How Word Problems Can Strengthen Language Development Using Math Talk

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a PK to Grade 6 math program

Please see my website karenfusonmath.net or karenfusonmath.com for 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, presentations, supports for teaching remotely, and extra papers and materials for *Math Expressions* users or for anyone.

Teaching Progressions on karenfusonmath.com or karenfusonmath.net

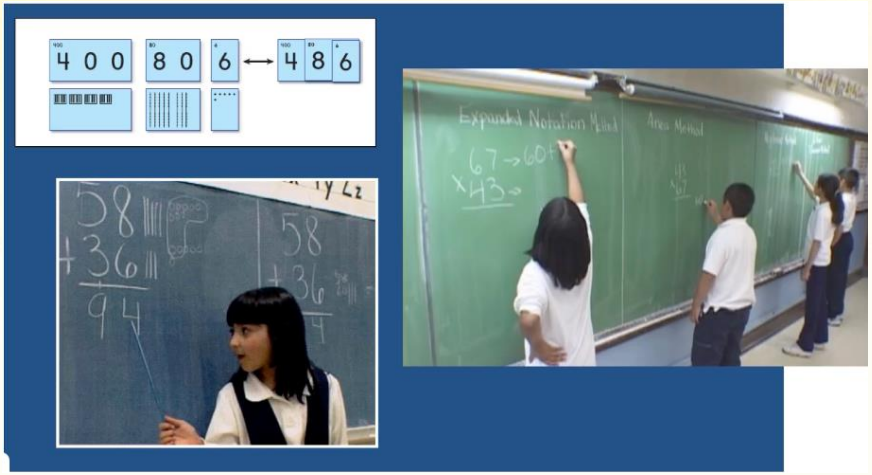
A Functioning Math Talk Community

Source of Math Ideas

- The students contribute their ideas as the teacher or other students are teaching, confident that their ideas are valued.
- The students spontaneously compare and contrast and build on ideas.
- The teacher is still engaged and deciding what is important to continue exploring.

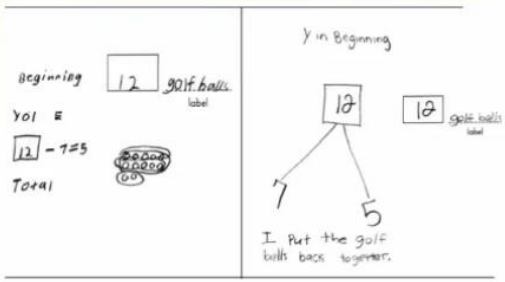
Responsibility for Learning

- The students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others.
- The students assist each other in understanding and correcting errors.
- The teacher supports students as they help one another.



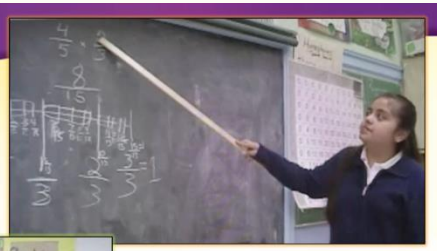
Yolanda has a box of golf balls. Eddie took 7 of them. Now Yolanda has 5 left. How many golf balls did Yolanda have in the beginning?

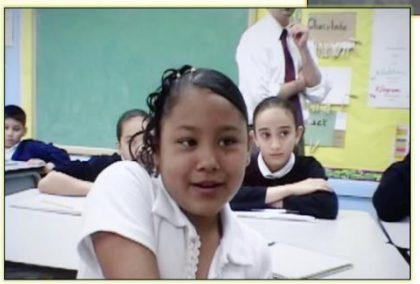
The key to solving story problems is understanding the situation. Students' equations often show the situation rather than the solution. Students drawings should be labeled to show which numbers or objects show which parts of the story situation.

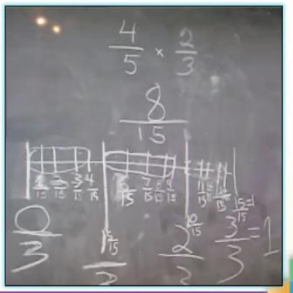


Math Talk Structure

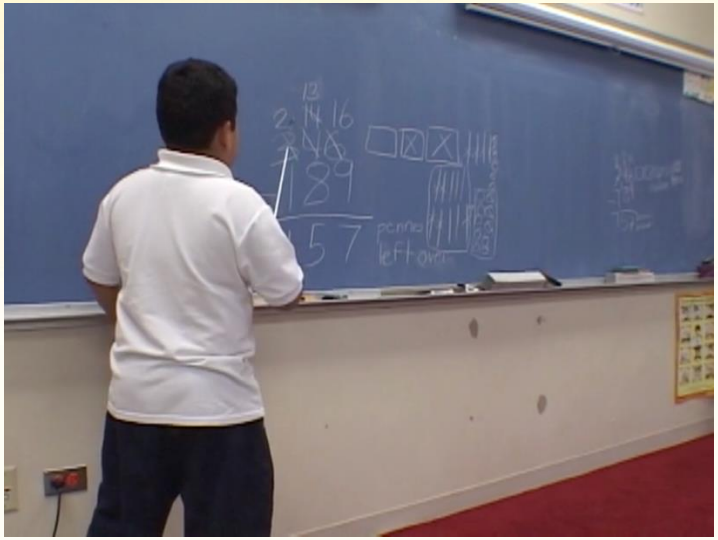
- Solve
- Explain
- Question
- Justify

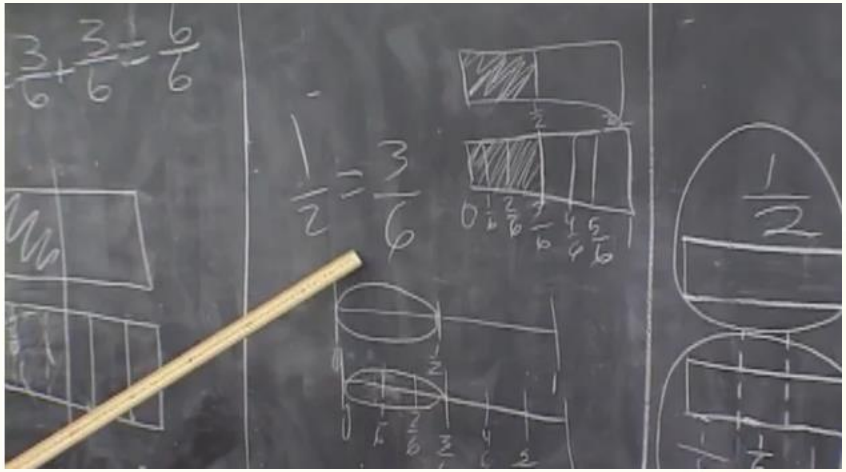






Classroom Videos on karenfusonmath.com or karenfusonmath.net





Building a Nurturing Math Talk Community

Representing and Solving Word Problems

Do Math Talk not Number Talks

Building a Nurturing Math Talk Community

The teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of students, using these **responsive** means of assistance that facilitate learning and teaching by all:

Engage and involve

Manage

Coach

model, clarify, instruct/explain, question, give feedback

The teacher helps students learn to use the responsive means of assistance with classmates.

A nurturing meaning-making visual Math Talk Community

is an inquiry-based teaching/learning environment, and has a continual focus on sense-making by all participants.

Students are expected:

- to understand what they are doing,
- come to be able to explain their thinking,
- understand the thinking of other students,
- learn to seek help when they need it, and
- help others who need it.

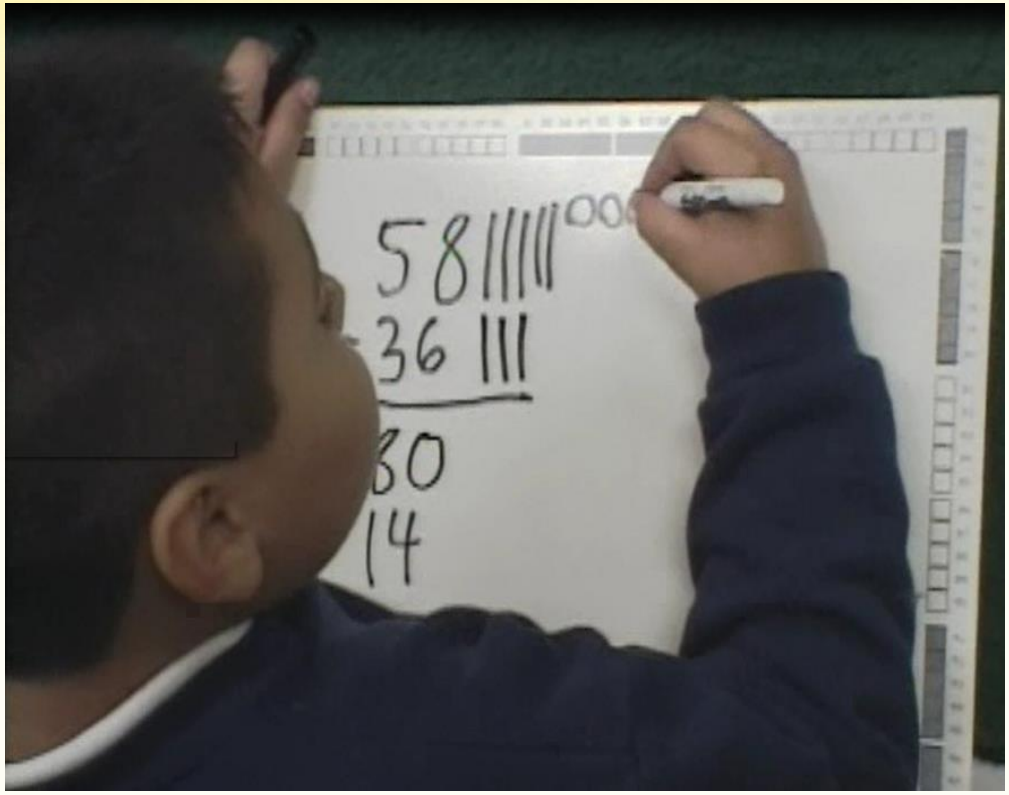
Solve and Discuss Classroom Structure

Solve	Explain	Question	Justify
All students solve. Some solve at the board, and the rest at their seats.	**One student at the board** explains and then asks, "Are there any questions?"	**Other students** ask questions to clarify or extend.	The original explainer **responds to the questions** by explaining more (justifying the original explanation).

Any student at any time can ask for help from anyone.

For more practice, Solve and Discuss can take place in pairs or small groups.

Make the math thinking visible

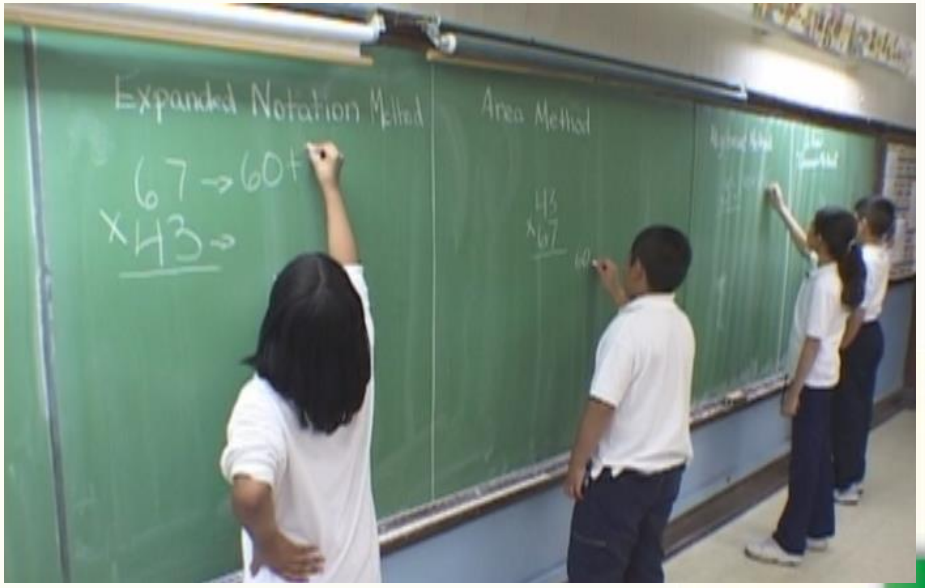


- Students must make some kind of math drawing related to the math symbols to show their thinking.
- This supports understanding by the listeners and promotes meaning.

Make the math thinking visible

- This is important for equity: less advanced students and English Learners are helped by the math drawing linked to the explanation by pointing.

- Be sure that important methods remain on the board or can be made visible again (e.g., on a Math Board) so they can be compared with other methods.



Students must speak and not just listen

2. “Bite your tongue” to provide wait time. Students will explain, ask questions, or add a comment if you wait.

1. Structure opportunities to explain to a partner and repeat what the partner says, if needed. Students eventually find their own words, but may need the security of saying an explanation they know is correct.

3. Help students speak to classmates by moving to the side or back of the room. Later remind students with a silent gesture to address each other.

Mathematical Practices			
Math Sense-Making	**Math Structure**	**Math Drawings**	**Math Explaining**
Make sense and use of appropriate precision.	**See structure and generalize.**	**Model and use tools.**	**Reason, explain, and question.**
MP1 Make sense of problems and persevere in solving them. **MP6** Attend to precision.	**MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	**MP4** Model with mathematics. **MP5** Use appropriate tools strategically.	**MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others.
Teachers continually assist students to do math sense-making about math structure using math drawings to support math explaining.			

Teachers continually assist students to do **math sense-making** about **math structure** using **math drawings** to support **math explaining**.

Representing and Solving Word Problems

Word problems often are not given to students who have weak English language skills or are achieving below grade level.

But that is the opposite of what is necessary.

Word problems are small fairly predictable linguistic situations that can help children develop mathematical and everyday language.

The 6 Situations

K — Add To / Take From

Start + Change = Result
Start – Change = Result

→ (becomes)

K — Put Together / Take Apart

Total

Addend (Partner) Addend (Partner)

T = A + A
$\overset{i}{=}$ (identical)

Gr1 — Additive Comparison

Big

Small Difference

Small + Difference = Big
Big – Difference = Small
Big – Small = Difference
$\overset{n}{=}$ (same number)

Gr3 — Equal Groups

Product

M x

G G G G

G + G + G + G = P
M times

M x G = P

→ (becomes)

Gr3 — Rectangular Everything Times Everything

Array

Factor

Factor Product

(Long Division Format)

Area

Factor

Factor Product

Factor

Product

F x F = P
P ÷ F = F

(table)

$\overset{i}{=}$ (identical)

Gr4 — Multiplicative Comparison

Big = 3 x Small

Big 5 5 5

Small 5

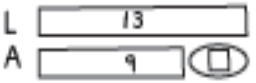
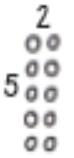
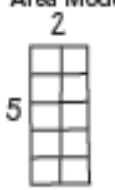
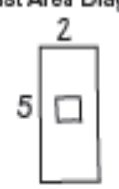
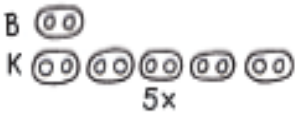
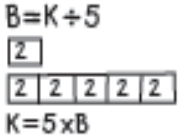
Small = $\frac{1}{3}$ x Big

Big ÷ 3 = Small

$\overset{n}{=}$ (same number)

CCSS Addition (top row) and Multiplication (bottom row) Word Problem Situations and *Math Expressions* Diagrams for Each

Math drawings, diagrams, and equations make word problems accessible:
Label these for sense making, correct solving, and language development.

Problem Type	Word Problem	Representation	
		Math Drawing	Diagram
Add To	Dan had 9 cherries. Then he picked 4 more. How many does he have now?		9+4=□ (situation/solution equation)
Take From	Dan had 13 cherries. Then he ate 9 of them. How many does he have now?		13−9=□ (situation/solution equation)
Put Together/ Take Apart	Ana has 9 dimes and 4 nickels. How many coins does she have in all?		Math Mountain Diagram 
Additive Comparison	Ali has 9 balloons. Lisa has 13 balloons. How many more balloons does Lisa have than Ali?	Matching Drawing 	Comparison Bars 
Equal Groups	Amy has 5 cousins. She is making 2 puppets for each cousin. How many puppets will Amy need to make?	Grouping Model 	Equal Shares Diagram 
Array	A garden has 5 rows and 2 columns of bean plants. How many plants are there in all?	Array Model 	Fast Array Diagram 
Area	The garden is 5 yards on one side and 2 yards on the side touching this. What is its area?	Area Model 	Fast Area Diagram 
Multiplicative Comparison	Bill has 2 apples. Kim has 5 times as many apples as Bill. How many apples does Kim have?	Grouping Model 	Comparison Bars 

The Problem Solving Process

Part A: Understand and represent: Conceptualize bottom up from the situation

Part B: Re-represent and solve: Use related problem types, representations, properties, and /or relationships between + - or x÷

A1. Understand the problem situation

Mathematize (and Storyize)

A2. Represent the problem situation in a drawing/diagram and/or an equation

Then focus on the question and:

B1. Re-represent to find the unknown

Do the solution actions

B2. Write the answer and check that it makes sense

Any of the three quantities can be the unknown quantity. This creates the Problem Difficulty Learning Path:

K is dark grey.
G1 is grey.
G2 is white.

	Result Unknown	Change Unknown	Start Unknown
Add To	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? A + B = ☐	*A* bunnies were sitting on the grass. Some more bunnies hopped there. Then there were *C* bunnies. How many bunnies hopped over to the first two? A + ☐ = C	Some bunnies were sitting on the grass. *B* more bunnies hopped there. Then there were *C* bunnies. How many bunnies were on the grass before? ☐ + B = C
Take From	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? C – B = ☐	*C* apples were on the table. I ate some apples. Then there were *A* apples. How many apples did I eat? C – ☐ = A	Some apples were on the table. I ate *B* apples. Then there were *A* apples. How many apples were on the table before? ☐ – B = A

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	*A* red apples and *B* green apples are on the table. How many apples are on the table? A + B = ☐	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? C = ☐ + ☐	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? A + ☐ = C C – A = ☐

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has *A* apples. Julie has *C* apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has *A* apples. Julie has *C* apples. How many fewer apples does Lucy have than Julie? A + ☐ = C C – A = ☐	*"More" version suggests operation.* Julie has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have? A + B = ☐	*"Fewer" version suggests operation.* Lucy has *B* fewer apples than Julie. Julie has *C* apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have? C – B = ☐ ☐ + B = C

The easiest problems: Representing them creates the solution.

Problem Type	Word Problem	Representation	
		Math Drawing	Diagram
Add To	Dan had 9 cherries. Then he picked 4 more. How many does he have now?		9+4=□ (situation/solution equation)
Take From	Dan had 13 cherries. Then he ate 9 of them. How many does he have now?		13−9=□ (situation/solution equation)
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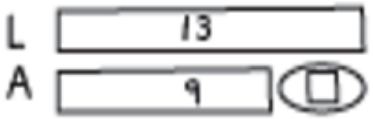
Additive Comparison	Ali has 9 balloons. Lisa has 13 balloons. How many more balloons does Lisa have than Ali?	Matching Drawing 	Comparison Bars 

Most word problems given to students in the United States are these easy kinds that can be solved by kindergarten children.

Students in Russian, East Asia, and many other parts of the world solve problems with each of the quantities unknown.

These are algebraic problems that require visual representation and then reflection on the representation.

Students from all backgrounds can solve these more difficult problems if they are supported to do so.

Types of Addition and Subtraction Situations

Add To/Take From Situations

Add To

Result Unknown $9 + 4 = \square$

Dan had 9 cherries. Then he picked 4 more. How many does he have now?

Change Unknown $9 + \square = 13$

Dan had 9 cherries. Then he picked some more. Now he has 13 cherries. How many did he pick?

Start Unknown $\square + 4 = 13$

Dan had some cherries. Then he picked 4 more. Now he has 13 cherries. How many did he start

Take From

Result Unknown $13 - 9 = \square$

Dan had 13 cherries. He ate 9. How many does he have now?

Change Unknown $13 - \square = 4$

Dan had 13 cherries. Then he ate some of them. Now he has 4 cherries. How many did he eat?

Start Unknown $\square - 9 = 4$

Dan had some cherries. Then he ate 9 of them. Now he has 4 cherries. How many did he start with?

Language Development in the Math Class

Change some words but tell almost the same situation.

Tell the same action but use different words.

Change one word to make it be a different situation.

How else can you ask the question?

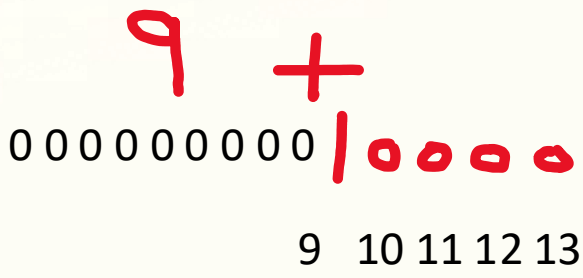
Changing which words do not change the equation?

Changing which words do change the equation?

Middle Difficulty Change Problems

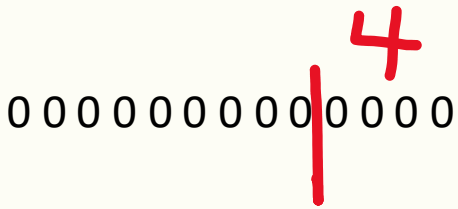
Change Unknown $9 + \square = 13$

Dan had 9 cherries. Then he picked some more. Now he has 13 cherries. How many did he pick?



Change Unknown $13 - \square = 4$

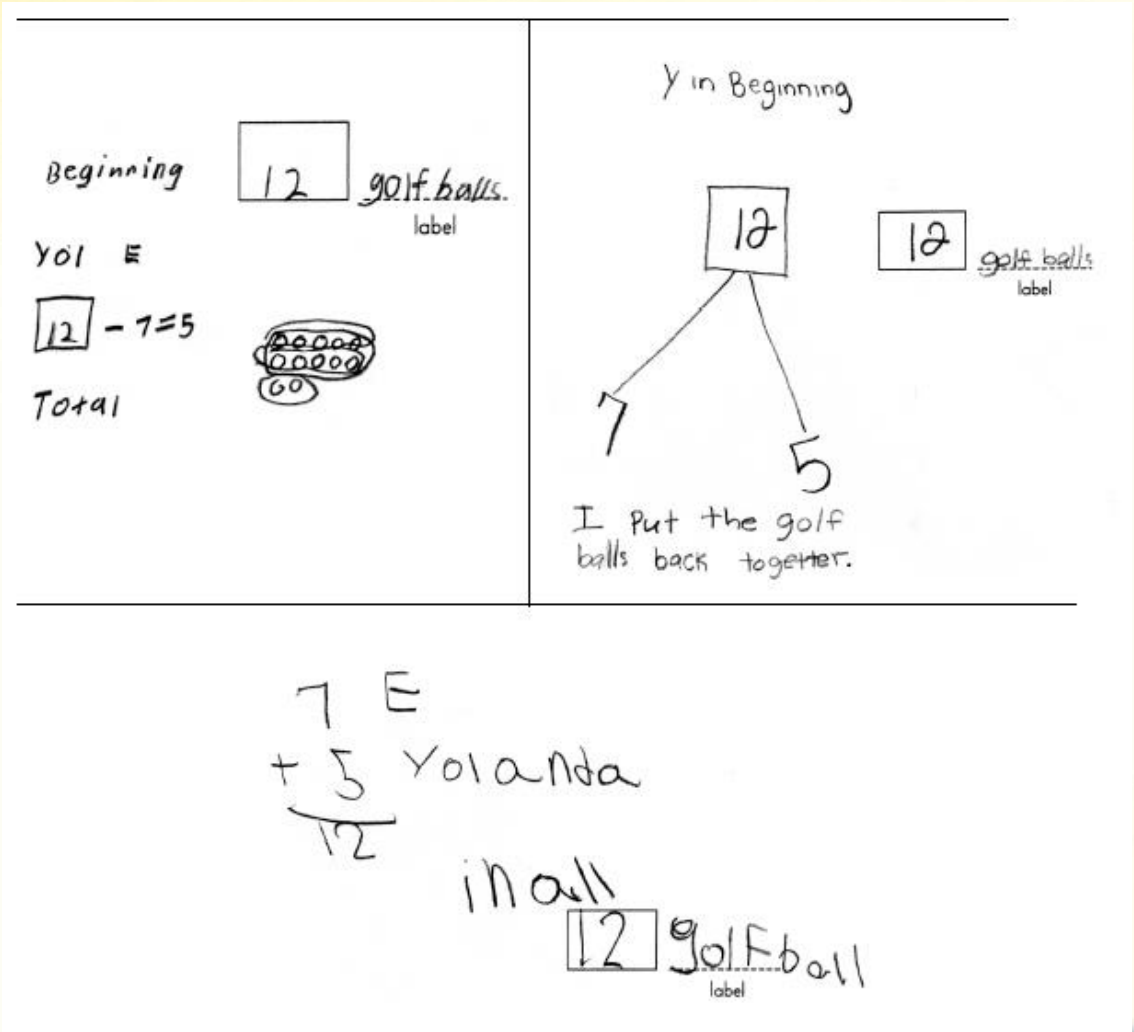
Dan had 13 cherries. Then he ate some of them. Now he has 4 cherries. How many did he eat?



Grade 2 Labeled Math Drawings for a Start Unknown Problem

Yolanda has a box of golf balls. Eddie took 7 of them. Now Yolanda has 5 left. How many golf balls did Yolanda have in the beginning?

The key to solving story problems is **understanding the situation.** Students' equations often show **the situation** rather than the solution. Students drawings **should be labeled** to show which numbers or objects show which parts of the story situation.



Grades 3 and 4: Represent a Start Unknown Situation

Situation Equation

Y	E	end
346	- 157	= 189

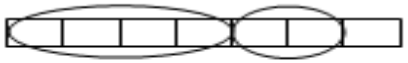
$$\begin{array}{r} 157 \\ +\ 189 \\ \hline 346 \end{array}$$

Y	E	left
$\frac{6}{7}$	$-\ \frac{4}{7}$	$= \frac{2}{7}$

total

$$\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}=\frac{6}{7}$$

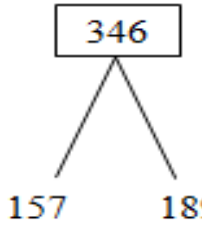
or



$$\frac{4}{7}+\frac{2}{7}=\frac{6}{7}$$

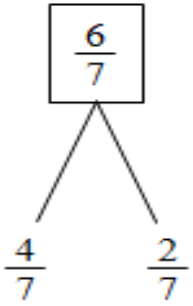
Numerical Relationships in Math Mountain

Y beginning



$$\begin{array}{r} 157 \\ +\ 189 \\ \hline 346 \end{array}$$

Y at first



Grades 3 and 4: Represent a Start Unknown Situation

Yolanda had a box of golf balls. Eddie took 157. Now Yolanda has 189 left. How many golf balls did Yolanda have in the beginning?

Yolanda had part of a submarine sandwich. Eddie took 4/7 of it. Now Yolanda has 2/7 left. How much of the submarine sandwich did Yolanda have in the beginning?

Solution

Computation

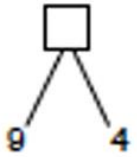
$$\begin{array}{rl} 157 & \text{Eddie} \\ +\ 189 & \text{Yolanda} \\ \hline 200 & \\ 130 & \\ 16 & \\ \hline 346 & \text{in all} \end{array}$$

$$\frac{4}{7} + \frac{2}{7} = \frac{6}{7} \quad \text{in all}$$

E Y B

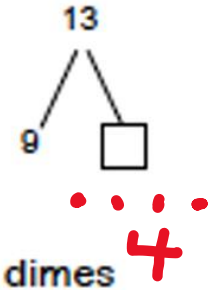
Put Together/Take Apart Situations

Total Unknown



Ana has 9 dimes and 4 nickels. How many coins does she have in all?

Addend Unknown



Ana has 13 coins. 9 are dimes and the rest are nickels. How many are nickels?

Either partner can be unknown.

This can be solved by counting on from 9 to 13:
9 10 11 12 13

Any of the three quantities can be the unknown quantity. This creates the Problem Difficulty Learning Path:

K is dark grey.
G1 is grey.
G2 is white.

	Result Unknown	Change Unknown	Start Unknown
Add To	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? A + B = ☐	*A* bunnies were sitting on the grass. Some more bunnies hopped there. Then there were *C* bunnies. How many bunnies hopped over to the first two? A + ☐ = C	Some bunnies were sitting on the grass. *B* more bunnies hopped there. Then there were *C* bunnies. How many bunnies were on the grass before? ☐ + B = C
Take From	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? C – B = ☐	*C* apples were on the table. I ate some apples. Then there were *A* apples. How many apples did I eat? C – ☐ = A	Some apples were on the table. I ate *B* apples. Then there were *A* apples. How many apples were on the table before? ☐ – B = A

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	*A* red apples and *B* green apples are on the table. How many apples are on the table? A + B = ☐	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? C = ☐ + ☐	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? A + ☐ = C C – A = ☐

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has *A* apples. Julie has *C* apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has *A* apples. Julie has *C* apples. How many fewer apples does Lucy have than Julie? A + ☐ = C C – A = ☐	*"More" version suggests operation.* Julie has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have? A + B = ☐	*"Fewer" version suggests operation.* Lucy has *B* fewer apples than Julie. Julie has *C* apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have? C – B = ☐ ☐ + B = C

Special Difficulties with Compare Language

The question can be asked in two ways.

The number is embedded next to the comparing word more than/fewer than:
Lisa has 4 more than Ali.
Lisa has more.
Those extra are 4 more.

Misleading language that suggests the wrong operation.

Compare Situations

Difference Unknown

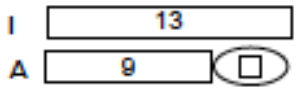
Ali has 9 balloons.
Lisa has 13 balloons.

How Many More?

How many more balloons does Lisa have than Ali?

How Many Fewer?

How many fewer balloons does Ali have than Lisa?



Bigger Amount Unknown

Ali has 9 balloons.

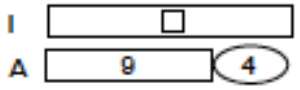
Leading Language

Lisa has 4 more than Ali.

Misleading Language

He has 4 fewer than Lisa.

How many balloons does Lisa have?



Smaller Amount Unknown

Lisa has 13 balloons.

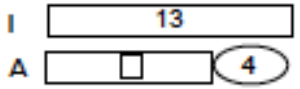
Leading Language

Ali has 4 fewer than Lisa.

Misleading Language

She has 4 more than Ali.

How many balloons does Ali have?



Situation Equations vs. Solution Equations

A situation equation shows the situation.

$\square - 7 = 5$ $\quad$ $189 + \square = 346$ $\quad$ $\square - 27 = 82$

A solution equation shows the solution operation.

$7 + 5 = \square$ $\quad$ $346 - 189 = \square$ $\quad$ $82 + 27 = \square$

Do Math Talk not Number Talks

The nurturing visual meaning-making Math Talk Community is all day every day.

It supports all students to represent, understand, and explain.

We have seen how important relating visual drawings and symbols are for sense making and problem solving.

Expanding Number Talks
to Build Equitable Math Talk Classrooms

by Karen Fuson and Steve Leinwand

The paper above appeared in the March 2023 NCTM journal Mathematics Teacher: Learning and Teaching PK to G12.

It describes how limiting Number Talks are and how they can be extended to Math Talk in all lessons by having students use and discuss drawings and written methods.

It discusses some kinds of drawings students can make for different math domains, so it is a good resource for you in starting your Math Talk Community.

Number Talks have these equity issues:

Only mental methods can be used so problems are often very easy.

Mental methods bias students toward counting on methods that do not generalize easily to larger numbers.

Students do not make drawings, so it is difficult for other students to see their thinking.

Students have to describe their methods in words. This is difficult for some.

Students do not write their own methods. The teacher writes their methods, implying that only the teacher can write or explain a method.

The talks are done in a separate part of the classroom away from regular math class. So what is happening in math class? Are students understanding, drawing, and explaining there? If so, why are Number Talks needed?

Some teachers who do Number Talks think that they have done Math Talk and teach the regular math class as a teacher demonstration.

Number Talks are not a Math Talk Community

If you are using Number Talks, then extend them by not doing the equity problems listed on the previous slide but instead start building your nurturing sense-making visual Math Talk Community in the regular math classroom.

If you are not using Number Talks, do not use them. It is much better not to do Number Talks at all but to start building your classroom to be a nurturing sense-making visual Math Talk Classroom. It is necessary to help children have some way of representing the math topic to be discussed. Math drawings are windows into minds, and they help everyone understand the Math Talk and develop language.

How Word Problems Can Strengthen Language Development Using Math Talk

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a PK to Grade 6 math program

Please see my website karenfusonmath.net or karenfusonmath.com for 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, presentations, supports for teaching remotely, and extra papers and materials for *Math Expressions* users or for anyone.