math expressions

Common Core State Standards (CCSS) Number and Operations—Fractions (NF) in Math Expressions

Building a New Standard of Success

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THE CCSS NF STANDARDS

The NF and CCSS standards in Grades 3, 4, and 5 focus on understanding and using the idea of a unit fraction 1/*b* as the unit underlying fraction computation and comparison. A unit fraction is the quantity formed by 1 part when a whole is partitioned into *b* equal parts. Any fraction *a*/*b* is the quantity formed by *a* unit fractions 1/*b*. The fraction progression moves from explaining equivalence in special cases and comparisons with the same numerator or same denominator (Grade 3) to a general understanding of equivalent fractions and comparing any two fractions (Grade 4). It also moves from adding and subtracting fractions and mixed numbers with like denominators (Grade 4) to finding like denominators to add and subtract fractions and mixed numbers with unlike denominators (Grade 5).

The fraction progression also extends previous understandings of whole number multiplication and division to multiply fractions using visual fraction models that use length and area first for special cases (Grade 4) and then for all fractions (Grade 5). Division of special cases using unit fractions (Grade 5) is followed by division with any fractions (Grade 6). Comparisons of and operations with fractions use visual fraction models that show unit fractions, and students are to understand and explain such comparisons and operations. Visual fraction models include length models (bar models such as fraction strips or drawn fraction bars and the number line diagram) and area models. The initial relationship between fractions and decimals and the introduction of decimals is in Grade 4 NF, but the rest of the treatment of decimals is in Number and Operations in Base Ten (NBT).

CONCEPTUAL DIFFICULTIES WITH FRACTIONS

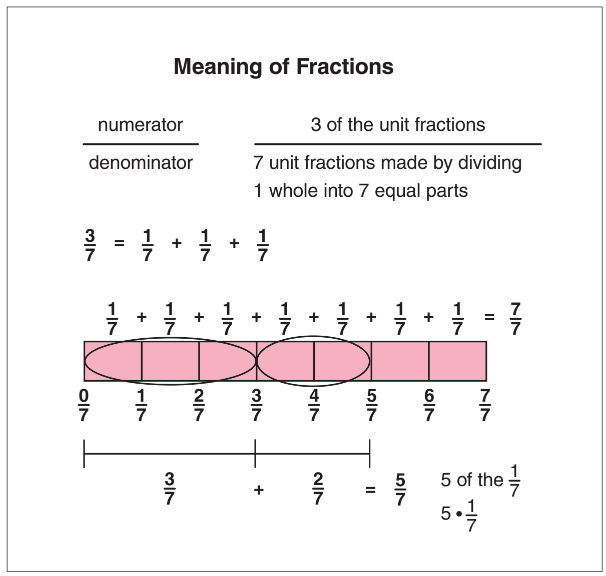
Students face many conceptual difficulties with fractions. Many of these difficulties come from or are exacerbated by fraction symbols and fraction words. Fraction symbols have two whole numbers. The top number (the numerator) has the usual whole number meaning as the number of units. But the denominator (the unit fraction) is misleading about the size of the unit fraction because it tells into how many equal parts the whole was divided. A larger number means a smaller unit fraction, and nothing in the symbol shows the size of the unit fraction. This opposite meaning of the denominator is supported in some languages such as Chinese, where you say 3/5 as "out of 5 parts (take) 3". This clarifies the meaning of both whole numbers in the fraction symbol. In English we use the same words for most fractions that are already used to mean the place in an order: third, fourth, fifth, etc. These words are unhelpful and confusing. Students also make wrong generalizations from their whole number knowledge.

OVERCOMING CONCEPTUAL DIFFICULTIES

Math Expressions used the unit fraction approach before creation of the Common Core State Standards (CCSS) and developed approaches and visual models to overcome conceptual difficulties. These have been honed through years of use and are overviewed below. Visual models and written methods are carefully related to build understanding. Teachers must continually support students to overcome their misleading generalizations based on their experiences with whole numbers by using fraction meanings and visual models within the Mathematical Practices to visualize and explain comparisons and operations with fractions. Quick Practice activities in ***Math Expressions*** help students remember and use important concepts and connections.

FRACTIONS IN GRADE 3

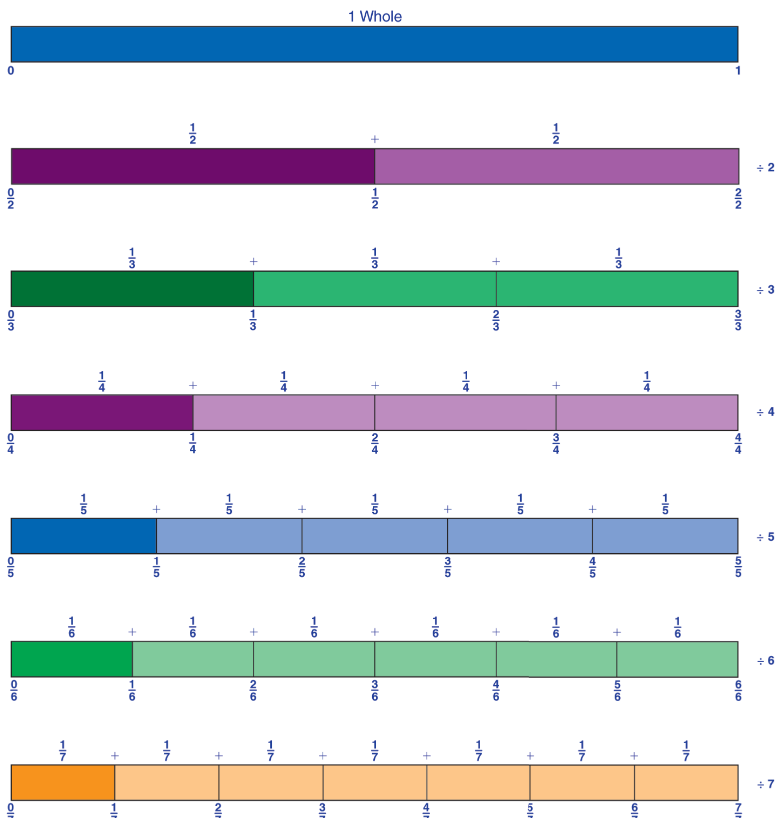
To understand the unit fractions composing fraction symbols and words, students fold fraction strips and see and label bar drawings with unit fractions. They also initially make fraction drawings in two steps: they first divide the whole into equal unit fractions (e.g., into five 1/5) and then circle or shade some (e.g., 2 of the 5 unit fractions). All of these approaches help students avoid the typical error of seeing a length separated into two groups (2 unit fractions and 3 unit fractions) and writing the fraction incorrectly as these numbers (2/3) rather than as 2/5. They need to use the total number of unit fractions they made in the whole, not the visible left-over parts.



To help students understand number line diagrams, students initially loop lengths and see bar drawings above the number line diagram. This helps students to focus on the lengths of the unit fractions to understand the difficult labeling of the number line as numbers showing the total lengths so far. The careful ***Math Expressions*** development of length as counting units of length introduced in earlier grades also contributes to understanding of fractions on a number line diagram as sums of unit fraction lengths. Without such focusing on unit lengths, students tend to see fractions as points on the number line diagram and make or label points with one too few unit lengths (ignoring the 0 mark) or one too many unit lengths (if the end-point marks are already there).

Students discuss and generalize the structure in unit fractions as they make and see more parts of the same whole: the unit fraction becomes smaller as the denominator becomes larger. This helps students avoid the typical error from whole-number comparisons: some students say that 1/5 > 1/3 because 5 > 3. Quick Practice with meaning can help students remember this opposite pattern for fractions: Do you want 1 of 3 equal parts or 1 of 5 equal parts (of the same whole)?

Comparing Unit Fractions



Students see drawings of fractions to support their explaining in simple cases the math structure in equivalent fractions: these have more but smaller parts (e.g., 3/6) or fewer but larger parts (e.g., 1/2). Relating the opposite numerical pattern for comparisons of fractions to equivalence is crucial for understanding the inverse pattern for equivalence: 1/5 < 1/3 because 5 > 3 and more parts means each part is smaller.

FRACTIONS IN GRADE 4

Students extend their understandings of equivalent fractions in special cases to recognize and generate equivalent fractions in general. They see and explain that they make related opposite changes in the numerator and denominator (see the drawing, part a) for equivalent fractions: you divide (equal-fracture) each unit fraction to get **more but smaller** unit fractions (change each 1/6 to 1/12) and use the visual model to see that we are **multiplying** the top and bottom by the same number (2). Students may need help to see that you **divide** the unit fractions physically, but you **multiply** numerically. Also the new unit fraction (10/12) shows that there are **more** unit fractions, but students have to realize that these unit fractions (1/12) are **smaller** (even though 12 > 10).

To find an equivalent fraction with a smaller denominator (see b in the drawing), students group unit fractions to get **fewer but larger** unit fractions (10/12 becomes 5/6, but each 1/6 is larger than each 1/12). Again students may need help to see that you **group** physically, but you **divide** numerically. The new unit fraction (5/6) shows that there are **fewer** unit fractions (1/6), but students have to realize that these unit fractions are **larger** than the 1/12. Students see chains of equivalent fractions as two rows from the multiplication table to emphasize the many equivalent fractions that are possible (see the chain below the drawing for a).

Students use their knowledge of equivalent fractions to compare two fractions with different denominators and different numerators by changing one or both to like denominators. They continue to use their Grade 3 understandings for cases where only the numerators or only the denominators are the same. Visual models continue to play central roles in explaining student thinking.

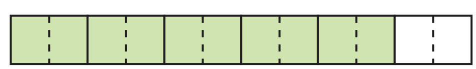
Students add and subtract fractions just as they add and subtract whole numbers and measures: they must add or subtract like units, so the unit fractions (the denominators) must be the same number. The presence of the unit fraction as a whole number in the fraction symbol may lead some students to make a typical error: adding (or subtracting) the denominators as well as adding (or subtracting) the numerators. Writing fractions as sums of unit fractions when adding and subtracting fractions helps to overcome this error because students learn to focus on the top numbers as telling how many and adding these numbers but they see the bottom number staying the same (see such unit fraction sums in the Meaning of Fractions drawings on page 2 for 3/7 + 2/7). This error may arise again after learning to multiply fractions because you do multiply tops and bottoms. So students may need to return at least briefly to the earlier supports to see why they do not add tops and bottoms.

Equivalent Fractions

Equivalent fractions are made by:

a. more but smaller parts

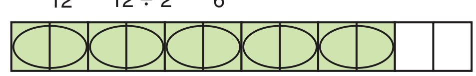
$$\frac{5}{6} = \frac{5 \bullet 2}{6 \bullet 2} = \frac{10}{12}$$



$$\frac{5}{6} = \underset{\bullet 2}{\overset{\bullet 2}{\frac{10}{12}}} = \underset{\bullet 3}{\overset{\bullet 3}{\frac{15}{18}}} = \underset{\bullet 4}{\overset{\bullet 4}{\frac{20}{24}}} = \underset{\bullet 5}{\overset{\bullet 5}{\frac{25}{30}}} = \underset{\bullet 6}{\overset{\bullet 6}{\frac{30}{36}}} = \underset{\bullet 7}{\overset{\bullet 7}{\frac{35}{42}}} = \underset{\bullet 8}{\overset{\bullet 8}{\frac{40}{48}}} = \underset{\bullet 9}{\overset{\bullet 9}{\frac{45}{56}}}$$

b. fewer but larger parts

$$\frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$



FRACTIONS IN GRADE 5

Students now add and subtract fractions with unlike denominators (including mixed numbers). To do so, they must use strategies for finding the common denominator. ***Math Expressions*** helps students focus on three cases:

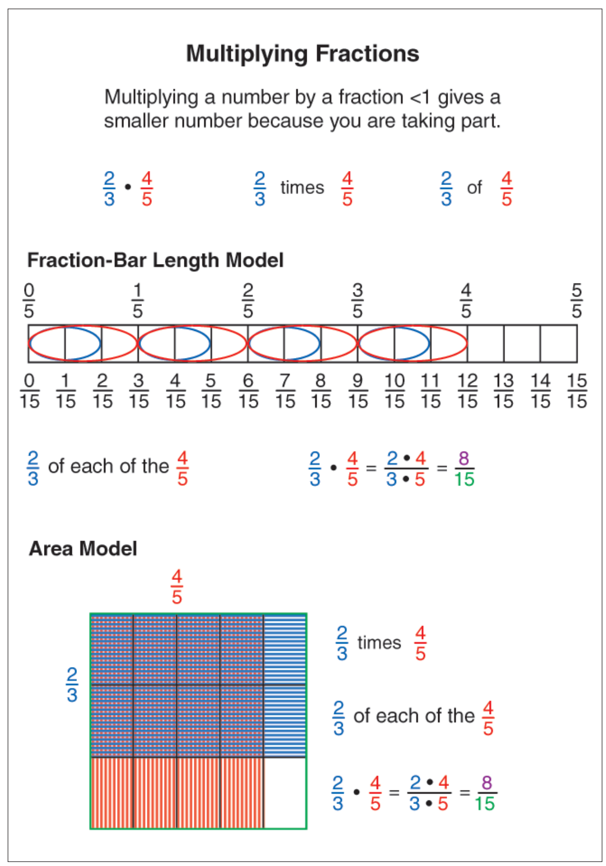
- a. **One denominator divides the other denominator:** Use the larger denominator as the common denominator.
- b. **No number except 1 divides both denominators:** Use the product of the denominators as the common denominator.
- c. **Some number divides both denominators:** You can use the product of the denominators as the common denominator, but first try to think of a smaller number that is a multiple of both (e.g., for 4 and 6, you could use 24 but you could also use 12).

It is faster and easier for students to use strategy c than to try to find the least common denominator, which can be time-consuming and is not necessary (especially because few fraction pairs involve strategy c).

The Grade 5 Math Board has fraction bars from halves to tenths that can be labeled with unit fractions above and as a number line diagram below (like the earlier Comparing Unit Fractions picture but without color and with three more rows for eighths, ninths, and tenths). These unit fractions can be partitioned or grouped into equivalent fractions to support the generalization of adding and subtracting to any numbers and to support other Grade 5 topics.

Fraction bars and number lines at the bottom of the bars can be used for finding the pattern in general multiplying of fractions. For example, students find 2/3 of each of the 4/5 and explain why the general structure is true: multiply the numerators and multiply the denominators to find *a/b x c/d = ab/cd* (see the drawing in Multiplying Fractions). Students also explain area models for the general pattern in multiplying fractions. The area model ensures that 2/3 of each of the 4/5 is found (see the drawing in Multiplying Fractions). Multiplying a fraction times a whole number is viewed as a special case of multiplying fractions in which you multiply the fraction times each 1 in the whole number *n*. With a visual model students see that this is just multiplying *n* times the fraction. Finding a fraction of a set is one example of such problems. Students sometimes make the error of multiplying the top and the bottom number by the whole number so that they find 2/3 x 5 = 10/15. Discussing why this conclusion is wrong and writing the whole number as a fraction with denominator 1 can help to overcome this error:

$$(23 \times 51 = \tfrac{2}{3} \times 5 = \tfrac{10}{15} \qquad \tfrac{2}{3} \times \tfrac{5}{1} = \tfrac{10}{3})$$

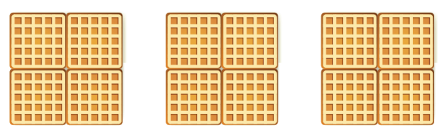


A major source of confusion for students is the fact that when multiplying or dividing by fractions or decimals less than one, multiplying results in a smaller number and dividing results in a larger number, opposite to the effects for multiplying or dividing with whole numbers. The CCSS call this multiplication as scaling (resizing), and this concept is carefully developed and emphasized in ***Math Expressions***.

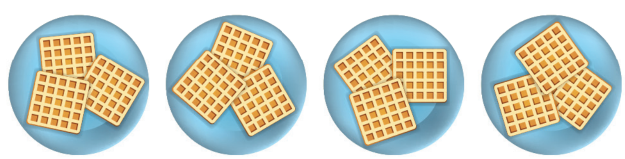
Students discuss and relate the division and fraction meanings of the fraction notation a/b as both $a \div b$ and the fraction a/b. They see that dividing a into b equal parts ($a \div b$) is just dividing each 1 in a into unit fractions $1/b$, so they get a unit fractions $1/b$, which is the fraction a/b. Students begin with the waffle situation below and then generalize it to other numbers and then to the algebraic summary for all numbers to see that $a \div b$ results in the fraction a/b. Division in Grade 5 focuses on cases involving a whole number and a unit fraction. Both cases are shown with visual models and real-world interpretations that students discuss and use to solve more numerical examples and then generalize as an equation using letters.

There are 4 people in the Walton family, but there are only 3 waffles. How can the Waltons share the waffles equally?

Divide each waffle into 4 pieces.

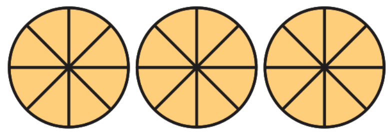


Each person's share of one waffle is $\frac{1}{4}$. Since there are 3 waffles, each person gets 3 of the $\frac{1}{4}$s, or $\frac{3}{4}$ of a whole waffle.

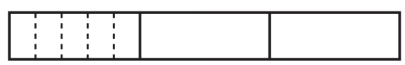


$3 \div 4 = 3 \cdot \frac{1}{4} = \frac{3}{4}$

Divide a Whole Number by a Unit Fraction Visual models are used to illustrate that for any whole number w and any unit fraction $\frac{1}{d}$, $w \div \frac{1}{d} = w \cdot d$. For example, the diagram below is used to model $3 \div \frac{1}{8}$. Because there are 8 eighths in each 1 whole, there are $3 \cdot 8$ eighths in 3 wholes. That is $3 \div \frac{1}{8} = 3 \cdot 8 = 24$.

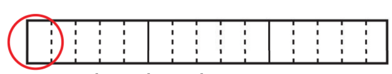


Divide a Unit Fraction by a Whole Number A real world situation and visual model are used to develop the idea of dividing a unit fraction by a whole number. Specifically, students consider the case of $\frac{1}{3} \div 5$. This can be interpreted as the question: If we divide $\frac{1}{3}$ into 5 parts, how big is each part? The diagram below shows $\frac{1}{3}$ divided into 5 parts:



$\frac{1}{3} \div 5$

To find the size of each part, we need to divide the other two thirds into five parts, which is like multiplying by $\frac{1}{5}$. Each part is $\frac{1}{15}$. So, $\frac{1}{3} \div 5 = \frac{1}{15}$.



$\frac{1}{3} \div 5 = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$

In general, for any unit fraction $\frac{1}{d}$ and any whole number w that is greater than or equal to 1, $\frac{1}{d} \div w = \frac{1}{d} \cdot \frac{1}{w} = \frac{1}{d \cdot w}$

FRACTIONS IN GRADE 6
(IN THE NUMBER SYSTEMS STANDARD 6.NS.1)

a/bb/a

▶ Unsimplify to Divide

$\frac{2}{3} \div \frac{5}{7} = ?$

We cannot divide the numerator of $\frac{2}{3}$ by 5 or the denominator by 7.

To be able to divide, we need to unsimplify $\frac{2}{3}$. To **unsimplify** we rewrite it as an equivalent fraction so the numerators and denominators divide evenly.

Step 1 $\frac{2}{3} \div \frac{5}{7} = (\overbrace{\underbrace{\frac{2}{3} \bullet \frac{5}{5} \bullet \frac{7}{7}}_{\frac{2}{3}\text{ unsimplified}}}^{\frac{2}{3} \bullet 1 \bullet 1}) \div \frac{5}{7}$

Step 2 $= \frac{2 \bullet 5 \bullet 7}{3 \bullet 5 \bullet 7} \div \frac{5}{7}$ ($5 \div 5 = 1$; $7 \div 7 = 1$)

Step 3 $= \frac{2 \bullet 7}{3 \bullet 5}$

Step 4 $= \frac{2}{3} \bullet \frac{7}{5}$

1. How is the number you divide $\frac{2}{3}$ by in the original division problem related to the number you multiply $\frac{2}{3}$ by in the final multiplication problem?
You multiply by $\frac{7}{5}$, the reciprocal of original divisor, $\frac{5}{7}$.

Multiplication Equation	Related Division Equation
3. $\frac{2}{3} \bullet \frac{\boxed{4}}{\boxed{5}} = \frac{8}{15}$	$\frac{8}{15} \div \frac{2}{3} = \frac{\boxed{4}}{\boxed{5}}$
4. $\frac{5}{7} \bullet \frac{\boxed{3}}{\boxed{8}} = \frac{15}{56}$	$\frac{15}{56} \div \frac{5}{7} = \frac{\boxed{3}}{\boxed{8}}$
5. $\frac{5}{8} \bullet \frac{\boxed{4}}{\boxed{9}} = \frac{20}{72}$	$\frac{20}{72} \div \frac{5}{8} = \frac{4}{9}$
6. $\frac{3}{4} \bullet \frac{\boxed{5}}{\boxed{9}} = \frac{15}{36}$	$\frac{15}{36} \div \frac{3}{4} = \frac{5}{9}$

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