Why Accessible Algorithms?

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Please see my website karenfusonmath.net or karenfusonmath.com for 22 hours of audio-visual Teaching Progressions for all CCSS domains and for my papers, classroom videos, presentations, supports for teaching remotely, and extra papers and materials for *Math Expressions* users.

Why do we need accessible algorithms?

Why don't we just teach the familiar "standard algorithms" that I learned?

Because those methods are difficult and lead students to make mistakes.

Furthermore, there is

no one "standard algorithm."

There are variations in ways

to record efficient, accurate,

and generalizable methods that

form the collection of standard algorithms.

These variations are used in different countries

and at different times.

Some standard algorithms are better than other standard algorithms.

My research is about these.

These are in classroom videos, papers, and Teaching Progressions on my website

karenfusonmath.net or karenfusonmath.com

These better standard algorithms are the mathematically desirable and accessible methods in Math Expressions.

Watch on my website videos from public school classrooms with children from backgrounds of poverty and many children who are not native English speakers as they explain the methods I discuss today.

These are on my website karenfusonmath.com or karenfusonmath.net under Classroom Videos and then

A. Classroom Components and Part 3 Math Talk has:

Math Talk Introduction

Grade 1 addition with regrouping invented method and

New Groups Below method

Grade 2 Subtraction with Ungrouping First

Grade 4 Expanded Notation Multiplication

On my website karenfusonmath.net and karenfusonmath.com are publications describing the research and these methods:

The Best Multidigit Computation Methods: A Cross-cultural Cognitive, Empirical, and Mathematical Analysis, Karen C. Fuson. Universal Journal of Educational Research 8(4): 1299-1314, 2020 DOI: 10.13189/ujer.2020. 080421

Fuson, K. C. & Beckmann, S. (Fall/Winter, 2012-2013). Standard algorithms in the Common Core State Standards. *National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14 (2)*, 14-30.

Fuson, K. C., Kiebler, S., & Decker, R. (2024). Accessible Standard Algorithms for Understanding and Equity. *Mathematics Teacher: Learning and Teaching K-12*. Volume 117, Issue 04, April, 268–275. DOI: 10.5951/MTLT.2023.0212

Fuson, K. C., Kiebler, S., & Decker, R. (2024). Accessible Standard Algorithms for Understanding and Equity Part 2: Multidigit and Decimal Subtraction, Multiplication, and Division. *Journal of Education and Development, Vol. 8, No. 2*, May, 2024. Online version.

The CCSS say in the critical area for each first year of a given computation: “Students develop, discuss, and use efficient, accurate, and generalizable methods.”

Standard algorithms are such methods, so these can and should be introduced early.

The CCSS do not say to wait until Grade 4 to do “standard algorithms.”

What Is the Standard Algorithm?

The NBT Progression document summarizes that ***the standard algorithm*** for an operation implements the following mathematical approach

with minor variations in how the algorithm is written:

- **decompose numbers into base-ten units and then carry out single-digit computations with those units using the place values to direct the place value of the resulting number; and**
- **use the one-to-ten uniformity of the base ten structure of the number system to generalize to large whole numbers and to decimals.**

To implement a standard algorithm one uses a systematic written method for recording the steps of the algorithm.

What are the best computation methods for multidigit addition, subtraction, multiplication, and division?

I today summarize 20 years of classroom research.

These methods were all created by students. They were then tried in many different classrooms to see if they are accessible. They are. Students, teachers, and parents understand them.

They are more mathematically desirable than other methods.

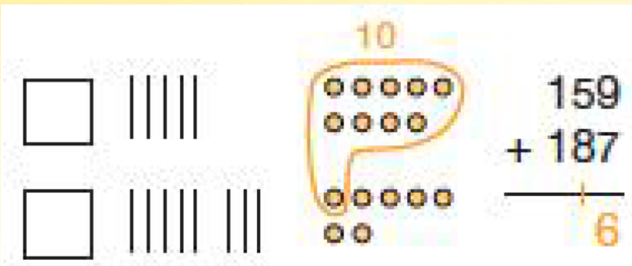
They are all standard algorithms.

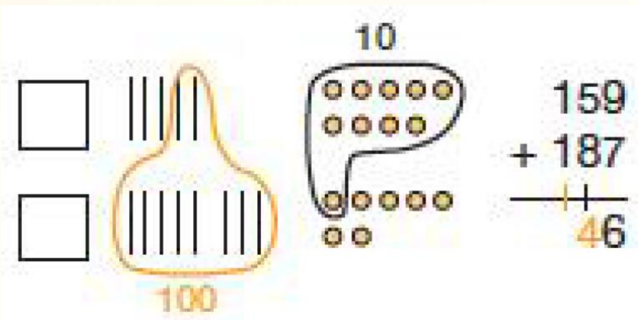
These standard algorithms are the accessible and mathematically desirable methods in *Math Expressions.*

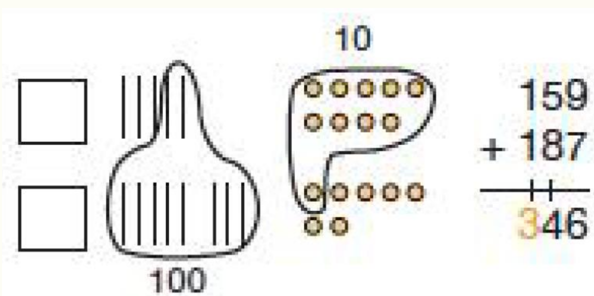
Multidigit Addition

Many of us are so familiar with writing the New Groups Above addition problem that we do not recognize how difficult it is for students. But when we contrast it to writing the New Groups Below the problem, we can see the difficult aspects of the New Groups Above method that is familiar to many of us but is not familiar to most of our students.

New Groups Below is the best multidigit addition method.







Think about why New Groups Below is better than writing the new one ten and new one hundred above the problem.

New Groups Above

```
  11
  159
+ 187
-----
  346
```

Fuson and Beckmann (NCSM, Fall/Winter, 2012-2013) identified criteria for algorithms that should be taught in the classroom.

I will exemplify these criteria by comparing the better New Groups Below to the problematic New Groups Above in several slides.

Ways in which New Groups Below is better than New Groups Above

Variations that support and use place value correctly:

It is easy to see the teen total in New Groups Below because they are close together. For example, see the 16 ones and the 14 tens.

In the New Groups Above method these teen numbers are widely separated and difficult to see as teen numbers.

```
  159
+ 187
  1 1
-----
  346
```

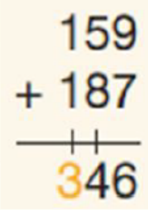
It is easy to see where to write the new unit: The 1 ten for the 16 ones is written in the column just to the left of the 6 below in the ones column and similarly for the 14 tens in the tens column.

In the New Groups Above Method, some children say that the separation of the teen numbers makes it more difficult to put the new 1 group in the next left column.

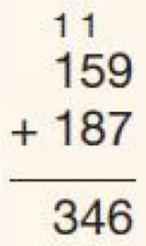
```
  1 1
  159
+ 187
-----
  346
```

Ways in which New Groups Below is better than New Groups Above

Variations that make single-digit computations easier:



In New Groups Below one adds the two larger numbers first: add the 5 tens and 8 tens to get 13 tens and then add the 1 new ten waiting below.



In New Groups Above children may forget to add the 1 new group above if they add the larger numbers first. And adding the 1 to the top number and adding that total to the second number means that the child has to add a number they do not see (6) and ignore a number they do see (5) in order to get 14 tens.

Ways in which New Groups Below is better than New Groups Above

Variations that allow children to write teen numbers in their usual order left to right, which is the one ten and then the ones:

This is easy to do for New Groups Below (9 plus 7 is sixteen which I can write as 1 then 6).

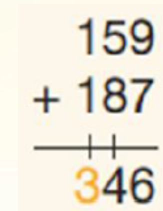
$$\begin{array}{r} 159 \\ +\ 187 \\ \hline {}_{1\,1} \\ 346 \end{array}$$

For New Groups Above, children are often told to write the 6 and carry/regroup the 1, the opposite order to their usual way of writing numbers, which is left to right. Sometimes children have the 6 above the tens place because they wrote the 1 ten first and then the 6 ones.

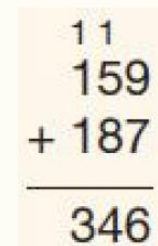
$$\begin{array}{r} {}^{1\,1} \\ 159 \\ +\ 187 \\ \hline 346 \end{array}$$

Ways in which New Groups Below is better than New Groups Above

Variations that keep the initial multidigit numbers unchanged because they are conceptually clearer:



New Groups Below does not change the original addends 159 or 187. Each addend and the total are in their own horizontal space.



For New Groups Above some children object to writing 1 above the top number because they say that you are changing the problem (and you are).

Show All Totals is also an accessible and mathematically desirable standard algorithm.

Show All Totals

	1	8	9
+	1	5	7
	2	0	0
	1	3	0
		1	6
	3	4	6

It has all 5 of the above criteria

and also two more:

It can be done from the left.

It does not alternate the adding of the places and the adding to make the total.

But it does get unwieldy for big numbers.

This is the most typical subtraction error. Many students make this error.

$$\begin{array}{r} 346 \\ -\,159 \\ \hline 213 \end{array}$$

What is this error?

Why do students make it?

What can we do about it?

What is this error? **Focusing on the vertical columns and subtracting the two numbers they see and not thinking of the whole multidigit numbers 346 and 159.**

$$\begin{array}{r} 346 \\ -\,159 \\ \hline 213 \end{array}$$

Why do students make it?

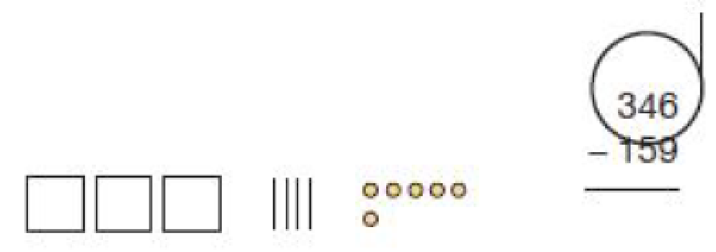
They may have problems with no ungrouping first so they get used to just looking at columns and subtracting.

What can we do about it?

$$\begin{array}{r} 346 \\ -\,159 \\ \hline 213 \end{array}$$

Here is a powerful approach to prevent this error.

Draw attention to the total in the subtraction situation by encircling it. We call this a magnifying glass because you are looking inside the total to find its parts.

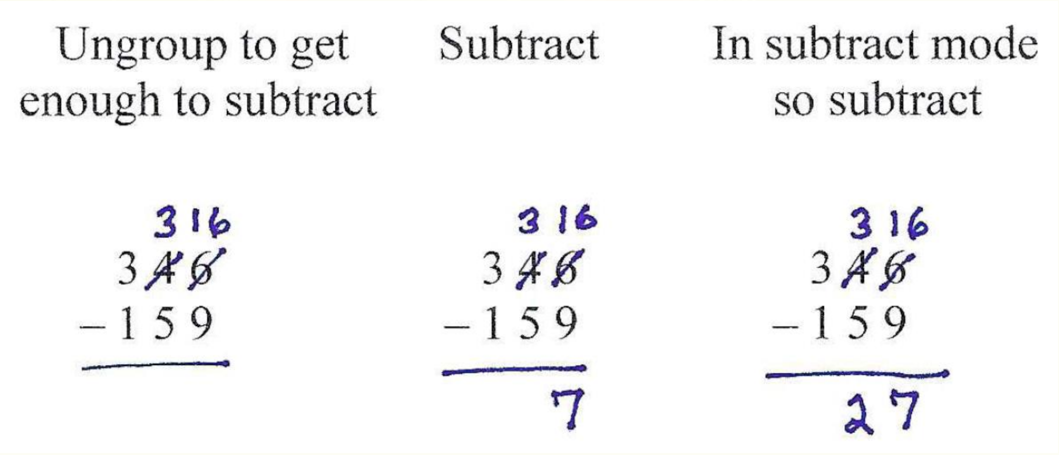


Do not draw or show the known addend 159 because it is part of 346.

The magnifying glass stops students from subtracting right away and reminds them to check each place to see if they need to get more in order to subtract in that place.

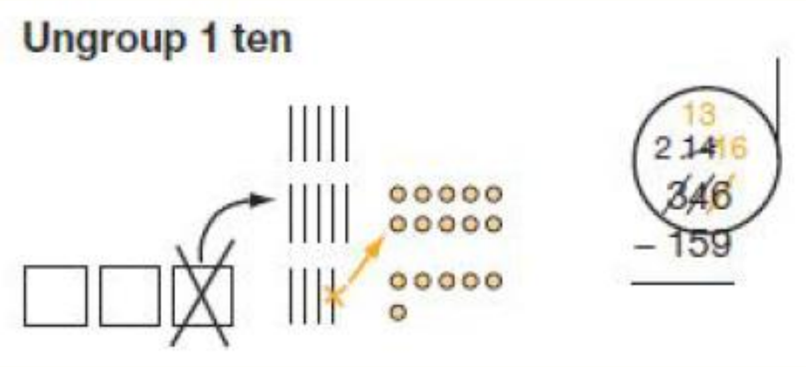
I drew three hundreds, four tens, and six ones to show three hundred forty six. I wrote one hundred,five tens, nine ones below three hundred forty six in my problem, but I did not draw it because it is already part of three hundred forty six. To subtract, we separate the total into two numbers, the number we are taking away and the number that is left. Here I drew my magnifying glass around the total to remind me to check if I need to ungroup to get more to subtract.

The Same Error Is Created by the Alternating Steps in the Common Standard Algorithm

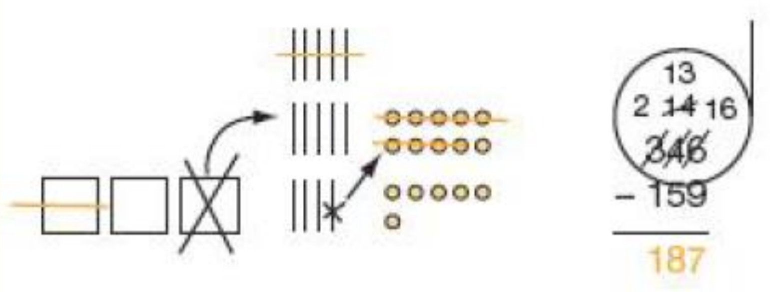


So the best multidigit subtraction method does not alternate ungrouping and subtracting. You do all necessary ungrouping first and then subtract in all places. Each of these processes can be done left to right or right to left.

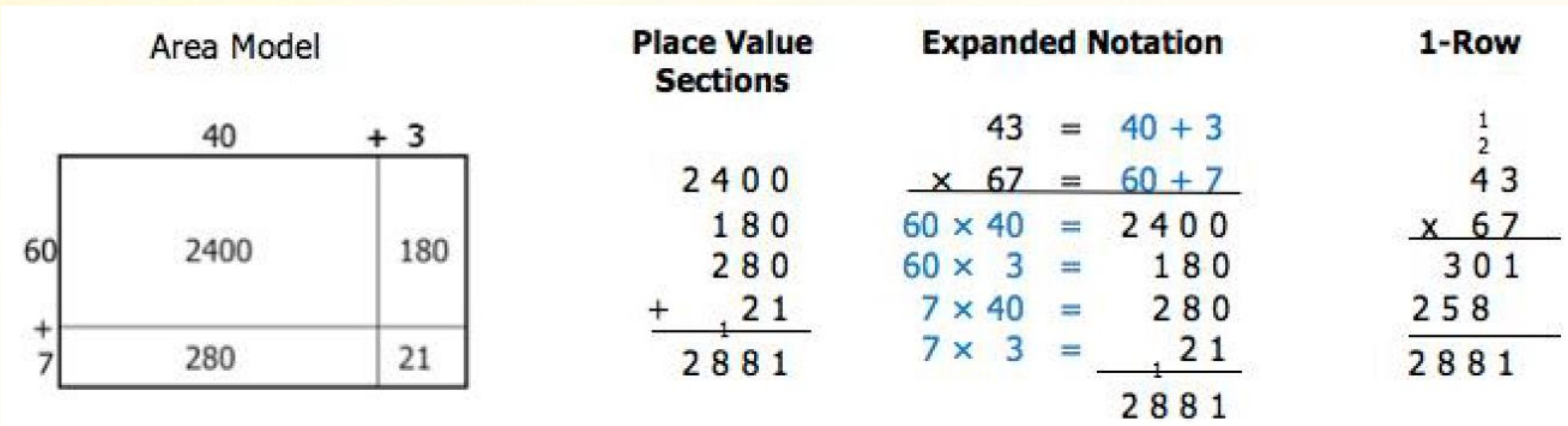
Step 2: Check to ungroup as needed, moving to the right to check the ones place.



Step 3: Subtract in each place moving from left to right or from right to left.

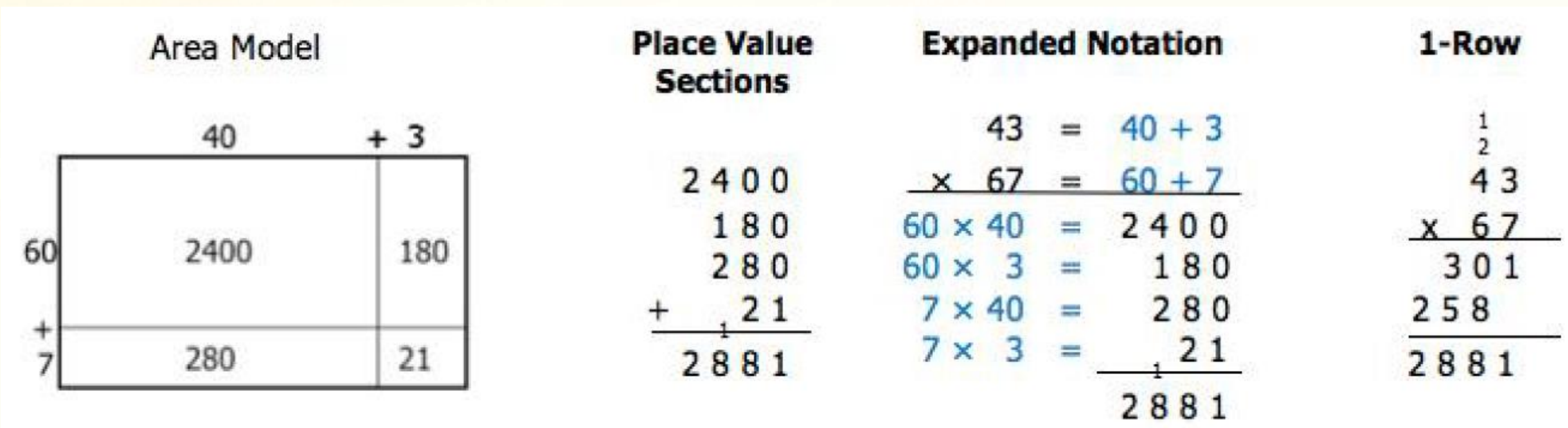


The 1-row method on the right is taken to be "the standard algorithm". But see how it uses wrong place-values for the step 60 x 3 = 180 but the 1 hundred is written above the tens place.



This method is also difficult because it alternates the multiplying step and the adding step. The other two standard algorithms shown here do all of the multiplying first and then all of the adding. This is much easier.

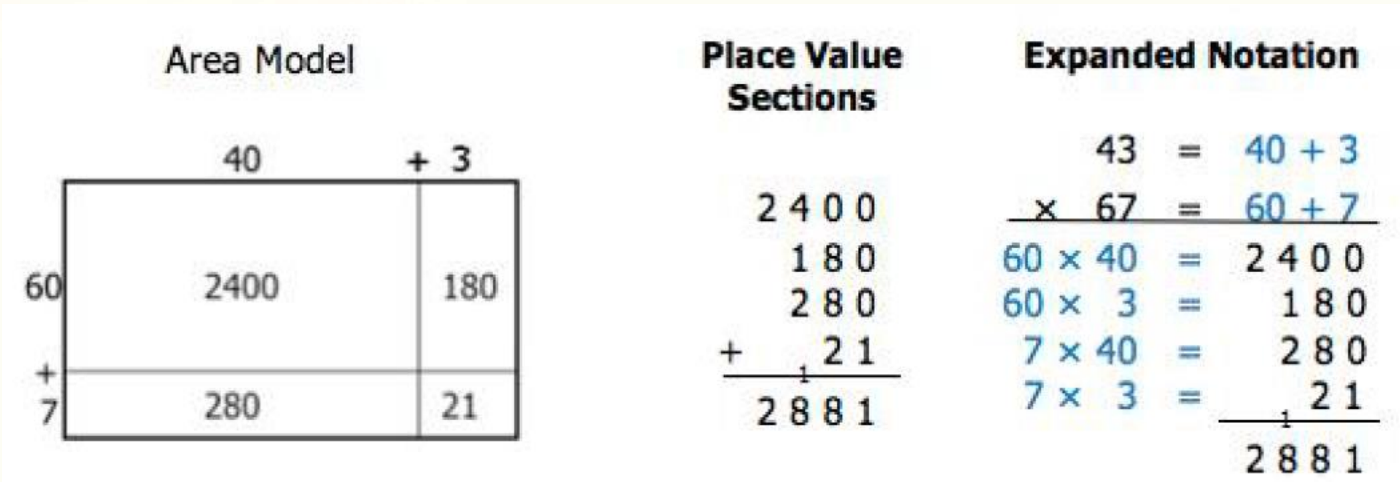
Notice how the area model is helpful for all methods to see what place in one factor gets multiplied by what place in the other factor.



Expanded Notation is the best multidigit multiplication standard algorithm.

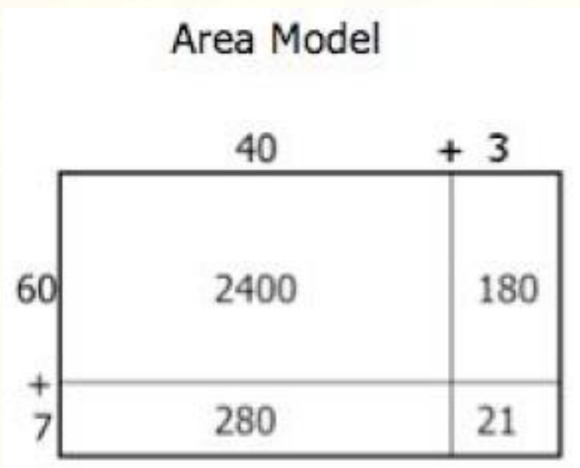
The steps in blue can be dropped whenever students are ready. Then the partial products can be written under the factors.

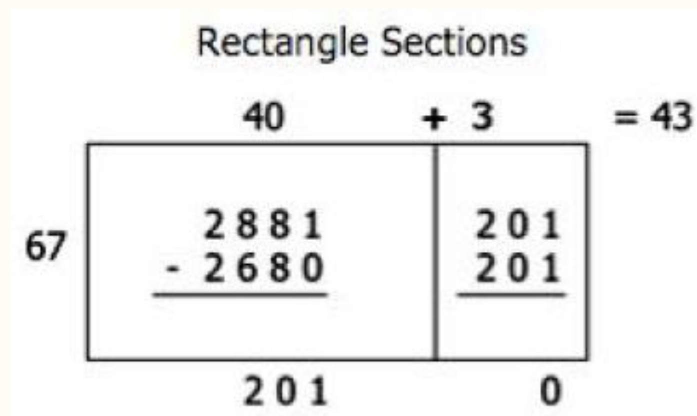
Some students cannot handle the complex lay-out of Expanded Notation, but they can make and understand an area model and add all of the partial products as shown in the Place Value Sections standard algorithm.



Both of these standard algorithms do all of the multiplying first and then all of the adding. This is much easier.

Relationships between multidigit multiplication and multidigit division are important. **The area model can be used for both operations with division seen as finding the place values 40 and 3 in the unknown factor along the top of the rectangle.**





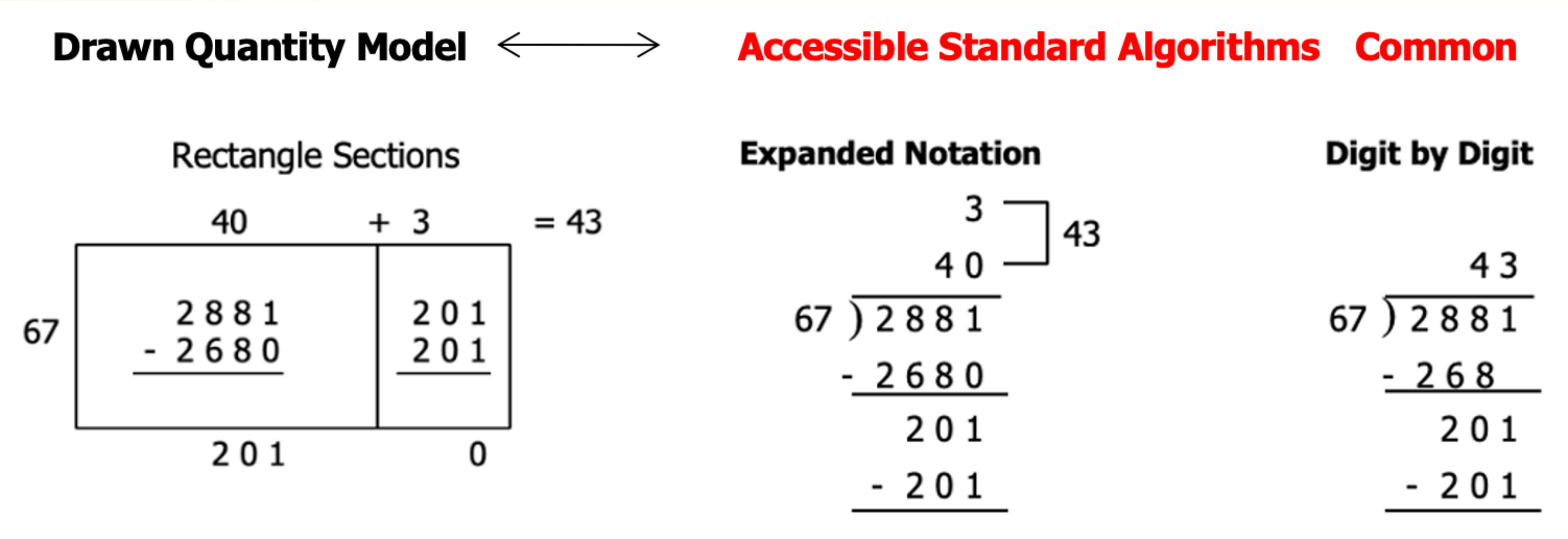
67 x 43 = 2881

67)2881 = 40 + 3 = 43

The two accessible and mathematically desirable **division standard algorithms** are:

Rectangle Sections finds each part of the unknown factor within the area model.

Expanded Notation puts the rectangle sections above each other to subtract successively.



The first two methods show the values of the places within the algorithms. Digit by Digit uses only single-digit values that can easily get placed in the wrong place and make an error.

Major steps in making computation meaningful:

Students make drawings to show place values and explain their drawings and algorithm.

Students relate drawings to numbers to make computations meaningful and do not use drawings just to find answers.

Students later do not use drawings and only use written methods but they can go from numbers to drawings sometimes to retain or recall meanings.

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