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AUTHOR NAMES:
Fuson, Karen C.; Kiebler, Shannon; and Decker, Robyn

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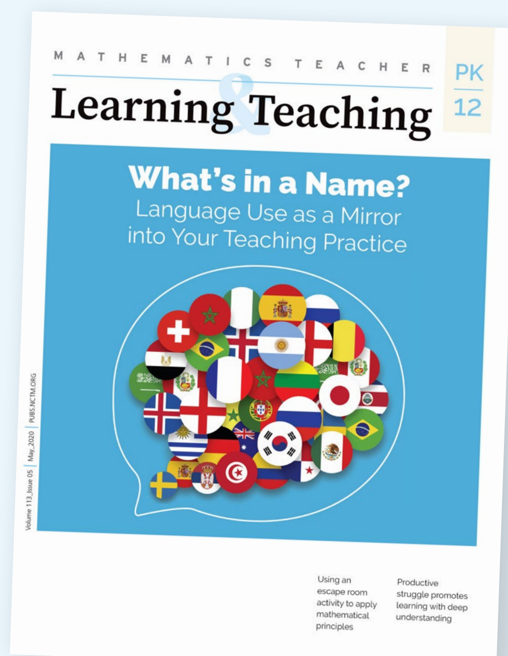
CONTACT: mtlt@nctm.org



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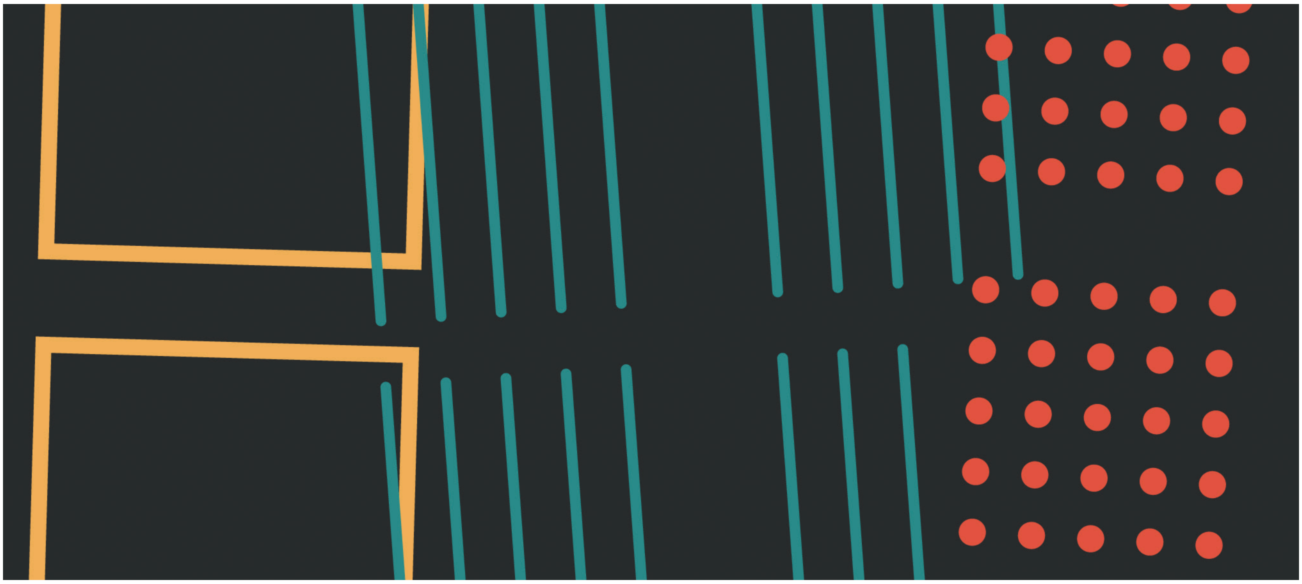

Accessible Standard Algorithms

Learn accessible standard algorithms that students can understand, carry out easily, and bring to fluency. Students initially make drawings that facilitate understanding and explaining.

Karen C. Fuson, Shannon Kiebler, and Robyn Decker

In our experience with hundreds of classrooms across the nation, we have found that having students learn accessible standard algorithms by explaining them using mathematics drawings increases students' sense of place-value numbers and enables students to articulate their understanding of what is actually happening with the numbers and why. We want to share our experiences to enable teachers everywhere to support their students to have similar understanding. Our goals here are to provide answers to these questions:

- What is a standard algorithm?
- When should students learn standard algorithms?
- Which algorithms should be learned and why?

The accessible algorithms we have been teaching have been recommended for many years (e.g., Fuson, 2003a, 2003b, 2020; Fuson & Beckmann, 2012/2013; National Research Council, 2001), but they have not been used widely, largely because of misunderstandings about the three questions listed above. We seek to eliminate these misunderstandings so that teachers can be freed to teach algorithms that are accessible to students, parents, and teachers. These standard algorithms can be understood, explained, and remembered.

WHAT IS A STANDARD ALGORITHM?

The Numbers and Operations in Base Ten (NBT) Progression document (The Common Core Writing Team, 2015) defines a standard algorithm as follows:

> Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit (p. 3).

The NBT Progression goes on to say, "Standards do not specify a particular standard algorithm for each operation" (p. 3). Over the years, there have been many algorithms taught in the United States, and many algorithms are now taught around the world. In this article, we will discuss three standard algorithms for multidigit addition: the common one and two options that we have found to be more accessible to students. The accessible standard algorithms that we will discuss were each originally invented by students with whom we were working. They were then successfully introduced to many other students by teachers in classroom settings from all parts of the U.S., ranging from high to low poverty, and including English speakers and multilingual learners.

WHEN SHOULD STUDENTS LEARN STANDARD ALGORITHMS?

In the initial grade in which a multidigit calculation is introduced, the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) specify that students are to:

- use concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction and relate the strategy to a written method and explain the reasoning used. [Grade 1 addition 1.NBT.4 and Grade 2 addition and subtraction 2.NBT.7 and 9];
- illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. [Grade 4 multiplication and division 4.NBT.5 and 6].

There is thus an important emphasis on meaning making and understanding by students when multidigit calculations are first introduced.

However, there is also an emphasis on generalizability in this same initial grade level for a given multidigit computation. The Common Core State Standards specify in the critical areas on the first page for these initial grade levels that students are to "develop, discuss, and use efficient, accurate, and generalizable methods" for adding in Grade 1, for subtracting in Grade 2, and for multiplying/dividing in Grade 4 (NGA Center & CCSSO, 2010). Therefore, from the very beginning grade for any

Karen C. Fuson is professor emerita of Northwestern University, has served on several National Research Council Committees, and worked on the Common Core State Standards and on its progressions. She has published over 80 research articles on mathematics teaching and learning and authored the PK–6 mathematics program *Math Expressions*. She has developed many learning resources for teachers that are free and available on her website, karenfusonmath.com.

Shannon Kiebler, shannon@empowerconsulting.org, is a former classroom teacher and mathematics coach turned professional development specialist, children's book author, and owner of an online professional learning platform. She dedicates her time to collaborating with teachers and teacher leaders. With a focus on making mathematics education engaging, visual, and conceptual, Shannon actively provides professional learning in elementary classrooms and online environments.

Robyn Decker, rdecker@oaisd.org, a mathematics and professional learning consultant based in Grand Haven, MI, collaborates with educators to enhance teaching strategies. Passionate about empowering teachers, she believes that impactful learning experiences for students stem from continuous and powerful professional learning.

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kind of multidigit calculation, teachers need to help students make sense of and explain written calculation methods that are efficient, accurate, and generalizable. Such methods are standard algorithms because their efficiency and accuracy come from using single-digit calculations with place value meanings, and their generalizability comes from the structure of place value as repeated groups of 10 from place to place. The sense making—from creating mathematics drawings and connecting steps in such drawings to steps in written methods—enables students to develop, discuss, and use accessible standard algorithms with understanding. We have seen, in hundreds of classrooms, students from a wide range of backgrounds succeed in explaining these accessible algorithms.

Historically, algorithms often have been taught as a meaningless, step-by-step procedure. Therefore, some educators think that algorithms are "bad" or "difficult" and that teaching algorithms should be delayed as late as possible or not taught at all. However, accessible standard algorithms can and should be taught with meaning from the beginning grade at which a kind of calculation is introduced.

WHICH ALGORITHMS SHOULD BE LEARNED AND WHY?

Fuson and Beckmann (2012/2013) and Fuson and Li (2014) identified criteria for choosing which algorithms should be emphasized in the classroom (see Table 1).

The accessible standard algorithms have most of these criteria. Their counterpart common algorithms that are often employed as the standard algorithms lack several of these criteria and also can stimulate errors or misunderstandings.

The accessible algorithms may initially seem strange to you because you have years of experience with other algorithms that are now very familiar to you. Engaging in productive struggle to understand these two accessible algorithms can be very empowering for teachers. As one group of urban teachers said when they were piloting teaching these algorithms, "Do you know what we call this program? We call it Math Therapy for Teachers, because we are finally understanding how algorithms work and we can explain them." Students learning these accessible algorithms as their initial algorithms do not have interference from having practiced different algorithms as teachers do. However, students who have

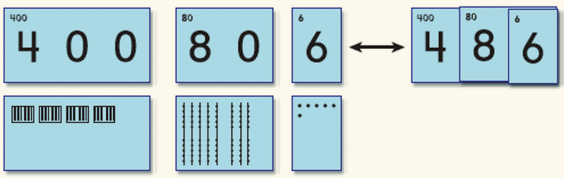
Table 1 Criteria and Reasons for Accessible Standard Algorithms

Criteria	Reason
1. Supports and uses place value correctly:	Because understanding and using place value is crucial.
a. Aligns like places (like units),	So that adding/subtracting like units is easier.
b. Makes it easy to see any -teen totals for the new grouped unit,	So that students understand that the teen total is added to the correct place values.
c. Makes it easy to see where to write the new grouped unit,	So that errors are not introduced by visual confusion.
2. Allows children to write -teen numbers in their usual order (e.g., for 14, write the '1' and then the '4'),	Because not doing so can introduce errors as digits are written in the wrong place.
3. Makes single-digit computations easier,	Because single-digit computation is so important in using algorithms.
4. All of one kind of step is done first, and then the other kind of step is done rather than alternating steps,	Because alternating steps can introduce errors as students continue one kind of step, instead of changing to the alternate step.
5. Keeps the initial multidigit numbers unchanged,	Because such algorithms are conceptually clearer.
6. Can be done left to right,	Because many students prefer to calculate from left to right in the same direction as they read words.

learned a common standard algorithm or any other method can understand the accessible algorithms if each of the algorithms is shown in mathematics drawings with steps in the drawings related to steps in the written algorithms, as shown in Figure 3. If these are older students, teachers can emphasize exploring and explaining different methods, rather than just learning a new method to get an answer.

Visual representations for quantities and for written numerals. Making algorithms accessible begins with using visual representations and connecting those representations with the written symbolic method. Connecting drawings to algorithms is addressed in the next section. Here, we highlight a tool that helps students understand the quantities in the written mathematics notation. We call these Secret Code Cards because they show the secret code of numbers. These cards can be layered on top of each other to show the number as it is written in single digits or taken apart to see the place values in the expanded notation of the number (see Figure 1). The numeral side has tiny numbers on the top left so that the place-value expanded notation version is shown even when the cards are layered to just show single digits. The Show All Totals standard algorithm can be developed by making layered cards to show the problem and then unlayering the cards to show written notation for hundreds (100 + 100), tens (80 + 50), and ones (9 + 7). The Secret Code Cards help students use both names for groups of ten, the English word *eighty* for 80 and the tens word *8 tens*. The backs of the cards show hundred-boxes, ten-sticks, and ones-circles so that when these backs are layered they show mathematics drawings, such as in Figure 3.

Figure 1 Secret-Code Cards Layer and Unlayer to Connect Single Digits to Place Value Expanded Notation

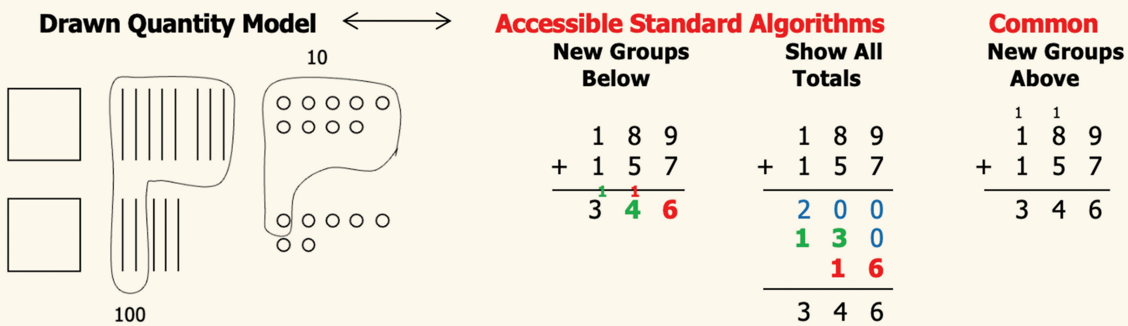


Note. Templates to make these are available for download; see Handout 1 (link online). From *Math Expressions, Common Core*. Copyright ©2018 by Houghton Mifflin Harcourt Publishing Company. Reprinted by permission of the Publisher. All rights reserved.

Accessible Algorithms for Multidigit Addition

The accessible standard algorithms for multidigit addition are shown in Figure 2, followed by the common standard algorithm; a mathematics drawing that can direct all of these algorithms is shown on the left of Figure 2. Students create math drawings on a centimeter grid of dots, with a vertical stick through ten dots and a box around ten of these ten-sticks, which also are 100 dots (a template is available for download; see Handout 2 [link online]). Students then begin to draw ten-sticks without dots on them, but they can imagine the ten dots on ten-sticks they draw if needed. Children may draw the ones horizontally or vertically; five-groups are emphasized in their program from the beginning because drawings using these five-groups

Figure 2 Multidigit Addition Math Drawings, Two Accessible Standard Algorithms, and a Common Standard Algorithm That Is Less Accessible

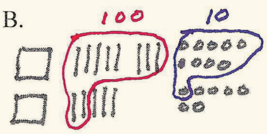
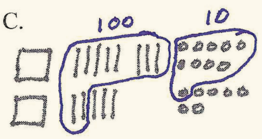


enable students to see the amounts in the math drawing clearly. These five-groups also support single-digit strategies, such as make-a-ten, where a student might say, "I can see that nine needs one more to make ten" and draw around the nine and one more to make a ten.

The New Groups Below accessible standard algorithm is a variation of the common standard algorithm: instead of writing the new one 10 and new 100 above the problem, these new groups are written below the problem on the line under it. This change makes the New Groups Below standard algorithm more understandable and easier to carry out. The steps in these two standard algorithms are shown in Figure 3. The steps are listed in Table 2 for each of these that meet the criteria outlined in Table 1.

Both the New Groups Below and the New Groups Above meet Criterion 1a. However, New Groups Below meets five other criteria (1b, 1c, 2, 3, 5), while New

Figure 3 Steps in the New Groups Below and in the New Groups Above Standard Algorithms Linked to Math Drawings of the Place-Value Quantities

Math Drawing for Sense-Making	New Groups Below	New Groups Above
A.  I add 9 ones and 7 ones. 9 needs 1 more to make a ten, so I get 1 from the 7 and that leaves 6.	189 + 157, written below: 1 6. I write 1 ten in the tens place and then 6 ones in the ones place.	1 above; 189 + 157 = 6. I write the 6 ones below and then write the 1 ten above in the tens place.
B.  8 tens and 2 more tens make ten tens so I circle these to to show my new 1 hundred in the 13 tens.	189 + 157, written below: 1 1, 46. 8 tens and 5 tens make 13 tens and 1 more ten makes 14 tens. I write the 1 for the 10 tens in the hundreds place and write the 4 tens in the tens place.	1 1 above; 189 + 157 = 46. 1 ten and 8 tens is 9 tens. 9 tens and 5 tens is 14 tens. I write 4 tens in the tens place and write the 1 for the ten tens in the hundreds place.
C.  I count the hundreds including the new hundred I made from ten tens.	189 + 157, written below: 1 1, 346. 1 hundred and 1 hundred make 2 hundreds and the 1 more hundred I got from the ten tens is 3 hundreds which I write in the hundreds place.	1 1 above; 189 + 157 = 346. 1 hundred up here above and 1 hundred make 2 hundreds and one more hundred make 3 hundreds which I write in the hundreds place.

Groups Above meets no other criteria. Clearly, New Groups Below is more accessible and warrants emphasis in the classroom, but does this mean that New Groups Above should not be taught? In our experience, students do bring algorithms—including New Groups Above—from home, and these algorithms all deserve discussion and explanation. Such an explanation validates the student using that algorithm and may bring up interesting mathematical issues. We believe that students should be free to use any algorithm that they can explain and use accurately.

The Show All Totals standard algorithm shown in Figure 2 meets the same criteria as does New Groups Below, but also meets Criteria 4 and 6. In Show All Totals, students do all adding of the single digits in each place value first, and then add those partial sums to find the total. Some students like this algorithm because they like to see the place values for each total written out. Students can do Show All Totals from the left or from the right. For larger numbers, they may make errors if the columns are not carefully aligned. This algorithm might be considered a Helping Step algorithm that helps students see that they are adding hundreds, tens, and ones in other algorithms as well, but this algorithm may need to be replaced with New Groups Below in Grades 3 or 4.

Accessible Standard Algorithms for Multidigit Subtraction, Multiplication, and Division

Fuson, Kiebler, and Decker (2024) describe accessible standard algorithms for multidigit subtraction, multiplication, and division, as well as drawings that support understanding of these accessible standard algorithms. For more about these algorithms, see Table 3.

Table 2 Steps in New Groups Below and in New Groups Above That Meet Criteria for an Algorithm That is Accessible and Should Be Taught

New Groups Below	New Groups Above
YES Criterion 1a: Aligning like places above each other makes adding easier.	**YES Criterion 1a:** Aligning like places above each other makes adding easier.
YES Criterion 1b: It is easy to see the teen totals 16 ones and 14 tens, because the digits are written next to each other.	**NO** The teen totals 16 ones and 14 tens are difficult to see as teen numbers, because they are written so far apart.
YES Criterion 1c: It is easy to see where to write the new one ten and new one hundred, because they are written below the ten and hundred columns. Students can visually trace down the column to write the new 10 or 100 below that column.	**NO** Students have more difficulty writing a number above a column than tracing down a column to write a number below; some say that it is just up there somewhere. Also, some students think they write all new groups above the leftmost column.
YES Criterion 2: Students can write teen numbers such as 16 and 14 in their usual order 1 ten and then the ones.	**NO** Students often are urged to write 6 ones and then carry (regroup) the 1 ten, so they write 16 in reverse order as 6 and then 1.
YES Criterion 3: Adding the tens is easy, because you add the two big numbers 8 and 5 first, and then increase that total by one.	**NO** So that students do not forget to add the new 1, teachers often say to add the 1 to the 8, and then add that 9 to the 5. So students have to add a number that is not there (9) and ignore a number that is there (8).
NO Criterion 4: Alternating steps are used.	**NO** Alternating steps are used.
YES Criterion 5: The problem is not changed, because the new groups of ten and hundred are below in the answer space.	**NO** Some students say that this algorithm changes the problem by writing the 1s above the problem. This bothers them.
NO Criterion 6: This method does not go left to right.	**NO Criterion 6**: This method does not go left to right.

The multiplication and division accessible standard algorithms, and the drawings for these algorithms, are described in more detail in Fuson and Beckmann (2012/2013).

More examples and explanations. Table 3 lists resources for watching students explaining math drawings and various standard algorithms. All the students shown in the videos are from Spanish-speaking backgrounds, and many experience poverty. They are wearing uniforms because their public school requires them to do so to decrease gang activity. The two supplementary files are also available in Fuson, Kiebler, and Decker (2024).

Whether, What, When, and Which Standard Algorithms?

Edwards and Robichaux-Davis (2022) reported that their middle school students cannot explain standard algorithms and thus these authors would eliminate standard algorithms entirely and have students use invented methods. We agree that the common standard algorithms discussed above and in Fuson, Kiebler, and Decker (2024) each have issues that can make them difficult to understand and can also increase errors. However, discussing accessible standard algorithms enables students to understand place value more deeply and be more accurate. These are valuable goals.

As discussed above, we suggest that students discuss accessible standard algorithms with step-by-step math drawings to support understanding and explanation. These should be taught in the initial grade for each multidigit computation: Grade 1 for addition, Grade 2 for subtraction, and Grade 4 for multiplication and division. Teachers should continue to discuss and explain these accessible standard algorithms in subsequent grades, and each student can and should stop making math drawings when they are no longer needed. Fluency is using a written algorithm without a supporting drawing, but then students should occasionally connect their accessible standard algorithm to a math drawing to keep understanding alive. We have found

Table 3 Resources for Teaching Accessible Standard Algorithms

What to see	Where to go
To watch first-graders explaining math drawings and the three multidigit addition standard algorithms in Figure 1	Visit https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play Grade 1.
To watch the first author explaining math drawings and the three standard algorithms in Figure 1 along with students from several grades solving problems	Visit https://karenfusonmath.net/classroom-videos/#B-Math-Explanations and play Multidigit Addition.
To watch third-graders explain 3-digit subtraction accessible standard algorithms	Visit https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play G3 Multidigit Subtraction.
To watch fifth-graders explain 7-digit subtraction accessible standard algorithms	Visit https://karenfusonmath.net/classroom-videos/#G-Place-Value-and-Multidigit-Addition-and-Subtraction and play the last video G4 Explaining 7-digit subtraction.
To watch fourth-graders explaining all three multiplication accessible standard algorithms	Visit https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play the fourth video 4 G4 Multidigit Multiplication.
To watch the first author explaining math drawings and the three multiplication accessible standard algorithms with examples of students using each algorithm	Visit https://karenfusonmath.net/classroom-videos/#B-Math-Explanations and play Multidigit Multiplication.
For a description of extensions of the multidigit accessible standard algorithms to decimals	Visit https://karenfusonmath.net/teaching-progressions/ and scroll down to Numbers Base Ten, Place Value Parts 4, 5, 6, 7.

that students readily recover their earlier understandings if they draw or imagine drawing quantities for a problem.

Student-Invented Methods

Some students invent their own multidigit methods, especially if they can make math drawings that allow them to make sense of the quantities. In our experience, these methods may not be very efficient or generalize well to larger numbers. For addition and subtraction, invented methods often involve extensions from small whole numbers, such as making groups for 35 and 27 and then counting all the objects. Other methods involve counting on or counting down from one whole number, such as 35, 45, 55, 56, 57, 58, 59, 60, 61, 62. These are not standard algorithms because they do not involve single-digit operations of the place values. We have found in many different classrooms that the counting on or counting down methods are mostly comprehensible for two-digit numbers but get difficult for many students with three-digit numbers and are even more difficult for larger numbers. Therefore, students using these methods with small numbers need to discuss and learn an accessible standard algorithm along with their invented method so that they can easily solve problems with larger numbers.

Accessible Standard Algorithms at Any Grade

Teachers can discuss these accessible standard algorithms at any grade level. Students need to learn to make math drawings and connect them to written algorithms to make these algorithms understandable. This method helps students build deeper understanding of place value and of written algorithms. Also, accessible standard algorithms can easily be extended to operations with decimals, making those operations more understandable and deepening understanding of place value on both sides of the decimal point (Beckmann & Fuson, 2014).

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