Building Understanding of Fraction Operations

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Details of fraction standards and these approaches can be found by grade level

in the Teaching Progressions for fractions

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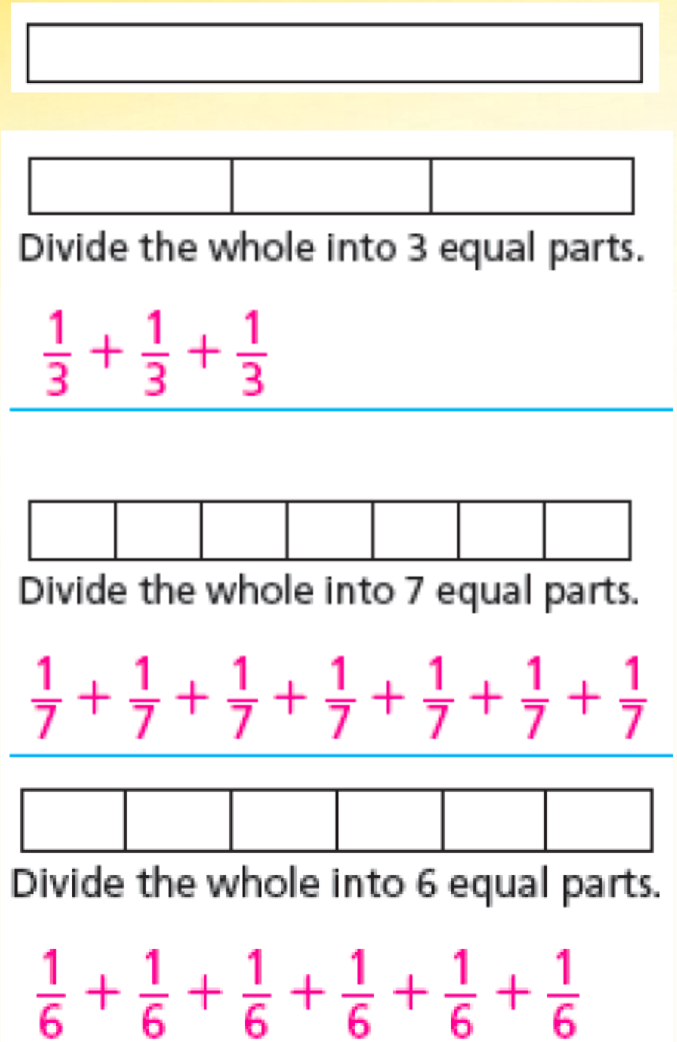
The typical fraction error is to call the above 2/3 instead of 2/5. This is because it is difficult for students to see the 2 parts embedded in the 5 wholes. They see part/part and not part/whole.

This is the same conceptual understanding that students needed to move from counting all to counting on: To solve 8 + 6 they needed to see the 8 embedded within the total and then they could count on 6 more.



Fractions are composed of unit fractions.

Unit fractions are what you count and add and subtract.

Students need to see unit fractions visually and in fraction notation 1/n and in fraction words.

They need to divide the 1 whole into n equal parts and name that part.

Fraction words do not help with fraction meanings.

Our fraction words are also used for ordinal number situations (except for second/half):

Third in a row or one third of a whole.

Fourth in a row or one fourth of a whole.

Fifth in a row or one fifth of a whole.

Sixth in a row or one sixth of a whole.

... or ...

4/6 is four sixths

Chinese students say: Of six parts (take) four.

Use two steps to make a fraction a/n.

First make unit fractions.

Then **make in a separate drawing** a given fraction **a**/n by shading or circling **a** unit fractions

Build Fractions from Unit Fractions

Write the unit fractions for each whole. Next, shade the correct number of parts. Then show each shaded fraction as a sum of unit fractions.

10 
Shade 2 parts.
Divide the whole into 3 equal parts.

$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

11 
Shade 5 parts.
Divide the whole into 7 equal parts.

$\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$ $\frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{5}{7}$

G4 Adding and Subtracting Common Error

Common error:

Add fractions by adding the tops and adding the bottoms

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{14}$$

Math Expressions **Solution**

1. Offer many experiences writing sums of unit fractions where the denominator stays the same number and include drawings.

2. Find partners of a whole to practice this concept and use drawings.

1. Show addition and subtraction models. Write the sum or difference above the unit fraction as a middle step.

Sums of Unit Fractions

Shade the fraction bar to show each fraction. Then write the fraction as a sum of unit fractions and as a product of a whole number and a unit fraction. The first one is done for you.

13. $\frac{4}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = 4 \times \frac{1}{7}$

$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

14. $\frac{7}{9} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 7 \times \frac{1}{9}$

$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

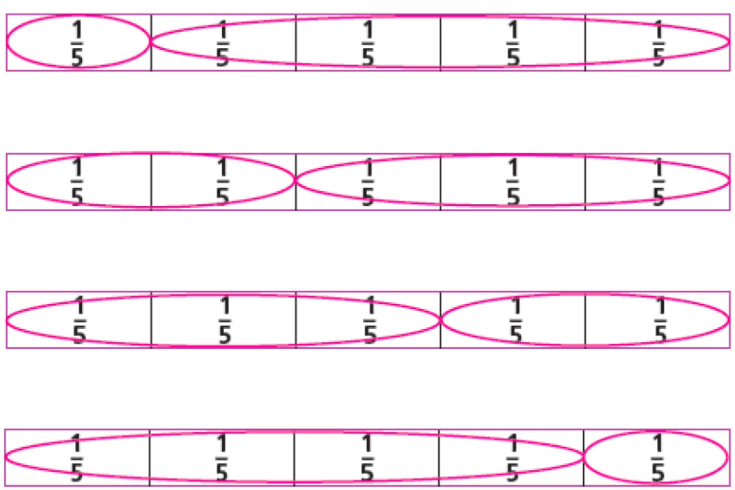
Fifths that Add to One

Every afternoon, student volunteers help the school librarian put returned books back on the shelves. The librarian puts the books in equal piles on a cart.

One day, Jean and Maria found 5 equal piles on the return cart. They knew there were different ways they could share the job of reshelving the books. They drew fraction bars to help them find all the possibilities.

1. On each fifths bar, circle two groups of fifths to show one way Jean and Maria could share the work. (Each bar should show a different possibility.) Then complete the equation next to each bar to show their shares.
Possible answers are shown.

1 whole = all of the books	1 whole	Jean's share	Maria's share



$\frac{5}{5} = \frac{1}{5} + \frac{4}{5}$

$\frac{5}{5} = \frac{2}{5} + \frac{3}{5}$

$\frac{5}{5} = \frac{3}{5} + \frac{2}{5}$

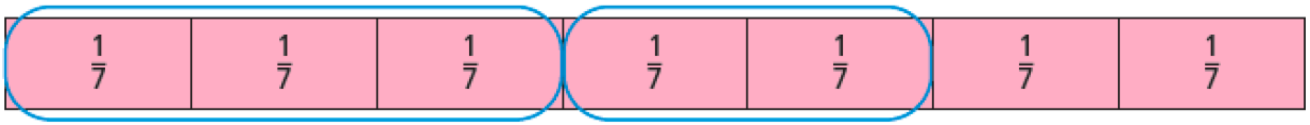
$\frac{5}{5} = \frac{4}{5} + \frac{1}{5}$

See the unit fraction (the denominator) stay the same but numerators add when decomposing one whole fraction as n/n.

Students saw the patterns in the numerators (the top numbers) in Kindergarten when they found partners of 5.

Add Fractions

The circled parts of this fraction bar show an addition problem.



1. Complete this addition equation to match the problem above.

$$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$$

Write the numerators to complete each addition equation.

2. $\frac{3}{9} + \frac{4}{9} = \frac{3+4}{9} = \frac{7}{9}$

3. $\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$

4. $\frac{2}{8} + \frac{5}{8} = \frac{2+5}{8} = \frac{7}{8}$

5. What happens to the numerators in each equation?

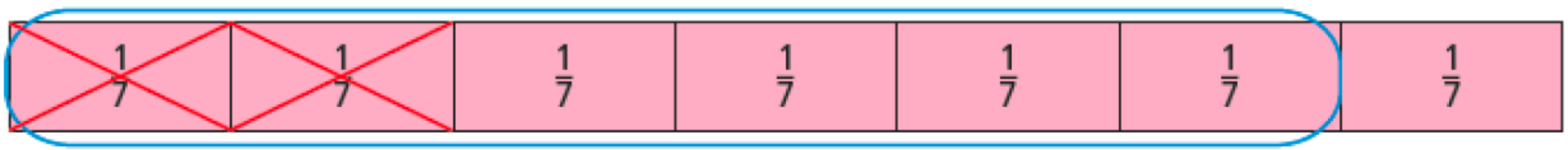
   The numerators are added together.

6. What happens to the denominators in each equation?

   The denominators stay the same.

Subtract Fractions

The circled and crossed-out parts of this fraction bar show a subtraction problem.

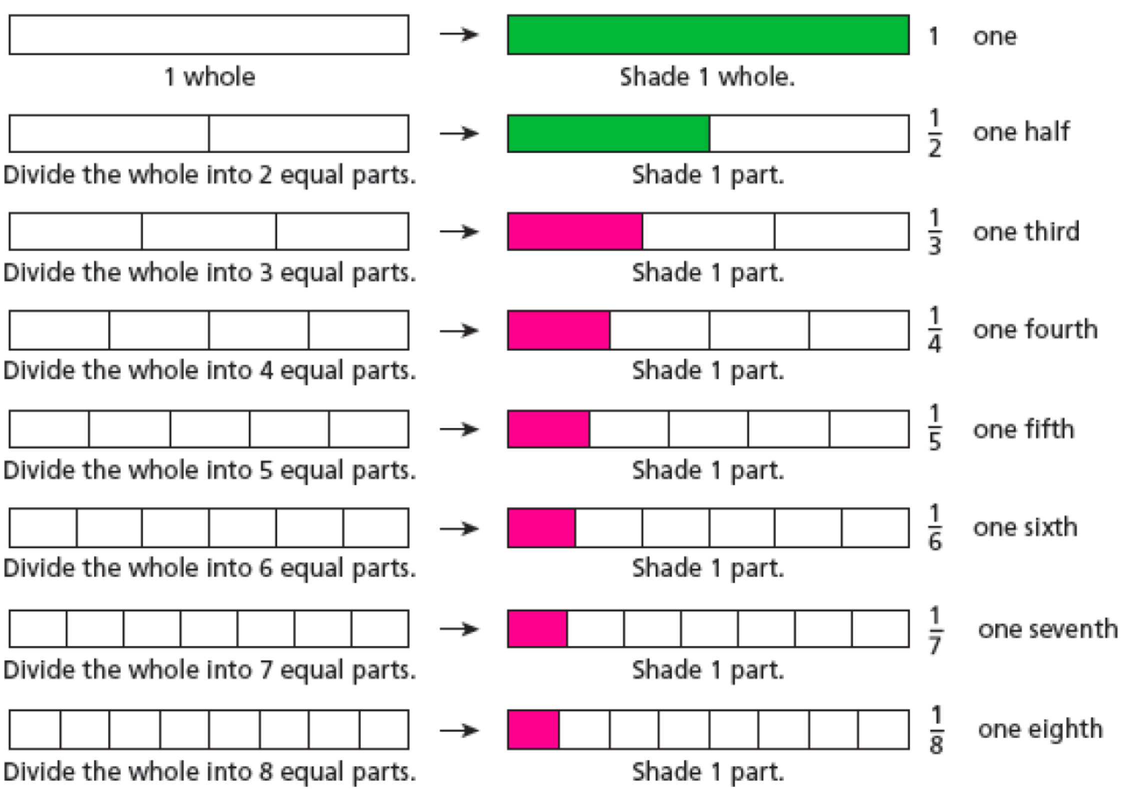


7 Write the numerators to complete the subtraction equation.

$$\frac{6}{7} - \frac{2}{7} = \frac{6-2}{7} = \frac{4}{7}$$

Fraction notation shows the **number but not the size** of unit fractions in one whole.

A unit fraction has a numerator of 1. Shade the rest of the fraction bars at the right below to represent unit fractions. What patterns do you see?



The **number** and the **size** of unit fractions are related **inversely**: a **larger denominator** makes a **smaller unit fraction**.

This is one of the most difficult aspects of fractions. You need to work on it visually and conceptually over and over again.

G3 Common Error When Comparing Fractions

Compare unit fractions 1/5 and 1/3.

Look at the denominators and use whole number knowledge:

Error: 5 > 3, so 1/5 > 1/3

Or same numerators 2/5 and 2/3:

Error: 5 > 3, so 2/5 > 2/3

Students need to look at the same whole divided into unit fractions in order to see the pattern: more unit fractions (a larger denominator) means **smaller** unit fractions,

so **1/5 <** 1/3.

Even when students understand this, they need practice to inhibit this error. Quick Practice with visual supports helps overcome this error:

Do you want 1 of 3 equal parts or 1 of 5 equal parts (of the same whole)?

Number lines are conceptually difficult.

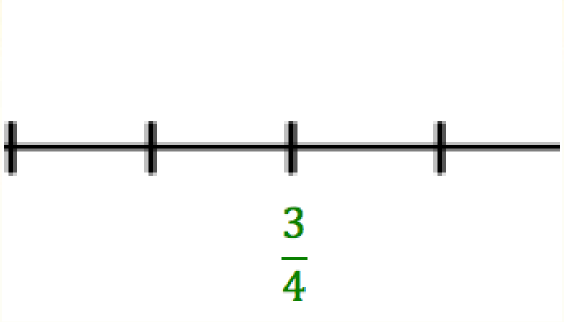
Students look at the points or marks but they need to look at the lengths that show unit fractions.

Students made the same errors in Grade 2 with measuring length.

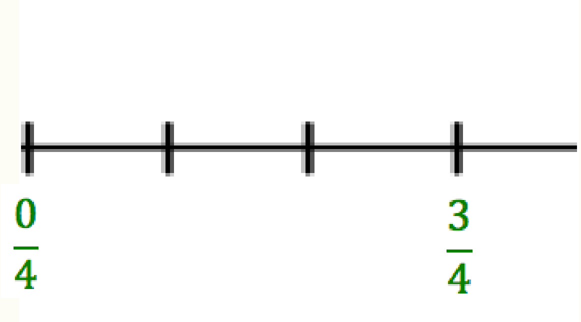
G3 Not Enough Unit Lengths

Errors when drawing or using number lines

Error: Not enough unit lengths-
student counts marks rather than lengths.



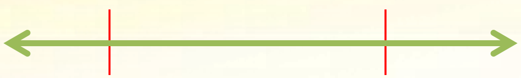
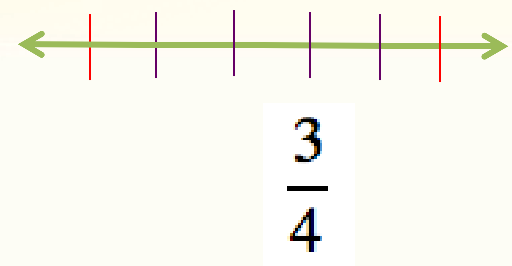
Correct: Count 3 unit lengths



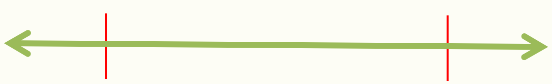
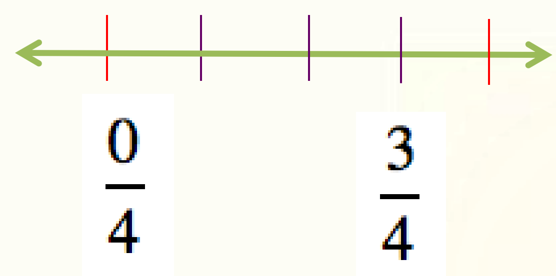
G3 Too Many Unit Lengths

Makes marks instead of lengths

Error: Makes 4 marks instead of 4 lengths.

Given:  , do 

Correct: Count 3 new marks to create 4 unit lengths.

Given:  , do 

G3 See Lengths on Number Lines

Math Expressions helps students see the lengths on number lines by

a) relating unit lengths on number lines to lengths more easily seen in fraction bars and/or

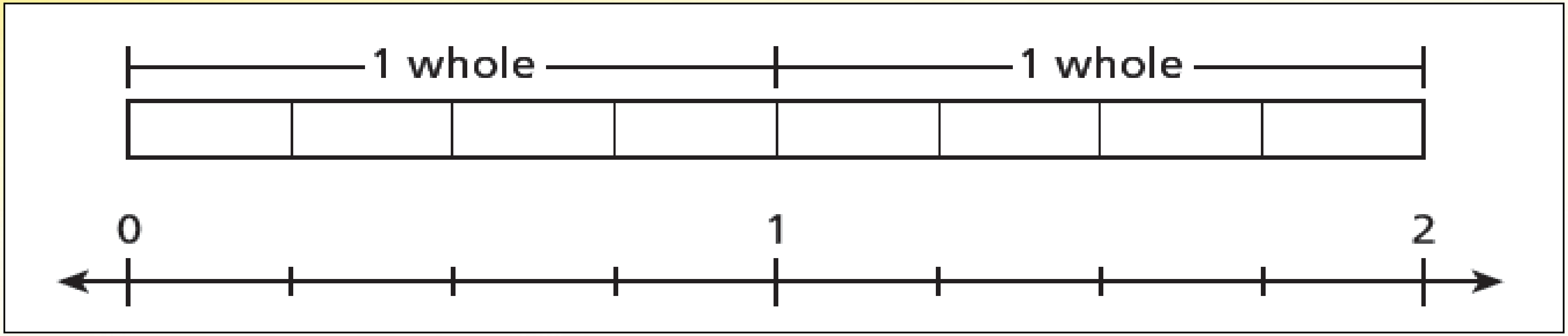
b) encircling the lengths on number lines and/or

c) moving a finger along lengths to see and count them.

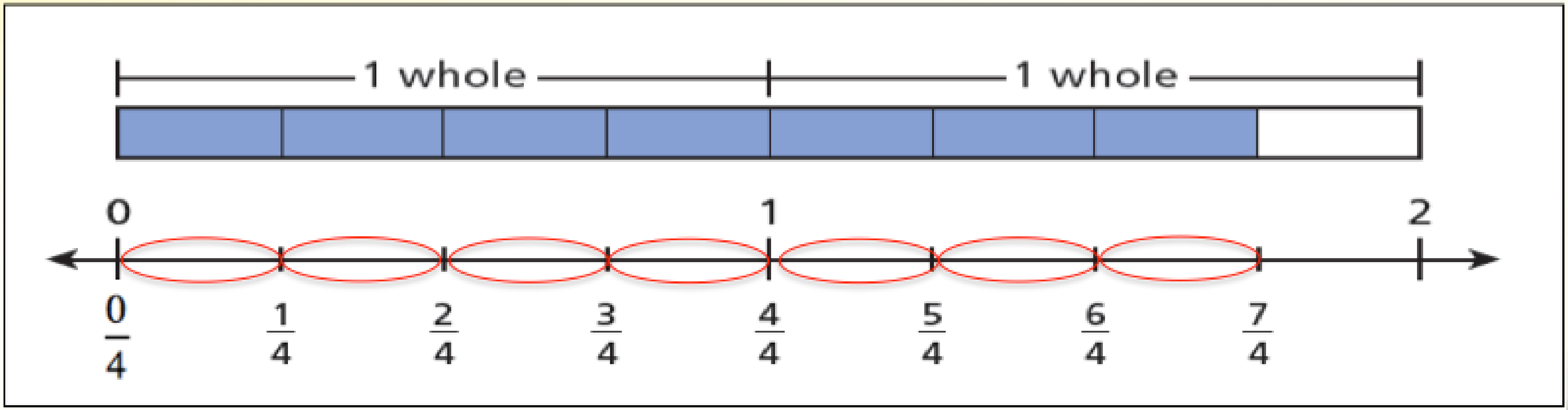
It is vital for the teacher to emphasize the lengths in all work with number lines.

G3 Unit Fraction Lengths

Step 1: Make the 4 unit fractions ¼ within each 1 whole.



Step 2: Shade or encircle 7 unit fractions and label the number line.



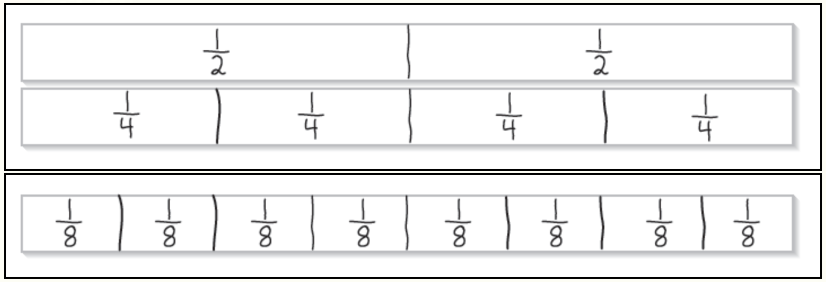
Many shapes can be a unit fraction.

But length models are by far the best.

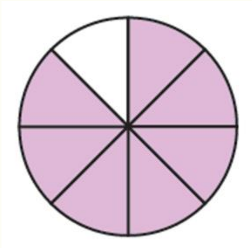
Folding fraction strips can be the start.

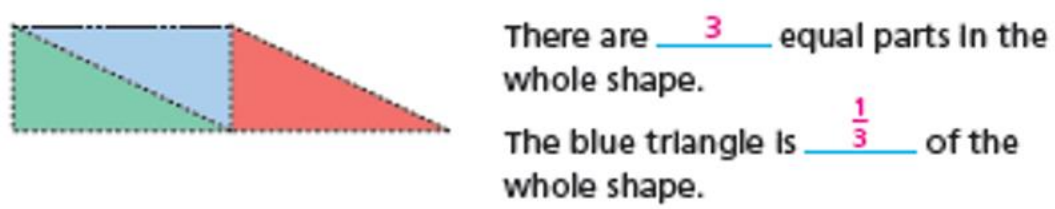
Then drawn fraction bars.

And finally fraction number lines because we have to.



Pizzas or pies are NOT a good fraction model. They are too difficult to draw accurately or compare.





Demonstrate several points by cutting fraction strips in the longer direction for half the class and in the shorter direction for the other half of the class.

a. My ½ is not the same length as your ½: Different wholes make different unit fractions.
b. Making more unit fractions makes them smaller no matter how big the whole is.
c. You can't change the whole by tearing off a piece after you have folded unit fractions, for example thirds.

G4 Make Related Opposite Changes for Equivalent Fractions

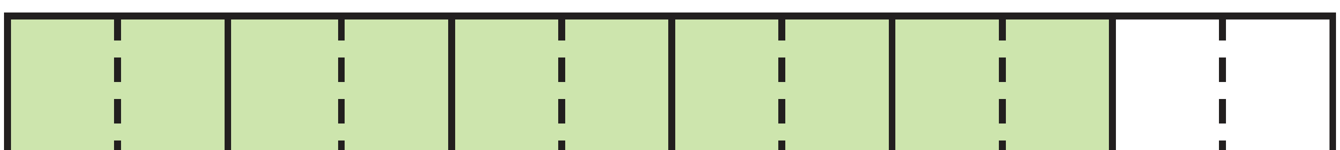
Equivalent fractions are made by **dividing physically**
but multiplying numerically to get more but smaller parts.

divide (equal-fracture) each unit fraction in the visual model to get **more but smaller** unit fractions in the numerator and the denominator.
multiply the top and bottom of the written fraction to get **more but smaller** unit fractions.

You see the numbers in the written fraction getting bigger,
but you **do not see the unit fractions getting smaller except in visual models.**
You have to **remember** that a larger denominator is a smaller unit fraction.

a. More but smaller parts

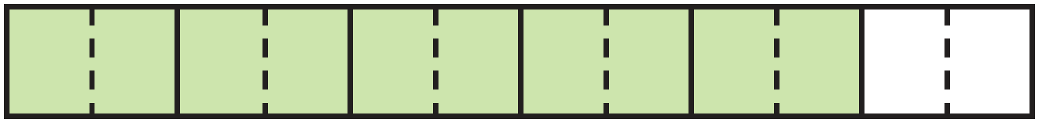
$$\frac{5}{6} = \frac{5 \bullet 2}{6 \bullet 2} = \frac{10}{12}$$



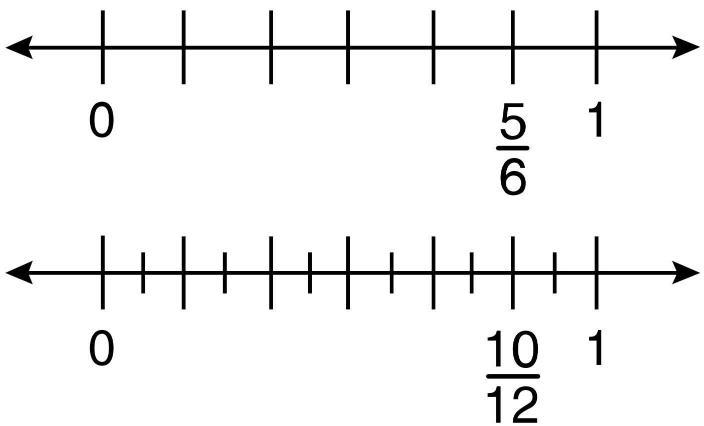
G4 More Visual Models for Equivalent Fractions

a. more but smaller parts

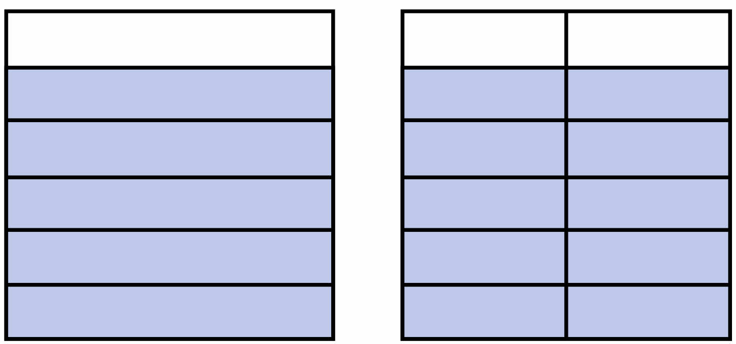
$$\frac{5}{6} = \frac{5 \bullet 2}{6 \bullet 2} = \frac{10}{12}$$



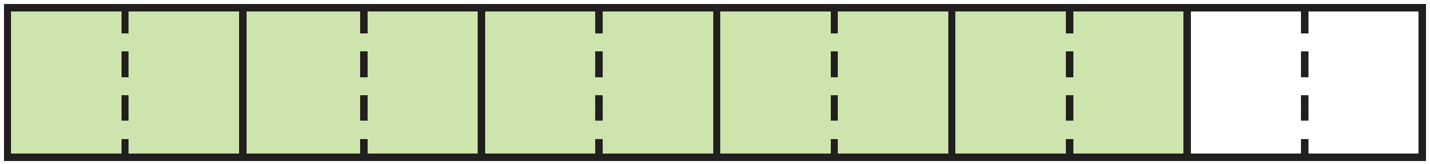
Number Line Model



Area Model



G4 Many Equivalent Fractions



$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12} = \frac{5 \cdot 3}{6 \cdot 3} = \frac{15}{18} = \frac{5 \cdot 4}{6 \cdot 4} = \frac{20}{24} = \frac{5 \cdot 5}{6 \cdot 5} = \frac{25}{30} = \frac{5 \cdot 6}{6 \cdot 6} = \frac{30}{36} = \frac{5 \cdot 7}{6 \cdot 7} = \frac{35}{42} = \frac{5 \cdot 8}{6 \cdot 8} = \frac{40}{48} = \frac{5 \cdot 9}{6 \cdot 9} = \frac{45}{56}$$

G4 Make Related Opposite Changes for Simplifying to an Equivalent Fraction

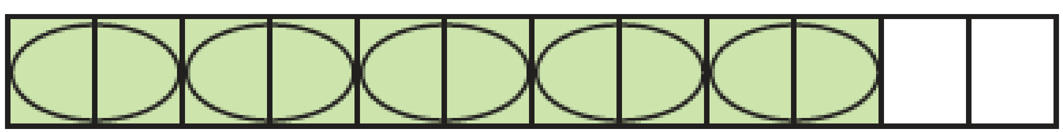
Simplifying to an equivalent fraction is done by by grouping physically but dividing numerically:

group unit fractions in the visual model to get fewer larger unit fractions in the numerator and the denominator.
divide the top and bottom of the written fraction to get fewer larger unit fractions.

You see the numbers in the written fraction getting smaller but you do not see the unit fractions getting larger except in visual models.
You have to remember that a smaller denominator is a larger unit fraction.

b. fewer but larger parts

$$\frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$



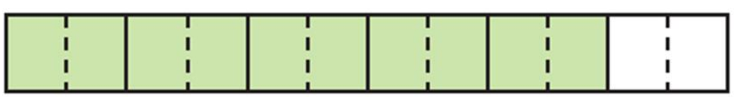
Equivalent fractions involve two inverse (opposite) changes:

Equivalent fractions are made by **dividing physically** but **multiplying numerically** to get **more but smaller parts** or by grouping physically but dividing numerically to get **fewer but larger parts**.

Equivalent fractions are made by

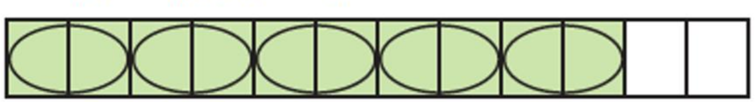
a. more but smaller parts

$$\frac{5}{6} = \frac{5 \bullet 2}{6 \bullet 2} = \frac{10}{12}$$



b. fewer but larger parts

$$\frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$



The importance of math drawings to show fraction equivalence:

The fraction symbols show the more but not the smaller parts or show the fewer but not the larger parts.

You need drawings to understand what is going on.

G4 Cases for Finding a Common Denominator

Strategies for finding the common denominator.

Analyze pairs of fractions into three classes:

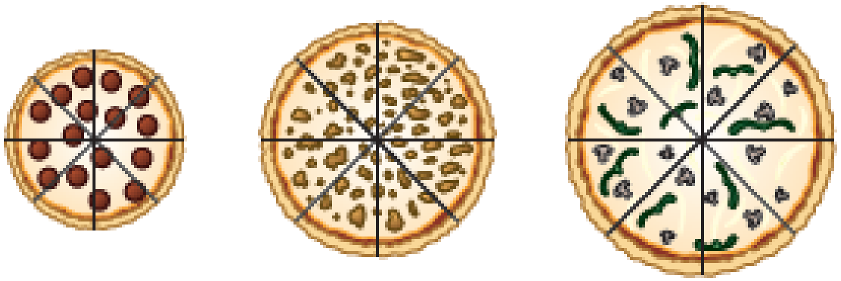
A. One denominator divides the other denominator:	B. No number except 1 divides both denominators (they are relatively prime).	C. Some number divides both denominators.
$\frac{3}{5} ? \frac{7}{10}$ Use the larger denominator as the common denominator. I'll use 10, multiply by 2 to make 5 be 10: $\frac{3\times2}{5\times2} = \frac{6}{10}$ $\frac{6}{10} < \frac{7}{10}$	Use the product of the denominators as the common denominator. Multiply each fraction top and bottom by the other denominator: $\frac{2}{3} ? \frac{4}{5}$ $\frac{2\times5}{3\times5} ? \frac{4\times3}{5\times3}$ $\frac{10}{15} < \frac{12}{15}$	I can use the product of the denominators as the common denominator, but first I'll think of a smaller number that is a multiple of both. $\frac{2}{4} ? \frac{5}{6}$ I'll use 12: $\frac{2\times3}{4\times3} ? \frac{5\times2}{6\times2}$ so $\frac{6}{12} < \frac{10}{12}$

G4 Comparisons Must Use the Same Whole

Jon and his five friends want sandwiches. They make two sandwiches: one on a short loaf of bread and one on a longer loaf. Jon cuts each sandwich into 6 pieces. His friends think the pieces are not the same size.

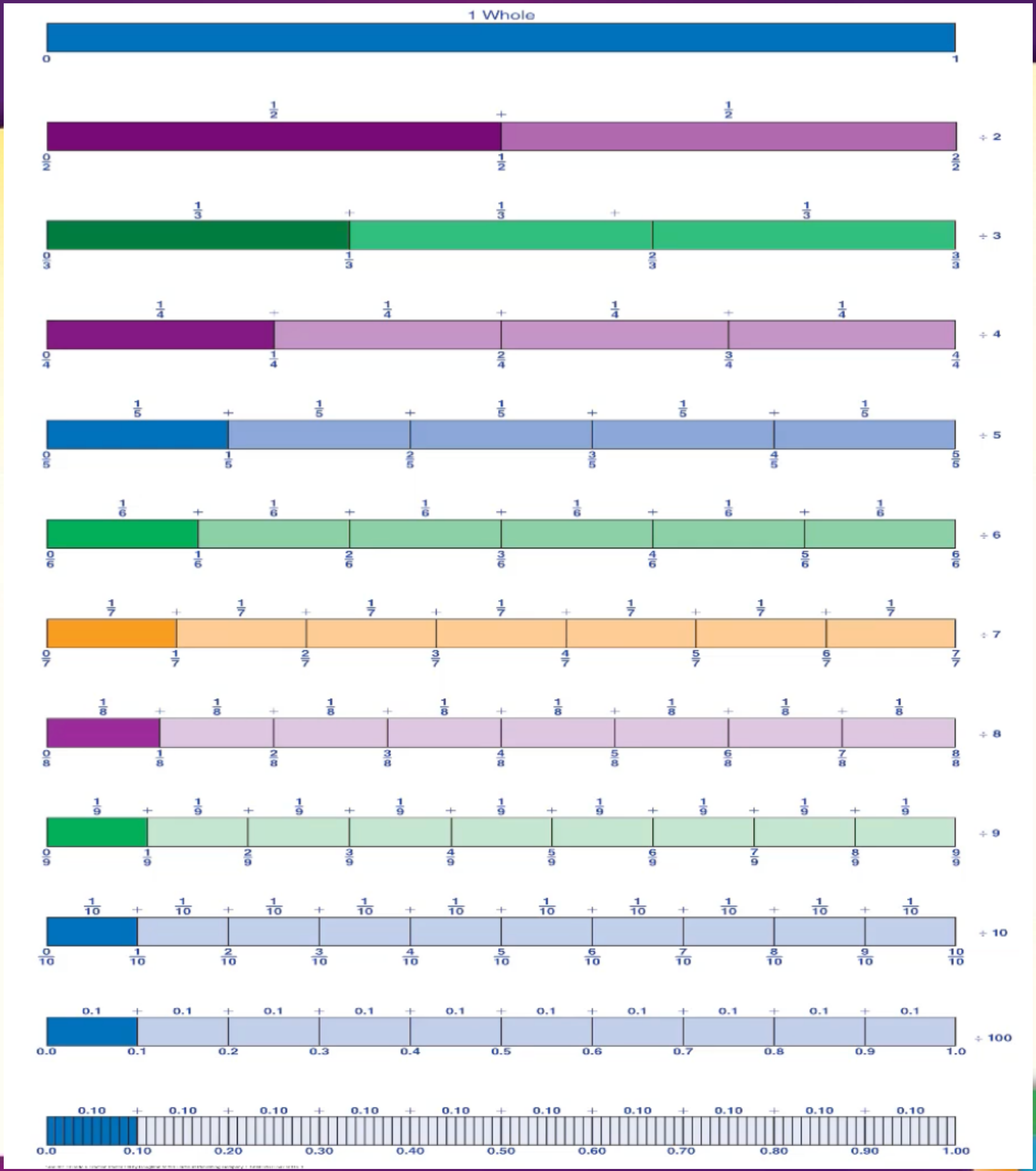
$$\frac{1}{6} ? \frac{1}{6}$$

Hattie's dad orders one small, one medium, and one large pizza. He divides each pizza into 8 equal pieces. Hattie takes $\frac{1}{8}$ of the small pizza and her friend takes $\frac{1}{8}$ of the large pizza.



$$\frac{1}{8} ? \frac{1}{8}$$

Grade 5 Math Board shows advantages of number bars versus number lines.



Operate piece by piece to find general methods.

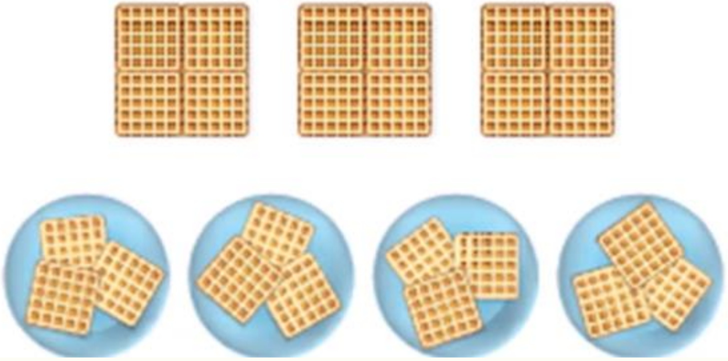
▶ Explore Fractional Shares

There are 4 people in the Walton family, but there are only 3 waffles. How can the Waltons share the waffles equally?

Divide each waffle into 4 pieces.

Each person's share of one waffle is $\frac{1}{4}$. Since there are 3 waffles, each person gets 3 of the $\frac{1}{4}$s, or $\frac{3}{4}$ of a whole waffle.

$3 \div 4 = 3 \cdot \frac{1}{4} = \frac{3}{4}$



G5 5.NF.4

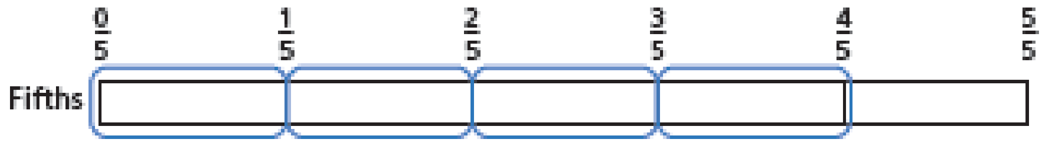
Any Fraction Times Any Fraction

Length Model

Use Bar Models to Multiply Fractions

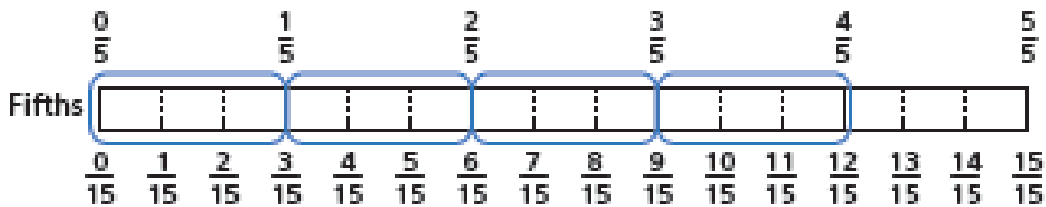
Miguel explains how to use fraction bars to find $\frac{2}{3} \cdot \frac{4}{5}$.

First, I circle 4 fifths on the fifths fraction bar.



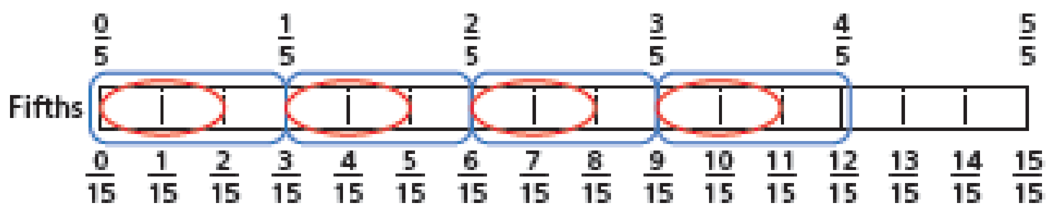
$$\cdot \frac{4}{5}$$

To find $\frac{2}{3}$ of $\frac{4}{5}$, I can circle $\frac{2}{3}$ of each fifth. But, first I have to split each fifth into three parts. After I do this, the bar is divided into fifteenths.



$$\frac{}{3} \cdot \frac{4}{5} = \frac{}{15}$$

Now, it is easy to circle 2 thirds of each of the 4 fifths.

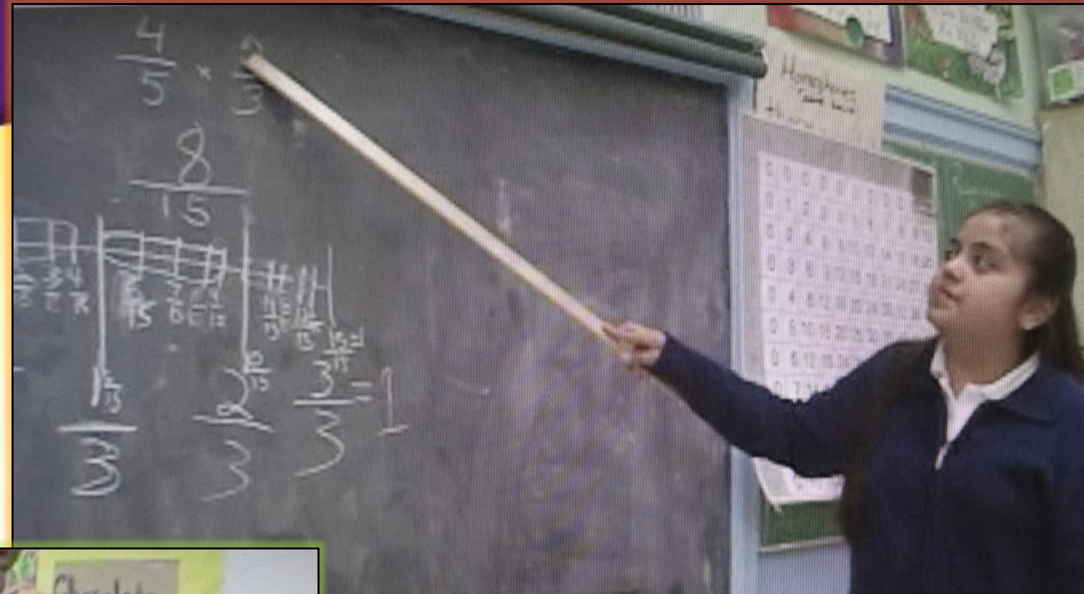


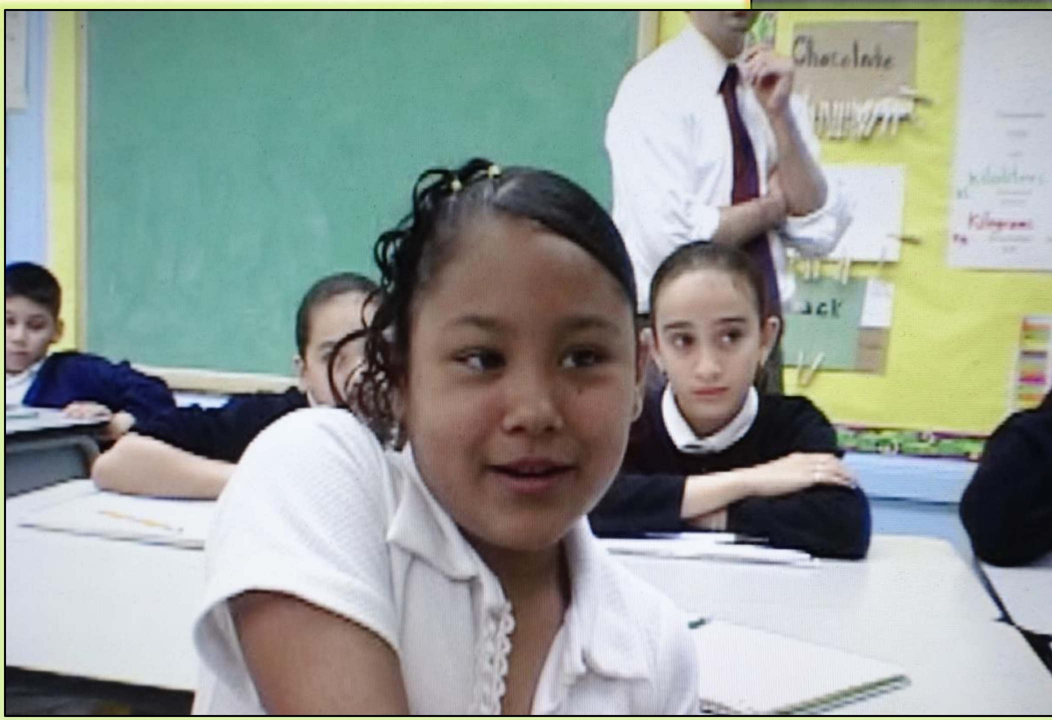
$$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$$

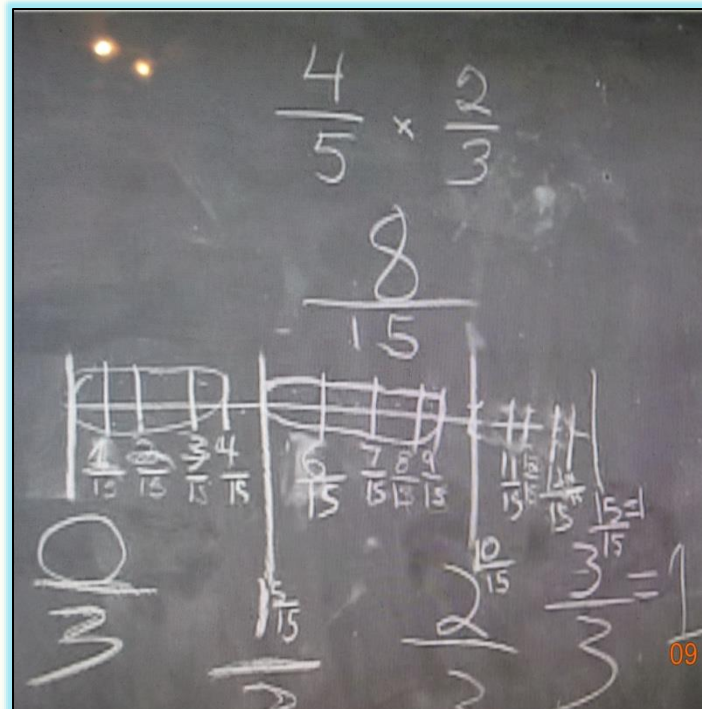
Each group I circled has 2 fifteenths, so I circled 4 groups of 2 fifteenths. That's 8 fifteenths in all. So, $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$.

Math Talk Structure

- Solve
- Explain
- Question
- Justify







G5 Common Error Multiplying by a Whole Number

The common error is to multiply the numerator and the denominator by the whole number (often mentally).

Fraction x Whole Number

$$\frac{2}{3} \times 15 = \frac{30}{45}$$

Whole Number x Fraction

$$15 \times \frac{2}{3} = \frac{30}{45}$$

Avoid the error by writing the whole number over 1.

Fraction x Whole Number

$$\frac{2}{3} \times \frac{15}{1} = \frac{2 \times 15}{3 \times 1} = \frac{30}{3} = 10$$

Whole Number x Fraction

$$\frac{15}{1} \times \frac{2}{3} = \frac{15 \times 2}{1 \times 3} = \frac{30}{3} = 10$$

Comparing, adding, and subtracting require the same unit fractions.

Multiplying may make a new smaller unit fraction and then you multiply numerators to find out how many of those new smaller unit fractions.

	$\frac{3}{5}$ and $\frac{4}{7}$
>	$\frac{3}{5} > \frac{4}{7}$ or $\frac{21}{35} > \frac{20}{35}$
+	$\frac{3}{5} + \frac{4}{7} = \frac{21}{35} + \frac{20}{35} = \frac{41}{35} = 1\frac{6}{35}$
−	$\frac{3}{5} - \frac{4}{7} = \frac{21}{35} - \frac{20}{35} = \frac{1}{35}$
·	$\frac{3}{5} \cdot \frac{4}{7} = \frac{12}{35}$

G5 Multiplication or Division Situation?

Decide whether you need to multiply or divide. Then solve. *Show your work.*

1. A turtle crawls 3 yards in an hour. How far will it crawl in 2 hours?

   Multiply; $2 \cdot 3 = 6$ yards

   How far will the turtle crawl in $\frac{1}{4}$ hour?

   Multiply; $\frac{1}{4} \cdot 3 = \frac{3}{4}$ yard

2. Emily has 2 tons of sand. She will move it by wheelbarrow to the garden. Her wheelbarrow holds $\frac{1}{10}$ ton. How many trips will she make?

   Divide; $2 \div \frac{1}{10} = 20$ trips

G6 Seeing Division as Finding the Unknown Factor Numerically

inverse operations

▶ Relating Multiplication and Division

Find the unknown factor in each equation. Then rewrite the multiplication as a division equation.

Multiplication Equation	Related Division Equation
3. $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$	$\frac{8}{15} \div \frac{2}{3} = \frac{4}{5}$
4. $\frac{5}{7} \cdot \frac{3}{8} = \frac{15}{56}$	$\frac{15}{56} \div \frac{5}{7} = \frac{3}{8}$

Multiplication and division are **inverse operations** for all whole numbers, decimals, and fractions. One operation undoes the other.

$$\frac{2}{5} \cdot \frac{1}{5} \div \frac{1}{5} = \frac{2}{5}$$

G6 Unsimplify to Divide and See That You Have Multiplied by the Reciprocal

unsimplify

▶ Unsimplify to Divide

$\frac{2}{3} \div \frac{5}{7} = ?$

We cannot divide the numerator of $\frac{2}{3}$ by 5 or the denominator by 7.

To be able to divide, we need to unsimplify $\frac{2}{3}$. To **unsimplify** we rewrite it as an equivalent fraction so the numerators and denominators divide evenly.

Step 1 $\frac{2}{3} \div \frac{5}{7} = \underbrace{\overbrace{\left(\frac{2}{3} \cdot \frac{5}{5} \cdot \frac{7}{7}\right)}^{\frac{2}{3} \cdot 1 \cdot 1}}_{\frac{2}{3}\text{ unsimplified}} \div \frac{5}{7}$

Step 2 $= \frac{2 \cdot 5 \cdot 7}{3 \cdot 5 \cdot 7} \div \frac{5}{7}$ ($5 \div 5 = 1$; $7 \div 7 = 1$)

Step 3 $= \frac{2 \cdot 7}{3 \cdot 5}$

Step 4 $= \frac{2}{3} \cdot \frac{7}{5}$

1. How is the number you divide $\frac{2}{3}$ by in the original division problem related to the number you multiply $\frac{2}{3}$ by in the final multiplication problem?

You multiply by $\frac{7}{5}$, the reciprocal of original divisor, $\frac{5}{7}$.

If you can, divide the top and bottom numbers (the numerator and denominator) of the product by the top and bottom numbers (the numerator and denominator) of the factor.

If you can't divide easily, flip the factor and multiply the product by it (multiply the product by the reciprocal of the factor).

Details of fraction standards and these approaches can be found by grade level

in the Teaching Progressions for fractions

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