings that connect* mathematical language and symbols to quantities and to actions in the world,
- *leading children's attention* across these crucial aspects to help them see patterns and make connections, and
- *supporting repeated experiences* that give children time and opportunity to build their ideas, develop understanding, and increase fluency.

B. The teacher creates a nurturing and helping *math talk community*—

- within which to *elicit thinking* from students, and
- to help students *explain and help* each other explain and solve problems.

C. For each big mathematics topic, the teacher leads the class through a *research-based learning path* based on children's thinking. This allows the teacher to differentiate instruction within whole-class, small-group, and center-based activities. This path provides the repeated experiences that young children need.

D. Children need to follow up activities with real three-dimensional objects by working with math drawings and other written two-dimensional representations that *support practice and meaning-making with written mathematical symbols.* This supports equity in math literacy. The use of such conceptual-visual-symbolic pages might be called *meaning-making and discussion* pages to emphasize that their use reflects teaching practices A and B above.

Note. This table is adapted from tables in the National Council of Teachers of Mathematics books *Focus in Kindergarten* (2010) and *Focus in Grade 1* (2009). Reston, Va.: NCTM and is used with permission. These tables summarized points in Cross, Christopher T., Taniesha A. Woods, and Heidi Schweingruber (Eds.) (2009). *Mathematics learning in early childhood: Paths toward excellence and equity.* National Research Council, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, D.C.: National Academy Press.

Prerequisite kindergarten learning relating counting words, written numerals, and quantities. Before kindergarten children can add or subtract, they need to learn the counting words in order, use these words to count quantities, and know that the last counted word tells the number of objects counted (K.CC.1, 4, 5). They also need to relate a written numeral to a counted quantity, write numerals, and learn the order of numerals (K.CC.1, 2). For each number, children need to form over time the basic conceptual triad of a *number word* related to its *written numeral* related to *that quantity of things*.

Four large classroom displays were used over time to help children see these conceptual triads and learn the order of words, numerals, and patterned quantities (see Figure 1). The Number Parade was used from the beginning of the year. Children discussed patterns they saw and counted from one to ten while a child pointed to each number and quantity. Figure 2 shows at the top Unit 1 activities used to build the basic triad knowledge and more complex relationships. These activities allow children to function at different levels of difficulty. Some

Figure 1. Poster visual supports for 5-groups and 10-groups in quantities in order

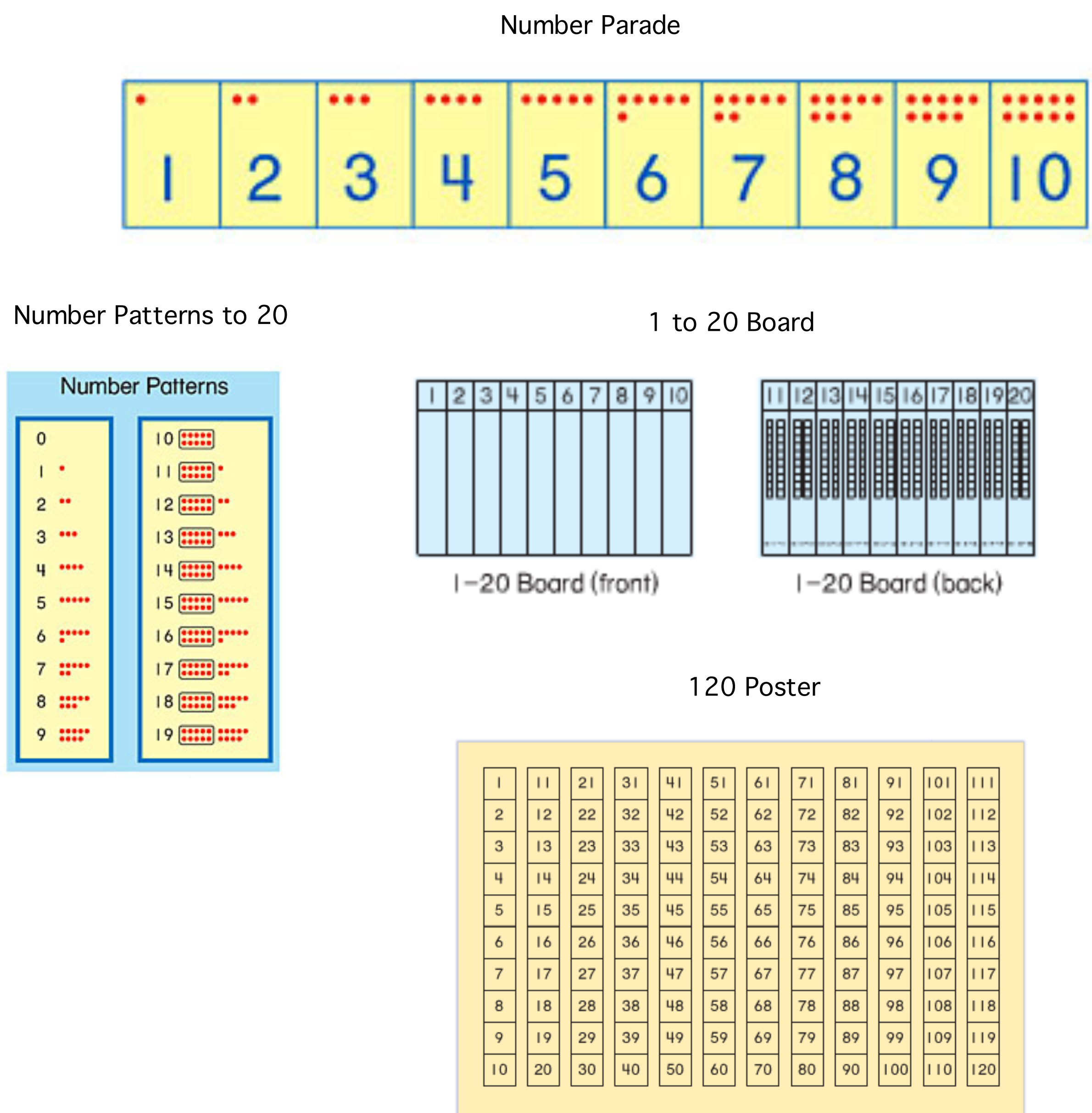
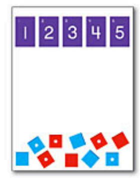


Figure 2. Counting Mat Activities for Triad and Partner Knowledge

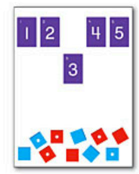
Unit 1: Activities to 5 and then to 10.

A) Put number tiles in order at top and 5 red and 5 blue tiles at bottom.
B) Pull down the number tile for the number said.
C) Show that number of tiles.
D) Have a Math Talk Discussion:
   a. relate the visual quantity to fingers, sounds, and body movements
   b. practice visual imagery (Close your eyes. Visualize.)
   c. describe different arrangements by color, dot/no dot, spatial relationships (e.g., 3 = 2 + 1)
   d. change your arrangement and discuss why you still have 3
   e. copy the arrangement of another person
   f. see partners of numbers already described in c and create new partners
   g. graph on a graph map (2 rows/columns of 10 empty squares).

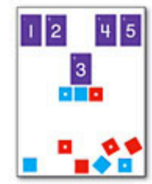
A)



B)

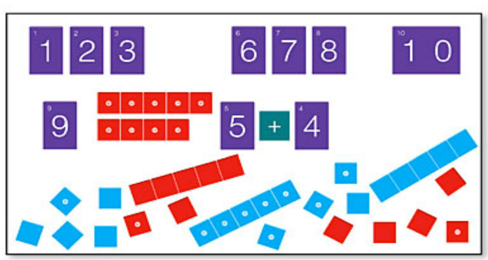


C)



Unit 2: Use 5-groups to show quantities, addition expressions, and total for numbers 6 to 10

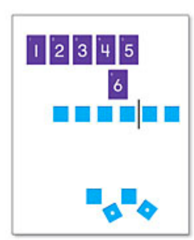
Use Unit 1 Steps A, B, C with a group of 5 and some units:
    one unit of 5 red or blue squares, each with a dot on one side or
    one unit of 5 pennies drawn in squares on a strip.
Children put tiles for the total to the left and for an addition expression for the partners (addends) to the right.



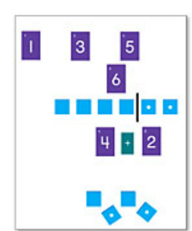
Unit 3: Partners of 2, 3, 4, 5, and 6 with tiles, break-apart stick, total, and addition expression

A) Make a number with a numeral tile and that many things.
B) Elicit partners of that number.
C) Use a break-apart stick to show the partners.
D) Use number tiles and the + tile to show an addition expression for the partners and say the partners:
    Six is four plus two. Show with fingers.
    Teacher writes equation 6 = 4 + 2.
E) Switch the partners with objects, stick, and tiles.
    Teacher writes 6 = 2 + 4 beside 6 = 4 + 2.
F) Repeat for different partners of the number.
G) Repeat all steps with a different number.

A-C)



D)



children were still learning how to put the numbers in order (they could look at the Number Parade on the wall) or how to count out a given number, while others were fast at these and were doing a lot of talking about the more-advanced tasks Dc, Dd, De, and Df (describing parts indetail and making and describing partners of numbers). Sitting near each other allowed children to help each other and to learn from watching what other students did and said. The back of the number tiles had the circle patterns using sub-groups of 5 shown at the top of the Number Parade, so children could be asked to order the tiles by quantity or by numeral.

Children learned and practiced the order of the counting words within triad tasks so that they were always focusing on at least two of the word-numeral-quantity connections. Teaching/learning activities focused on these tasks occurred in three settings: (a) short Quick Practice activities that began the math period and lasted for half of a unit (about 13 days), (b) lesson activities with objects and later with student activity pages that had drawings of things and of circles (or other simple shapes), and (c) homework pages like the student activity pages done in class or at home. Quick Practice and some lesson activities used Giant Number Cards like the Number Parade but with the circles on the back of the cards. To a numeral or a dot pattern, children showed that number of fingers, or made that many sounds and movements like an animal they chose, or used pretend instruments for a marching band and made that many sounds. A Student Leader led many of the Quick Practice activities; these Student Leaders varied over days.

Unit 2 focused on the 5-group patterns for 6 through 10 shown in the Number Parade. A 5-group pattern was used to help children visualize these larger numbers and to facilitate seeing the partner to ten (as specified in CCSS K.OA.4). Children made drawings using the 5-group pattern in all multidigit work in later grades; this helped in showing and understanding composing and decomposing a ten. Children had 5-groups of 5 red and 5 blue squares in a strip (see Figure 2 Unit 2 above), and they also used paper strips with a row of 5 pennies on one side and one nickel on the back. Children made numbers with the blue and red tiles or with the penny strips and pennies using a 5-group tile or penny strip and loose strips and pennies. They also made expressions using a + tile that had a – on the back for later units (see Unit 2 Figure 2 above).

In Unit 2 children discussed the patterns in the Number Patterns to 20 chart (see Figure 1) and worked to connect teen words, written teen numerals, the quantities of ten and some ones shown on the chart and with fingers (ten fingers flashed to the left then some fingers flashed to the right saying *ten and one make eleven*, etc.). They made teen numbers as a ten and some ones with various kinds of objects. The 1 to 20 Board was used in Unit 4 to connect a single vertical group of ten ones and some extra ones. The 120 Poster was used all year to build connections among count words, place value numerals, and quantities. These three visual supports in Figure 1 that go beyond ten reinforced and used the basic triad of number words, numerals, and quantities from 1 to 10, facilitating fluency for all with this knowledge.

Children wrote numerals from 1 to 5 in Unit 1, from 6 to 10 in Unit 2, from 11 to 20 in Unit 3, and 1 to 30 in Unit 4. They began writing numbers to tell how many in Unit 3, and wrote numbers in expressions in Unit 3 and as parts of an equation in Units 4 and 5 for taking apart situations (see Table 3 above) and in Unit 4 for word problems.

Unit observational assessments carried out by teachers found 20 to 30% of the children did not completely master a task in the specified unit, but all children whose attendance was reasonable did master those tasks within 2 months of the unit end. These triad tasks all build over units, and children continued to learn and become more fluent with practice.

There were five major units, with the sixth shorter unit including discussion and practice on experiences that were building all year (money, length units, write to 100, and time). The half-day kindergarten classrooms in this study got through Unit 4 and part of Unit 5, which continued tasks in earlier units and went more deeply into tens and ones and partners for 7, 8, 9.

Kindergarten learning supports for addition and subtraction single-digit word problems and numeral problems. Figure 3 (next page) shows the extensions from the simple triad connections for a single number discussed above to four aspects of addition and subtraction that kindergarten children must learn and then relate to understand addition and subtraction operations. They must

- mathematize real-world situations by focusing on the quantities and mathematical actions (operations),
- learn mathematical language,
- learn written mathematical marks (expressions and equations), and
- understand and make situation or solution representations of the real-world situation.

Learning progressions in mathematizing and in representing are identified as steps moving in the two columns from A up to E (mathematizing) or from A up to D (representing). In these experiences children move from totals to 5 to totals of 6 to 10. In each of the five units there were multiple lesson segments that focused on these word problem activities and the numerical work related to it.

Mathematizing moved from concrete work with familiar situations acted out and shown in pictures to less familiar situations: from family meals to stories in a park situation to a grocery store to any situation. Mathematizing involves considerable language work relating mathematical terms, real-world language, focusing on/relating/extending and retelling a situation, and working especially on the question and varying its language (these are the most challenging for children). Initially the teacher tells or structures the situations, but soon children retell word problems in their own words and then they move to telling their own situations.

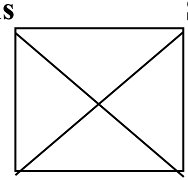
Representing moved from the teacher and children acting out a situation with objects and with fingers to the teacher making math drawings using circles and then the children making math drawings and also solving with fingers. Because kindergarten children can confuse the parts of an equation, initially in the lessons only the teacher wrote an equation, and children made expressions with number tiles (see Figure 2 Unit 2 above). Eventually in the lessons children wrote expressions or equations. However, we found that in some classrooms teachers ignored this careful development and wrote equations from the beginning and had children write them early on.

Figure 3. Learning Progressions in Teaching Level 1 Word Problems in Kindergarten

Children must learn and relate the four aspects of addition and subtraction that label the corners of the square.

Mathematizing Real-World Situations	Totals of	Situation and Solution Repesentations
E) Children tell and retell word problems for any situation; continue to focus on language. ↑		D) Children make math drawings and also solve by fingers. Children write expressions or equations. Teacher writes equations and elicits and summarizes all connections. ↑
D) Continue approaches below using new familiar settings. Retell and work on language and on the question. ↑	6 to 10	
C) Elicit word problems from a familiar setting and focus on/relate/extend the language: Children retell in different words and practice the question. ↑	6 to 10	C) Each child solves with fingers or objects and shows the expression with number tiles. Children share and discuss solutions. Teacher makes math drawings and writes equations and relates all aspects. ↑
B) Mathematize situations from student lives and from pictures of scenes to describe word problem situations and ask questions. ↑	1 to 5	B) Teacher and children act out each situation with objects and fingers. ↑
A) See and draw groups of 1 to 5 things in a real-world scene, then 6 to 10 things.		A) Work on numerical triad relationships for 1 to 5 and then 6 to 10 with 5-groups. o o o o o six 6 o

Mathematizing Real-World Situations **Situation and Solution Repesentations**



Mathematical Language **Written Mathematical Marks**

Five has three and two inside.
Four plus three equals seven.
Seven minus four equals three.
Put together, add, and, in all, total, together, altogether, in the beginning, then, at the end, take away, left, rest

Expressions	Equations	
3 + 2	5 = 3 + 2	
4 + 3	4 + 3 = 7	4 + 3 = □
7 – 4	7 – 4 = 3	7 – 4 = □

Teacher writes → child makes with number and +/- tiles → child writes

Note. After full integrated word problem activities that connect the four aspects, children also begin solving equations where the answer is unknown: 4 + 1 = □ and 5 – 4 = □, first with totals ≤5 and then ≤10.

Notice that all possible connections among the four aspects of learning for word problems are indicated in Figure 3 by the line segments connecting the four aspects. This is a huge amount of necessary specific knowledge for children to build and relate. To adults including teachers the word problems given in kindergarten seem simple and not very different. But there are subtleties in these situations for these young children and especially in the meaning of the equals sign. Many studies have identified such subtleties (see the reviews in Sarama & Clements, 2009; Verschaffel, Greer, & DeCorte, 2007; Fuson, 1992). We overview these here and add conceptual relations with the new type of situation, Take Apart: Both Addends Unknown.

Table 5 (see next page) shows an example of each of the four types of word problems in the OA Common Core State Standards for kindergarten: Add To, Take From, Put Together, and Take Apart. The Add To and Take From situations each involve three action steps over time that create the answer quantity (see the top sections of Table 5). When you add the second addend to the first addend, the total is made and is visible: 4 (+3) → 7. When you take an addend from the initial total, the other addend is made and is visible: 7 (– 4) → 3. For these situations, the = in the equation representing the situation means an arrow: the left side of the equation *becomes* the right side (the result), and the quantities for the left side may no longer be present. This is a restrictive meaning of = but it is a meaning that children do need to learn and that is easy for them because the situations if acted out create this meaning. These three steps in acting out the meaning of Add To and Take From situations give the names Start, Change, and Result to the three quantities involved. Here the structure of the situation and of the solution representations are closely related: acting out the situation by counting or drawing objects produces the solution.

In the Put Together and Take Apart compose/decompose situations shown in the middle rows of Table 5, the roles of the addends are not so different from each other. The addends are two partner quantities that can be put together to make all of the quantities, or all of the quantities can be taken apart to make the two partner quantities. The direct modeling solution representations are the same as those shown in the top row for the three-step action situations Add To and Take From, but only because one must make one addend before the other. Conceptually the meaning of the equals sign in equations representing these situations is "are the same objects as": The 4 girls and the 3 boys are the same as the 7 children. You either think of them as 4 and 3 or as the total 7. The double-headed arrow 4 + 3 ←→ 7 shows this meaning of the 4 and 3 being the same as the 7 and vice versa.

The bottom left section of Table 5 shows the conceptual relationships between Put Together and Take Apart situations as each moving back and forth between both addends and the total. This meaning is shown by what was called in *Math Expressions* the math mountain drawing: one can imagine that the 4 and 3 slide up to be 7 or the 7 slides down and breaks apart to become the two partners 4 and 3. Such situation representations allow children to move rapidly between the total and the addends and begin to conceptualize the addends as embedded in the total. This *Math Expressions* drawing is not in Fuson's earlier papers. Fuson (1988)

Table 5

Level 1 Equation Meanings for Kindergarten Word Problems and Situation/Solution Representations That Show/Count Each of the Three Quantities

Three-Step Action Situations	
Add To: Result Unknown	Take From: Result Unknown
1) Four people were at the table. **4** Count/draw 4: o o o o 1 2 3 4 2) Then my three cousins came. 4 + **3** Count/draw 3 more: o o o o o o o 1 2 3 3) How many people were at my table? Count all of them: o o o o o o o 1 2 3 4 5 6 7 There are seven. 4 + 3 **= 7** 4 (then + 3) becomes 7 4 (+3) → 7	1) Mrs. Garcia had seven lemons at her store. **7** Count/draw 7: o o o o o o o 1 2 3 4 5 6 7 2) My grandfather bought four of them. 7 – **4** Take away 4: o o o → [o o o o] 1 2 3 4 3) How many lemons does Mrs. Garcia have left? Count the rest: o o o 1 2 3 Three are left. 7 – 4 **= 3** 7 (then – 4) becomes 3 7 (– 4) → 3
Compose/Decompose Situations	
Put Together: Total Unknown	Take Apart: Addend Unknown
1) There were four girls and three boys playing at the park. **4 + 3** Count/draw 4 and then 3 more: o o o o o o o 1 2 3 4 1 2 3 2) How many children were playing at the park? Count all of them: o o o o o o o 1 2 3 4 5 6 7 There are seven. 4 + 3 **= 7** 4 + 3 are the same objects as 7 4 + 3 ←→ 7	1) Dad made seven pancakes. **7** Count/draw 7: o o o o o o o 1 2 3 4 5 6 7 2) I got 4 pancakes. My sister got the rest. Separate into 4 and the rest: o o o o o o o 1 2 3 4 3) How many did my sister get? Count the rest. o o o o o o o 1 2 3 My sister gets three. 7 = 4 + 3 7 are the same objects as 4 + 3 7 ←→ 4 + 3
Relating Put Together and Take Apart	Relating Take From and Take Apart
The addends become the total or the total becomes the addends, but they are exactly the same objects in the situation: You think of these objects as addends or as the total. (o o o o) (o o o) (o o o o o o o) (o o o o o o o) (o o o o) (o o o) 4 + 3 ←→ 7 7 ←→ 4 + 3 The legs in the Math Mountain diagram show the putting together or taking apart action. 7 / \ 4 3 Both representations help the addends gradually to become conceptually embedded within the total.	1) Count/draw 7 as in above two situations. 2) Draw through first 4 ~~o o o o~~ o o o to take away 4: 1 2 3 4 3) Count the rest: 1 2 3 Taking away with objects as in the Mrs. Garcia example on the top right leaves only the result addend. It is difficult to reflect on the relationships among the parts of the situation. Drawing through the first circles as above in step 2) lets children see both addends and relate Take From and Take Apart situations as both having a total separated into two addends.

showed the usual Part-Part-All diagram as a rectangle with a horizontal line segment across the middle of the rectangle to make the All at the top, and the bottom two parts separated by a vertical line segment to make the two Parts. Fuson (1992) showed two rectangles each labeled Part followed by an arrow and a rectangle labeled All. The problem with both of these drawings is that the objects or numbers put into the drawing are double the number in the situation: 7 is written or put into the horizontal half, and 4 and 3 into the fourths. Such static diagrams do not show the movement of the 4 and 3 to become the 7 or vice versa. We have seen children become confused by the 14 things in such part-part-whole drawings when only 7 exist in the situation. The math mountain with the implied action in both directions seems stronger conceptually, and we have not seen children having difficulties with it. This drawing will be discussed further in the next section.

For Add To there is not much difference between solving by counting objects or drawing objects. But for Take From, drawing objects is better because drawing can leave the total at the end of the problem and show how the total has been taken apart to make the two addends (see the bottom right step 2 where one draws through the first 4 to take away 4). With such drawings, children can begin to connect the total to the addends in the Take From subtraction situation and thus to relate the Take From to the Take Apart situation as both involving a total separated into addends.

Most of the children given the word problems in Table 2 (discussed above) made simple math drawings using circles. The circles were in some cases elaborated a bit more, for example drawing simple designs on the circles for the marble problem and drawing dots on the circles for the cookie problem . Most of the addition drawings looked like those in the top two left sections of Table 5, with the circles for the second addend in the same row with or without a space or preceded by + or below. In Table 5 we show how taking away the first objects enables children to relate the Take From and Take Apart situations (and also prepare for counting on in grade 1). In the two classes with almost all correct subtraction drawings, 65 of the drawings crossed out objects from the beginning compared to 24 drawings that crossed out objects from the end. In the class with many subtraction errors, the correct drawings were split evenly, 7 to 7. Four subtraction drawings showed correct strategies but the answers given were off by one.

Kindergarten learning supports for taking apart a number in more than one way (K.OA.3 and 4). From the beginning of the year children had experiences seeing a number *broken apart into two partners hiding inside that number*. The words in italics are the words used in *Math Expressions* for such decomposing situations to help children understand this complex task; these words seemed to be helpful for understanding this situation. The Unit 1 counting mat activities discussion topic Dc asked children to describe how they showed a given number as made from two numbers using red/blue or dot/no dot tiles (see the top of Figure 2 above). These began with small numbers such as 3 separated into 1 and 2 that children could subitize (see quickly) and moved to larger numbers where children might begin to see the two partners hiding inside larger totals. The teacher and children also made up and solved Take

Apart: Both Totals Unknown word problems during the Unit 2 word problem activities focused on the four types of word problems in Table 5.

In Unit 3 children focused on break apart actions and representations for particular numbers (see the bottom of Figure 2 above). Children made partners with a break-apart stick and represented these partners with an addition expression using tiles. The teacher represented children's examples by a decomposing equation 6 = 4 + 2 with the total on the left, as in CCSS K.OA.3. As discussed above for Table 5, this helps children form a broader concept of equals instead of the limited "arrow as becomes" idea in Add To/Take From situations. In the third day of such activities, the teacher listed all examples in order as an equation on the board, and children discussed patterns they saw, such as the first addend decreases as the second addend increases. Children began to do reflection sheets with items like Task 1 in Table 3 for numbers 2 through 6. Later they played unknown partner games using the Table 3 Task 1 lay-out. One child made partners and then took one partner, and the other child had to tell the partner that had been taken and make the partner addition expression with the tiles.

Table 3 shows how this Take Apart numerical work increased in difficulty and added new representations across Units 4 and 5. Pictures of things were sometimes used on discussion pages, and children used the square tiles and pennies with a break-apart stick or with physical moving apart. Over the Units 3, 4, and 5 three formats were used: items in a row separated by a break-apart stick with objects and a vertical line segment in drawings, shapes in a pattern separated by shading, and a math mountain drawing (an upside-down V with addends shown as circles on the legs and as numbers at the bottom). Expressions or equations were included in each format. Tasks increased in difficulty in several ways:

a. from writing the partners shown in the drawing in boxes in an expression (with the total written above the row) to filling in the boxes in an equation (from Task 1 to Task 2) and then to writing the whole expression in the equation (Task 4);
b. from being given the partners already separated (by a line segment) in Task 1 to drawing such segments to make partners (Task 2) to shading groups to make partners (Task 5);
c. from showing different partners in any order (Task 2) to showing all possible partners in order (Task 5);
d. from drawing circles on the sides of the math mountain to show the given partners (addends) at the bottom of each side (Task 3) to counting on up to 10 (or knowing or finding the unknown partner to 10 of the partner on the left) and writing in that number and drawing that many on the right (Task 6).

Items such as Task 4 were usually above each other because it was easier to see the numerical patterns across each drawing and equation. Sometimes things were used, and sometimes just circles. Except for Tasks 1 and 3 where children might have just focused on the addends and ignored the totals written above, these tasks supported children to see or count the total and then make the addends (partners) and to move between these addends and totals when discussing answers.

The math mountain drawing shown in Table 3 was introduced in Unit 4 with a story about special math mountains where Tiny Tumblers lived in a cozy house. Each day they would wake up, eat a good breakfast, brush their teeth, do their chores, and then go out to play. Some would play on one side of the mountain and the rest would play on the other side of the mountain. Children drew circles to show how many Tiny Tumblers were on each side of the mountain.

Many partner activities in Unit 4 concentrated on partners of ten (K.OA.4). This focus was continued in Unit 5 with two special classroom displays (see the PPT for this paper in Visual presentations). The Night Sky showed the partners of ten in order using big stars and little stars with an equation below (10 = 1 + 9). The 10-Partner Showcase was made from 10-partner drawings children had made on a page with boxes of two rows of 5 squares. Ten such pages were arranged in order in two rows of five, each with an equation below. Partner switches (commuted pairs like 10 = 1 + 9 and 10 = 9 + 1) were below each other. A student leader covered one partner in an equation and led classmates in the Partner Peek activity by asking for the covered partner (e.g., Six and how many make ten?).

First graders in East Asian schools do take-apart activities for specific numbers to build prerequisites K.OA.3 and K.OA.4 for the make-a-ten methods these children will learn later in grade 1 (Fuson & Kwon, 1992; Fuson & Li, 2009; Murata, 2004; Murata & Fuson, 2006). These *how many and how many* pairs are often recorded with equations and sometimes with a drawing like the math mountain called a Number Bond drawing. This drawing is sometimes used in the teaching of the make-a-ten method to show the breaking apart of the second addend into the number to make ten and the number added to ten, but in textbook materials we found that this drawing is not used to represent or solve word problems.

All of these partner numerical activities help children see partners hiding inside a number. They gain experience quickly shifting from the total to the partners (addends) and back using objects, drawings, equations, and math mountains. They begin to expect numbers to have numbered parts and can generate these, and commonly refer to a number as 'having' numeric parts (specifically having numeric 'partners' that 'make' the total number). They begin to build an elaboration of their basic Level 1 number concepts, representations, and actions on these such that numbers now have parts within them. These experiences are helpful for the coming focus in grade 1 moving from kindergarten three-step Level 1 action solution methods to Level 2 methods requiring the partners to be embedded within the total.

We use the word *addends* in this paper (for example, *addends-within-total*), but *Math Expressions* encouraged teachers to use the word *partners* as well as *addends* because *partners* was a meaningful word to many children. It seemed to "glue" the numbers or parts involved together and help to link them within the total. The *Math Expressions* Teacher Edition explained that *total* was suggested for use in the classroom because of the unfortunate identical sound of *sum* and *some*. Discourse using *some* and *total* within the classroom is clear, whereas discourse using *some* and *sum* is often incomprehensible. Children in grade 2 also learned the word *sum*.

Grade 1

Research has established relative problem difficulties for word problem subtypes that depend on the main type and the unknown (e.g., see summary reviews in Sarama & Clements, 2009; Verschaffel, Greer, & DeCorte, 2007; Fuson, 1992). These can be generally grouped into three levels: easy, intermediate, and difficult. Figure 4 (see next page) shows these three difficulty levels by labelling problem subtypes with grades (K, 1, 2). These are the grades at which the Common Core State Standards OA progression (The Common Core Standards Writing Team, 2011) specifies mastery for those problem subtypes (for totals ≤10 for K and for totals ≤18 for 1 and 2). Situation equations that represent the problem situation are given in the table for intermediate and difficult problems. In these equations the unknown is not alone on the right as it is in the easy problems. For the easy problems (marked by K) the situation equation and the solution equation are the same because the situations give the result at the end of the actions.

We first report the performance of first graders on the easy word problems discussed above for kindergarteners and examine the strategies used to solve numeric addition and subtraction problems. We then examine how first graders in another high poverty district represented and solved word problems of easy and medium difficulty. Of particular interest was the extent to which the tools that students used successfully for Level 1 easy problems – equations, drawings using circles, and math mountain diagrams – could be used for intermediate difficulty problems. We then discuss how children in *Math Expressions* can move from level 1 count all methods to level 2 count on methods for easy addition situations and how the count on conception and tools support and are supported by reresenting and solving the intermediate word problems with unknown addends. The conceptual move to relate subtraction situations to unknown addend situations is also discussed.

(Scroll down to see Figure 4.)

Figure 4. Common Core State Standards Addition and Subtraction Situations by Grade Level

	Result Unknown	Change Unknown	Start Unknown
Add To	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$ **K**	*A* bunnies were sitting on the grass. Some more bunnies hopped there. Then there were *C* bunnies. How many bunnies hopped over to the first two? $A + \square = C$ **1**	Some bunnies were sitting on the grass. *B* more bunnies hopped there. Then there were *C* bunnies. How many bunnies were on the grass before? $\square + B = C$ **2**
Take From	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? $C - B = \square$ **K**	*C* apples were on the table. I ate some apples. Then there were *A* apples. How many apples did I eat? $C - \square = A$ **1**	Some apples were on the table. I ate *B* apples. Then there were *A* apples. How many apples were on the table before? $\square - B = A$ **2**

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	*A* red apples and *B* green apples are on the table. How many apples are on the table? $A + B = \square$ **K**	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$ **K**	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$ **1**

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has *A* apples. Julie has *C* apples. How many more apples does Julie have than Lucy? **1**	*"More" version suggests operation.* Julie has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Julie have? **1**	*"Fewer" version suggests operation.* Lucy has *B* fewer apples than Julie. Julie has *C* apples. How many apples does Lucy have? **1**
	"How many fewer?" version. Lucy has *A* apples. Julie has *C* apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$ **1**	*"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have? $A + B = \square$ **2**	*"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$ **2**

Adapted from the OA Progression (2011) which is adapted from Table 1, page 88, of the Common Core State Standards (CCSS). Used with permission. In that Table 1 the footnote for the new problem situation Take Apart: Both Addends Unknown says "These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as."

Method

Setting and participants. The data for addition and subtraction numeric and easy word problems came from four classrooms in School A (described above for kindergarteners), *n* = 90. Also children were randomly selected from two classrooms for individual interviews concerning their addition and subtraction strategies (14 children from each classroom for *n* = 28).

For the addition, subtraction, and unknown addend equations and easy and medium difficulty word problems, participants came from an elementary school in a rural town with a persistently high poverty rate (since the1980's, when local factories moved out). 65% of the children are on the free lunch program, and 21% of the children had IEP's. The school district was on the state's warning list for not making 'adequate yearly progress' in math and reading. The school used the Responsive Classroom program to create a positive classroom learning environment, so teachers could carry out math talk about student drawings in their classrooms. Performance is reported for all students in the three math classes at first grade (*n* = 53).

Tasks and procedures. The addition and subtraction word problems and numeral problems given are shown with the results in Table 2. Problems with totals between 11 and 18 (teen totals) were given to Grade 1 children. Procedures were as described above for kindergarten.

A spring interview was given to identify solution methods. This consisted of 5 numeric addition problems and their 5 subtraction inverses (2 + 6, 4 + 4, 7 + 8, 6 + 9, 7 + 7; 8 – 2, 8 – 4, 17 – 8, 15 – 9, 14 – 7). One problem below and above 10 was a double (4 + 4 and 7 + 7), and one problem could be solved with a doubles +or-1 method (7 + 8). These problems were said aloud by the interviewer, and the method was observed. Students had pencil and paper available and were asked follow-up questions if the method was not clear. Interviewers were experienced and had been trained to a high level of reliability with respect to coding methods. Methods were coded as Level 1 count all/count left, Level 2 count on/up/down, Level 3 recompose to a related problem (make-a-ten or doubles +or-1), or immediate recall; subtraction problems also could be coded as using a related addition.

On the Unit 3 test in December students were given numeric problems with totals ≤10 in equation format: addition equations, subtraction equations, and unknown addend equations (a + _ = c equation forms).

Unit tests were gathered for the units that had a focus on word problem solving. The unit tests were made by the publisher and were not designed to gather data on all of the intermediate or difficult problem subtypes because these were beyond most state standards at the time. Most children made drawings or wrote more than just an answer; these written responses were considered in the coding. Two coders recursively classified errors using all of the student's written work until there was complete agreement. Errors were first differentiated as being correct strategy or incorrect strategy because it is possible for a child to build a correct problem solving strategy yet make a small counting or calculating error in finding the unknown quantity or writing it in the response box. Building a correct strategy indicates understanding of the problem situation, and the small errors involved are relatively easier to eliminate or reduce when teaching. Further error types were differentiated within correct and incorrect strategy and will be described in the results. Problem types and number sizes in the unit test are given in a note at the bottom of each table. To show the range of problem representations and solutions children drew, we made figures that show children's work for two of the three classes (there is not space to display all three classes on a single page). We chose the two classes that display the fullest range of types of representation, so the reader can see the variation. Drawings of the third class are described briefly in the text.

Results

Addition and subtraction single-digit easy word problems and numeral problems. Performance is given in Table 2 along with significance levels for the chi-square tests. On the two word problems with totals between 11 and 18 (teen totals), chi-square analyses comparing the number of *Math Expressions* grade 1 children answering correctly with the number of the Stigler, Lee, & Stephenson (1990) grade 1 U.S. children answering correctly indicated sigificantly more *Math Expressions* grade 1 correct answerers, 90% vs. 64% for addition and 89% vs. 30% for subtraction. Significantly more of the *Math Expressions* grade 1 children were correct on each of these tasks than were the grade 1 Chinese children, 90% vs. 76% and 89% vs. 38%. On the subtraction but not on the addition task, significantly more of the *Math Expressions* grade 1 children were correct than were the grade 1 Japanese children, 89% vs. 66%

Children were quite accurate on the Grade 1 interviews: 99% correct on addition and 94% on subtraction. For the 4 problems with totals ≤10, 14% of the solution methods could not be clearly identified (they were all rapid). This number was 2% for the problems with totals ≥10. Strategies varied by number size and double/not-double. On the doubles problems, the percentages of children immediately giving the number and saying they knew because it was a double were 82% for 4 + 4 and 64% for 8 – 4 (children said they knew that 4 + 4 was 8); the percentages immediately giving an answer were 61% for 7 + 7 and 43% for 14 – 7 (knowing 7 + 7 = 14). Other methods for 7 + 7 and 14 – 7 were counting on to find the total or counting up to find the unknown addend (32% and 57%). The predominant method for non-doubles problems was Level 2 counting on to find the total or counting up to find the unknown addend: 89% for 7 + 8 and 6 + 9 and 98% for 17 – 8 and 15 – 9. A few children (7%) used a make-a-ten method on these addition problems. For the simplest problems 2 + 6 and 8 – 2, 29% and 7% of the methods were immediate recall. Most of the other methods were the Level 2 methods counting on or up. No child counted down.

Equations and easy and intermediate difficulty non-compare word problems. On the Unit 3 test in December students had for totals ≤10 high levels of correct solutions for addition equations (93%), subtraction equations (89%), and unknown addend equations (88% for a + _ = c equation forms). Chi-square tests indicated that significantly ($p<.05$) more of these *Math Expressions* first graders answered correctly for addition and for subtraction than did the Stigler, Lee, & Stephenson (1990) U.S. first graders: 93% vs. 77% and 89% vs. 52% (see Table 2).

Table 6 shows at the top Grade 1 performance on easy and intermediate word problems by difficulty and number size. Performance on the easy problems with totals ≤10 is high (93% correct answer), and only 3% of these problems showed an incorrect strategy. On the more difficult intermediate problems with totals ≤ 10, 71% of the children show a correct answer, and

Table 6

Grade 1 Answer, Strategy, and Error Results for Easy and Intermediate Non-Compare Word Problems

Problem Difficulty	Correct Answer	Correct Strategy/ Incorrect Answer	Incorrect Strategy/ Incorrect Answer
Totals ≤ 10 (from Unit 3 Test, given in December)			
Easy Problems	93%	4%	3%
Intermediate Problems	71%	18%	11%
Totals 11 to 18 (from Unit 5 Test, given in March)			
Easy Problems	83%	9%	8%
Intermediate Problems	83%	6%	11%

Note: Easy problems were Add-To: Result Unknown, Take-From: Result Unknown, Put-Together/Take-Apart: Total Unknown. Intermediate problems were Put-Together/Take-Apart: Addend Unknown.

an additional 18% have a correct strategy but make small calculating or other non-strategic mistakes. Only 11% show an incorrect strategy. By the March Unit 5 test for totals 11 to 18, correct answer percentages for intermediate problems were higher (83% instead of 71%) and again, only 11% show an incorrect strategy. For easy problems with the higher totals 11 to 18, 83% show correct answers and 8% show incorrect strategy errors. In summary, incorrect strategies were only 3% and 11% for easy and intermediate problems in December for totals ≤ 10, and 8% and 11% for easy and intermediate problems in March for totals 11 to 18. So most children were understanding and representing, and many were also solving all types correctly.

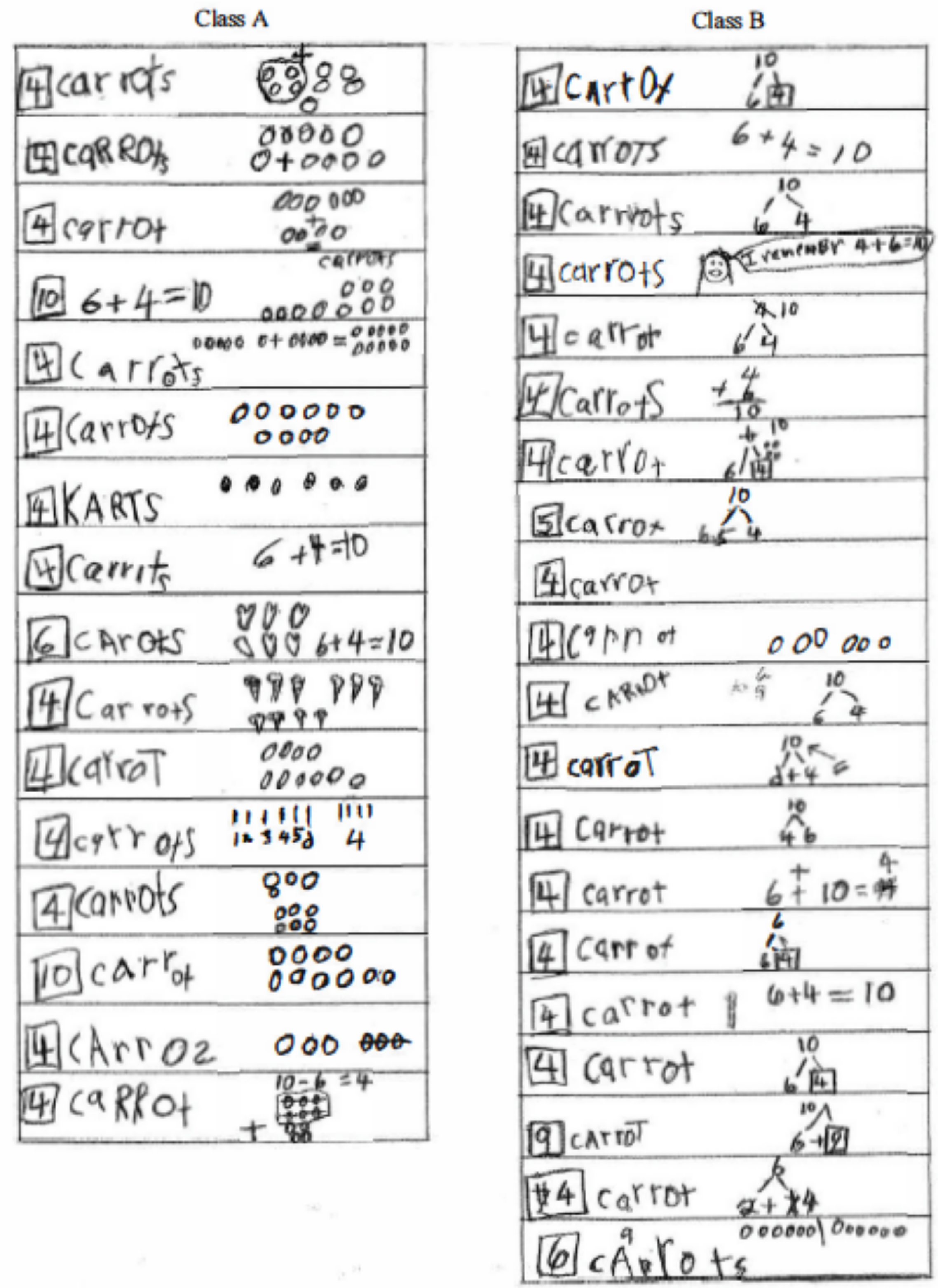
Children made three kinds of correct strategy errors: a calculation error resulting in a close but incorrect answer, miswrote a number from the problem but solved that representation correctly, or entered the wrong answer in the answer box, usually a number from the problem, even though the representation shown had been solved correctly. The total might have been entered because students were more used to problems with the total unknown. Or a student might have remembered that an addend was unknown but forgotten which addend was unknown by the end of problem solving. These errors highlight the importance of putting an unknown box in a equation or a math mountain during initial problem representation to ensure that the needed unknown will be written in the answer box at the end of solving. The major incorrect strategy error was doing the wrong operation (2% to 6% of solutions across problem types). Only one child made more than one such error. There were also infrequent different strategy errors, each made once. All but one of these errors were made by just two children.

Figure 5 (see next page) shows all of the written work for two classes of children on an intermediate difficulty problem Put Together: Addend Unknown. Many children in Class A drew six things and drew four things. These drawings had many different lay-outs, and some used equation symbols like + or =. Several children also wrote situation equations 6 + 4 = 10, and one child wrote a solution equation 10 – 6 = 4. The work was more varied in Class B. More than half the children drew a math mountain, often with a box for the unknown addend. A few children wrote situation equations 6 + 4 = 10, two made circle drawings, one wrote nothing but the answer, and one wrote *I remembered 4 + 6 = 10*. All but one child labeled their answer box as carrots. The third class (not shown in Figure 5 because of space limitations) had the most even distribution of representational types. Four of the 17 drawings represent quantities with a number of circles, but do so in 4 different ways (4 circles separated spatially from 6 circles, 10 circles with 4 crossed out, only 6 circles, and 10 circles shown with no separation). Five of the 17 drawings use math mountains. Three drawings use equations (6 + [4] = 10, 6 + 4 = 10, and 10 – 6 = 4), and four children wrote in the correct answer but made no drawn representation.

The tools children had available--equations, drawings using circles, and the math mountain diagram--were adequate to enable them to represent and solve addend unknown word problems. Most children were able to move from hearing/reading an addend unknown word problem to making a visual representation of that situation using one or more of the written tools. They then could move within that written representation to find the unknown addend and

(Scroll down to see Figure 5.)

Figure 5. Grade 1 Math Drawings and Solutions for the Put Together: Addend Unknown Problem "Rosa picked 6 carrots. Her sister picked some too. Together they picked 10 carrots. How many did Rosa's sister pick?"



(usually) reflect on what was written to find the unknown number and write it into the answer space. This requires the coordination of a great deal of information.

Compare problems. Compare problems are the most difficult type of word problem. The situation and the language are more complex than for the other situations. The Figure 4 compare difference unknown situation example is *Lucy has A apples and Julie has C apples*. These two groups of apples are the only two quantities in the situation. The third quantity, the difference, is created by the problem question *How many more apples does Julie have than Lucy?* This difference quantity is either embedded within the larger quantity as part of it, or it is represented by more added to the smaller quantity to equal the larger quantity. The difference question can also be asked by using the word *fewer* instead of *more* to compare in the opposite direction: *How many fewer apples does Lucy have than Julie?* English language comparisons for the bigger and smaller unknown types have especially difficult language (we use now 7 and 4 instead of A and C). The sentence *Julie has 3 more apples than Lucy* has two kinds of important information in it: Julie has more apples than Lucy, and she has 3 more. Children compared numbers in kindergarten (K.CC.6 and 7), but they did not find how many more or fewer. So many first graders initially do not even hear the number 3 in the question; they only hear that *Julie has more*. They need help seeing and comparing the two quantities in a compare situation.

Common Core State Standards for measurement and data can provide helpful visual supports for representing and solving compare problems. In grade 1 children are to represent and interpret data (1.MD.4) and ask and answer questions about the data including how many more or less are in one category than another. Picture graphs show quantities lined up for easy physical or visual matching. Once two quantities are matched, you can see the *how many more quantity* embedded within the larger quantity because there are no pictures matching those items (they are *the extra things*). Using picture graphs was the approach taken in *Math Expressions*. Picture graphs were on many state standards at the time, but compare problems were not. There were five lessons on picture graphs including answering compare unknown difference questions about the picture graph. Then followed two lessons on picture graphs and tables. Then numeric comparisons were modeled in one lesson with the dot sides of the linear Stairstep manipulative used for various concepts in grade 1 (these were in lengths from 1 to 10 inches, with the inch lengths marked and numbers at the end of the final length). Children then spent one lesson exploring how to model and solve compare word problems given without any picture graphs (three problems to solve and discuss in class and three homework problems).

The grade 1 Unit 6 test had five compare problems with each unknown (difference, bigger, smaller) given at least once; some had totals ≤10 and some had totals between 11 and 18. On these problems 87% of the students had a correct strategy, 76% had a correct answer, and only 13% had an incorrect strategy (more than half of these were in one of the three classes). This percentage of correct strategy (87%) is about the same as children achieved in the intermediate (unknown addend) problems (89%). Children's drawn/written strategies indicated that relatively few of them modeled the comparing situation with a matching representation.

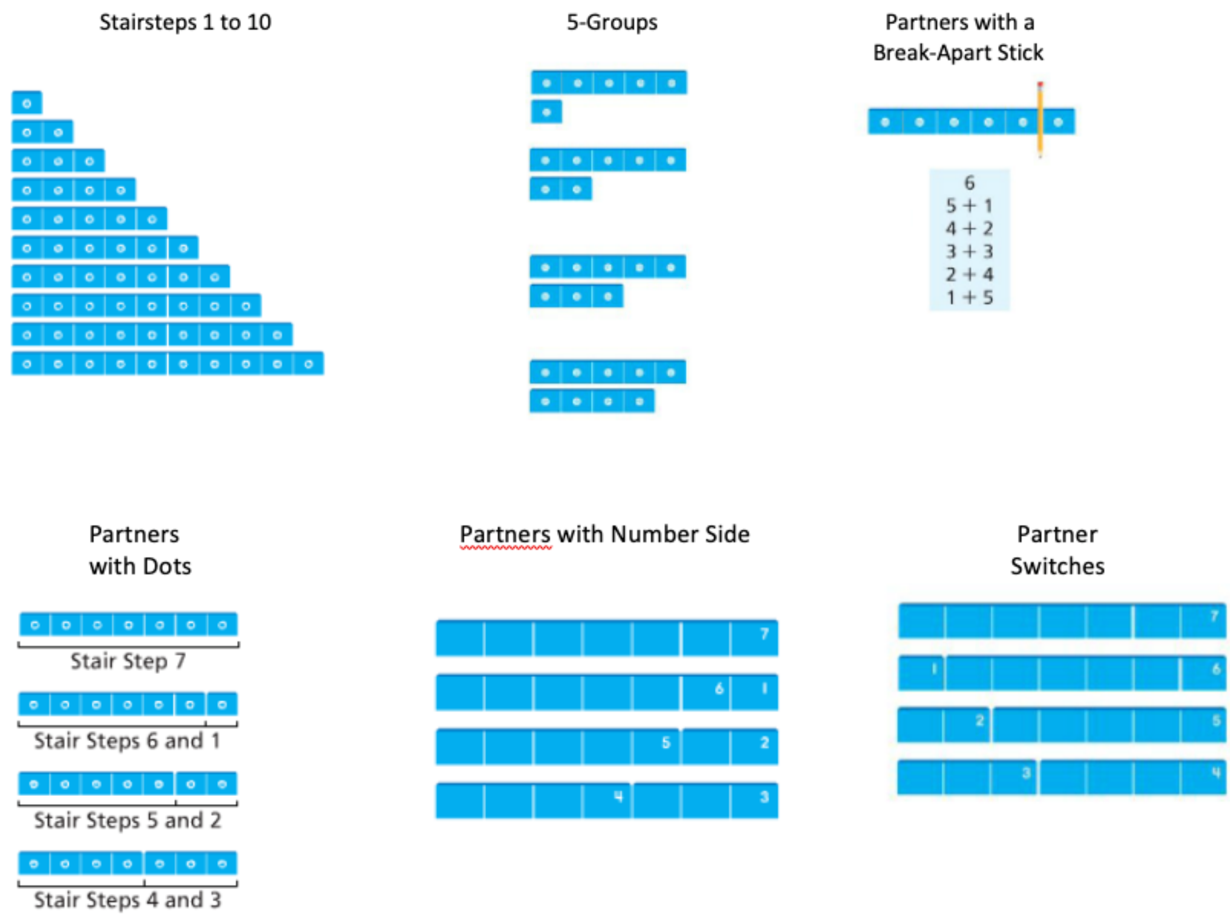
Instead most of them used their Level 2 embedded number understandings about the three quantities in the problem to embed the two smaller numbers (the smaller compared quantity and the difference) within the larger compared quantity. They then used equations or made embedded drawings to solve. They were able to adapt previously familiar and simple equation and object representations to use Level 2 interpretations that place the smaller and the difference quantity within the larger quantity.

Learning Paths and Visual Learning Supports

We know from the interview data reported above that most numeric problems were solved by the *Math Expressions* grade 1 children by counting on to find the total (addition) or to find the unknown addend (subtraction). These methods were taught conceptually and practiced in *Math Expressions* in three phases. The first phase concentrated on numerical situations and on counting on in easy addition word problem situations. Table 7 (see next page) shows how the Level 1 methods that represent each of the three quantities separately can progress to become the Level 2 counting on methods. Repeated experiencing of Level 1 three-step counting all enables children to internalize and abbreviate to just the final part of the counting all of the total to do Level 2 counting on (shown on the top of Table 7). Children know the first addend, shift from the cardinal meaning of that addend (here, 4) to the counting meaning, and begin the final total count with that addend. Conceptually this requires that the counter embeds both addends within the total to consider them simultaneously (shown by the brackets around each addend 4 and 3): the 4 and the 3 compose the total in that final count, but they also exist as separate addends within the final count of the total.

Figure 6 shows a new grade 1 manipulative that quickly summarizes key concepts for children who did not have *Math Expressions* kindergarten and also provides a lead into lengths (CCSS 1.MD.2). The Stairsteps are blue foam strips with inch-long segments. A dot is on one side to make each segment an easily countable unit, and on the other side numbers are at the end of each strip. Children use Stairsteps to show how the numbers from 1 to 10 increase by one unit, to show numbers 6 to 10 as 5-groups, to break apart a number into its partners and discuss that pattern, and to show all of the partner pairs of a number. All of the strips can be placed on top of each other in order (smallest on top) aligned on the left; this creates an inch ruler made from unit lengths which supports Grade 1 Common Core State Standards length standards (1.MD.1 & 2). Partners and partner switches can also be seen with this number side. Stairsteps also are used to support make-a-ten methods with teen totals for adding and for finding an unknown addend in addition or subtraction situations. Children also use number cards to discuss patterns in the counting sequence that produce partner pairs. All of these activities, like the partner activities in kindergarten, help children form conceptions of a number as having two addends (partners) embedded within it and examine specific numbers involved for totals ≤10. This conception of embedded numbers (the partners hiding inside a given number) is crucial for moving to Level 2 counting on.

Figure 6. G1 Stairsteps for Levels 1, 2, 3 Adding/ Subtracting



Level 2 counting on from an addend requires some method of keeping track of the second addend being counted on so that the counter can stop when that addend has been counted. This can be done by drawing circles as one counts on and having a feedback loop that stops the counting when the counter sees 3 (top left in Table 7). The last word counted 7 tells the total. Children move on to a method in which the counting words have become the objects representing the addends (see Table 7 top right). Then the second addend counted on is matched to fingers or head bobs or a mental image, stopping when 3 has been matched to the words counted on (5, 6, 7).

Counting on is introduced conceptually in *Math Expressions* grade 1 Unit 2 for totals ≤10 by discussing with children whether it is necessary to count the first addend in the final count of all of the objects; some children usually point out that they can just do the final part of the counting. They can "trust that number" and count on from it keeping track of how many they have counted on to know when to stop. Children then use small cards for numbers 1 through 10 with circles on the back in the patterns as shown in Figure 2 for the Number Parade. They make an addition with two cards, count all of the circles, and then count on by turning over the first

Table 7

Level 2 Count On Methods to Add or Subtract for the Easy Kindergarten Word Problems

Add To: Result Unknown: Four people were at the table. Then my three cousins came. How many people were at my table? 4 (+3) → ☐ Put Together: Total Unknown: There were four girls and three boys playing at the park. How many children were playing at the park? 4 + 3 ←→ ☐	
Conceptually embed addends within total: ([4][3]) so can abbreviate the final count all of objects to count on 3 more from 4 to find the total 7	
Draw 3 circles as say 5 6 7. Stop when have made 3 (keep track of second addend). I have already counted 4, [ ] [o o o] **4** 5 6 7 Answer is last counted ([ ] [o o o]) word seven. **7**	Count words can be the counted objects. Keep track of second addend by raising a finger with each word or by head bobs or some other method matching the words said. I have already counted 4, ([4][3]) **4** 5 6 7 Answer is last counted ([4][3]) word seven. **7**
Take From: Result Unknown: Mrs. Garcia had seven lemons at her store. My grandfather bought four of them. How many lemons does Mrs. Garcia have left? 7 (– 4) → ☐ Take Apart a Total to Make Any (All) Addends: Dad made seven pancakes. I got four and my little sister Toni got the rest. How many did my sister get? 7 ←→ 4 + ☐	
Conceptually embed addends within total: ([4][☐]) so can abbreviate the final count all of objects to count on some more from 4 to the total 7 to find the unknown addend 3	
7 – 4 = ☐ and 7 = 4 + ☐ 7 / \ 4 ☐ Count on to find <u>an addend</u> looks and sounds like count on to find the total but you stop when you hear the total 7. I took 4 from 7 ([ ] [o o o]) Or I got 4 of the 7 **4** 5 6 7 The unknown addend is the number of objects you have drawn, three. 7 – 4 = **3** and 7 = 4 + **3**	Count words can be the counted objects. Keep track of the second addend by raising a finger with each word or by head bobs or some other method matching the words said. Stop when you hear 7. I took 4 from 7 ([4][☐]) Or I got 4 of the 7 **4** 5 6 7 Answer is the number of fingers you raised or head bobs or other method to track the unknown addend: **3** words counted on
7 has 4 and 3 hiding inside. A focus on three numbers helps to relate the other tools.	

card to show just the numeral and count on from that number. Problems initially all have the larger number given first. Later children discuss how it is faster to count on from the larger number even if it is given second. Children use conceptual Addition Count On flash cards later in the unit to practice for totals ≤10; these cards show counting on (a number and then dots for the smaller addend) as well as the answer on the back so children can do supported counting on.

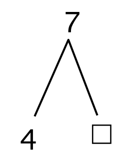
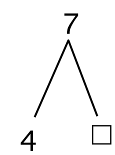
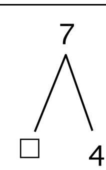
In Unit 2 children discuss the kindergarten easy addition and subtraction situations and represent them with expressions and equations. For both operations they see pictures and later circles in two groups with the objects all in a row. The addition expression is written above and the total below to support flexible relating of these components of an equation. Addition and subtraction equations are written in the usual way with the result alone on the right because this models the three-step actions for counting all solutions. Children then discuss subtracting situations, expressions, and equations by drawing a horizontal segment through pictures/circles at the beginning of the row as in Table 5 bottom right. This allows children to see both partners (addends) and so supports a Level 2 conception of addends-within-total. Subtraction stories are told about such drawings. The total is written above and the two addends below as in the math mountain.

The second phase of Level 2 counting on related the counting on done for numeric and easy addition problems to the more difficult Add To and Put Together problems with an unknown addend (see the top of Table 8 on the next page). Children start Unit 3 by seeing math mountains that show an unknown addend. Children discuss how to find the unknown addend. Some count on from the known addend and draw circles under the unknown box, stopping when they reach the known total (see Table 8 top right). Others use count words as the counted objects and keep track by raising a finger with each word (or use head bobs or some other method) for the unknown addend. Children then use various methods to represent such problems and practice counting on to find the unknown addend. In Figure 5 children made circle drawings like those on the left of Table 8, and they made math mountains and equations to find the unknown addend. For several lessons children hear and tell unknown addend situations and see, write, and solve unknown addend equations such as 5 + ☐ = 9. Then they use conceptual Unknown Addend Count On flash cards to practice finding unknown addends (e.g., 4 + ☐ = 7). These cards have a number and dots for the second addend on the back to practice counting on if needed.

The third phase of Level 2 counting on begins by relating subtraction situations and drawings to unknown addend situations and to counting on to find the *unknown addend*. For subtraction situations, children in the U.S. often move to Level 2 by counting back from the total. But this is difficult and prone to errors (see the research summarized in Fuson, 1992). In many countries children instead learn to think of subtracting situations as finding an unknown addend, as described in the Common Core State Standard 1.OA.4. Fuson found that using the forward counting on method to find the unknown addend in a subtraction problem reduced errors and was easier for children (Fuson, 1986; Fuson & Fuson, 1992; Fuson & Willis, 1988). This approach

Table 8

Types of and Representations for Intermediate Change or Addend Unknown Word Problems Using Level 2 Addends-Within-Total Conceptions

Add To: Change Unknown: Four people were at the table. Then some cousins came. Now there are seven people at my table. How many cousins came? 4 (+☐)→ 7 **A + A = Total** Put Together: Addend Unknown: There were four girls and some boys playing at the park. Seven children were playing at the park. How many boys were playing at the park? 4 + ☐ ←→ 7 **A + A = Total**	
1) Count/draw 4. ([o o o o] [ ]) 1 2 3 4 2) Draw/count more to make 7. ([o o o o] [o o o]) 5 6 **7** 3) Count/see second addend: **3** ([o o o o] [o o o]) 1 2 **3** 4 + ☐ = 7 and 7 = 4 + ☐	1) Count on to make an addend. ([ ] [o o o] **4** 5 6 7 2) Count/see second addend: **3** ([ ] [o o o]) 1 2 **3**  4 + ☐ = 7 and 7 = 4 + ☐
Take Apart: Addend Unknown: Dad made seven pancakes. I got four and my little sister Toni got the rest. How many did my sister get? 7 ←→ 4 + ☐ **Total = A + A**	
1) Count/draw 7. (o o o o o o o) 1 2 3 4 5 6 7 2) Separate into 4 and some. ([o o o o] | [o o o]) 1 2 3 4 3) Count/see second addend: **3** ([o o o o] | [o o o]) 1 2 **3** 7 = 4 + ☐	Counting on is the same as above.  Situation equation 7 = 4 + ☐ Solution equation 4 + ☐ = 7 or 7 – 4 = ☐
Take From: Change Unknown: Mrs. Garcia had seven lemons at her store. My grandfather bought some of them. Mrs. Garcia has four lemons left. How many lemons did my grandfather buy? 7 (– ☐) → 4 **Total – A = A**	
Representations are the same as above. Situation equation 7 (– ☐) → 4 Solution equation 4 + ☐ = 7 or 7 – 4 = ☐	Counting on is the same as above.  Situation equation 7 (– ☐) → 4 Solution equation 7 – 4 = ☐ or 4 + ☐ = 7

was taken in *Math Expressions*. Table 5 at the bottom right shows how a drawing can support such counting on to find an addend even in a taking away situation; one draws a line segment through the first objects. Then children can move to just thinking of taking away as shown on the bottom of Table 7. The child can think *From 7 I took 4, so 5, 6, 7 tells me how many are left. I see three circles (or see 3 fingers raised or feel 3 head bobs). So 4 plus 3 is 7 and 7 minus 4 is*

3. Children make math mountains as shown on the bottom left of Table 7 and show and discuss how these relate to the subtraction and unknown addend equations. Children solve Take From and Take Apart subtraction situations by using these visual model and then finding the unknown addend. They then practice using conceptual Subtraction Count On flash cards that have subtraction equations on both sides but the known addend and dots for the unknown addend on one side to support counting on to the known total to find the unknown addend.

The Level 2 counting on methods for addition and subtraction sound identical when they are spoken aloud. What is different is whether the child is stopping when the second addend has been counted on (adding by counting on to find an unknown total) or when the total has been reached (subtracting by counting on to find an unknown addend). Subtracting by counting on (see Table 7 bottom left) uses an easier method of keeping track than does counting on for adding: one just listens for the total 7 rather than monitoring the size of the second addend *(Have I made 3 yet?*). But finding the answer, the unknown addend, can be more difficult because one has to focus on the addend one made and not the total. Counting on to find the unknown addend can also be done using the count words as objects (see Table 7 bottom right) and keeping track with fingers or some other method.

In Unit 4 children discuss and practice counting on for addition and for subtraction for single-digit totals 11 to 18. Level 2 counting on is crucial for solving such problems rapidly because such problems cannot be put on the fingers easily using Level 1 counting of all three quantities. The interviews indicated that almost all children counted on to solve the subtraction problems; no child counted back, so counting on teaching was successful.

Students also discuss and experience make-a-ten methods (1.OA.6) for adding by seeing a counting on situation (e.g., 9 followed by 5 circles) turned into a make-a-ten solution by encircling the 9 and one circle and writing 10 above it, and then writing the new problem below: 10 + 4 = 14. Grade 1 children who did not have the extensive kindergarten experience with tens as teens were helped to build this understanding in quick Daily Routines in Unit 3. Children used Addition Make-a-Ten flash cards that have a number and dots on the back like the Count On flashcards (e.g., 9 and 5 dots), but these dots for the second addend are separated into the amount to make ten with the known addend numbers and the rest (e.g., 9 o oooo). The new problem 10 + 4 is written below this. In Unit 5 make-a-ten methods are discussed for unknown addend problems and subtraction problems, and all make-a-ten methods are discussed and related. Children later practice with Unknown Addend and Subtraction Make-a-Ten flashcards with visual models like those for addition.

Grade 2

Method

Setting and participants. All participants came from the same school as described above for the medium word problems in grade 1. Performance is reported for all students in the three math classes at second grade (n = 56).

Tasks and procedures. These were the same as described above for the equations and easy and medium word problems in grade 1 except that difficult word problems were also given.

Results

Easy, intermediate, and difficult non-compare problems. Table 9 shows Grade 2 performance on easy, intermediate, and difficult non-compare word problems for totals 11 to 18. Performance on easy and intermediate problems was high (96% and 89%), with only a mean of 5% using an incorrect strategy. Incorrect strategies rose to 20% for difficult problems. These were mostly wrong operation errors. Every student had a correct strategy on at least one intermediate or difficult problem, and most students making a wrong operation error did so on only one problem. The error analysis indicated that the grade 1 errors of finding the answer but writing a different number in the answer box had practically disappeared. Most correct strategy/incorrect answer responses resulted from a minor calculation error. Students at the beginning of the year on the Unit 1 test showed high levels of correct solution of equations (half with totals ≤10 and half with totals 11 through 18): 99% for addition equations, 96% for subtraction equations, and 96% for unknown addend equations (a + _ = c equation forms).

Table 9

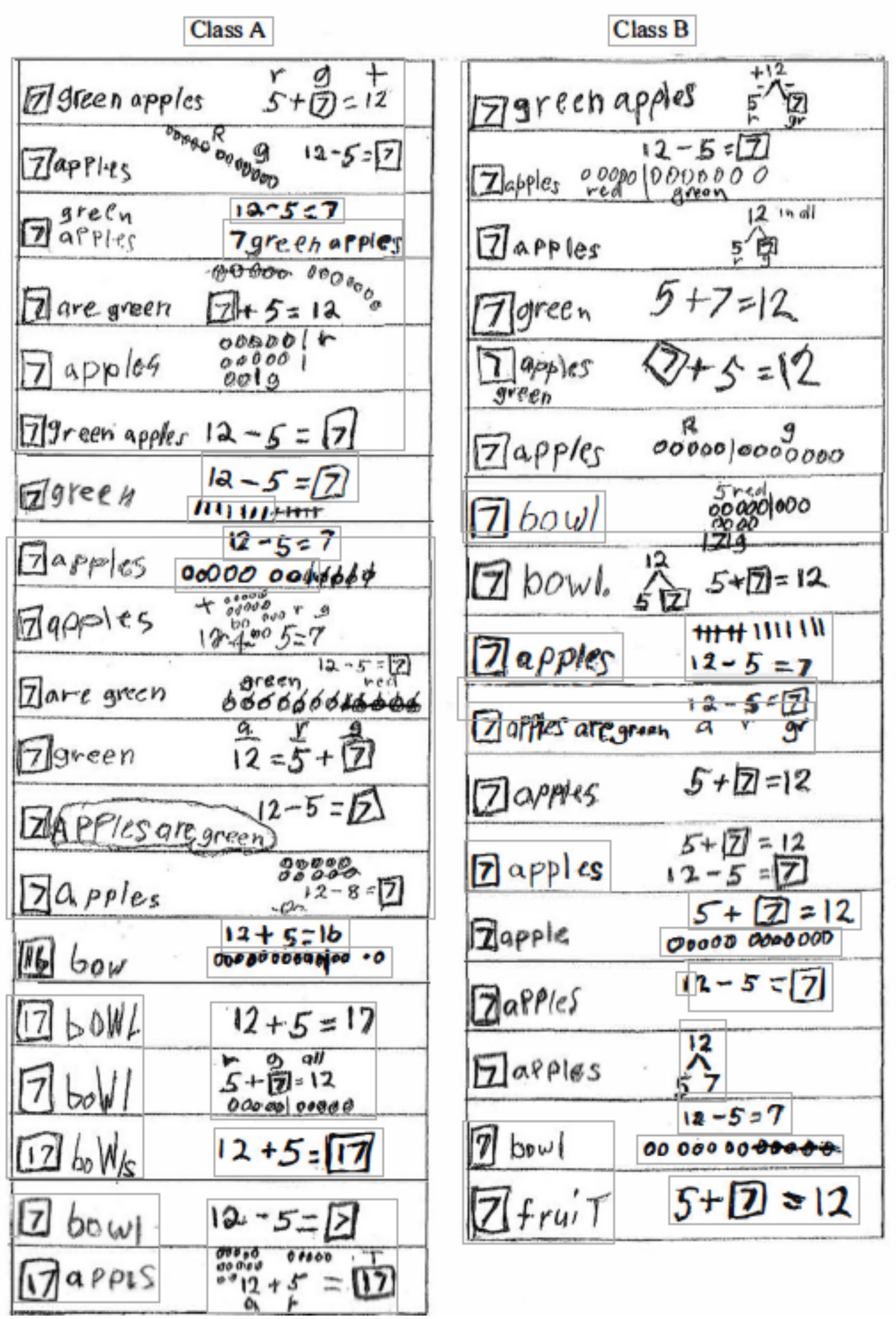
Grade 2 Answer, Strategy, and Error Results for Easy, Intermediate, Compare, and Difficult Word Problem Types with Totals 11 to 18

Problem Difficulty	Correct Answer	Correct Strategy/ Incorrect Answer	Incorrect Strategy/ Incorrect Answer
Easy Problems	96%	0%	4%
Intermediate Problems	89%	5%	5%
Compare Problems	82%	8%	10%
Difficult Problems	77%	4%	20%

Note: Items are from the Unit 2 test given in November.
Easy problem was Add-To: Result Unknown.
Intermediate problems were Put-Together/Take-Apart: Addend Unknown and Take-From: Change Unknown.
Compare problems were Compare: Difference Unknown and Compare: Bigger Unknown.
Difficult Problem was Take-From: Start Unknown.

In Figure 7 (see next page) the work of two classes of second graders is shown for a Take Apart: Addend Unknown problem. In both Grade 2 classes there was considerable variability in which tool a student used. Equations were used by almost all students in both classes. Many students in Class A and some students in Class B made circle drawings that separated the total objects into parts. Some students in Class B made a math mountain drawing. More than in Grade 1, equations, circle drawings, and math mountains had one or more parts labeled with letters to relate to the word problem situation. Students wrote many different correct equation forms, most with a box into which they filled the unknown 7. There were three different addition equation

Figure 7. Grade 2 Math Drawings and Solutions for the Take-Apart: Addend Unknown Problem "There are 12 apples in a bowl. 5 of them are red. The rest are green. How many apples are green?"



forms (5 + [7] = 12, [7] + 5 = 12, 12 = 5 + [7]) and one subtraction form (12 – 5 = [7]). Almost as many students wrote a correct addition equation as wrote the subtraction equation (11 vs. 14). Many of the subtraction equations were accompanied by circle drawings taking away five objects to leave 7 (both addends were visible). The addition equations tended to have no drawings or a math mountain. The class not shown in Figure 7 mostly (18 of the 20 cases) used an explicit subgrouping representation: a total number of circles is drawn and one of the addends (usually the given five red apples) is boxed or encircled, with addends often (11 of the 18 cases) labeled (red, green). All in that class showed a correct strategy, with only one off-by-one counting error. So children used many of the tools they had related and discussed in class and that are shown in Tables 5, 7, and 8.

In Figure 8 (see next page) the work of two classes of second graders is shown for a different kind of unknown addend intermediate problem, a Take From: Change Unknown problem. Again students use all possible tools, and there is considerable variability across solutions. Most students write an equation, sometimes with some drawing. There are many situation equations that show the problem situation 11 – ☐ = 7, and there are some solution equations that show an equation to solve 11 – 7 = ☐ and 7 + ☐ = 11 (recall that students could solve subtraction situations or equations by finding the unknown addend). In the third class, subgrouping representations enclosing one addend were drawn by 15 of the 20 children, most representations labelled an addend (13 out of 20), and all showed correct strategies and answers. These reflect the representations and solutions shown in Table 8.

In Figure 9 (see the page following Figure 8) the work of two classes of second graders is shown for a difficult problem, a Take From: Start Unknown problem. These problems are difficult precisely because it is the starting amount, which is also a total, that is unknown. So it can be difficult for students to get started representing or solving this problem type. In Class A 14 of the 20 students made circle drawings of both addends with no equations. Six of these were in the order of the numbers in the problem (8 then 7). These seem to be situation drawings that ended up reconstructing the original beginning bag of peanuts. Eight drawings reversed the order of the problem situation, starting with the 7 left and adding the 8 peanuts taken to make the original total bag of peanuts. These seem to be solution drawings in which children knew that the unknown total had been composed over time into an amount ‘taken away’ and an amount ‘left’ and that the total could be made by composing (adding) these amounts. In Class B children use all possible tools with considerable variability across solutions. Most students write an equation, often with some drawing. There are six situation equations ☐ – 8 = 7, and five equations 8 + 7 = ☐ that might be situation equations like the drawings that begin with 8 or might be solution equations. Four students wrote words (*in all, add, now*) to show that they were adding the two addends to make the unknown total. Four equations mixed solution and situation equations (8 – 7 = 1) to produce incorrect strategies and answers. The third class (not shown in Figure 9) displayed the same range of correct and incorrect representational forms as did Class B.

Figure 8. Grade 2 Math Drawings and Solutions for the Take-From: Change Unknown Problem "Jenna has 11 goldfish. She gives some to her friend. Now she only has 7 goldfish. How many goldfish did she give to her friend?"

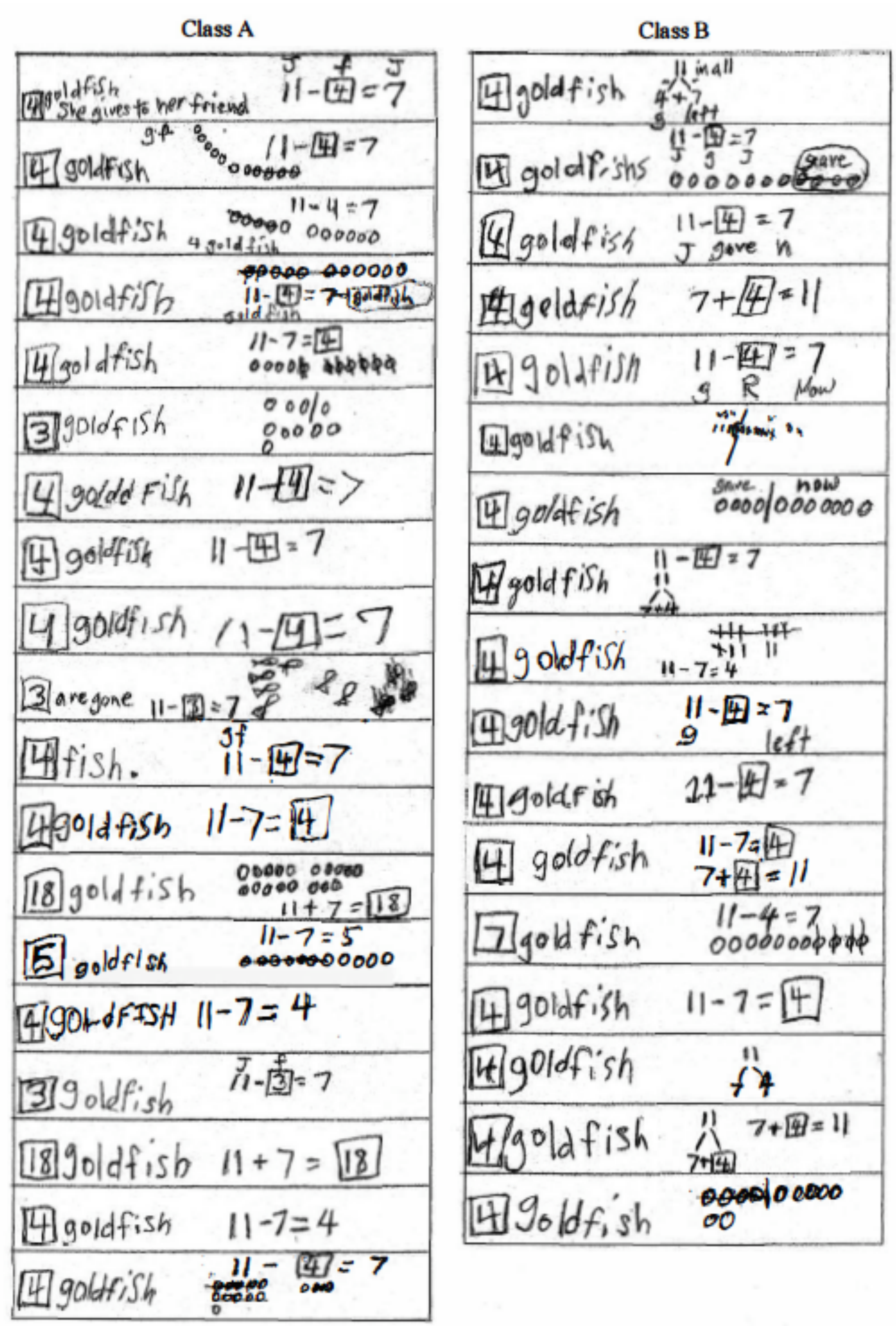
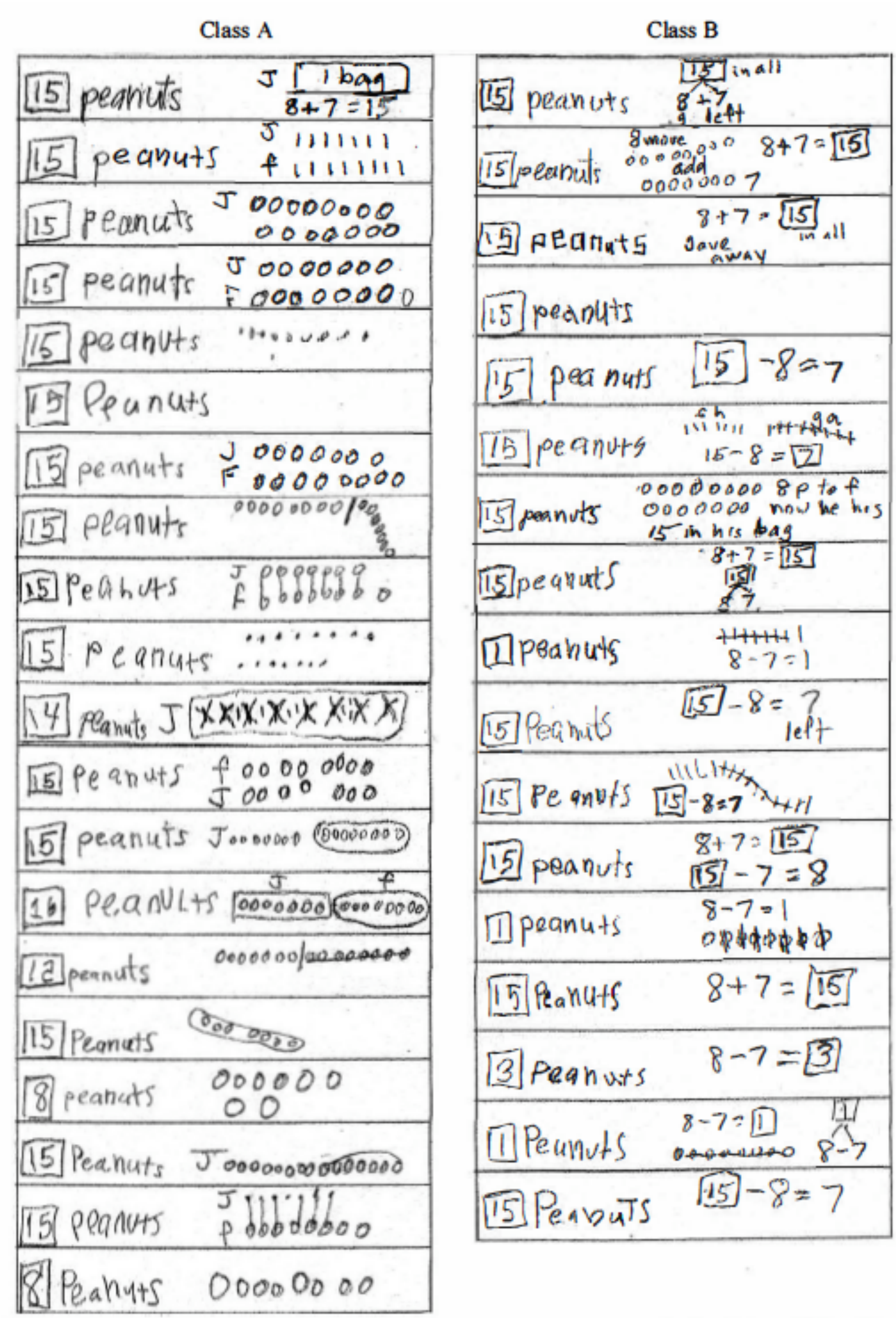


Figure 9. Grade 2 Math Drawings and Solutions for the Take-From: Start Unknown Problem "Joey had a bag of peanuts. He gave 8 peanuts to his friends. Then he had 7 left. How many peanuts were in the bag?"



Compare problems. Table 9 shows that a high percentage of second graders (90%) solved compare problems with a correct strategy (82% correct answer + 8% correct strategy/incorrect answer). The other 10% showed an incorrect strategy, with 9% using a wrong operation and 1% answering with the given addend.

Table 9

Grade 2 Answer, Strategy, and Error Results for Easy, Intermediate, Compare, and Difficult Word Problem Types with Totals 11 to 18

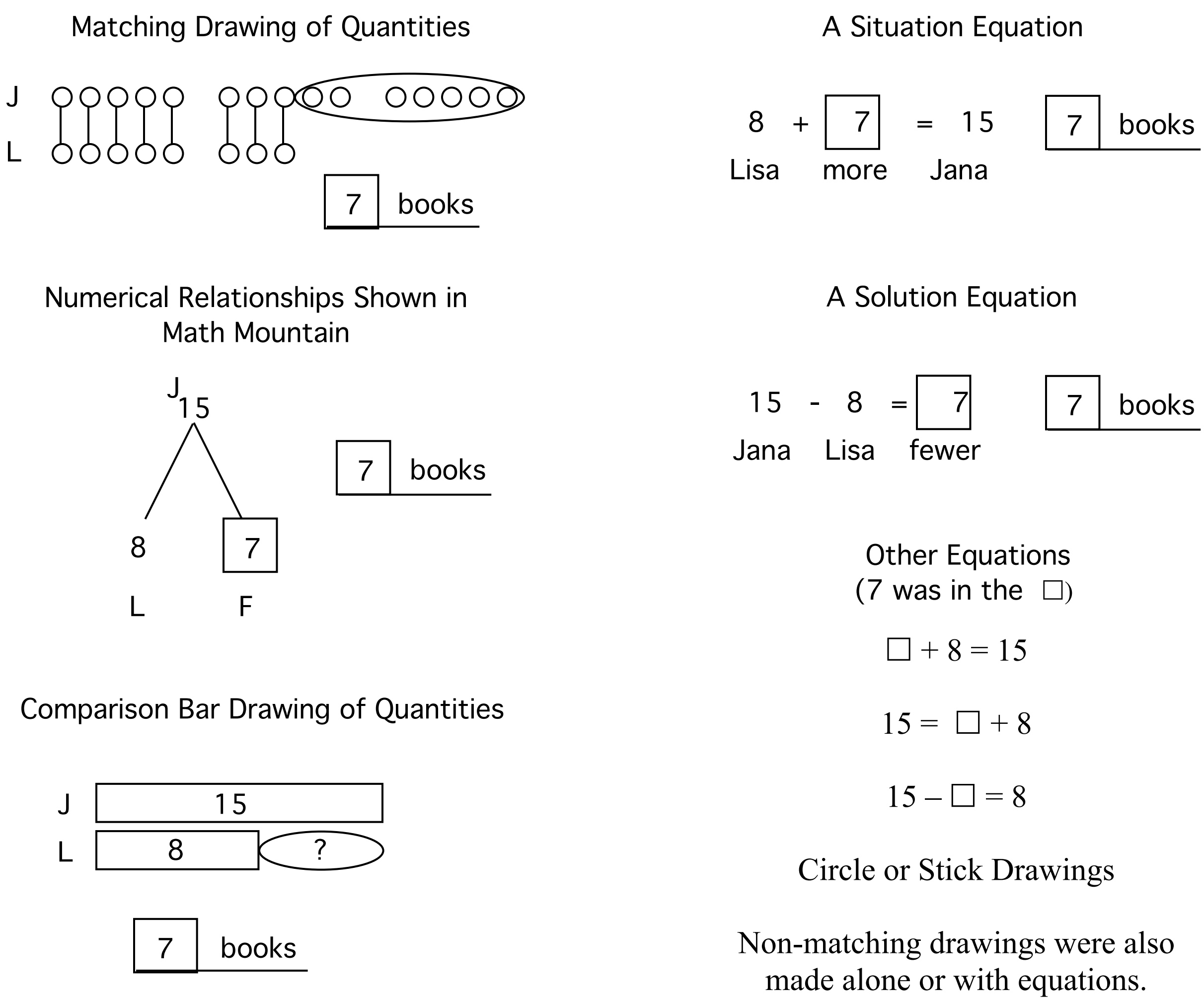
Problem Difficulty	Correct Answer	Correct Strategy/ Incorrect Answer	Incorrect Strategy/ Incorrect Answer
Easy Problems	96%	0%	4%
Intermediate Problems	89%	5%	5%
Compare Problems	82%	8%	10%
Difficult Problems	77%	4%	20%

Note: Items are from the Unit 2 test given in November.
Easy problem was Add-To: Result Unknown.
Intermediate problems were Put-Together/Take-Apart: Addend Unknown and Take-From: Change Unknown.
Compare problems were Compare: Difference Unknown and Compare: Bigger Unknown.
Difficult Problem was Take-From: Start Unknown.

These children had two lessons on compare problems after the lessons on the non-compare problems discussed above. The teacher elicited student solutions for a compare difference unknown problem. If no one suggested a matching drawing, the teacher showed such a drawing (see top left of Figure 10 on the next page where the extra books that Lisa read are encircled). The teacher also showed a comparison bar drawing of a longer and shorter bar, each labelled and with a number inside (see the bottom left of Figure 10). The difference quantity is represented with an oval to differentiate it from the two actual quantities in the situation (the difference quantity is not actually present in the situation but is created by the compare question). The difference oval is drawn after the shorter bar so that together the two smaller quantities are as long as the larger quantity. These drawings relate to picture graphs where things are visually matched and to bar graphs where bars are visually matched using a scale. These drawings differ from those shown in Fuson (1988, 1992), where the difference quantity is a solid or dashed line rectangle. The oval seems to be a clearer representation because it points to the difference as different from the two compared quantities. After the two lessons on compare problems, students solved mixed problems of all of the types shown in Table 9 and then had 4 more lessons on more difficult problems (not enough information, too much information, and two-step problems).

Figure 10. Grade 2 Solution Approaches to a Compare: Difference Unknown Problem

In March Jana read 15 books. Lisa read 8 books.
How many fewer books did Lisa read than Jana?



Problem representations varied by class for the Compare: Difference Unknown problem on the unit test. In Class A, 18/20 children used circle or stick matching representations in which the drawn quantities were aligned across from each other; in 14 of these 18 drawings short sticks matched the quantities. One child drew compare bars labeled with D and A and with 14 and 9 filled in. One child used a circle/sub-grouping representation. All of the children in this class show correct strategies; the one error is an off-by-1 error. Students in the other 2 classes used circle or tally matching representations in only 5/36 cases. The main representations were equations in the variety of forms shown in Figure 9, used in 28/36 cases. Other representations were math mountains and circle drawings of the larger quantity 14 with 9 crossed out. Children wrote more than one representation of the problem in 36% of the cases: circle or tally drawings and equations, circle drawings and math mountains, matching models and equations, and two different equations.

The (31/36) cases in the two classes that did not use a matching representation of comparison do not fully display the situational comparison of a larger and smaller quantity to identify the difference between them, but they do show that children have managed to place the smaller quantity within the larger to identify the unknown difference as part of that larger quantity. This is what the majority of first graders did for comparison problems. The most frequent cluster of equation and drawing representations place the smaller within the larger quantity and take it away or separate it. The additive equation forms also place the smaller within the larger, but involve counting on the difference between the smaller and larger. Math mountain representations show the larger quantity separated into parts (the smaller quantity and the difference), as can circle representations of the larger quantity, separated by blank space or a stick. These representational forms and quantity separating techniques were used by children in prior Level 1 and 2 problem solving contexts for other problem types (see Figures 6, 7, and 8 above). It appears that most of these children were able to adapt their prior modes of separating quantities and prior Level 2 embedded number understandings to place the smaller quantity (and the difference) within the larger quantity and identify the unknown difference, even without making an explicit external match to scaffold the comparison of the larger to the smaller quantity.

Second graders' representations of compare situations are more highly detailed than first graders representations, just as for other problem types. Most of the equation representations (19/28) showed the unknown quantity specially boxed. Components of equations, drawn quantities, and matching circles were often labeled, anchoring reference to quantities. More than one representation of a problem was made in more than a third of the cases.

Learning Paths and Visual Learning Supports

Grade 2 children used the grade 1 tools to review counting on to find the total numerically and in easy word problem situations that required adding. They related subtraction and unknown addition situations and discussed counting on to find an addend. Children solved all of the intermediate difficulty problems using teen totals. Children also discussed make-a-ten methods for teen totals because most of the problem solving in Grade 2 involved teen totals rather than totals ≤10. Children solved the difficult Add To/Take From: Start Unknown problems. Such problems could be solved by Level 2 representations and methods if a child could use a Level 2 addends-within-total conception, leave a blank space or use a ☐ for the start initially, and then fill in the situation information. The resulting equation (☐ + 8 = 14 or ☐ – 8 = 7), math mountain, or circle drawing could lead to a solution by counting on by starting with the known addend. Students did use such situation equations (see Figure 8), where we also saw many solution equations adding the two addends or drawings of both addends that were counted or added to make the start total.

Relating equations and math mountains within a word problem situation enables students to begin to understand where the total is in an addition equation (alone on one side) and in a subtraction equation (in front of an addend). Students did activities to support such understanding. They discussed relationships they saw in the eight related forms of equations made from the four equations with one number on the left (e.g., 6 = 8 – 2, 2 = 8 – 6, 8 = 2 + 6, 8 = 6 + 2) as well as the usual four forms with only one number on the right (e.g., 8 – 2 = 6, 8 – 6 = 2, 2 + 6 = 8, 6 + 2 = 8). Children labeled quantities in such related equations with T (total), P, P (partner). In Figure 6 children showed their flexibility with equations for a Take Apart: Addend Unknown problem because across children, four different equation forms were written with roughly as many addition forms as subtraction forms. Knowing relationships among equation forms is especially helpful for difficult word problems. For an Add To: Start Unknown problem, knowing where addends and the total are within an equation can help children move from a situation equation such as ☐ + 6 = 14 to a solution equation such as 6 + ☐ = 14 (one can just switch the addends). For a Take From: Start Unknown problem, children could move from a situation equation such as ☐ – 6 = 8 to a solution equation such as 6 + 8 = ☐ or 8 + 6 = ☐ by knowing that these equations were related.

Discussion

Question 1: Drawings, Situations, and Equations

These results illustrate the power of children making math drawings that show the situation and also support a solution method. Math drawings can be simple circle drawings that show the quantities and various ways of relating or separating the quantities. Relating such drawings to situation equations and to solution equations is also important. Both kinds of equations are used by children in problem solving. Math mountains are also simple drawings used effectively by children. Math mountains were used primarily to support numerical work with decomposing a number into addends in different ways, but children also used them in problem solving. Comparison bars are a third kind of drawing/diagram that show the structure of comparing situations but can also show the numerical relationships of the larger quantity equalling the smaller quantity plus the difference. Math mountains and the comparison bars can be used with all numbers (multidigit numbers, fractions, and decimals), so they and the situation and solution equations form a suite of tools that provide power and coherence through grade 6 and even beyond. Helping children understand where the addends and the totals are in each of these tools enables the tools to be used to represent and solve more difficult problems and to relate all of the addition and subtraction problem unknowns.

The problem representation tools used by students in this study can be considered as representing particular word problem types but also as representing conceptual levels of thinking in the meaning of the equals sign they use. The Add To and Take From situations use a Level 1 meaning of objects on one side of an equation changing to become the objects on the other side of the equation (the "arrow" meaning shown in Table 7). The Take Apart and Put Together situations facilitate a Level 2 conception of addends embedded within the total shown by the math mountain rapid movement of the total becoming the addends and vice versa. The Compare situations use Level 3 quantities where the total (the bigger quantity) is outside of and separate from the addends (the smaller quantity and the difference). However, evidence in this study indicates that children use these tools in individual ways to facilitate their own meaning-making in problem solving. The relationships among the tools extend and deepen children's problem representing and solving in ways discussed for Tables 5, 7, and 8.

In reporting our data we used correct strategies as well as correct answers because correct strategies mean that the user has partial and perhaps even very strong understanding. We have found that emphasizing correct strategies with teachers helps to shift them to thinking more deeply about the thinking of the children in their class. They begin to focus on exactly how to help particular students move forward, based on what is correct and what is not correct in the child's method. The error analyses reported in the tables show the kinds of analyses teachers can learn to do. It can be a useful learning experience for teachers, and an effective means of gathering data for formative assessment, for teachers to record the results of a quiz or some problems worked in class by recording any wrong answers and incorrect drawings.

Question 2: Children's Levels of and Variations in Tool Use

We outlined in tables and the text the learning progression of problem situations and use of equation and drawing tools and how these build from Level 1 representing and solving of easy word problems to Level 2 grade 1 use of counting on to represent and solve the same easy word problems. The next step is how the counting on concepts and the unknon addend siuations support representing and solving intermediate difficulty problems including solving to find the total and solving to find an addend.

It is important to emphasize that the Level 2 addend-within-total representations and solution methods are not simply applied as a whole and used as given. Representations are not given in problem solving contexts. One of the main purposes of problem solving teaching is for children to form their own representations of problem situations and to discuss, explain, and vary them, continually, in their everyday problem solving processes. The extensive variation in the grade 1 and 2 drawn problem representations reported above shows the many ways children modify, mix, and invent, as well as use introduced representational forms and solution methods. For example, children find many ways to separate quantities to isolate an unknown addend other than the vertical stick or the math mountain addends: They may put one quantity (either a numeral or some number of drawn objects) on paper and work with another in their mind or on fingers; separate quantities spatially on the page; circle (or otherwise bound) one quantity; show one quantity as a numeral and the other quantity as drawn objects; and use '+' as a separator. The partner numeric Take Apart: Both Addends Unknown explorations prior to Level 2 problem solving introduce the importance of separating to find a part, show one way to do it, and support addend-within-total representations and conceptions, but children thereafter find many different ways to separate quantities to represent and find unknowns.

Children also make hybrids of equation and drawn quantity forms. They semantically label equation and drawn model components in grade 2. Some mentally represent the problem and then 'know' the unknown quantity without using an external representation at all. As this variation shows, the Level 2 addend-within-total representational forms children have already constructed do not support only through being reproduced and used as given. They offer resources for children's own varying representations of problem situations and of solution methods. They also support through their use as exemplars and in orienting attention and supporting interactive explanation, discussion, and teaching. So we see that representing and solving Level 2 intermediate and difficult problems require a teacher and a child to view and operate from multiple perspectives *within* an addends-within-total model, from the perspective of any unknown and sometimes also from various model/equation links and equation/equation links, such as the relating of $7 - 4 = \square$ and $4 + \square = 7$ discussed above for Table 7. Children were offered extensive opportunities to discuss all such relationships in *Math Expressions* grade 1 and grade 2.

The Common Core State Standards include make-a-ten as a possible grade 1 strategy (1.OA.6) and they include the three prerequisites for this method as kindergarten standards (K.OA.3 and 4; K.NBT.1). But the third prerequisite K.NBT.1 (the final step in the make-a-ten method $10 + 5 = 15$) is more difficult in the irregular teen English words than in the regular East Asian languages where 15 is said *ten five*. The extent to which and when in what grade children in the U.S. will be able to use the make-a-ten methods is an open question. We reported that fairly high levels of performance were possible on the first two prerequisites for this method (K.OA.3 and K.OA.4), and performance on the third prerequisite was also high (Fuson, 2017). But our results in this paper suggest that Level 2 conceptions are enough to handle the OA demands in grades 1 and 2 for single-digit addition and subtraction problem solving with any unknown. Because the English number words make the make-a-ten method more difficult in English than in East Asia, it is not clear whether this method will be used by large numbers of children before multidigit adding and subtracting, when it becomes useful to recompose the totals above ten. Even for large multidigit addition and subtraction computation problems, using counting on to find teen totals or unknown addends (e.g., 15 – 8) can be rapid and accurate for most students. So less-advanced students may not need to move on to make-a-ten methods at any time.

Question 3: OA Performance by Children from Backgrounds of Poverty

The data here indicate that children from backgrounds of poverty can do well on the types of word problems in the Common Core State Standards if they have rich and extended opportunities to engage with the mathematics in these standards. The kindergarten children had high levels of performance on the easy problem types. They outperformed U.S. first graders from a range of backgrounds on these problems and equaled the performance of Japanese and Chinese first graders. Grade 1 children learned and used the Level 2 addition and subtraction counting on strategies specified in the standards (1.OA.6). Grade 1 children had high levels of performance on easy problems, outperforming U.S. first graders from a range of backgrounds, Chinese first graders on addition and subtraction, and Jahanese first graders on subtraction. They had fairly high levels on problems of intermediate difficulty. Grade 2 children had high levels of performance on easy and intermediate problems and fairly high levels on difficult problems and on compare problems. These results were obtained even though children did not have as much time on the non-easy problems as they would have now when such problems are standards for most states.

Kindergarten children also did well on the new K.OA.3 task decomposing numbers ≤10 into pairs in more than one way and recording each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1). They learned to represent and solve such situations with circle drawings separated into partners (addends), expressions and equations, and math mountain diagrams. Grade 1 and grade 2 children also used the decomposing circle drawings and math mountains to represent various kinds of problems. Analyses of conceptions used in representing and solving Level 2 conceptions that embed the addends within the total suggested the importance of this decomposing task in the crucial move to Level 2 counting on solutions and Level 2 problem representing and solving. Many grade 2 children were able to use these Level 2 conceptions to represent and solve even difficult problems.

Question 4: How to Improve on the Present Results

More class time should be available now that these problems and strategies are in the Common Core State Standards, so the problem soving and discussion phases can be expanded for intermediate problems beyond the time available in this study. Explicitly focusing on, discussing, and relating multiple ways of representing problems with each unknown would seem to help ensure that all students saw central ways to represent and solve each kind of problem and discussed how these representations and the problems are alike and different. More time for representing and solving mixed types of problems would also seem to be useful. First graders could also begin to label drawings and equations as soon as they could do so to relate quantities to the problem situation.

The grade 2 Common Core State Standards include all types of problems with all unknowns for 2-digit addends with a total≤100. The equations, math mountains, and compare bars that students used for totals to 18 in Figures 6 to 9 can extend easily to problems with 2-digit addends. These three tools then can be used in later grades with larger whole numbers and with decimals and fractions, forming a coherent representational framework for all compare problems in the elementary grades. The understandings described earlier about the eight different related forms of an equation and about where the addends and the total are in each equation form will become even more useful and can be discussed during problem representing and solving.

These flexible equation understandings build strong algebraic intuitions and understandings that can make algebraic equation manipulations in middle school comprehensible or even unnecessary as students move among equation forms they know are equivalent.

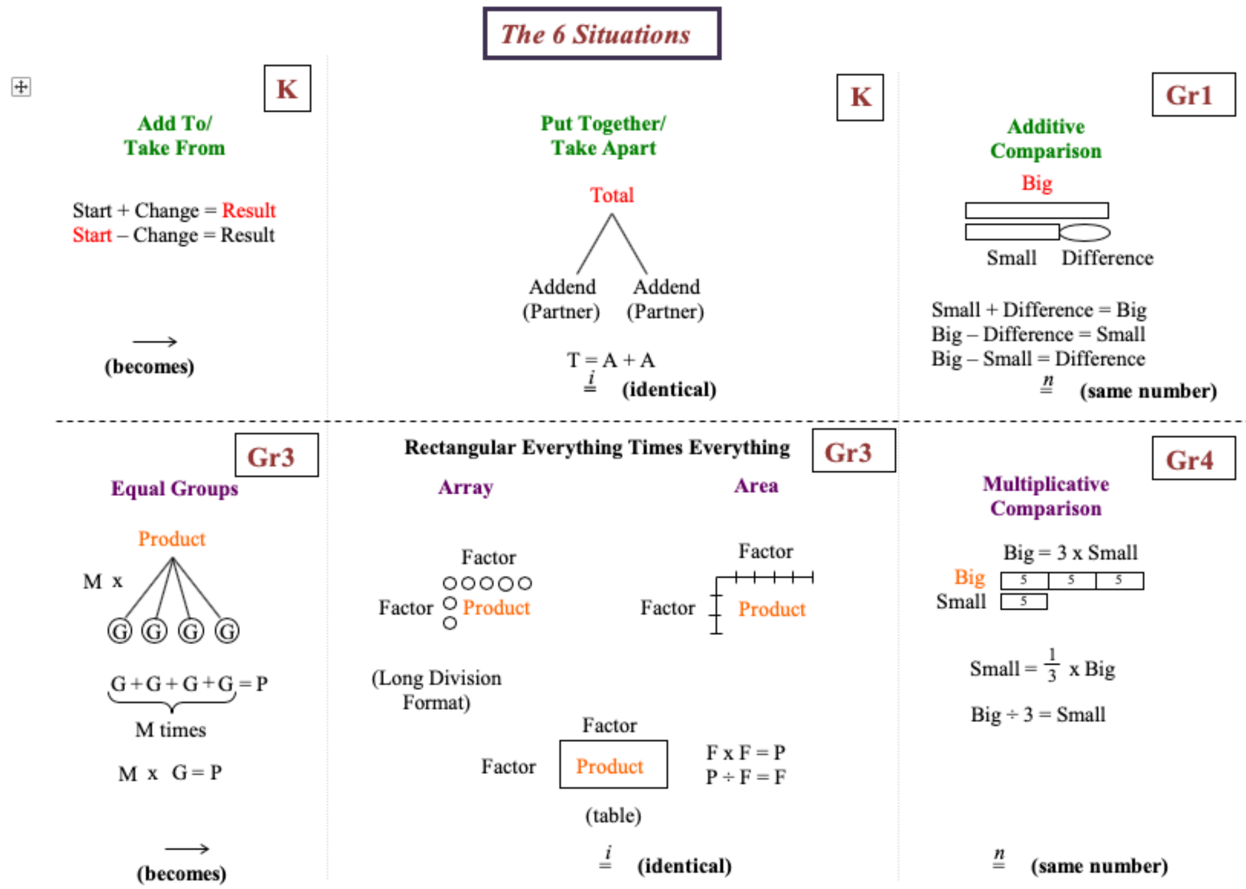
Also included in the grade 2 standards are two-step problems using all of the problem types. The OA progression (The Common Core Standards Writing Team, 2011) specifies that most two-step problems should involve single-digit addends and not use the difficult subtypes because students are still working to master these in grade 2. So the circle drawings as well as math mountain and compare bars could be used to represent and solve 2-step problems. Labelling will continue to be important for such problems. Emphasizing labelling in grade 1 could result in problem representations and solutions that look more like the grade 2 solutions in Figures 6, 7, and 8. With this background, students would more readily advance to the larger numbers and two-step problems in grade 2.

How Do the Drawings for Addition and Subtraction Extend to Drawings for Multiplication and Division?

It is important that the drawings for addition/subtraction situations used in kindergarten through grade 2 extend readily to multiplication/division situations so that students can relate these operations numerically and situationally. The Common Core State Standards identify in Table 1 page 88 the three major situations for addition/subtraction we have discussed in this paper. This document identifies in Table 2 page 89 the three common types of multiplication/ division situations: Equal Groups, Arrays/Area, and Multiplicative Compare. The drawings used in *Math Expressions* for these six operations are shown below in Figure 11. The grade placement of each kind of situation in the standards is identified in the small box by each type.

(Scroll down to see Figure 11.)

Figure 11. CCSS Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each



The Equal Groups drawing is an extension of the Put Together/Take Apart situation in which the multiple addends are counted by the multiplier so that 3 x 5 is three groups of five (or in some countries, a group of five taken three times). The roles of the factors in Equal Groups situations are different: one is the size of the group and the other is how many groups there are. The Multiplicative Comparison drawings are an extension of the Additive Comparison drawings in which the smaller group is repeated a number of times. The Array/Area situations are a new kind of situation in which the factors have similar roles as counting the length and height (or width) of an array of objects or of a rectangle of square units pushed together. These drawings relate to the visual lay-out of a division method in which the product and factors are as shown in Figure 11.

Division in all situations has one factor as the unknown. The two kinds of Equal Groups division situations are modeled differently depending on whether the size of the group is known (then equal groups can be made and counted or added on) or the number of groups is known (then the product can be dealt out to the known number of groups to find how many in a group). The product in an Equal Groups situation can also be made into an array, where the rows or the columns are the known groups. This allows students to see more readily that multiplication obeys the Commutative Property (*a x b = b x a*), which provides flexibility in finding particular unknown factors or products. Multiplicative Comparison situations relate easily to Equal Groups situations when the Small quantity is considered as an equal group.

Concluding Remarks

We want to emphasize that the high performance in kindergarten required intensive and extended learning path opportunities (repetitive experiencing) that increased in difficulty in the ways specified in the figures and tables. For grades 1 and 2 there were also intensive and extended learning path opportunities, although more such is necessary for mastery by all of the difficult problem types, as suggested above. For all grades, when a teacher demonstrates a visual tool or a solution method such as counting on, many children will not immediately see and use the relationships between quantities involved in those tools and methods. We have found that, especially where a new 'level' of complexity is involved such as Level 2 conceptions and tools, it takes an extended and elaborated phase of guided introducing in which children's individual internal forms are elicited and discussed to make these relational models accessible, meaningful, usable, and even 'seeable'. And then a protracted collective meaning making and explaining phase about individually varying use of the methods and tools in problem solving is required.

Now is the time for a substantive national discussion of instructional approaches that support the learning of concepts. The Common Core State Standards give us a learning progression of grade levels at which to support such understanding and so can focus grade-level instructional research and discussion. Details about what works is needed for widespread effective implementation of the Common Core State Standards. It was in this spirit that we analyzed the *Math Expressions* program used in this design research and provided details about the learning supports and activities that the children experienced. We are not claiming that these are the best approaches, but merely that they are effective approaches. We welcome participating in continued dialogue about effective instructional supports for understanding mathematical concepts, especially for children from backgrounds of poverty. We also emphasize how crucial it is to use the effective teaching-learning practices summarized in Table 4 and the Common Core Standards for Mathematical Practice, which can be briefly summarized as children focus on *sense-making* about *mathematical structure* using *math drawings* (visual models) to *support math explaining* (MP.1 + 6, MP.7 + 8; MP.4 + 5, MP.2 + 3).

References

CCSSO/NGA (2010). *Common core state standards for mathematics.* Washington, DC: Council of Chief State School Officers and the National Governors Association Center for Best Practices. Available at http://corestandards.org/ [Accessed 1 July 2010]

The Common Core Standards Writing Team (2011). *The operations and algebraic thinking (OA) progression for the Common Core State Standards in mathematics.* (http://ime.math.arizona.edu/progressions/).

Cross, Christopher T., Taniesha A. Woods, and Heidi Schweingruber (Eds.) (2009). *Mathematics learning in early childhood: Paths toward excellence and equity.* National Research Council, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, D.C.: National Academy Press.

Fuson, K. C. (1986). Teaching children to subtract by counting up. *Journal for Research in Mathematics Education, 17,* 172-189.

Fuson, K. C. (1988). *Children's counting and concepts of number.* New York: Springer-Verlag.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York: Macmillan.

Fuson, K. C. (2006). *Math Expressions, Grades K, 1, 2, 3, 4, 5, 6*. Boston: Houghton Mifflin Harcourt Publishers.

Fuson, K. C. (2009). Avoiding misinterpretations of Piaget and Vygotsky: Mathematical teaching without learning, learning without teaching, or helpful learning-path teaching? *Cognitive Development, 24*, 343-361. doi:10.1016/j.cogdev.2009.09.009

Fuson, K. (2011). The common core mathematics standards as supports for learning and teaching early and elementary school. In J. S. Carlson & J. R. Levin (Eds.), *Instructional strategies for improving student learning: Focus on early math and reading* (pp. 177-186). Vol. 3 in *Psychological perspectives on contemporary educational issues.* Charlotte, NC: Information Age Publishing.

Fuson, K. (2017). U.S. English-speaking kindergarten and first-grade children from backgrounds of poverty can understand place value. Paper presented at the Annual Conference of the National Council of Supervisors of Mathematics, San Antonio, Texas, April.

Fuson, K. C. & Fuson, A. M. (1992). Instruction to support children's counting on for addition and counting up for subtraction. *Journal for Research in Mathematics Education, 23,* 72-78.

Fuson, K. C., & Kwon, Y. (1992). Korean children's single-digit addition and subtraction: Numbers structured by ten. *Journal for Research in Mathematics Education, 23,* 148–165. doi:10.2307/ 749498.

Fuson, K. C. & Li, Y. (2009). Cross-cultural issues in linguistic, quantitative, and computational supports for mathematical thinking. *ZDM – The International Journal on Mathematics Education, 41, 793-808.* DOI 10.1007/s11858-009-0183-7

Fuson, K. C., & Secada, W. G. (1986). Teaching children to add by counting on with finger patterns. *Cognition and Instruction, 3*, 229-260.

Fuson, K. C., & Willis, G. B. (1988). Subtracting by counting up: More evidence. *Journal for Research in Mathematics Education, 19,* 402-420.

Fuson, K. C., & Willis, G. B. (1989). Second graders' use of schematic drawings in solving addition and subtraction word problems. *Journal of Educational Psychology, 81,* 514-520.

Geary, D. C., Bow-Thomas, C. C., Fan, L., & Siegler, R. S. (1993). Even before formal instruction, Chinese children outperform American children in mental addition. *Cognitive Development, 8,* 517-529.

Geary, D. C, Bow-Thomas, C. C, Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development, 67*, 2022-2044.

Kilpatrick, J., Swafford, J., and Findell, B. (eds.) (2001). *Adding It Up: Helping Children Learn Mathematics*. Mathematics Learning Study Committee. Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Murata, A. (2004). Paths to learning ten-structured understanding of teen sums: Addition solution methods of Japanese Grade 1 students. *Cognition and Instruction, 22*, 185–218. doi:10.1207/s1532690xci2202_2.

Murata, A., & Fuson, K. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using ten. *Journal for Research in Mathematics Education, 37*, 421–456.

National Council of Teachers of Mathematics (NCTM) (2010). *Focus in Kindergarten.* Reston, Va.: NCTM.

National Council of Teachers of Mathematics (NCTM) (2009). *Focus in Grade 1.* Reston, Va.: NCTM.

Sarama, J. & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children.* New York and London: Routledge.

Secada, W., G., Fuson, K. C., & Hall, J. W. (1983). The transition from counting-all to counting-on in addition. *Journal for Research in Mathematics Education, 14*, 47–57.

Stigler, J. M., Lee, S.-Y., & Stevenson. H. W. (1990). *Mathematical knowledge of Japanese, Chinese, and American elementary school children.* Reston, VA: National Council of Teachers of Mathematics.

Verschaffel, L., Greer, B. & DeCorte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Charlotte, NC: Information Age Publishing.