Chapter 9
Building on Howe's Three Pillars in Kindergarten to Grade 6 Classrooms

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Abstract Howe (2014, Three pillars of first grade mathematics, and beyond. In: Li Y. & Lappan G. (eds), Mathematics curriculum in school education, Springer, Dordrecht, pp 183–207) identified three pillars of first grade mathematics and beyond that described central mathematical and sense-making aspects of major Common Core State Standards Math (National Governors Association Center for Best Practices, Council of Chief State School Officers. 2010) domains. This chapter builds on each pillar by sharing visual models that have been powerful in helping students learn the aspects identified by Howe. Visual models are central core ideas and practices in the CCSS–M and deserve attention and discussion. The research-based examples discussed here are simple math drawings that students can make and use in their own ways in problem solving and explaining of thinking. Such drawings support the math talk discussions that are at the heart of the CCSS–M and of the mathematical practices. They enable (Howe's, 2014, Three pillars of first grade mathematics, and beyond. In: Li Y. & Lappan G. (eds), Mathematics curriculum in school education, Springer, Dordrecht, pp 183–207) three pillars to come to life in the classroom. Teachers and students can come to appreciate all of these pillars: Pillar I, the power of robust understanding of the operations of addition and subtraction including situations that give meaning to the operations and levels of single-digit addition and subtraction; Pillar II, an approach to arithmetic computation that intertwines place value with the addition/subtraction facts; and Pillar III, making connections between counting number and measurement number.

Howe (2014) identified three pillars of first grade mathematics and beyond that describe central mathematical and sense-making aspects of major Common Core State Standards Math (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) domains. The Howe paper circulated before it was posted on the website in 2010, and it was influential in the design of the Common Core State Standards Math. This chapter builds on each pillar by sharing visual models that have been powerful in helping students learn the aspects

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identified by Howe. I draw on 35 years of research in classrooms to discuss strengths and limitations of various visual models. This research has shown me the power of research-based math drawings that students make. This research was carried out in independent research studies and in funded research as part of the Children's Math Worlds Project that led to the publication of these models in the K to grade 6 math program *Math Expressions*. The research studies and experience with classrooms using the *Math Expressions* programs have provided extensive teacher data about the effectiveness of these visual models. Visual models are central core ideas and practices in the CCSS–M and deserve attention and discussion. But which visual models should we be using and why? We need discussion of this issue for various math domains. This chapter is a contribution to such discussion.

9.1 Pillar I: A Robust Understanding of the Operations of Addition and Subtraction

9.1.1 Situations that Give Meaning to the Operations

The major real-world situations that give meaning to addition/subtraction and to multiplication/division have been the focus of much research (e.g., see National Research Council, 2001, 2009). Problem classifications of these situations drawn from research are given in the Common Core State Standards on pages 88 and 89 (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). I spent many years trying different diagrams students could use especially with larger numbers, fractions, and decimals (Fuson, 1988; Fuson, Carroll, & Landis, 1996; Fuson & Smith, 2016; Fuson & Willis, 1989; Willis & Fuson, 1988). Students can make their own drawings. But specially designed diagrams provide a common visual language to support discussion, and they provide consistency across kinds of numbers.

The final set of research-based diagrams that were successful in hundreds of classrooms is shown in Fig. 9.1. Students must learn meanings of equations, so equations were chosen as the visual support for the simplest kind of problems add to/take from. These situations show action over time, so it is natural for students to write each step of an equation as a step in the problem situation over time. Put together/take apart diagrams with the total on the top and two legs for the addends were found to help students understand these situation actions. Comparison bars show the two compared quantities in an additive comparison situation, and the difference quantity created by information about the situation is shown as an oval that makes the smaller quantity as long as the bigger quantity. The equal groups multiplication/division situations use the put together/take apart drawing repeatedly, and the multiplicative comparison situations draw the repeated quantity bar repeatedly. The array/area situations begin by drawing all of the objects or squares but quickly get abbreviated to a drawn rectangle in which the factors are along the sides and the product is inside. This model also reflects the traditional long division format.

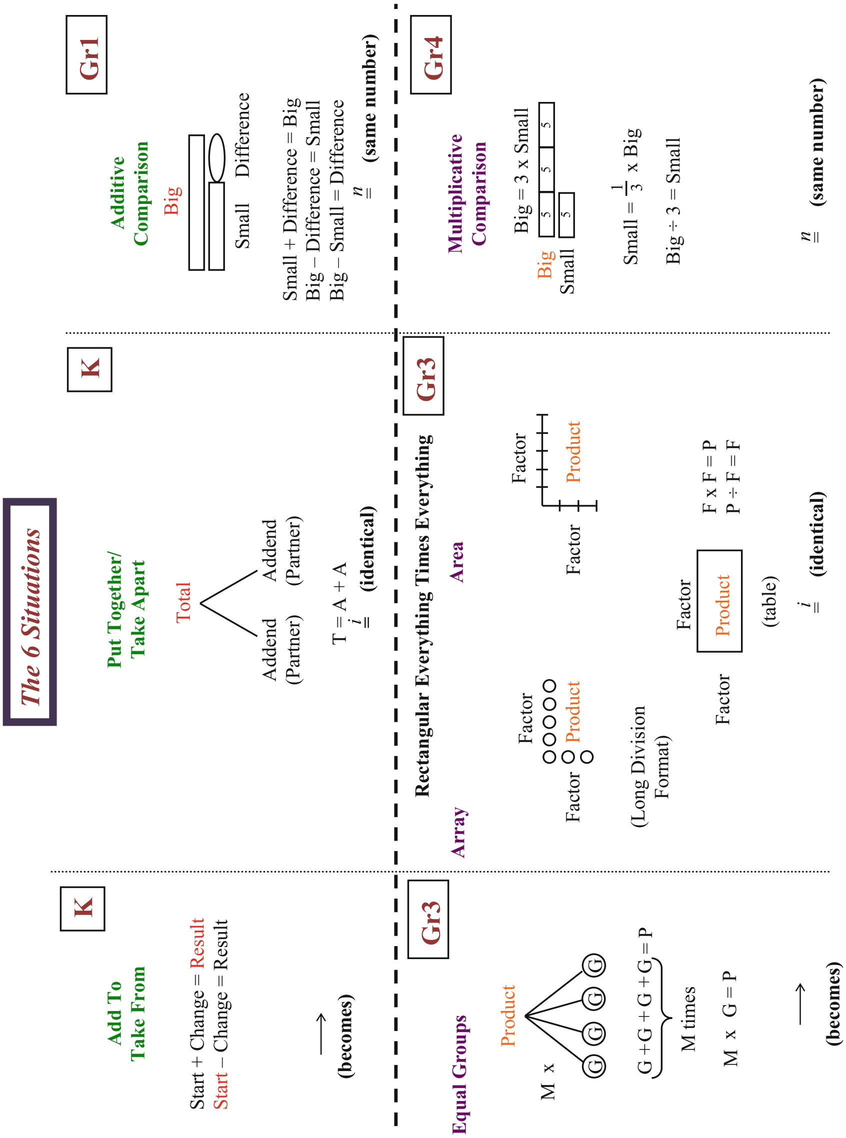


Fig. 9.1 CCSS addition (*top row*) and multiplication (*bottom row*) word problem situations and *Math Expressions* diagrams

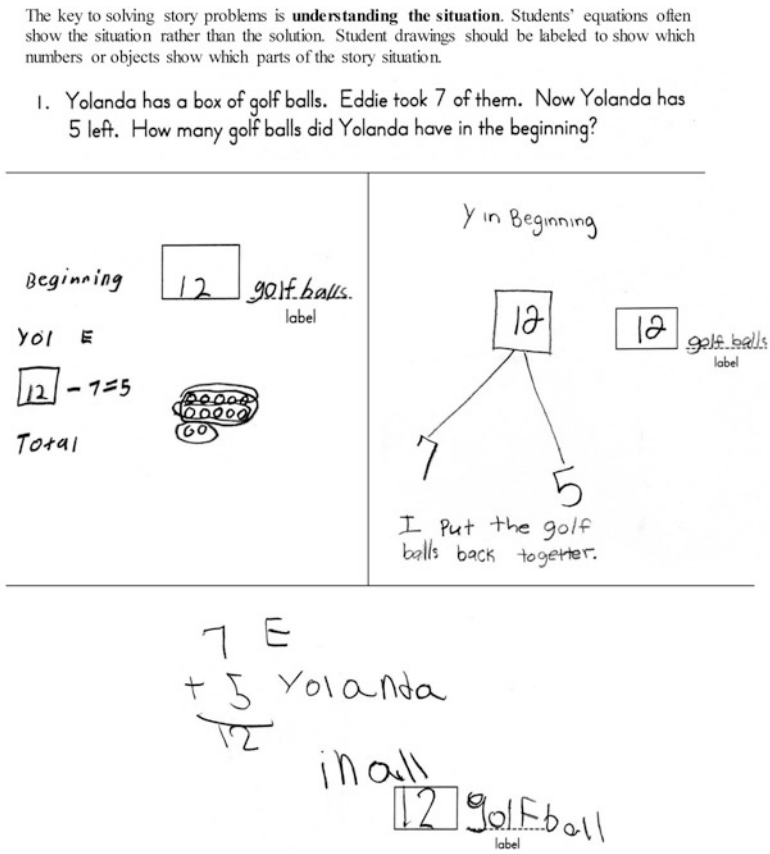


Fig. 9.2 Labeled math drawings for an unknown start problem

Each type of situation has three quantities, and each quantity can be the unknown. Some unknowns are more difficult than other unknowns. These differences create the learning path of difficulty across addition/subtraction situations that extend from kindergarten to grade 2. The key to solving word problems is understanding the situation and then making a labeled drawing if needed. Students' equations often show the situation rather than the solution. They then think about their drawing or equation to solve the problem. A difficult take from:start unknown problem is shown in Fig. 9.2. At the top left, the equation shows the situation, and the student then draws quantities to show the adding of 5 and 7 to make 12. Students often represent and solve in different ways. Two other approaches are shown in Fig. 9.2. Older students can use the same diagrams to support varied approaches for problems with multi-digit numbers and fractions.

For more information about the learning path of difficulty of the problem types and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Operations and Algebraic Thinking (OA) in the CCSS: Part 1 Problem Situations and Problem Solving at http://www.karenfusonmath.com.

9.1.2 Levels in Adding and Subtracting Single-Digit Numbers

Students worldwide go through three levels of conceptualizing and carrying out adding and subtracting (e.g., Fuson, 1986, 1992; Fuson & Fuson, 1992; Fuson & Willis, 1988). At level 1 they can only think of one number at a time. So adding is a three-step adding-to process in which they focus on the first addend, then on the addend added to the first addend, and then on the total of both addends. Subtracting is the reverse taking-from process that begins with the total, takes one addend from that total, and then focuses on the remaining other addend. At level 2, students can conceptually embed both addends within the total so that they can begin the counting of the total with the counting of the second addend: to add, they count on from the first addend to find the total, keeping track of how many are counted on and saying the last counted word as the total. To subtract, they count on from the first addend to find the unknown second addend, keeping track of and stopping as they say the total, and then seeing how many they counted on. At level 3, students can decompose and recompose addends within the total. So, for example, they can carry out the general method for single-digit adding and subtracting in which one addend is decomposed to make a ten with the other addend: for example, $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. Methods from all three levels are in the CCSS–M, level 1 at kindergarten and levels 2 and 3 at grade 1.

Howe (2014) discussed in his Pillar II the importance of decomposing a number into two addends in different ways. In the CCSS–M, such decompositions are a kindergarten standard:

K.OA.A.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).

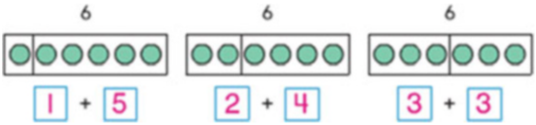
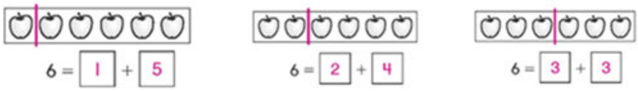
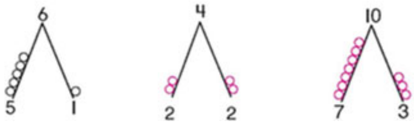
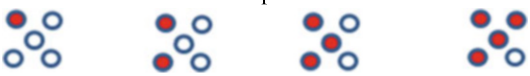
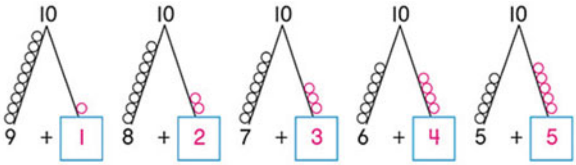
Notice that the equations to record these decompositions have the total alone on the left and the addends are added on the right side. This reflects the taking apart action in the situation and is helpful in overcoming the prevalent view by older students that an equation must have two numbers on the left and one number on the right. It is helpful for kindergarten and grade 1 children to see equations of this form that show the meaning of the situation.

In my own research, I have found that decomposing numbers into two addends helps children move to the level 2 methods of single-digit adding and subtracting that require the addends to be embedded within the total: counting on to find a total or to find an unknown addend. Such level 2 embedding of the addends within the total also allows children to solve the more difficult problem subtypes like add to or take from change unknown and start unknown problems (Fuson & Smith, 2016). Students can represent change unknown situations by a situation equation such as

8 +? = 14 or 14 −? = 8 and can solve them by level 2 counting on from 8 to 14 to find the unknown addend. Start unknown problems such as the problem in Fig. 9.2 also require an understanding of where totals and addends are in equations or diagrams and how to relate these three quantities to find the unknown.

Visual supports for decomposing that I have found to be effective in kindergarten are shown in Table 9.1. Students count out things to make a given number and partition these in various ways with a break-apart stick. Later, as in the tasks in Table 9.1, the partitioning is shown in drawings on paper by a break-apart stick and by shading. Students write the two addends that are created. These addends are called *partners* because this word was found to help students relate these two numbers. In grade 1 (see Fig. 9.3), students move on to using these visual supports to decompose larger numbers and to relate decompositions that reverse the order of the addends (using the commutative principle). The decompositions also become

Table 9.1 Percentage correct on partner (addend) tasks for kindergarten children

Unit	%	Task
3	90	1. Write the partners. 
4	92	2. Draw a line to show the partners. Write the partners. 
4	92	3. Draw tiny tumblers on the math mountain. 
4	85	4. Write the partner equation. 
5	88	5. Shade to show all the five partners in order. Write the five partners. 
5	83	6. Draw tiny tumblers on the math mountain and write the partner. 

Note. These tasks fall centrally within the following CCSS: K.OA.3

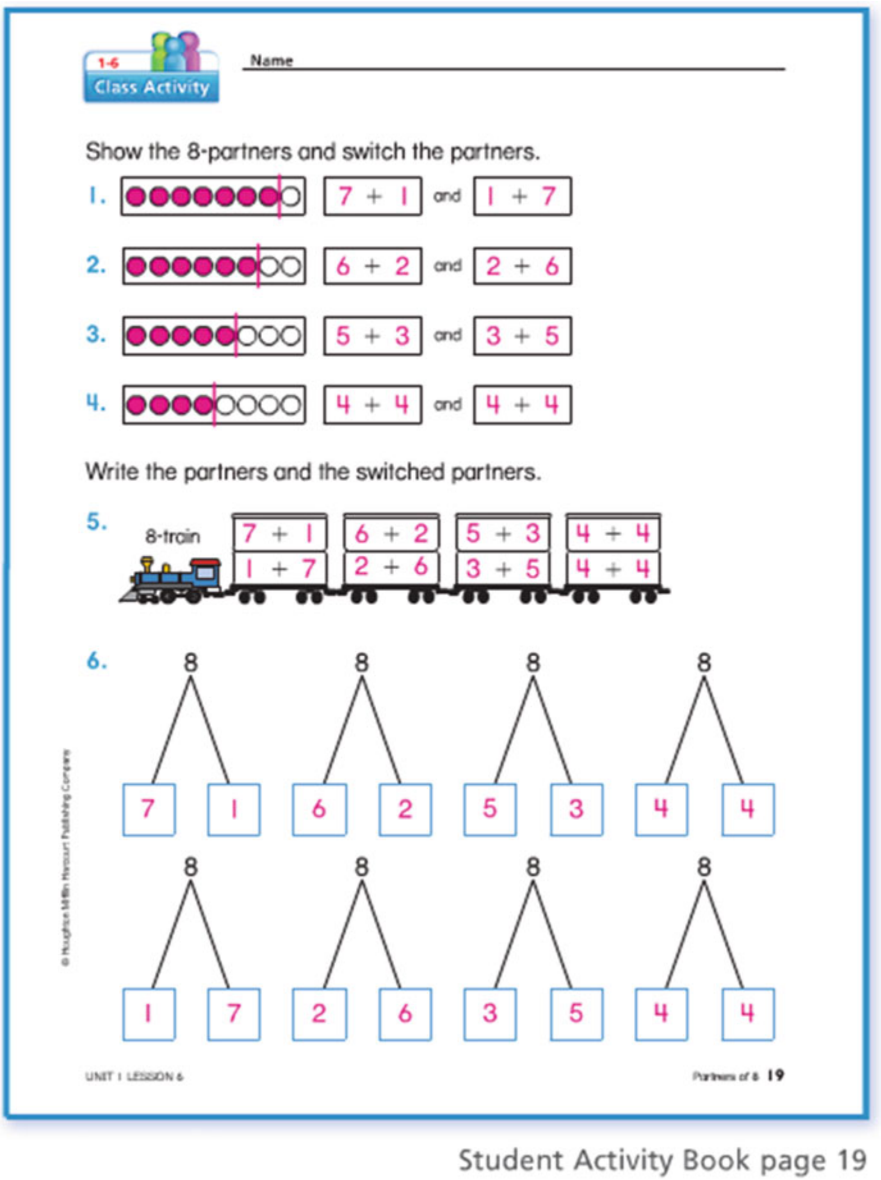


Fig. 9.3 Grade 1 partner switches

mostly numerical for first graders, as these small numbers take on quantitative meanings from extensive work in kindergarten. Such decompositions appear again in grade 4 as CCSS–M standard 4.NF.B.3b. This work helps students understand that unit fractions obey the same principles as whole numbers, reduces the common

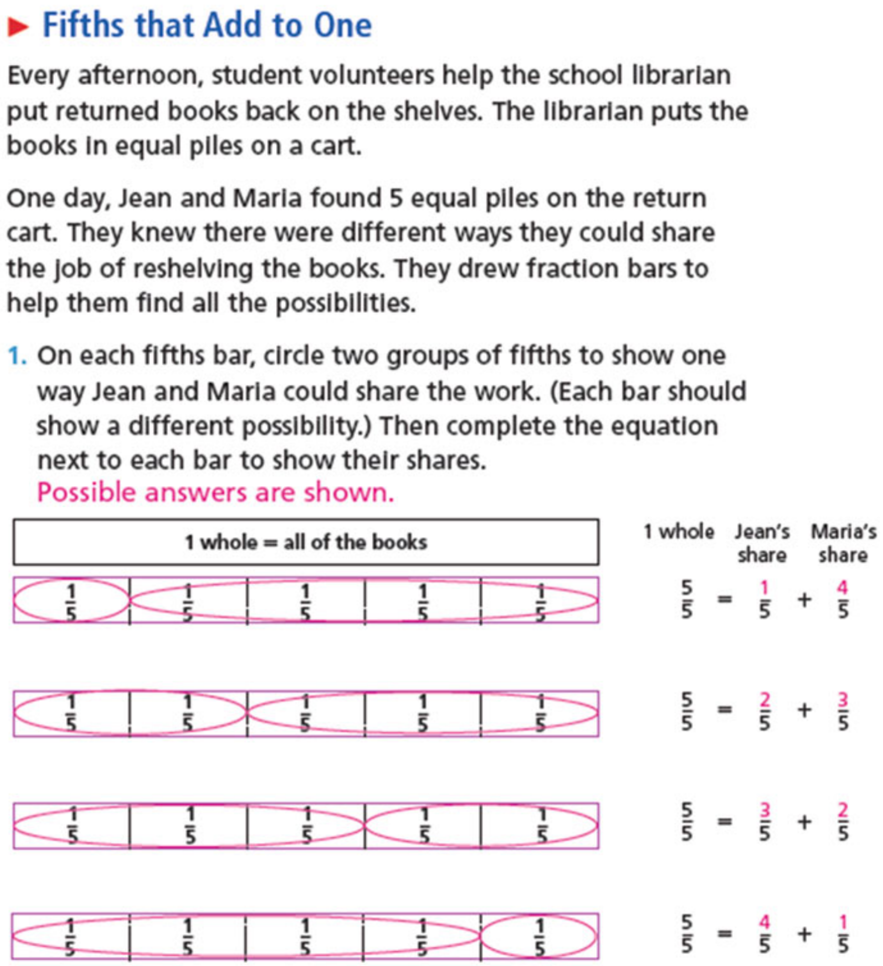


Fig. 9.4 Decomposing fractions into addends/partners

error of adding tops and bottoms when adding fractions (because students see that only the top numbers are added and that the unit fraction number does not change), and generalizes decomposing a number into addends (see Fig. 9.4).

Decomposing a number into addends is the second step in doing the general level 3 make-a-ten method: $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. In the first step ($8 + ?$), one must know the number that makes ten with the first addend. In the second step, one decomposes the second addend into the number added to ten and the rest of the second addend: $8 + (2 + ?)$, where $2 + ? = 6$. In the third step, one must know $10 + 4$, a total made with ten. All three prerequisites for the make-a-ten method are kindergarten CCSS–M:

K.OA.A.4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.A.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).

K.NBT.A.1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., $18 = 10 + 8$); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

This method and these prerequisites are emphasized in East Asian countries but have not been emphasized in this country, especially the second step of decomposing a number discussed by Howe (2014). As kindergarteners have time to learn these prerequisites, understanding and carrying out the make-a-ten method will become easier.

However, this method is more difficult in English than in East Asian languages based on Chinese that say 14 as *ten four*. Saying a number between ten and twenty as *a ten* and *some ones* helps with all three steps in the make-a-ten method. In contrast, an English word such as *fourteen* has a reversal in the ten and the ones that complicates the relationship with the written numeral 14. *Ten* is not said clearly (how many adults know that teen means ten?). And the number of ones is not said clearly in *eleven*, *twelve*, *thirteen*, and *fifteen*. For these reasons, the level 2 counting on methods may be enough for CCSS–M OA problem solving in grades 1 and 2. But make-a-ten methods can be helpful in CCSS–M NBT multi-digit adding and subtracting, as is discussed in the next section. For more information about the learning path of three levels of adding/subtracting and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Operations and Algebraic Thinking (OA) in the CCSS: Part 2 The K, 1, 2 Learning Paths for OA + and – (at http://www.karenfusonmath.com).

9.2 Pillar II. An Approach to Arithmetic Computation that Intertwines Place Value with the Addition/Subtraction Facts

Howe's (2014) Pillar II involves two major conceptions:

(a) Understanding that a two-digit number is made of some tens and some ones.
(b) In adding or subtracting, you work separately with the tens and the ones, *except* when regrouping is needed.

Both of these concepts extend to larger numbers with more places. Howe pointed out that it would be useful to have a term for the numbers created by a decomposition into place value numbers, for example, in 243 = 200 + 40 + 3. He suggested that such numbers be termed *single-place numbers*; the 200, 40, and 3 would be called single-place numbers. This is a helpful observation and might make it easier for students to conceptualize and discuss such parts. But I suggest instead the term *place value parts* for such numbers because they are parts and they explicitly name place values. Howe's two concepts above form the basis for general methods of adding and subtracting for any number of places. Students need to be able to add and subtract the single-digit addends discussed in Pillar I. And they need to understand how to think about and have a written method to record grouping when adding and ungrouping when subtracting. For two-digit numbers, students will group ten ones to make one ten whenever their total for the place value parts in a given place is ten or more. In general, they will group ten of one kind of place value parts to make one of the place value parts in the next-left column. And they will ungroup one of one kind of place value parts to make ten of the place value parts in the next-right column.

I did research for many years to ascertain what visual supports would help students understand these two vital conceptions that underlie multi-digit adding/subtracting and what general written methods were easy for students to understand and explain and relate to visual supports (Fuson, 1998, 2003; Fuson & Li, 2009; Fuson & Smith, 1997; Fuson, Smith, & Lo Cicero, 1997; Fuson et al., 1997). Students need to see and understand the quantities that make the place value parts for any number. Secret-code cards that can be layered to show place value parts and math drawings that students can make to show the quantities for each place value are both very helpful to students. The fronts and the backs of the secret-code cards are shown in Fig. 9.5. Unlayered, the cards show the place value parts (400 and 80 and 6). When layered on top of each other, the cards show the usual single-digit form of our base-ten numerals (486). But the little numbers on the top left of each card remind students of the place values for each part and of the zeroes that are hiding under the other digits. These cards are called secret-code cards because they show the secret code of our numbers, and students love the term. The quantities named by the place value parts are shown on the back of the cards: 4 hundreds, 8 tens, and 6 ones. Students learn how to draw these quantities by drawing on columns of ten dots to make a ten-stick and making a box around ten such columns of ten dots (or ten such ten-sticks) to make a hundred-box. Soon students make quick-hundred and quick-ten drawings that are just a hundred-box and a ten-stick,

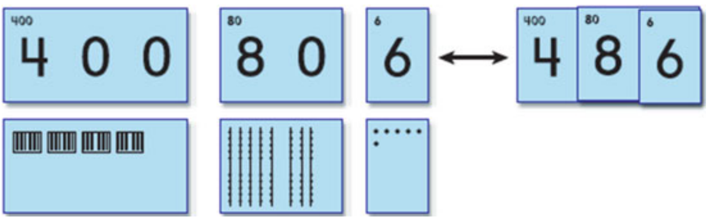


Fig. 9.5 Secret-code cards for 486

but they understand the quantities involved. Secret-code cards can be used on a millions frame to show the groups of three numbers in millions, thousands, and ones. Secret-code cards can also be extended in the opposite direction to show decimal place value parts.

These visual models support working separately with the place value parts, as described in Pillar IIb and in the CCSS–M. The CCSS–M critical areas for each grade at which new multi-digit computation is introduced specify that students are to "develop, discuss, and use efficient, accurate, and generalizable methods" for that computation. They further specify that students are to understand that adding and subtracting involve adding or subtracting place value parts, composing or decomposing these parts as needed. Importantly, the CCSS–M also specify that students use concrete models or drawings, relate strategies based on place value to a written method, and explain why the methods work. For example, in grade 2, NBT standards 7 and 9 state:

2.NBT.B.7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, and ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

2.NBT.B.9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[3] [[3]Explanations may be supported by drawings or objects.]

There are different ways to write generalizable methods that meet the above specifications. There is no such thing as "a standard algorithm" in spite of the widespread use of this term. Many different methods have been used historically in this country and in other countries, often several at the same time. The National Research Council report adding it up made this point and showed and discussed many methods (National Research Council, 2001). Fuson and Beckmann (2012) followed the lead of the NBT Progression document (the Common Core Writing Team, 7 April 2011) and summarized that *the standard algorithm* for an operation implements the following mathematical approach with minor variations in how the algorithm is written:

- Decomposing numbers into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number
- Using the one-to-ten uniformity of the base-ten structure of the number system to generalize to large whole numbers and to decimals

Fuson and Beckmann then identified variations in written methods for recording the standard algorithm for each operation, showed visual models that supported understanding of the written methods, and discussed criteria for evaluating which variations might be used productively in classrooms. A similar discussion for teachers of advantages and disadvantages of various written methods for addition and subtraction is given in National Council of Teachers of Mathematics (NCTM) (2011). Fuson and Li (2009) identified and analyzed a number of variations of written methods for multi-digit addition and subtraction found in textbooks in China, Japan, and Korea.

These analyses converge on one method of addition and one method of subtraction that are superior to others. The addition method is shown in Fig. 9.6, where drawings and a student explanation are shown for each step in adding using place value parts. Questions by other students follow at the bottom. This classroom example implements the CCSS–M and Pillar II. Notice, as you read, the example of how the drawings can support listeners' understanding of the explanation and of the questions by other students and can clarify both aspects of multi-digit adding identified above.

This method, often called New Groups Below, has several conceptual and procedural advantages compared to the current common method in which the new groups (the little 1 s) are written above the columns. It supports place value understanding by:

- Making it easier to see the teen sums for the ones (16 ones) and for the tens (14 tens), rather than separating these teen sums in the space above and below the problem so that it is difficult to see the 16 or the 14.
- Allowing students to write the teen numbers in the usual order as 1 then 6 (or 1 then 4) instead of writing the 6 and then "carrying" or grouping the 1 above.
- Making it easier to see where to write the new 1 ten or 1 hundred in the next left place instead of above the left-most place (a well-documented error that arises more with problems of 3 or more digits and is easier to make when one is separating the teen number below and above the problem).
- Making it easier to carry out the single-digit additions because you add the two larger numbers you see and then increase that total by 1, which is waiting below. When the 1 is written above the column, students who add the two numbers in the original problem often forget to add the 1 on the top. Many teachers emphasize that they should add the 1 to the top number, remember that number and ignore the number they just used, and add the mental number to the other number they see. This is more difficult than adding the two numbers you see and then adding 1.

Notice in Fig. 9.6 how the drawings use five groups to support the level 3 make-a-ten methods. When adding 9 ones and 7 ones, you can see that the 9 needs one more to make ten; this one ten can be written below in the tens column waiting for it to be added. The 7 has been decomposed into 1 to make ten and 6 left, so the 6 ones can be written below as the total number of ones. Similarly for the tens, 8 tens

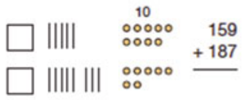
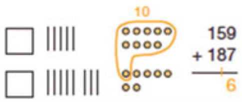
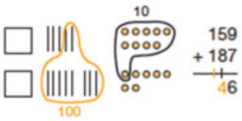
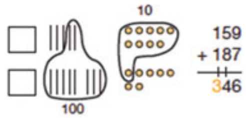
Math Drawing and Problem	Explanation Using Place-Value Language About Hundreds, Tens, and Ones
a. 	I drew one hundred, five tens, and nine ones to show one hundred fifty nine, and here below it I drew one hundred, eight tens, and seven ones for one hundred eighty seven. I put the ones below the ones, the tens below the tens, and the hundreds below the hundreds so I could add them easily.
b. 	See here in my drawing, nine ones need one more one from the seven to make ten ones that I circled here, and I wrote 10. That leaves six ones here. With the numbers the seven gives one to the nine to make ten that I write over here in the tens column, see one ten. And I write six ones here in the ones column.
c. 	With the tens, I start with eight because it is more than five so it is easier. I get two tens from five tens to make ten tens, see here, and I write one hundred here to remind me that the ten tens make one hundred. There are three tens left in the five tens and I have one more ten from my ones (see here in my drawing and the one ten at the bottom of the tens column). That makes four tens and the one hundred. So in my problem I write the one hundred below in the hundreds column and the four tens in the tens column.
d. 	There are three hundreds, two in the original numbers I'm adding and one new hundred from the ten tens. I write three hundreds here in the hundreds column. Are there any questions? Yes, Stephanie.
Student Question	**Explainer Answer**
Stephanie: For the tens, you never said fourteen tens as the total of the tens. Why not?	Because when I'm making ten tens, I just can write that one hundred over here with the hundreds and just think about how many tens I need to write. But I can think eight tens and five tens is thirteen tens and one more ten is fourteen tens, so that is one hundred and four tens. You can do it either way. (Aki)
Aki: Do you still need to make the drawings or did you just make them so you could explain better?	I don't have to make the drawings, but I can explain better with a drawing because you can see the hundreds, tens, and ones so well. (Jorge)
Jorge: Do you do make-a-ten in your head or just know those answers?	I just know all of the nine totals because of the pattern: the ones number in the teen number is one less that the number added to nine because it has to give one to nine to make ten. So nine plus seven is sixteen. I just know that pattern super fast. For eight plus five, I do make-a-ten fast, sort of just thinking five minus two is three, so thirteen. (Sam)
Sam: I know five and eight is thirteen, so why did you write a four in the tens column, Karen?	Because I had one more ten from the ones. See here in the drawing: nine ones and one one from the seven ones make ten ones. I wrote 10 here to remind me, and here in the problem I wrote the new one ten below where I can add it in after I find thirteen. You have to write your new one ten big enough to be sure you see it.
Sam: Oh yes, I see it now. I can see the new one ten when I write it, but I couldn't see yours.	OK, thanks. I'll write it bigger next time so everyone can see it.

Fig. 9.6 Three-digit addition using New Groups Below with student drawings, explaining, and questioning. The explainer stands to the side and points with a pointer to parts of the math drawing or to parts of the problem as they are mentioned. Pointing is a crucial part of the explanation. Reprinted with permission from Focus in Grade 2: Teaching with Curriculum Focal Points, copyright 2011, by the National Council of Teachers of Mathematics. All rights reserved

can be seen to need 2 more to make ten tens; this new one hundred can be written below the hundreds column. The 5 tens are decomposed into 2 to make ten with 8 tens and 3 tens left; the 3 can be added to the 1 ten waiting below and then the 4 tens written below the tens column. With experience, the make-a-ten method can be done mentally in this multi-digit adding context and then perhaps in other contexts. Such five-group visual models are used widely in East Asian classrooms. They can be used from the first day of kindergarten displayed on a poster with numerals to help children build understanding of single-digit numbers. These five groups are used on the backs of the secret-code cards shown in Fig. 9.5 to help children see how many hundreds, tens, and ones more easily.

Two written methods for subtracting after decomposing into place value parts are shown in Fig. 9.7. The better method is shown first. Before you subtract a given kind of place value part (a given column), you need to check if you can subtract the bottom number from the top number: *Is the top number greater than or equal to the bottom number?* If not, you need to get more of those units in the top number by ungrouping one unit from the left to make ten more of the units in the target column. All of these "checking and ungrouping if needed" steps can be done first, either from the left or from the right. Then all of the subtracting can be completed either from the left or from the right. These subtractions can actually be completed in any order, but going in one direction systematically creates fewer errors. This taking care of all needed ungrouping first is shown in Fig. 9.7 as method A with math drawings for a three-digit example and then without drawings for a six-digit number at the bottom to show how the method generalizes. Students can stop making drawings as soon as they understand and can explain the steps.

Ungroupings from the left and from the right are shown for the six-digit example. You can see how these ungroupings differ by looking at the ungroupings in the second and fifth columns. In ungrouping from the left, the 6 hundred thousands give 10 ten thousands to the 2 ten thousands, making 12 ten thousands and leaving 5 hundred thousands. Then the 3 thousands need more thousands (to subtract the 6 thousands), so the 12 ten thousands give 10 thousands to the right making 13 thousands and leaving 11 ten thousands. In ungrouping from the right, you ungroup moving to the left, and when you get to the ten thousands place, you have taken 1 ten thousands to give 10 thousands to the 3 thousands to make 13 thousands. The steps of ungrouping involve the same quantities, but they are done in different orders as shown by the ungrouped numbers above the problem.

Separating the two major kinds of steps involved in multi-digit subtracting as in this method is conceptually clear and makes it easier to understand that you are not changing the total value of the top number when you ungroup. You are just moving units around to different columns. Many students prefer to move from left to right, as they do in reading, and productive mathematical discussions can take place as students explain why they can go in either direction and still get the same answer.

The method B variation in Fig. 9.7 involves following the same steps but alternating between ungrouping and subtracting. Alternating steps is more difficult for students, and this method sets up the common subtraction error of subtracting the top from bottom number when it is smaller (e.g., for 94–36, get 62). Even when you know you should check and ungroup if needed, alternating steps prompts errors. For example, in the three-digit number in step 2, you have just subtracted 6 ones from 13 ones to get 7 ones. You look at the next column and see 1 and 5, and 4 pops into your head (if you are only in second grade). You write 4 and move left. In the

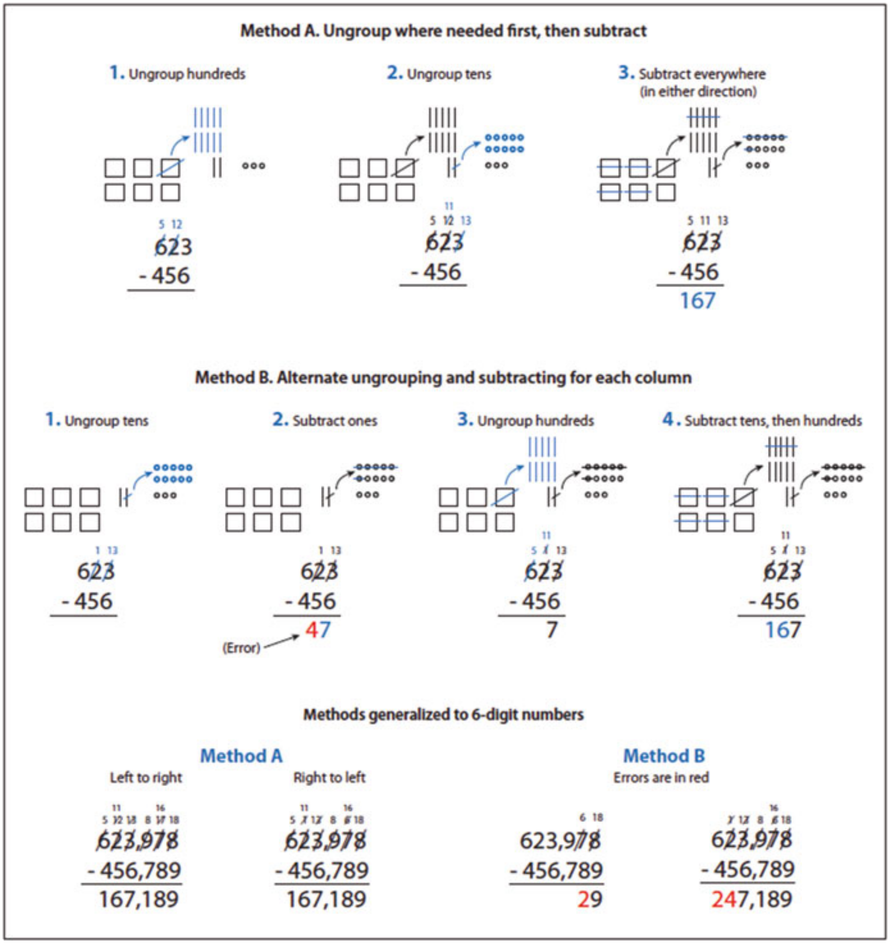


Fig. 9.7 Multi-digit subtraction methods

six-digit problem, the three errors that can be created by alternating ungrouping and subtracting in method B are in red. Although this alternating method can be used for numbers of any size, it is not as easy or conceptually clear as method A. For two-digit numbers, the alternating method B and non-alternating method A are the same because there is no iteration of the steps.

For more information about how to support students through the learning path of understanding place value parts and making drawings to show them and use them in explaining multi-digit addition and subtraction, see the Teaching Progression on *Math Expressions* and Number and Operations in Base Ten (NBT) in the CCSS: Part 2 Place Value and Multi-digit Addition and Subtraction in K to G4 (at http://www.karenfusonmath.com).

9.3 Pillar III. Making Connections Between Counting Number and Measurement Number

9.3.1 Limitations of Length for Showing Place Value and Addition and Subtraction

Physical and Practical Issues Howe (2014) suggested that students use trains of 100-rods, 10-rods, and 1-cubes to show place value. Length does show how the place value parts get big quickly. But length is not practical for use in a classroom. Length is too long for students to use to show or add or subtract even two-digit numbers. Base-ten blocks have ten-sticks 10 cm long and 1-cubes 1 cm long. Length trains of ten-sticks and 1-cubes do not fit across most student desks, and most rooms do not have enough tables on which all students can work. Base-ten blocks use a 10 cm by 10 cm square for hundreds rather than length; this is more practical. But the blocks present other difficulties. They are expensive, leave no record of the steps in using them, cannot be used for homework, are difficult to show the whole class, and are cumbersome to relate to written methods. The drawings shown in Figs. 9.6 and 9.7 have none of these disadvantages.

Drawings that just use length are also problematic. The CCSS–M 2.MD.B.5 and CCSS–M 2.MD.B.6 specify that students should relate addition and subtraction to length by solving word problems involving lengths and by representing whole numbers and whole number sums and differences within 100 as lengths from 0 on a number-line diagram. However, to get 100 units across a page even horizontally, each unit is about 1.6 mm long. This is small. Consequently, a number-line diagram to 100 is too short to see numbers clearly and is too complex for students to draw even semi-accurately. So, students can work with a few examples already drawn on a page to see that their count models do extend to length models. They can use meter sticks marked into centimeters and decimeters in demonstrations for the whole class of these lengths related to their place value parts. But the tools for adding and subtracting that can actually be used by each student are drawings of hundreds, tens, and ones related to written methods as shown in Figs. 9.6 and 9.7.

Length Models Constrain the Addition and Subtraction Methods Students Can Easily Use Length models do not support Pillar II or general CCSS–M methods that compose separate place value parts because they keep one multi-digit number together and add to or take from that number. Such methods require advanced sequence counting skills as students add on or take from hundreds or tens or ones from a whole two-digit or three-digit number. I tried these methods in classrooms for several years, but I found that it was difficult for less-advanced or non-native English speakers to learn these sequence counting skills. These methods can be done using drawn place value parts as in Figs. 9.6 and 9.7 instead of length models. They still require the same sequence counting skills, but they do not require learning to use a different visual model. Further problems with these length model methods are discussed in Fuson and Beckmann (2012) and NCTM (2011). Among other issues, they are not generalizable to larger numbers.

9.3.2 Counting Number and Measurement Number Do Relate Well to Show Multi-digit Multiplication and Division

Count models of drawn place value parts as used for addition and subtraction lead into the array models (count models using things as units) and area models (measure models using units of measure) commonly used to visualize multiplication and division. For example, the known factors are the numbers of rows and of columns in an array or the lengths of the sides, and the product is the number of total things in the array or the number of unit squares in the area model. I have found with many classrooms that students can make such array or area drawings for small numbers on a dry-erase Math Board that shows 100 by 50 dots, each 4 mm apart. Students can draw around the dots to make arrays, or they can draw on the lengths between the dots to show area. Such drawings (e.g., for 24 × 37) show all of the drawn place value parts accurately to scale. Then students can move to drawing sketches and relate them to a written method. Eventually students drop the sketches and just do a written method.

There are written variations for multiplication and division that record the place value parts in somewhat different ways. Advantages and disadvantages of many of these are discussed in Fuson and Beckmann (2012). Approaches that I have found to be understandable by many students are shown in Fig. 9.8. The area model is shown on the left. For multiplication, students know and draw the lengths of the two sides, separating the place value units tens and ones. They draw line segments inside the rectangle to make subareas for the products of the place value parts and fill in the products for each subarea.

The expanded notation method shown in the top middle for multiplication is a common approach. But there are tricky parts of this method, so students in one classroom added the blue steps to help all of them see what was happening in each step and avoid their errors, and the multiplying was written for the largest place value unit product (the tens × the tens) first so that the other products could be aligned underneath. The blue steps can drop out when they are not needed. This fuller method is helpful to many students initially.

But I found in many classrooms that some students had difficulty with this method: they could not see what to multiply by what. The area model was clearer about what to multiply by what, so they would draw a little rectangle, record the products inside the subareas, and add them up on the right as shown in the place value sections method.

The one-row method shown on the top right is a common embedded method that alternates multiplying and adding and that writes the added-in value for the tens × ones step in the wrong place: 60 × 3 is 180, but the 1 hundred is written above the tens column (above the 4 and the 6). Better methods are discussed in Fuson and Beckmann (2012).

The rectangle sections method, on the bottom left for division, helps students relate multiplication to division as they see how the same area model can be used for both. Students first draw a length 40 for the tens part of the unknown factor

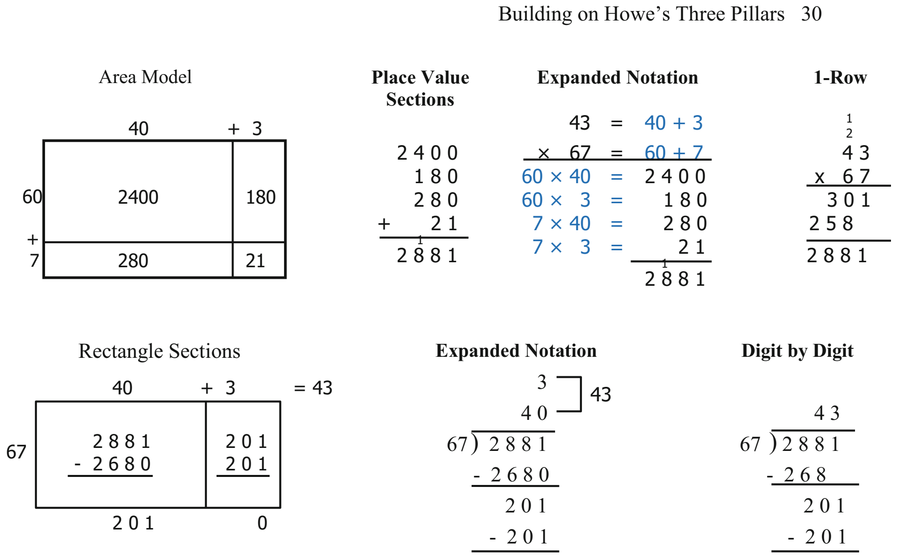


Fig. 9.8 Drawings and written variations of standard algorithms for multiplication and division

and multiply 67 by that number 40. They subtract the resulting 2680 from the total product to find the area of the subarea for the ones unit, getting 201. They draw the ones length 3, multiply 3 × 67, and subtract that from the area of the ones subarea. This problem has no remainder, but many problems do have a remainder. The other methods in the bottom row of Fig. 9.8 show the same steps of finding the tens and then the ones values of the unknown factor. These methods can be related to the area model so that students understand what they are doing, and students can discuss how all three methods relate to each other.

To return to the issue of length models with which this section began, alternating square and long shapes shows place values more easily than do just length models. The hundred square discussed earlier is the new larger square unit, ten of which can be composed in a tall column to make a thousand. Ten of these tall columns can be composed to make the new large square unit of ten thousand. Ten of these ten thousand units can be composed in a tall column to make a hundred thousand. Finally, ten of these hundred thousand long shapes can be composed to make a huge million square. The units for these models can be count numbers (dots) or measure numbers (tiny unit squares). Making such a display in the hallway has been a productive activity for many classrooms.

For more information about the learning path of multi-digit multiplication/division and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Number and Operations in Base Ten (NBT) in the CCSS: Part 3 Place Value and Multi-digit Multiplication and Division in G3 to G6 (at http://www.karenfusonmath.com).

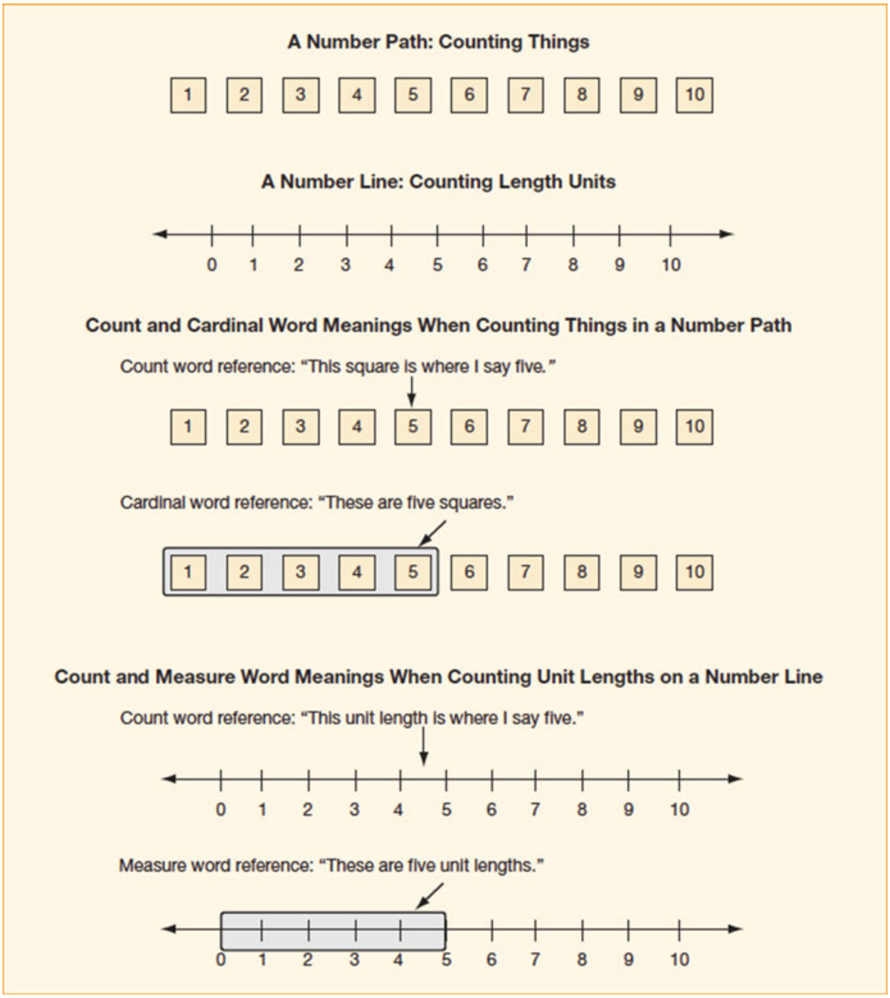


Fig. 9.9 Relationships between counting number, cardinal number, and measure number

9.3.3 Numbers on the Number Ray Tell Distances from the Endpoint/Origin

Howe's (2014) final major point concerning counting number and measurement number is the understanding that the numbers on the number ray tell *distances* from the endpoint/origin. This is a crucial understanding that provides a sound basis for placing whole numbers and fractions on the number-line diagram. Some people refer to whole numbers and fractions on the number line as points on the number line. Thinking only about points does not provide meanings for adding

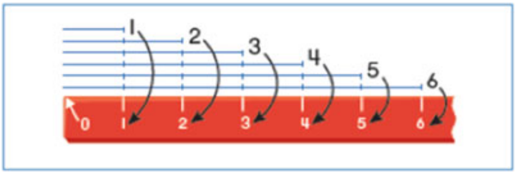


Fig. 9.10 Seeing length units on a ruler by drawing successive lengths

and subtracting. How can you add one point and another point and get a third point? This is only possible if the points are actually endpoints of distances from zero created by length units. The CCSS–M 3.NF.A.2a and CCSS–M 3.NF.A.b use this relationship between interval/distance/length units and the endpoint of the interval/distance/length from zero to describe representing fractions on a number-line diagram.

Seeing the length/distance units on a number line is difficult because our brains are wired to see things not lengths. In Fig. 9.9, counting numbers are shown at the top, each within a square to make it easy to count them. Below that is a number line where the numbers represent the number of length units from 0. Notice how your eye is drawn by the numbers below the line and the little vertical marks for the ends of each unit. It is difficult to see the unit lengths on the line that lie between the numbers. In the middle are shown the relationships between count and cardinal meanings of number described in K.CC.B.4b: the last counted word tells how many things there are. Below that are shown the similar relationships between count and measure meanings of number: the last counted word tells how many unit lengths there are. Because of the visual difficulty and the off-by-one errors induced by number lines, the National Research Council reports (2001, 2009) conclude that number lines are not appropriate for PK, K, or grade 1 children. Visual count models like the number path shown at the top of Fig. 9.9 are appropriate. The CCSS–M is consistent with these recommendations, first introducing number lines at grade 2.

Rulers and bar graph scales have the same structure as a number line. Figure 9.10 shows a ruler. Notice how the eye is drawn by the points marked by the short vertical segments and by the numbers below these. We have to work hard to help students see and use the distances/lengths in rulers, bar graph scales, and number lines. One way is shown in Fig. 9.10. Students can draw one length unit and write a 1 after it, then very close below they draw two length units with a 2 after it, then three length units followed by a 3, etc. They then can think of a ruler as all of these lengths pushed together to make a single line with all of these lengths on it; the number of lengths so far is written at the endpoint of each of the lengths. They also can make little vertical segments as they measure lengths initially and then count those lengths to emphasize that they are measuring length units.

In Fig. 9.11, we can see three other ways to see the lengths on fraction number lines. First, at the top, a fraction bar in which one can see lengths is drawn above a number line in which the eye is drawn to points instead of lengths. The lengths in the fraction bar help one see the lengths in the number line. At the bottom, the number of unit fractions (seven) is shaded in the number bar and encircled in the

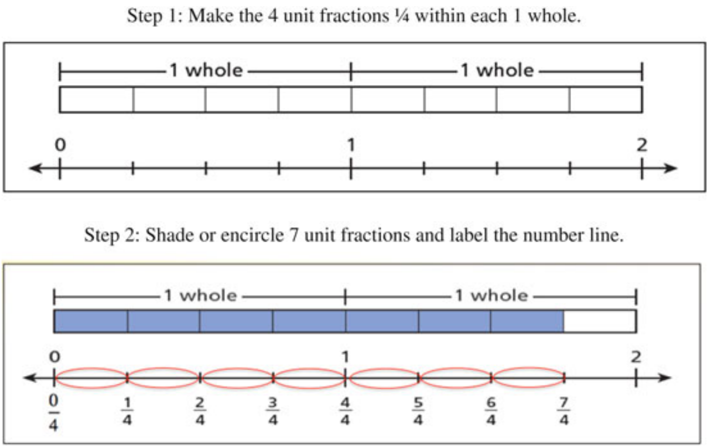


Fig. 9.11 Seeing the unit fraction lengths by shading or encircling

number line. This helps the viewer see the lengths. Students can also be asked to slide their finger along each length as they count the seven unit fraction lengths.

Making unit fraction drawings in these two steps also helps students make sense of unit fractions. Usually students just see the second step with some of the unit fractions shaded or otherwise marked. But then they do not see the total number of unit fractions, here four in one whole. They just see the two parts of the fraction embedded inside the whole. If only the second whole had been shown, students would see three parts shaded and one part not shaded in that second whole. Many students then say that the fraction is 1/3 because they see the parts 1 and 3 but not the total four parts. But in the top drawing in which four unit fractions are made in one whole, students can see the four unit fractions. So the right-hand bottom half of the drawing shows three parts shaded of the total four parts, so ¾. Here, to see that the bottom shows 7/4 and not 7/8, the top unit fractions could each have been labeled ¼.

Without consistent support to see the lengths in number lines, students make errors when drawing or labeling number lines for whole numbers or fractions. They may count the points beginning with the first point as 1 instead of as 0 and get one too few unit lengths. If they have a number line with the starting and end marks already made, they may make as many new marks as unit fractions, resulting in one too many unit lengths.

Number lines are an important mathematical tool, and students must come to understand the relationships between the distance/length units and the endpoints of these units that are labeled on the number line. For more information about the learning path of fraction conceptions and computation and how to support students through this learning path, see the Teaching Progression on *Math Expressions* and Number and Operations—Fractions (NF) in the CCSS–M (at http://www.karenfusonmath.com).

9.4 Visual Models Are Central Core Ideas and Practices in the CCSS–M and Deserve Attention and Discussion

We close by summarizing the importance of visual models for building understanding and explaining in classrooms. As the research-based examples here have shown, models can be simple math drawings that students can make and use in their own ways in problem solving and explaining of thinking. They support the math talk discussions that are at the heart of the CCSS–M. The CCSS–M specify eight mathematical practices that are to be implemented with the standards. These eight can be formed into four pairs (practices 1 and 6, practices 7 and 8, practices 4 and 5, practices 2 and 3) and given names to support their use in the classroom. A teacher can ask every day: "Did I support students to focus on math sense-making about math structure using math drawings (visual models) to support math explaining? And can I do this better tomorrow?" These mathematical practices, and the visual models that support their implementation, can help Howe's (2014) three pillars come to life in the classroom. Teachers and students can come to appreciate the power of robust understanding of the operations of addition and subtraction including situations that give meaning to the operations and levels of single-digit addition and subtraction (Pillar I), an approach to arithmetic computation that intertwines place value with the addition/subtraction facts (Pillar II), and making connections between counting number and measurement number (Pillar III). These are crucial aspects of CCSS–M OA, NBT, and NF standards.

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